Double Pendulum - Equations of Motion

Miruna and Venus

February 2026

1. System Description

We consider a double pendulum consisting of:

- Masses: m_1, m_2
- Massless rod lengths: ℓ_1, ℓ_2
- Angles from vertical: $\theta_1(t)$, $\theta_2(t)$

2. Coordinates

Mass 1

$$x_1 = \ell_1 \sin\theta_1 \tag{1}$$
$$y_1 = -\ell_1 \cos\theta_1 \tag{2}$$

Mass 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \tag{3}$$
$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 \tag{4}$$

3. Velocities

Mass 1

$$\dot{x}_1 = \ell_1 \cos\theta_1 \, \dot{\theta}_1$$
$$\dot{y}_1 = \ell_1 \sin\theta_1 \, \dot{\theta}_1$$

The squared velocity becomes:

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2$$

$$\boxed{v_1^2 = \ell_1^2 \dot{\theta}_1^2} \tag{5}$$

Mass 2

$$\dot{x}_2 = \ell_1 \cos\theta_1 \dot{\theta}_1 + \ell_2 \cos\theta_2 \dot{\theta}_2$$
$$\dot{y}_2 = \ell_1 \sin\theta_1 \dot{\theta}_1 + \ell_2 \sin\theta_2 \dot{\theta}_2$$

The squared velocity becomes:

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$\boxed{v_2^2 = \ell_1^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)} \tag{6}$$

4. Kinetic Energy

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Substituting:

$$T = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(\ell_1^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

Rearranged:

$$\boxed{T = \frac{1}{2}(m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)} \tag{7}$$

5. Potential Energy

$$V = m_1 g y_1 + m_2 g y_2$$

Substituting:

$$\boxed{V = -(m_1 + m_2) g \ell_1 \cos\theta_1 - m_2 g \ell_2 \cos\theta_2} \tag{8}$$

6. The Lagrangian

$$L = T - V$$

$$\boxed{\begin{aligned} L = \frac{1}{2}(m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \\ + (m_1 + m_2) g \ell_1 \cos\theta_1 + m_2 g \ell_2 \cos\theta_2 \end{aligned}} \tag{9}$$

7. Euler–Lagrange Equations

For each coordinate:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0} \tag{10}$$

8. Equation for θ_1

After computing derivatives and simplifying:

$$(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin\theta_1 = 0$$

9. Equation for θ_2

$$\ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \ell_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0$$

10. Equations of Motion for the Double Pendulum

$$\boxed{\ddot{\theta}_1 = \frac{-g(2m_1 + m_2)\sin\theta_1 - m_2 g\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)\, m_2(\dot{\theta}_2^2\ell_2 + \dot{\theta}_1^2\ell_1\cos(\theta_1 - \theta_2))}{\ell_1\left(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))\right)}} \tag{11}$$

$$\boxed{\ddot{\theta}_2 = \frac{2\sin(\theta_1 - \theta_2)\left(\dot{\theta}_1^2\ell_1(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \dot{\theta}_2^2\ell_2 m_2\cos(\theta_1 - \theta_2)\right)}{\ell_2\left(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))\right)}} \tag{12}$$