Review of Probability Theory Part 1

Fundamentals of Reinforcement Learning

Institut für Nachrichtentechnik
Fachgebiet Kommunikationstechnik
Prof. Dr.-Ing. Anja Klein
Dr. Sabrina Klos & Dr. Andrea Ortiz

Lecture Overview

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Chapter 1
Introduction

Chapters 2-3
Probability Theory

Chapters 4-6
The Multi-Armed Bandit Problem

Decisions do not influence future data

With/without context

Chapters 7-12
The full RL Problem

Decisions may influence future data

With/without knowledge of dynamics

Chapter 13
Case Study

Chapter 14
Extensions

Literature

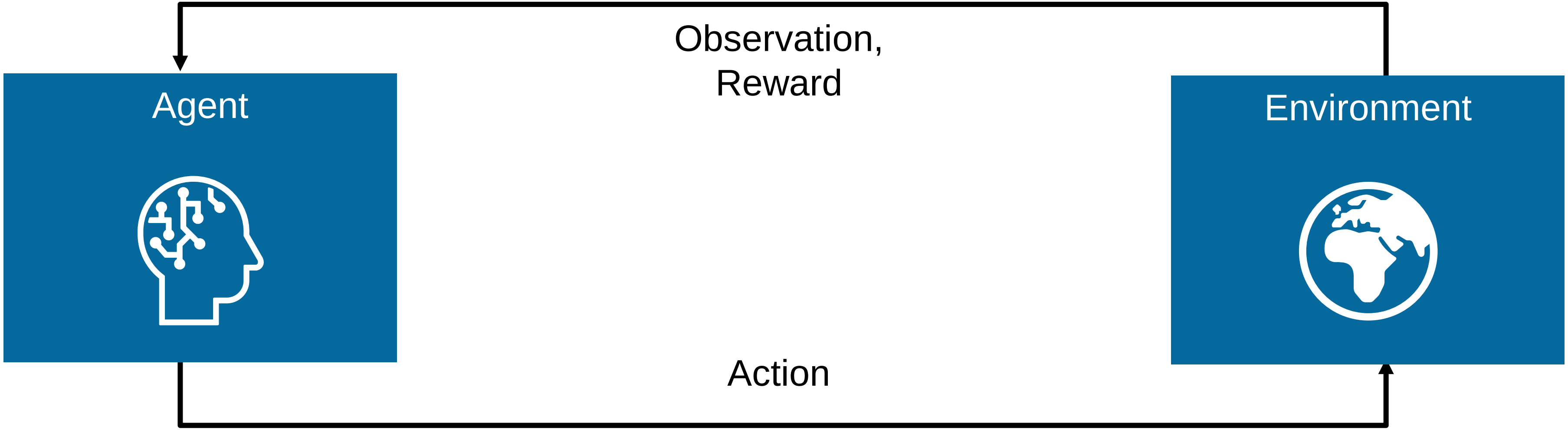
This review of probability is based on the following books

- Larry Wassermann, “All of Statistics – A Concise Course in Statistical Inference”, Springer Texts in Statistics, Springer, 2004.
- Scott L. Miller and Donald Childers, “Probability and Random Processes – With Applications to Signal Processing and Communications, Elsevier Academic Press, 2004.
- Judith Eckle-Kohler, Michael Kohler, “Eine Einführung in die Statistik und ihre Anwendungen”, 2. Auflage, Springer, 2011.

Recap: Idea of RL

RL deals with goal-directed learning from interaction

Agent–environment interaction



Motivation

RL needs probability theory due to environment‘s dynamics

Uncertainty in environment’s dynamics


https://www.hso.com

Manufacturing


https://media.istockphoto.com

Wireless Communications


https://de.wikipedia.org

Rover in Mars

Learning Goals

- You can apply the formulas of basic probability theory to compute probabilities and derive properties of events.
- You can model a suitable random variable for a given experiment and determine its distribution.
- You can determine characteristics of discrete random variables and relate important examples to their applications.

Outline

- Basic Probability
- Random Variables
- Discrete Random Variables

Outline

- **Basic Probability**
- Random Variables
- Discrete Random Variables

Experiment, Outcome, Sample Space, Event

The starting point are experiments and their outcomes

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Definition (Experiment, Outcome, Sample Space, Event)

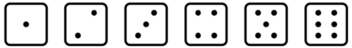
- **Experiment:** Procedure we perform that produces some result.
- **Outcome/Atomic event:** A possible result of an experiment.
  Outcomes are mutually exclusive.
- **Sample Space Ω:** The set of "all possible" distinct outcomes of an experiment.
- **Event:** A certain set of outcomes of an experiment.

Experiments, Sample Spaces and Events

Example: Basic concepts explained for rolling a dice

- **Experiment:** A fair dice is rolled once. 

- **Outcome:** “The result is a six.“  $A = \{6\}$

- **Sample Space:** All possible dice rolls.  $\Omega = \{1, 2, 3, 4, 5, 6\}$

- **Event:** “The result is an even number.“ 

$$B = \{2, 4, 6\}$$
$$= \{2\} \cup \{4\} \cup \{6\}$$

Picture source: www.flaticon.com

Frequentist Probability

Probability is a natural way to model uncertainty

We want to assign a number $\mathbb{P}(A)$ to each event with

- If A never occurs, then $\mathbb{P}(A) = 0$
- If A always occurs, then $\mathbb{P}(A) = 1.$

Intuitive understanding of probability: If a random experiment is performed a number of n times and the event A occurs n_A times, the probability $\mathbb{P}(A)$ is the limit of the relative frequency of the occurrence of event A, i.e.

$$\mathbb{P}(A) = \lim_{n \to \infty} \frac{n_A}{n} .$$

This is not a formal mathematical definition!

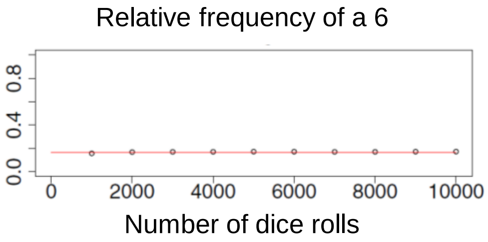
Frequentist Probability

Example: Probability of throwing a six when rolling a dice

- **Experiment:** A fair dice is rolled once.

- **Outcome:** “The result is a six.“  $A = \{6\}$

- **Frequentist Probability:**
  - Roll the dice a large number of n times.
  - Count the number n_A of times in which a six occurs.
  - The relative frequency of rolling a six approaches $\mathbb{P}(A) = \lim\limits_{n \to \infty} \frac{n_A}{n} = \frac{1}{6}$.



Picture sources: www.flaticon.com
Judith Eckle-Kohler, Michael Kohler, “Eine Einführung in die Statistik und ihre Anwendungen”, 2. Auflage, Springer, 2011, p. 69.

Frequentist Probability

Probability of event is volume of event compared to volume of sample space

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We can extend the intuitive understanding of probability to compute probabilities as follows:

If Ω is finite and all atomic events are equally likely, we have

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} .$$

- The probability of an event corresponds to: $\dfrac{\text{“volume” of the event}}{\text{“volume” of the sample space}}$

Question

What are the probabilities of the following events?

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- **Experiment:** A fair dice is rolled once.
- **Sample Space:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Events:**
  - “Even number“
  - “Prime number“
  - “Even prime number“

Answer

Compute probability of event by counting its number of outcomes

- **Experiment:** A fair dice is rolled once.
- **Sample Space:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Events:**
  - "Even number" $A = \{2, 4, 6\}$ $\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$
  - "Prime number" $B = \{2, 3, 5\}$ $\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$
  - "Even prime number" $A \cap B = \{2\}$ $\mathbb{P}(A \cap B) = \frac{1}{6}$

Event Space

The event space is the set of all possible events

Definition (Event Space)

The **event space** Σ is the set of all possible events.

For a collection of events, we want that

- Any intersection and union of events should also be events.
- The complementary events $A^C = \Omega \backslash A$ should be events.

Formally, this leads to the concept of a σ-algebra.

σ-Algebra

A σ-algebra is a set of events with certain characteristics

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Definition (σ –Algebra)

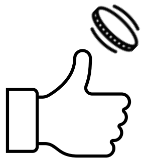
Let Ω be a set. A collection Σ of subsets of Ω is called a σ**-algebra** on Ω if

1. $\Omega \in \Sigma$
2. $A \in \Sigma \quad \Rightarrow \quad A^C \in \Sigma$
3. $A_1, A_2, \ldots \in \Sigma \quad \Rightarrow \quad \bigcup_n A_n \in \Sigma.$

The event space is a σ–algebra!

Question

What are the sample space and the event space of tossing a fair coin?



- **Experiment:** A fair coin is tossed once.
- **Sample Space:**
- **Event Space:**

Answer

List all outcomes for sample space and all events for event space

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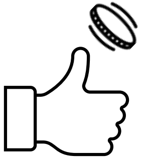

- **Experiment:** A fair coin is tossed once.
- **Sample Space:** $\Omega = \{H, T\}$ H: “Head“, T: “Tail“
- **Event Space:** Set of all possible events

$$\Sigma = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$$

empty set atomic events sample space

Probability Function

Mathematical probability is based on an axiomatic formulation

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The following axiomatic formulation defines mathematical probability:

Definition (Probability Function)

Let Ω be a sample space and Σ be a σ-algebra on Ω.
A **probability function** is a map $\mathbb{P} : \Sigma \to [0, 1]$ with

1. $\mathbb{P}(A) \geq 0$ for all $A \in \Sigma$
2. $\mathbb{P}(\Omega) = 1$
3. Any countable sequence of disjoint events $A_i \in \Sigma$ satisfies

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i)\,.$$

Probability Space

A probability space provides a formal model of an experiment

Together with a probability function, a sample space and a σ-algebra become a probability space, which provides a formal model of an experiment.

Definition (Probability Space)

A **probability space** is a triple $(\Omega, \Sigma, \mathbb{P})$ where

1. Ω is the set of all possible outcomes (sample space)
2. Σ is a σ-algebra on Ω
3. $\mathbb{P}$ is a probability function on Σ.

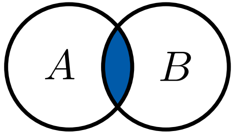
In the remainder of this lecture:
Let $(\Omega, \Sigma, \mathbb{P})$ a probability space.

Joint Probability

We can use the axioms to compute the probability of an intersection of events

Definition (Joint Probability)

We call $\mathbb{P}(A \cap B)$ the **joint probability** of two events A and B.



Using the axioms, we can derive the following formula for the joint probability:

Lemma

For two events $A, B \in \Sigma$, the joint probability is given by

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)\,.$$

Joint Probability

Example: Verify formula for joint probability in dice rolling example

- **Experiment:** A fair dice is rolled once.
- **Sample Space:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Events:**
  - “Even number“ $A = \{2, 4, 6\}$ $\mathbb{P}(A) = \frac{1}{2}$
  - “Prime number“ $B = \{2, 3, 5\}$ $\mathbb{P}(B) = \frac{1}{2}$
  - “Even prime number“ $A \cap B = \{2\}$ $\mathbb{P}(A \cap B) = \frac{1}{6}$
  - “Even or prime number“ $A \cup B = \{2, 3, 4, 5, 6\}$ $\mathbb{P}(A \cup B) = \frac{5}{6}$

→ The results are related via $\mathbb{P}(A \cap B) = \frac{1}{6} = \frac{1}{2} + \frac{1}{2} - \frac{5}{6} = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$.

Conditional Probability

Observing one event can change the probability of occurrence of another event

Definition (Conditional Probability)

For $\mathbb{P}(B) > 0$, the probability of event A given that event B has occurred is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

- The equation above is often stated in a form known as the **product rule of probability**:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A).$$

- The conditional probability $\mathbb{P}(\cdot \mid B)$ satisfies the axioms of probability and can thus be seen as a **probability function on the reduced sample space** B.

Conditional Probability

Example: Compute conditional probability in dice rolling example

- **Experiment:** A fair dice is rolled once.
- **Sample Space:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **What is the probability to get an even number given that the result is a prime number?**

  - "Prime number" $B = \{2, 3, 5\}$ $\mathbb{P}(B) = \frac{1}{2}$
  - "Even prime number" $A \cap B = \{2\}$ $\mathbb{P}(A \cap B) = \frac{1}{6}$

→ Apply formula for conditional probability $\Rightarrow$ $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{1/2} = \frac{1}{3}$

Independence

Observing an event not always changes the probab. of occurrence of another event

Definition (Independent Events)

Two events A and B are called independent events if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)\,.$$

- As a consequence, $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ and $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

Independence

Example: Verify independence of two events in dice rolling example

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- **Experiment:** A fair dice is rolled once.
- **Sample Space:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Events:**
  - “Even number“ $A = \{2, 4, 6\}$ $\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$
  - “Smaller or equal 4“ $B = \{1, 2, 3, 4\}$ $\mathbb{P}(B) = \frac{4}{6} = \frac{2}{3}$
  - “Even number smaller or equal 4“ $A \cap B = \{2, 4\}$ $\mathbb{P}(A \cap B) = \frac{2}{6} = \frac{1}{3}$

→ The results above are related via $\mathbb{P}(A \cap B) = \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A) \cdot \mathbb{P}(B).$

→ Events $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ are independent.

Law of total probability

This result is also known as the sum rule of probability

Definition (Partition)

A **partition** $\{A_i : i = 1, 2, \ldots\}$ of a set Ω is a non-empty collection of pairwise disjoint subsets $A_i \subset \Omega$ such that $\bigcup_i A_i = \Omega$.

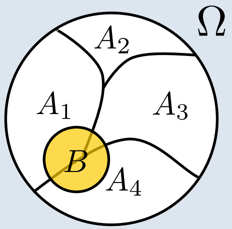
Proposition (Law of Total Probability)

For a partition $\{A_i : i = 1, 2, \ldots\} \subseteq \Sigma$ of Ω and an arbitrary event $B \in \Sigma$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \cap A_i)$$

or equivalently using the product rule

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)\,.$$



Bayes Theorem

This result is useful for computing conditional probabilities

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Theorem (Bayes Theorem)

- For two events $A, B \in \Sigma$ with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, the conditional probabilities are related via
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)} \,.$$
- For an event $B \in \Sigma$ and a partition $\{A_i : i = 1, 2, \ldots\} \subseteq \Sigma$ of Ω, one gets
$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B \mid A_j)\mathbb{P}(A_j)} \,.$$

- We call $\mathbb{P}(A_i)$ the *prior probability* of A_i and $\mathbb{P}(A_i \mid B)$ the *posterior probability* of A_i.
- **Proof** for two events: Using the definition of conditional probability and the product rule yields
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A)P(A)}{\mathbb{P}(B)} \,.$$

Question

What is the probability of an email with the word „free“ to be spam?

- An email is categorized as S = “Spam“ or L = “Low Priority“ or H = “High Priority“ with prior probabilities

$$\left.\begin{aligned}\mathbb{P}(S) &= 0.7\\ \mathbb{P}(L) &= 0.2\\ \mathbb{P}(H) &= 0.1.\end{aligned}\right\} \qquad \mathbb{P}(S) + \mathbb{P}(L) + \mathbb{P}(H) = 1$$

- Let B denote the event that an email contains the word “free“. The conditional probabilities of event B given the email category are

$$\begin{aligned}\mathbb{P}(B \mid S) &= 0.9\\ \mathbb{P}(B \mid L) &= 0.01\\ \mathbb{P}(B \mid H) &= 0.01.\end{aligned}$$

When receiving an email with the word “free“, what is the probability that it is spam?

Answer

Apply Bayes Theorem

$$\mathbb{P}(S \mid B) = \frac{\mathbb{P}(B \mid S)\mathbb{P}(S)}{\mathbb{P}(B \mid S)\mathbb{P}(S) + \mathbb{P}(B \mid L)\mathbb{P}(L) + \mathbb{P}(B \mid H)\mathbb{P}(H)}$$

$$= \frac{0.9 \cdot 0.7}{0.9 \cdot 0.7 + 0.01 \cdot 0.2 + 0.01 \cdot 0.1}$$

$$= 0.995$$

→ When receiving an email with the word “free“, it is spam with a probability of 99.5%.

Outline

- Basic Probability
- **Random Variables**
- Discrete Random Variables

Random Variables (RVs)

The concept of random variables links sample spaces and events to data

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Definition (Random Variable)

A **random variable (RV)** is a function $X : \Omega \to \mathcal{X}$ that assigns an element of $\mathcal{X}$ to each $\omega \in \Omega$.

- **Notation:**
  - The random variable X takes a particular value x: $X(\omega) = x$.

    a random variable

    a particular realization of the random variable
  - Set of outcomes $\omega \in \Omega$ for which X takes values less than or equal to x :

$$\{X \leq x\} = \{\omega : X(\omega) \leq x\}$$

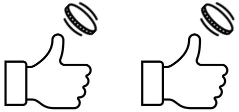
- Typically, we consider
  - **Discrete random variables** with $\mathcal{X} = \mathbb{N}$ or $\mathbb{Z}$
  - **Continuous random variables** with $\mathcal{X} = \mathbb{R}$.

Will be formally defined later.

Random Variables

Example: Number of heads in two coin tosses

- **Experiment:** A fair coin is tossed two times.



2 fair coin tosses

- **Sample Space:** $$\Omega = \underbrace{\{H, T\} \times \{H, T\}}_{\text{2 coin tosses}}$$

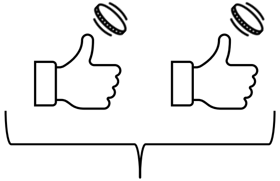
- **Random Variable:** Let $X(\omega)$ be the number of “heads“ in the sequence $\omega \in \Omega$.

  For example, if $\omega = HH$, then $X(\omega) = 2$.

Question

How is the RV “number of heads in two fair coin tosses“ distributed?

- **Random Variable:** Let $X(\omega)$ be the number of “heads“ in the sequence $\omega \in \Omega$ of two fair coin tosses, where $\Omega = \{H, T\} \times \{H, T\}$.



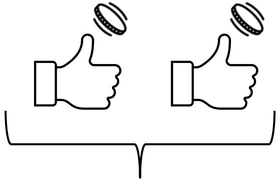
2 fair coin tosses

  - **Which values can the RV take?**
  - **What are the probabilities for each of these values?**

Answer

Map all outcomes to the corresponding value of the RV

- **Random Variable:** Let $X(\omega)$ be the number of “heads“ in the sequence $\omega \in \Omega$ of two fair coin tosses, where $\Omega = \{H, T\} \times \{H, T\}$.



2 fair coin tosses

→ The RV can take the values 0, 1, 2 and the probabilities are

$$\mathbb{P}(X = 0) = \mathbb{P}(TT) = \frac{1}{4}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(HT, TH) = \frac{1}{2}$$

$$\mathbb{P}(X = 2) = \mathbb{P}(HH) = \frac{1}{4}\,.$$

Distribution Functions

Distribution functions characterize the distribution of random variables

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Definition ((Cumulative) Distribution Function)

Let $\mathcal{X} = \mathbb{Z}$ or $\mathcal{X} = \mathbb{R}$. The function $F_X(x) := \mathbb{P}(X \leq x)$ is called the **cumulative distribution function (CDF)** of the random variable $X : \Omega \to \mathcal{X}$.

Proposition (Properties of the CDF)

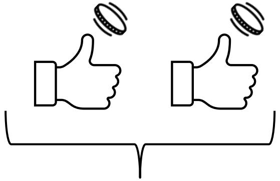
- $0 \leq F_X(x) \leq 1$ with $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.
- $F_X(x)$ is continuous from the right, i.e., $\lim_{\epsilon \to 0} F_X(x + \epsilon) = F_X(x)$.
- $F_X(x)$ is non-decreasing, i.e., $F_X(x_1) \leq F_X(x_2) \quad \forall x_1 < x_2$.
- For an interval $[a, b]$, we have
$$\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a).$$

- **Notation:** We write $X \sim F$ if X follows a particular distribution F.

Distribution Functions

Example: CDF of RV “Number of heads in two fair coin tosses“

- **Random Variable:** Let $X(\omega)$ be the number of “heads“ in the sequence $\omega \in \Omega$ of two fair coin tosses, where $\Omega = \{H,T\} \times \{H,T\}$.



2 fair coin tosses

We already found:

$$\mathbb{P}(X=0) = \mathbb{P}(TT) = \frac{1}{4}$$

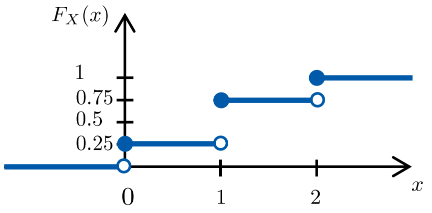
$$\mathbb{P}(X=1) = \mathbb{P}(HT,TH) = \frac{1}{2}$$

$$\mathbb{P}(X=2) = \mathbb{P}(HH) = \frac{1}{4}$$

$\Rightarrow$

The CDF $F_X(x) := \mathbb{P}(X \leq x)$ is given by:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



Outline

- Basic Probability
- Random Variables
- **Discrete Random Variables**

Discrete RVs and Probability Mass Functions
RVs with countably many values

We often consider RVs with values in $\mathcal{X} = \mathbb{Z}$.

Definition (Discrete Random Variable)

A random variable X is **discrete** if it takes only countably many values $\{x_1, x_2, ...\}$.

Definition (Probability Mass Function)

For a discrete random variable X, we define the **probability mass function (PMF)** of X by

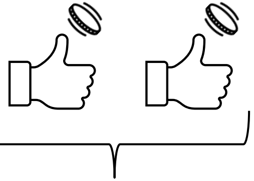
$$f_X(x) := \mathbb{P}(X = x).$$

- Thus, $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and $\sum_i f_X(x_i) = 1$.
- The CDF of X is related to f_X by $F_X(x) = \mathbb{P}(X \leq x) = \sum_{x_i \leq x} f_X(x_i)$.

Discrete RVs and Probability Mass Functions

Example: PMF of RV “Number of heads in two coin tosses“

- **Random Variable:** Let $X(\omega)$ be the number of “heads“ in the sequence $\omega \in \Omega$ of two fair coin tosses, where $\Omega = \{H,T\} \times \{H,T\}$.



2 fair coin tosses

We already found:

$$\mathbb{P}(X=0) = \mathbb{P}(TT) = \frac{1}{4}$$

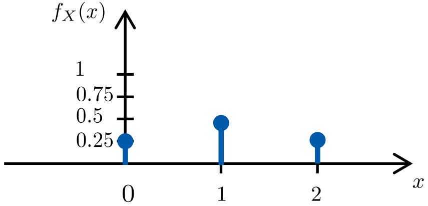
$$\mathbb{P}(X=1) = \mathbb{P}(HT,TH) = \frac{1}{2}$$

$$\mathbb{P}(X=2) = \mathbb{P}(HH) = \frac{1}{4}$$

$\Rightarrow$

The PMF is given by:

$$f_X(x) = \begin{cases} \frac{1}{4} & x=0 \\ \frac{1}{2} & x=1 \\ \frac{1}{4} & x=2 \\ 0 & x \notin \{0,1,2\} \end{cases}$$



Important Discrete Random Variables

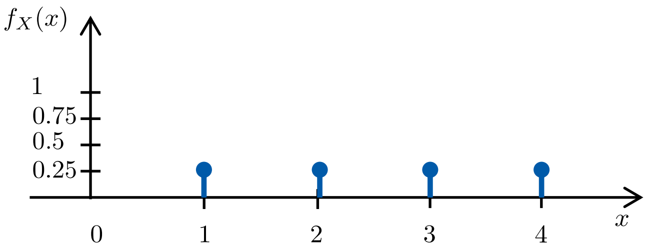
Example: Discrete Uniform Distribution

Definition (Discrete Uniform Distribution)

Let $k > 1$ be a given integer. Suppose that X has PMF given by

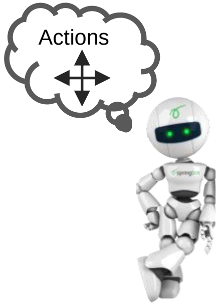
$$f_X(x) = \begin{cases} 1/k & \text{for } x = 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

We say that X has a **uniform distribution** on $\{1, \ldots, k\}$.



uniform distribution on $\{1, 2, 3, 4\}$

- **Application:** Can be used to model a stochastic policy of a robot taking a step in N, S, E, W with probability 0.25.



Important Discrete Random Variables

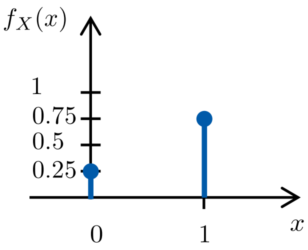
Example: Bernoulli Distribution

Definition (Bernoulli Distribution)

Let X be a binary random variable, i.e., $\mathcal{X} = \{0, 1\}$ with $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$ for a parameter $p \in [0, 1]$. Then, the PMF can be written as

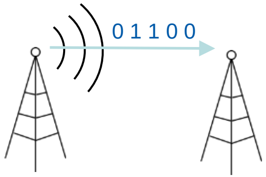
$$f_X(x) = p^x (1 - p)^{1-x}, \qquad x \in \{0, 1\},$$

and we write $X \sim \mathrm{Bernoulli}(p)$.



Bernoulli distribution with $p = 0.75$

- **Application:** Can be used to model whether a bit in a digital communication system was received correctly.



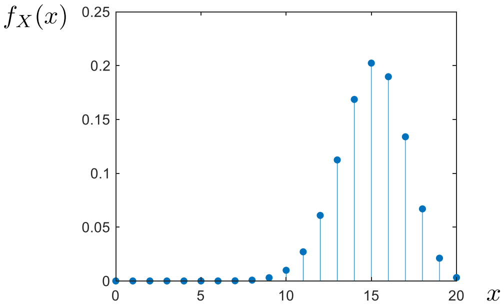
Example: Binomial Distribution

Definition (Binomial Distribution)

Let $n > 1$ be a given integer and parameter $p \in [0,1]$. Suppose that X has PMF given by

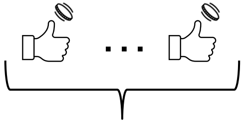
$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

We say that X has a **binomial distribution** and we write $X \sim \mathrm{Binomial}(n,p)$.



Binomial distribution with $n = 20, p = 0.75$

- **Application:** Number of heads in n independent coin tosses with probability $p \in [0,1]$ of heads follows $\mathrm{Binomial}(n,p)$.



n coin tosses

Important Discrete Random Variables

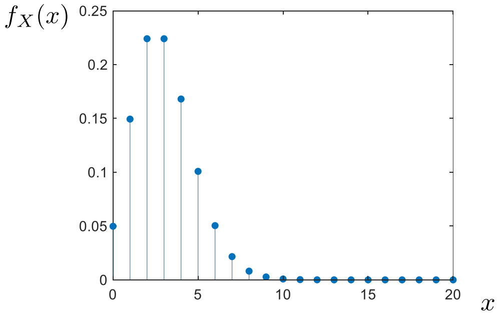
Example: Poisson Distribution

Definition (Poisson Distribution)

Let $\mathcal{X} = \mathbb{N}$ and X be a random variable with PMF

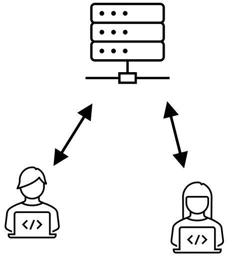
$$f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x \geq 0\,,$$

where $\lambda \in [0, \infty)$. We say that X has a **Poisson distribution** with parameter λ and we write $X \sim \mathrm{Poisson}(\lambda)$.



Poisson distribution with $\lambda = 3$

- **Application:** Can be used to model event counts, e.g. the number of accesses to a server.



$n = 2$

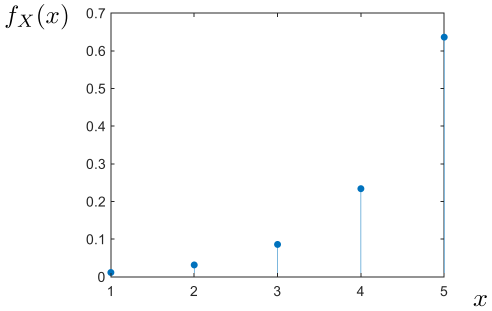
Important Discrete Random Variables
Example: Soft-Max Distribution

Definition (Soft-Max Distribution)

Let $k > 1$ be a given integer and let $z_1, \ldots, z_k$ be k real numbers. Suppose that X has a PMF given by

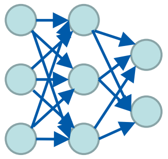
$$f_X(x) = \begin{cases} \dfrac{\exp z_i}{\sum_{j=1}^{k} \exp z_j} & \text{for } x = z_i,\ i = 1, \ldots, k \\ 0 & \text{otherwise.} \end{cases}$$

We say that X has a **Soft-Max (or Gibbs or Boltzman) distribution** on $\{z_1, \ldots, z_k\}$.



Soft-Max distribution on $\{1, 2, 3, 4, 5\}$

- **Application:** Is used as activation function in neural networks and for preference-based action selection in some RL algorithms.



$\longrightarrow$ Chapter 11

Learning Goals

- You can apply the formulas of basic probability theory to compute probabilities and derive properties of events.
  - → Joint Probability; Conditional Probability; Independence; Law of Total Pr; Bayes Theorem.
- You can model a suitable random variable for a given experiment and determine its distribution.
  - → RV= Maps outcomes of experiment to real numbers; Cumulative Distribution Functions.
- You can determine characteristics of discrete random variables and relate important examples to their applications.
  - → Probability Mass Function; important discrete RVs discussed in lecture and exercise.

Lecture Overview

Next week: Continuous RVs, Multiple RVs, Operations on RVs, Statistics

Chapter 1 — **Introduction**

Chapters 2-3 — **Probability Theory**

Chapters 4-6 — **The Multi-Armed Bandit Problem**

Decisions do not influence future data

With/without context

Chapters 7-12 — **The full RL Problem**

Decisions may influence future data

With/without knowledge of dynamics

Chapter 13 — **Case Study**

Chapter 14 — **Extensions**