Digital Signal Processing
Tutorial 4

Prof. Dr.-Ing. A. Zoubir
Signal Processing Group

TECHNISCHE UNIVERSITÄT DARMSTADT

Winter Term 2024/2025

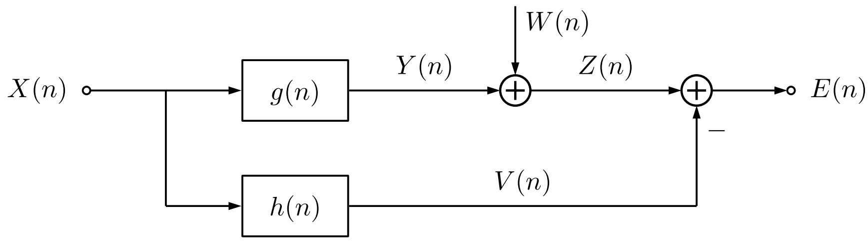
Task 1: Second Order Moment Function and Power Spectral Density

a) The second order moment function (SOMF) $r_{XX}(k)$ of a real process $X(n)$ is given in terms of constants $a, b \geq 0$:

$$r_{XX}(k) = \begin{cases} a & \text{for } k = 0, \\ b & \text{for } |k| = 1, \\ 0 & \text{elsewhere,} \end{cases}$$

Find the relationship between a and b so that $r_{XX}(k)$ is a correct SOMF. Recall that a correct SOMF has to fulfill several requirements, e.g. it has to correspond to a non-negative power spectral density (PSD), $S_{XX}(e^{j\omega}) \geq 0$.

b) The error process $E(n)$ is as given in the figure below. Determine $E(n)$ and its z-transform $E(z)$.



c) The SOMF of the input process $X(n)$ is as described in part (a) with $a = 3$ and $b = 1$. Furthermore, the impulse response $g(n)$ is given by

$$g(n) = \begin{cases} 3 & \text{for } n = 0, \\ 2 & \text{for } n = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Determine the cross-SOMF $r_{YX}(k)$.

(ii) Determine the SOMF $r_{YY}(k)$.

(iii) Calculate the PSD $S_{XX}(e^{j\omega})$ and $S_{YY}(e^{j\omega})$.

(iv) Calculate the cross-PSD $S_{XY}(e^{j\omega})$.

d) Find the SOMF $r_{EE}(k)$. Assume the process $W(n)$ to be white noise with zero mean and variance σ^2, and uncorrelated with the input $X(n)$. Provide the result in terms of $r_{XX}(k)$, $g(n)$, $h(n)$ and σ^2.

Task 2: Mean and Variance of the Periodogram

We consider a stationary random process $X(n)$ with zero mean and an unknown covariance function $c_{XX}(k)$. Assume we have N samples of $X(n)$ for $n = 0, \ldots, N-1$ available and we want to use the periodogram to estimate the spectrum $C_{XX}(e^{j\omega})$.

a) Provide the definition of the periodogram $I_{XX}(e^{j\omega})$. What are its advantages?

In the following we want to assess the statistical performance of the periodogram in terms of its mean, variance and mean square error (MSE).

b) Determine the mean of the periodogram. What happens if the sample size is very large?

c) Use the manuscript to obtain the formula of the variance of the periodogram. Determine the approximate variance for $\omega \neq 0$ if the sample size is very large?

d) Determine the MSE of the periodogram for $N \to \infty$. Is the periodgram a consistent estimator?

Task 3: Averaging Periodograms

In Problem 2, we have pointed out that the periodogram is not a consistent estimator. A strategy to reduce the variance of spectrum estimators when N is large, is to average periodograms of segments.

a) Explain the principle of averaging periodograms.

b) We want to estimate the spectrum of process $X(n)$ which is observed for $n = 0, \ldots, 2047$. Construct an asymptotically unbiased Welch estimate that shows a spectral resolution of 0.1π, using an overlap of 50% and a Bartlett window. Determine the block size M and the normalization factor A of the estimate. The following identity may be useful.
$$\sum_{i=0}^{I} i^2 = \frac{I(I+1)(2I+1)}{6}$$

c) Give an expression for the variance of this estimate, assuming that the above given sample size is considered to be very large.

Task 4: Data Examples

Consider the stochastic process

$$X(n) = \cos(\omega_1 n + \phi_1) + \cos(\omega_2 n + \phi_2) + V(n)$$

where ω_1 and ω_2 are fixed frequencies, ϕ_1 and ϕ_2 are uniformly distributed phase variables on $[-\pi, \pi)$ and $V(n)$ is a white noise process with zero mean and variance σ^2. Assume $V(n)$, ϕ_1 and ϕ_2 to be mutually uncorrelated.

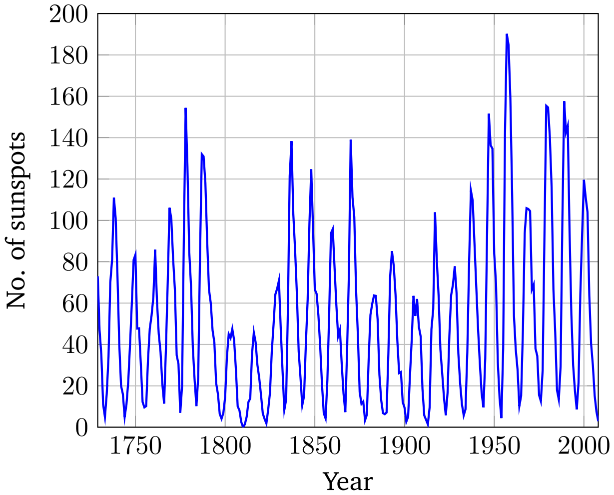
a) Determine μ_X, $c_{XX}(k)$ and the true spectrum $C_{XX}(e^{j\omega})$.

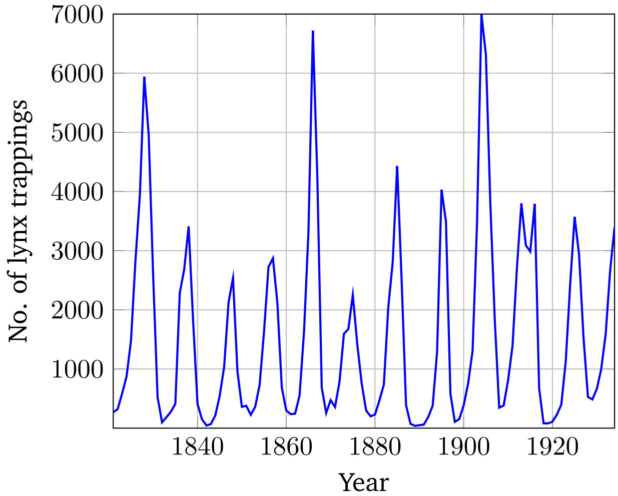
b) Ignore the random variables ϕ_1 and ϕ_2, namely we have

$$X(n) = \cos(\omega_1 n) + \cos(\omega_2 n) + V(n).$$

Use MATLAB to generate a realization of $X(n)$ for $n = 0, \ldots, N-1$ with $\omega_1 = 0.24\pi$ and $\omega_2 = 0.4\pi$. Use noise variances $\sigma^2 = 0$ and $\sigma^2 = 1$ and sample sizes $N = 16$ and $N = 64$. Calculate the periodogram of the generated realizations and compare your results with the true spectrum.

Two important examples for time series analysis are the annual sunspot number and lynx trappings. Sunspots are phenomena on the surface of the sun that appear as dark spots which is due to a lower temperature than the remaining surface. The sunspot activity can be measured using solar telescopes, a reference number is determined annually. Furthermore, the number of lynx trappings has been measured near the Mackenzie River in Canada in order to monitor the local lynx population. Both data sequences are shown below and are provided through moodle.





c) Use MATLAB to calculate the periodogram of both data sequences after possible preprocessing. Try to answer the following questions:

   (i) Why is it useful to remove the mean?

   (ii) What is the sample rate and the corresponding frequency unit?

   (iii) Are there sinusoidal components (periodic structure) in the data? If so, how many components are there and at what frequencies do they occur?