TECHNISCHE UNIVERSITÄT DARMSTADT

DSP

Prof. Dr.-Ing. Abdelhak M. Zoubir

SIGNAL PROCESSING GROUP

Copyright

Literature (1/2)

TECHNISCHE UNIVERSITÄT DARMSTADT

Filter Design:

- Zoubir, A. M. *Manuscript: Digital Signal Processing*, 2020
- Oppenheim, A. V. and Schafer, R. W. *Discrete-Time Signal Processing*, Prentice-Hall, 3rd ed, 2009.
- Proakis, J. G. and Manolakis, D. G. *Digital Signal Processing Principles, Algorithms, and Applications*, Maxwell MacMillan Editions, 1992.
- Mitra, S. K. *Digital Signal Processing: A Computer-Based Approach*, Prentice-Hall, 1998.
- Porat, B. *A Course on Digital Signal Processing*, J. Wiley & Sons Inc., 1997
- Diniz, P. , da Silva, E. and Netto, S. *Digital Signal Processing, System Analysis and Design*, Cambridge University Press, 2nd ed, 2010.

Literature (2/2)

TECHNISCHE UNIVERSITÄT DARMSTADT

Stochastic Processes and Spectrum Estimation:

- Kammeyer, K. D. and Kroschel, K. *Digitale Signalverarbeitung*, Teubner Studienbücher, 1998
- Böhme, J. F. *Stochastische Signale*, Teubner Studienbücher, 1998
- Hänsler, E. *Statistische Signale*, Springer Verlag, 3. Auflage, 2001

For more detailed study, see:

- Brillinger, D. R. *Time Series - Data Analysis and Theory*, Holden-Day Inc., 1981
- Priestley, M. B. *Spectral Analysis and Time Series*, Academic Press, 2001
- Welch G., Bishop, G. *An introduction to the Kalman Filter*, SIGGRAPH 2001.

Course structure (1/2)

TECHNISCHE UNIVERSITÄT DARMSTADT

- Discrete-Time Signals and Systems
- Digital Signal Processing of Continuous-Time Signals
- Filter Design
- Finite Impulse Response Filter Design
- Design of Infinite Impulse Response Filters

Course structure (2/2)

TECHNISCHE UNIVERSITÄT DARMSTADT

- Introduction to Digital Spectral Analysis
- An introduction to Random Variables and Stochastic Processes
- The Finite Fourier Transform
- Estimation of the Spectrum
- Non-Parametric Spectrum Estimation
- Parametric Spectrum Estimation
- Advances in Digital Signal Processing

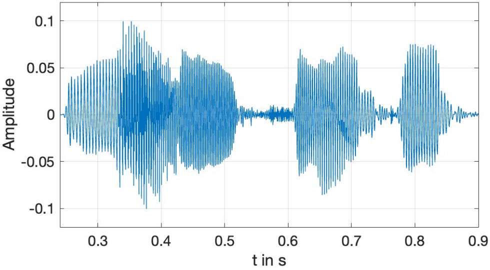
Outline

1. **Discrete-Time Signals and Systems**
   - Signals
   - Systems
   - Convolution
   - Stability and Causality
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Signals

Most signals in practice can be classified into three broad groups:

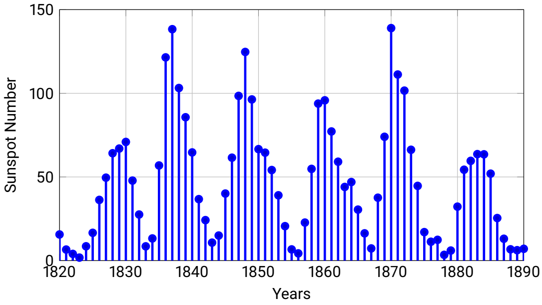
1. Analog or continuous-time
   - Signals are continuous in both time and amplitude.
   - Examples: speech (left), images (right), seismic, radar signals, etc.



Signals

2. Discrete-time

- Signals are discrete in time and continuous in amplitude.
- Examples: Average yearly sunspot numbers, lynx trapping, stock market data, etc.



Signals

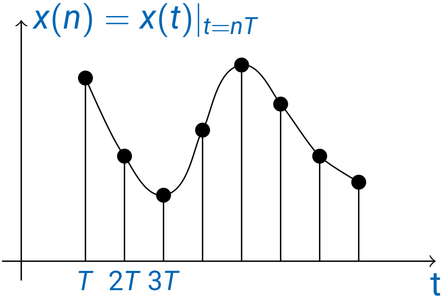
TECHNISCHE UNIVERSITÄT DARMSTADT

3. Digital Signals
   - Signals are discrete in both time and amplitude.
   - May be generated through amplitude quantization of discrete-time signals.
   - Signals used by computers are digital, i.e. discrete in both time and amplitude.

- Digital signal processing of quantized signals may be difficult and tedious.
- For this reason, digital signal processing concepts are based on discrete-time signals and continuous-time signals, where the amplitude is continuous.

Signals

Example: $x(n)$ is defined for all integer values of n. If n is not integer then $x(n)$ is undefined. $x(n)$ refers to the discrete-time function of the value of the function x, at a specific n.



Example of a sampled continuous-time signal with sampling interval T

Signals

TECHNISCHE UNIVERSITÄT DARMSTADT

The unit sample sequence

$$\delta(n) = \begin{cases} 1, & n{=}0, \\ 0, & \text{otherwise.} \end{cases}$$

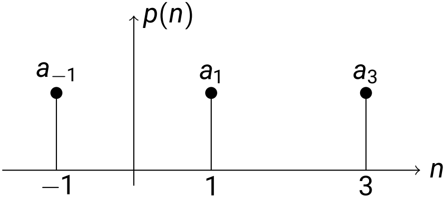
- Also known as Kronecker's delta.
- May be regarded as an analogon to the impulse function (Delta function) used in continuous-time system analysis.
- Any sequence can be represented as a linear combination of delayed impulses. In general,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\,.$$

Signals

TECHNISCHE UNIVERSITÄT DARMSTADT

Example: $p(n) = a_{-1}\delta(n+1) + a_1\delta(n-1) + a_3\delta(n-3)$



Signals

The unit step sequence

$$u(n) = \begin{cases} 1, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Related to $\delta(n)$ by

$$u(n) = \sum_{k=-\infty}^{n} \delta(k) = \sum_{k=0}^{\infty} \delta(n-k)\,,$$

and

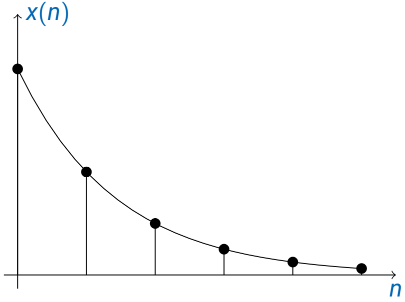
$$\delta(n) = u(n) - u(n-1) = \sum_{k=-\infty}^{n} \delta(k) - \sum_{k=-\infty}^{n-1} \delta(k)\,.$$

TECHNISCHE UNIVERSITÄT DARMSTADT

Exponential sequences

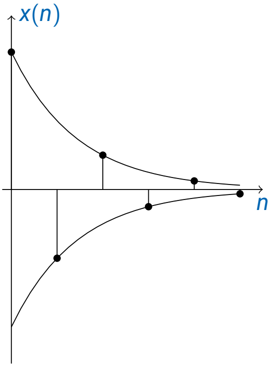
$$x(n) = A \cdot \alpha^n \,, \qquad \alpha, A \in \mathbb{C} \,.$$

Example 1: $A, \alpha \in \mathbb{R}$, $A > 0$ and $0 < \alpha < 1$.

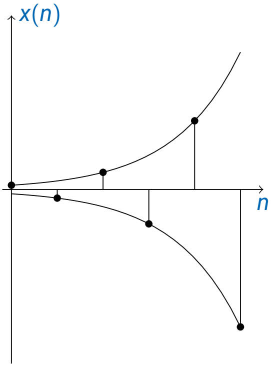


Signals

Example 2:
$A, \alpha \in \mathbb{R}$, $A > 0$ and $-1 < \alpha < 0$.



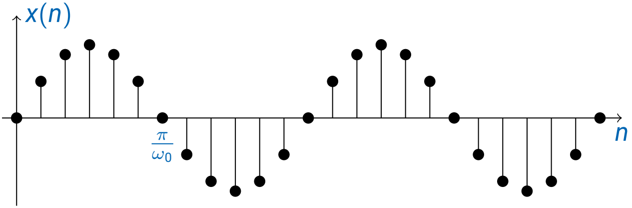
Example 3:
$A, \alpha \in \mathbb{R}$, $A > 0$ and $\alpha < -1$.



Sinusoidal sequences

$$x(n) = A \cdot \cos(\omega_o n + \phi)\,, \qquad A, \phi \in \mathbb{R}\,.$$

Example: $\phi = \frac{3\pi}{2}$



Signals

Now $A, \alpha \in \mathbb{C}$ so that $A = |A|e^{j\phi}$ and $\alpha = |\alpha|e^{j\omega_0}$, then

$$\begin{aligned} x(n) = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{jn\omega_0} \\ &= |A| \cdot |\alpha|^n e^{j(\omega_0 n + \phi)} \\ &= |A| \cdot |\alpha|^n \left[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right] \end{aligned}$$

For $|\alpha| = 1$, $x(n)$ is referred to as the complex exponential sequence, and has the form

$$x(n) = |A| \cdot e^{j(\omega_0 n + \phi)} = |A| \left[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right] ,$$

i.e., real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n.

Signals

TECHNISCHE UNIVERSITÄT DARMSTADT

Note:

- In the continuous-time case, the quantity ω_0 has the dimension radians per second.
- In the discrete-time case n is a dimensionless integer.
- Sometimes, we also specify the units of n to be samples, then ω_0 is of dimension radians per sample.
- So we can define the period of a complex exponential sequence as $2\pi/\omega_0$ which has the dimension samples.

TECHNISCHE UNIVERSITÄT DARMSTADT

Periodic Sequences

For positive integers N, a periodic sequence satisfies

$$x(n) = x(n+N) \quad \forall\, n\,.$$

Example: $x(n) = \cos(\pi n)$ is periodic with period $2k$, $k \in \mathbb{Z}$, but $x(n) = \cos(n)$ is not periodic.

- For complex exponential sequences $C \cdot e^{j\omega_0 n}$, i.e. periodicity with period N requires
$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \qquad \text{with } \omega_0 N = 2\pi k, \quad k \in \mathbb{Z}\,.$$
- Complex exponential and sinusoidal sequences are not necessarily periodic with period $2\pi/\omega_0$.

Systems

System T maps an input $x(n)$ to an output $y(n)$.

$$y(n) = \mathrm{T}[x(n)]$$

Represenation of a discrete-time system

Example: The ideal delay system is defined by $y(n) = x(n - n_d)$, $n_d \in \mathbb{Z}^+$. The system would shift the input to the right by n_d of samples corresponding to a time delay.

Systems

TECHNISCHE UNIVERSITÄT DARMSTADT

Linearity

If T satisfies

$$\mathrm{T}[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)\,,$$

where $\mathrm{T}[x_1(n)] = y_1(n)$, $\mathrm{T}[x_2(n)] = y_2(n)$, and a and b are any scalar constants, T is said to be linear.

Properties:

1. Homogeneity: $\mathrm{T}[ax(n)] = a\mathrm{T}[x(n)]$
2. Additivity $\mathrm{T}[x_1(n) + x_2(n)] = \mathrm{T}[x_1(n)] + \mathrm{T}[x_2(n)]$

- Additivity and homogeneity are collectively known as the principle of superposition.

Systems

Shift Invariance

A system is said to be shift invariant (SI) if

$$\mathrm{T}[x(n-n_0)] = y(n-n_0)$$

where $y(n) = \mathrm{T}[x(n)]$ and n_0 is any integer.

- If a system is linear and shift invariant, it is called LSI System.

Example 1: An accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

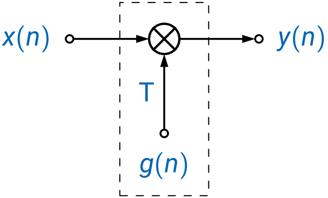
is linear and shift invariant.

Systems

Example 2:

$$y(n) = \mathrm{T}[x(n)] = x(n) \cdot g(n)\,,$$

where $g(n)$ is an arbitrary sequence.



Systems

TECHNISCHE UNIVERSITÄT DARMSTADT

Linearity:

$$\begin{aligned} \mathrm{T}\left[ax_1(n) + bx_2(n)\right] &= g(n)\left[ax_1(n) + bx_2(n)\right] \\ &= ag(n)x_1(n) + bg(n)x_2(n) \\ &= a\mathrm{T}\left[x_1(n)\right] + b\mathrm{T}\left[x_2(n)\right] \,, \end{aligned}$$

is satisfied.

Shift invariance:

$$\mathrm{T}\left[x(n-n_0)\right] = x(n-n_0)g(n) \neq x(n-n_0)g(n-n_0)\,,$$

is not satisfied.

Systems

Example 3:

$$y(n) = \mathrm{T}\left[x(n)\right] = x(n)^2$$

Linearity:

$$\mathrm{T}\left[x_1(n) + x_2(n)\right] = \left[x_1(n) + x_2(n)\right]^2 \neq y_1(n) + y_2(n)$$

is not satisfied.

Shift invariance:

$$\mathrm{T}[x(n-n_0)] = x(n-n_0)^2 = y(n-n_0)$$

is satisfied.

Systems

The unit sample response of linear systems

Consider a linear system T. Using the relationship $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$,

$$
\begin{aligned}
y(n) &= \mathrm{T}[x(n)] \\
&= \mathrm{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\
&= \mathrm{T}[\cdots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \cdots] \\
&= \sum_{k=-\infty}^{\infty} x(k)\mathrm{T}[\delta(n-k)]\,.
\end{aligned}
$$

- A linear system is completely characterised by its $T[\delta(n-k)], \ \forall k$.

Systems

TECHNISCHE UNIVERSITÄT DARMSTADT

Moreover, if the system is also shift-invariant, i.e. LSI, we have:

$$h(n - n_0) = \mathrm{T}[\delta(n - n_0)]$$

where

$$h(n) = \mathrm{T}[\delta(n)]$$

is the unit sample response of a system T to an input $\delta(n)$.

Convolution

TECHNISCHE UNIVERSITÄT DARMSTADT

Input-output relation of LSI system T:

$$y(n) = \mathrm{T}[x(n)] = \sum_{k=-\infty}^{\infty} x(k)h(n-k)\,.$$

The knowledge of $h(n)$ completely characterises the system, allowing us to determine the output $y(n)$ for any given input $x(n)$.
Convolution

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \end{aligned}$$

Convolution

TECHNISCHE UNIVERSITÄT DARMSTADT

Remark 1: The unit sample response $h(n)$ loses its significance for a nonlinear or shift-variant system.

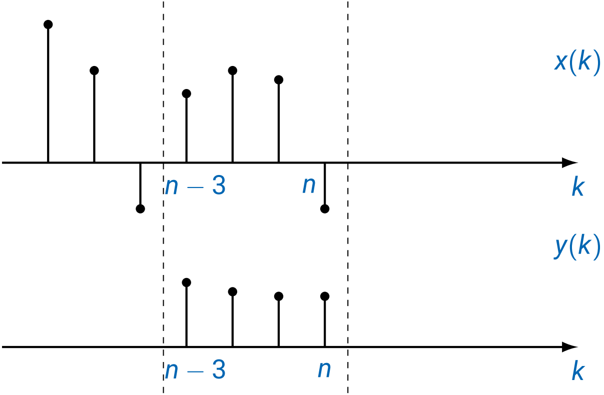
Remark 2: The choice of $\delta(n)$ as an input in characterising an LSI system is the simplest, both conceptually and in practice.

Example 1:

$$
\begin{aligned}
y(n) &= \sum_{k=0}^{q} x(n-k)a(k) \\
&= a(0)x(n) + a(1)x(n-1) + \ldots + a(q)x(n-q)
\end{aligned}
$$

For $a(0) = a(1) = \ldots = a(q) = \frac{1}{q+1}$, the output sequence is an average of $q+1$ samples of the input around the nth sample.

Convolution

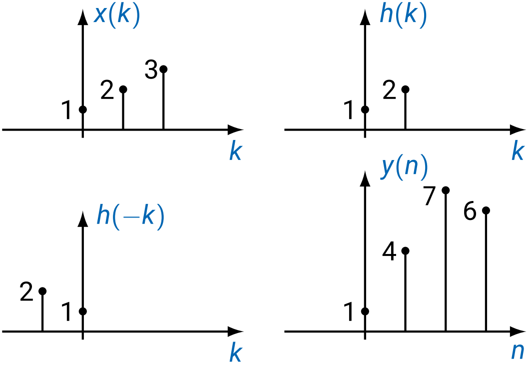


Transversal filter input and output for $q = 3$

Convolution

TECHNISCHE UNIVERSITÄT DARMSTADT

Example 2: Graphical convolution



Stability and Causality

TECHNISCHE UNIVERSITÄT DARMSTADT

Stability

- A system is considered stable in the Bounded-Input Bounded-Output (BIBO) sense if and only if a bounded input always leads to a bounded output.

Necessary and sufficient condition for an LSI system:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \,.$$

Causality

- We say that a system is causal if and only if the current output $y(n)$ does not depend on any future values of the input such as $x(n+1),\ x(n+2),\ \ldots$.

Necessary and sufficient condition for an LSI system:

$$h(n) = 0, \qquad \text{for } n < 0$$

Proof:

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Causality holds for any practical system regardless of linearity or shift invariance.

Outline

1. Discrete-Time Signals and Systems
2. **Digital Signal Processing of Continuous-Time Signals**
   - Periodic Sampling
   - Reconstruction of Band limited Signals
   - Discrete-time Processing of Continuous-time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Periodic Sampling

$x_c(t)$ a continuous-time signal and its Fourier transform $X_c(j\Omega)$

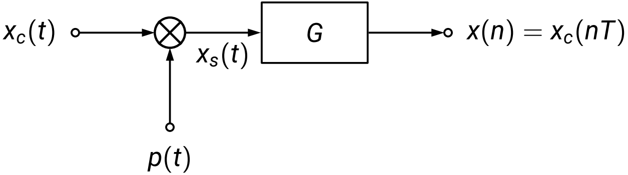
- $X_c(j\Omega) = \int_{-\infty}^{+\infty} x_c(t) e^{-j\Omega t} dt\,.$
- $x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_c(j\Omega) e^{+j\Omega t} d\Omega\,.$

discrete-time representation through periodic sampling

- $x(n) = x_c(t)|_{t=nT} = x_c(nT), \qquad n = 0, \pm 1, \pm 2, \dots, \qquad T > 0.$

Periodic Sampling

Sampling Model :



Continuous-Time to digital converter.

with $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, where $\delta(t)$ is the Dirac function

Periodic Sampling

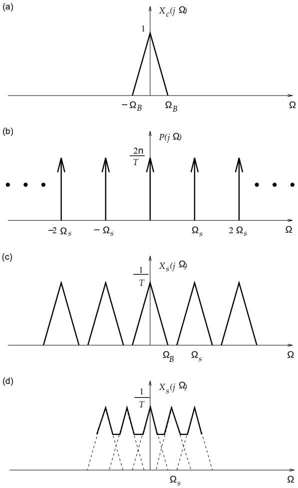
Consequently, we have

$$x_s(t) = x_c(t) \cdot p(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT).$$

and

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\Omega - \frac{2\pi k}{T} \right) \right] = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left(j\Omega - kj\Omega_s \right).$$

Periodic Sampling



Effect in the frequency domain of sampling in the time domain:

(a) Spectrum of continuous-time signal

(b) Spectrum of sampling function

(c) Spectrum of sampled signal with $\Omega_s > 2\Omega_B$

(d) Spectrum of sampled signal with $\Omega_s < 2\Omega_B$

- $\Omega_s - \Omega_B > \Omega_B$, or $\Omega_s > 2\Omega_B$

Reconstruction of Band limited Signals

Reconstruction of the continuous-time signal with an ideal lowpass $H_r(j\Omega)$

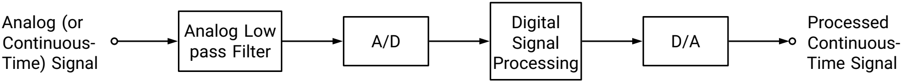
$$x(n) \longrightarrow \boxed{h_r(t)} \longrightarrow x_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)h(t-nT) \quad \text{with} \quad h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

Consequently, the relationship between $x_r(t)$ and $x(n)$ is given by

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Discrete-time Processing of Continuous-time Signals



Discrete-time processing of continuous-time signals.

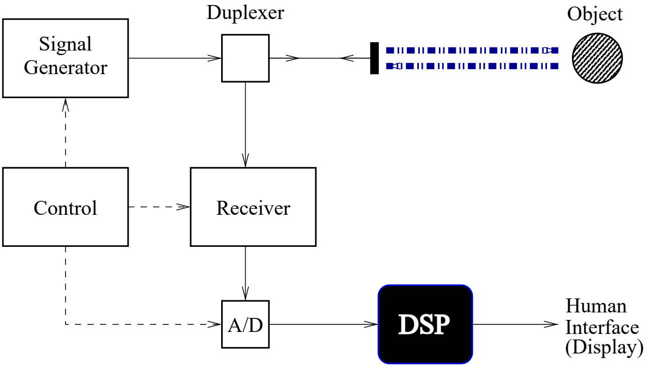
Advantages

- Flexibility: If we want to change the continuous-time filter because of change in signal and noise characteristics, we would have to change the hardware components. Using a digital approach, we only need to modify the software.
- Accuracy: Better control of accuracy requirements. Tolerances in continuous-time circuit components makes it extremely difficult for the system designer to control the accuracy of the system.
- Storage: The signals can be stored without deterioration or loss of signal quality.
- Cost: Lower cost of the digital implementation.

Outline

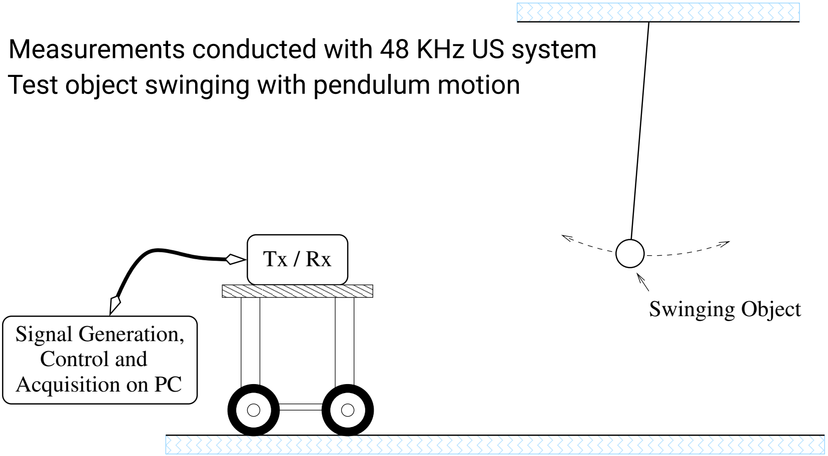
1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. **Application: Ultrasound**
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Ultrasound System

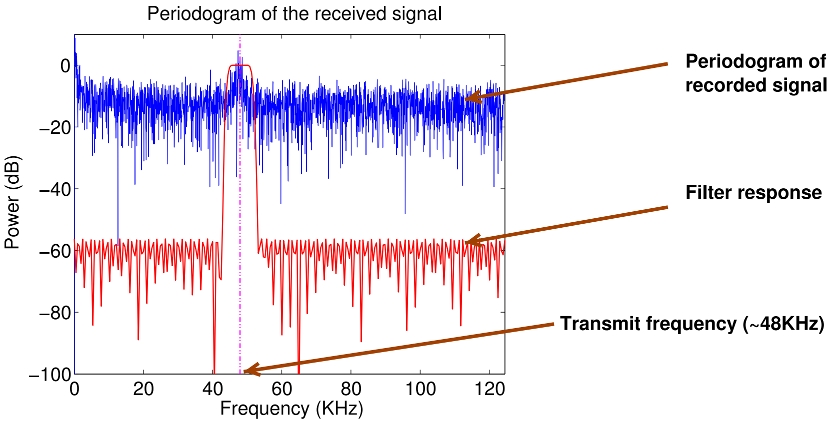


Ultrasound Experiment

- Measurements conducted with 48 KHz US system
- Test object swinging with pendulum motion

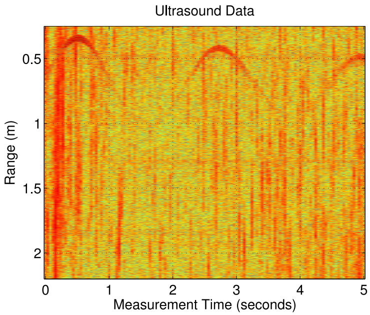


Bandpass Filtering

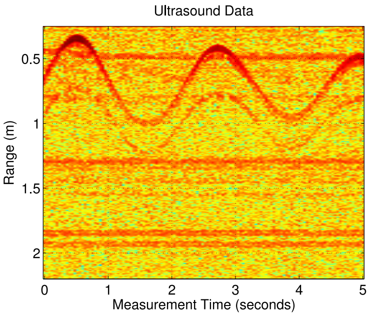


Ultrasound Range Images

- Band-pass filtering helps for noise reduction in range images:



Before filtering



After filtering

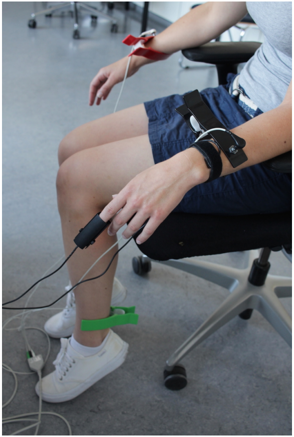
SPG

Outline

TECHNISCHE UNIVERSITÄT DARMSTADT

1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. **Application: Photoplethysmography (PPG)**
5. Filter Design
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Photoplethysmography (PPG) Example

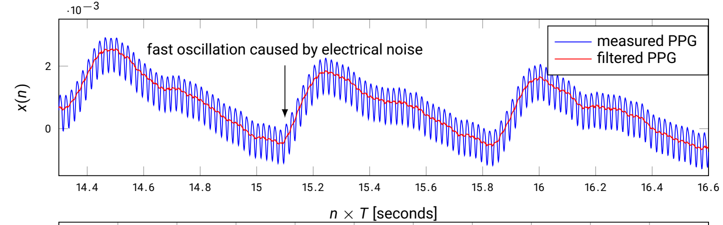


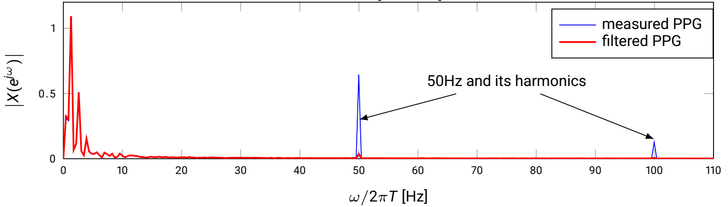
- The PPG signal can contribute information, e.g., about heart beat rate.
- Measurements are contaminated, e.g., by electrical power noise.
- Measuring PPG signal using a finger clip sensor in the SPG Biomedical Lab.

Photoplethysmogram (PPG) Example

TECHNISCHE UNIVERSITÄT DARMSTADT

Band-stop filtering attenuates noise without distorting the PPG pulse.





Outline

TECHNISCHE UNIVERSITÄT DARMSTADT

1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. **Filter Design**
   - Digital Filters
   - Linear Phase Filters
   - Filter Specification
   - Properties of IIR Filters
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Introduction

TECHNISCHE UNIVERSITÄT DARMSTADT

Three steps to design a digital filter:

1. Filter specification
2. Filter design
3. Filter implementation

Because of practical properties we only consider:

- $h(n)$ is real.
- The system is causal, i.e., $h(n) = 0$ for $n < 0$.
- The system is BIBO stable, i.e, $\sum_{n=0}^{\infty} |h(n)| < \infty$.

Introduction

For a band-limited signal $x_c(t)$:

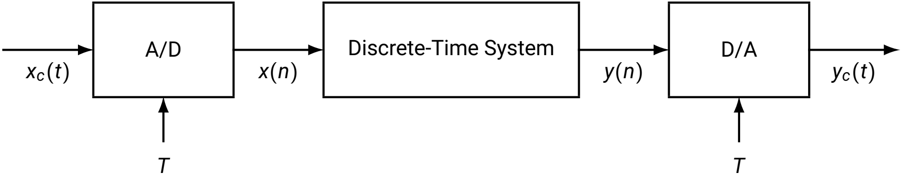


Figure: A basic system for discrete-time filtering of a continuous-time signal.

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| > \pi/T \end{cases}$$

Digital Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

A linear time-invariant discrete-time system can be characterised by a linear constant coefficient difference equation:

$$\sum_{k=0}^{M} a_k y(n-k) = \sum_{r=0}^{N-1} b_r x(n-r)\,.$$

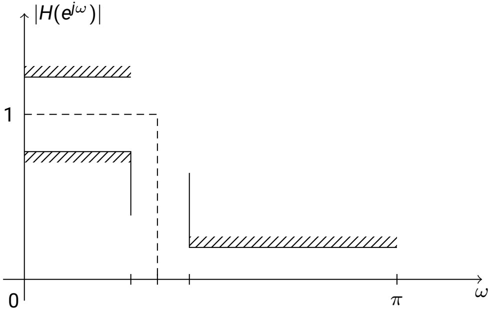
IIR Infinite impulse response system has $M > 0$

FIR Finite impulse response system has $M = 0$

Digital Filters: Motivation

TECHNISCHE UNIVERSITÄT DARMSTADT

- Problem: Find filter coefficients such that $H(e^{j\omega})$ satisfies given constraints
- Desired: small M, N



- Different approaches: FIR and IIR

FIR case ($M = 0$) :

$$y(n) = \frac{1}{a_0} \sum_{r=0}^{N-1} b_r x(n-r)$$

Comparison with the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

shows that

$$h(n) = \begin{cases} \frac{b_n}{a_0}, & n = 0, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Linear Phase Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

A digital filter $h(n)$ has a linear phase if

$$H(e^{j\omega}) = H_M(e^{j\omega})e^{-j\omega\alpha} \qquad |\omega| < \pi$$

with $H_M(e^{j\omega}) \in \mathbb{R}$ and α being a constant.

Why linear phase?

with $H_M(e^{j\omega}) = 1$ follows $y(n) = h(n) * x(n) = x(n - \alpha)$.

$\Rightarrow$ *a linear phase preserves the shape of the signal.*

Linear Phase Filters

Generalized linear phase system with:

$$H(e^{j\omega}) = H_M(e^{j\omega})e^{-j\omega\alpha}e^{j\beta} = H_M(e^{j\omega})e^{-j\phi(\omega)}\,,\ \phi(\omega) = -\beta + \omega\alpha$$

Which conditions must hold true for $H_M(e^{j\omega}) \in \mathbb{R}$?

$$H_M(e^{j\omega}) = H(e^{j\omega})e^{+j\phi(\omega)} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n + j\phi(\omega)}$$

$$H_M(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\left[\cos(-\omega n - \beta + \omega\alpha) + j\sin(-\omega n - \beta + \omega\alpha)\right]$$

Linear Phase Filters

It can easily be seen that the condition is met with:

$$\sum_{n=-\infty}^{\infty} h(n) \sin[\omega(n-\alpha)+\beta] = 0 \;\; \forall\, \omega \in \mathbb{R} \tag{1}$$

It is a necessary but not a sufficient condition on $h(n)$, α, and β for $H_M(e^{j\omega}) \in \mathbb{R}$. The condition in (1) can be achieved with either

$$\beta = 0 \;\text{ or }\; \pi, \;\; 2\alpha = N-1, \; h(2\alpha - n) = h(n)$$

or

$$\beta = \tfrac{\pi}{2} \;\text{ or }\; \tfrac{3\pi}{2}, \;\; 2\alpha = N-1, \;\; h(2\alpha - n) = -h(n).$$

Linear Phase Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

There are two possible causal FIR Systems which meet the conditions:

- $$h(n) = \begin{cases} h(N-1-n), & 0 \leq n \leq N-1, \\ 0, & \text{otherwise}, \end{cases}$$

  then $H(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega(N-1)/2}$

- $$h(n) = \begin{cases} -h(N-1-n), & 0 \leq n \leq N-1, \\ 0, & \text{otherwise}, \end{cases}$$

  then $H(e^{j\omega}) = jH_o(e^{j\omega})e^{-j\omega(N-1)/2} = H_o(e^{j\omega})e^{-j\omega(N-1)/2+j\pi/2}$

Generalised Linear Phase for FIR Filters

We classify FIR filters into four types, depending on whether $h(n)$ is symmetric ($h(n) = h(N-1-n)$) or anti-symmetric ($h(n) = -h(N-1-n)$), and whether the filter length is odd or even.

Type I: symmetric impulse response and N is odd.
Type II: symmetric impulse response and N is even.
Type III: antisymmetric impulse response and N is odd.
Type IV: antisymmetric impulse response and N is even.

Symmetric Filters

$$h(n) = h(N-1-n), \qquad 0 \leq n \leq N-1$$

Type I FIR linear phase filter (N is odd)

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} = e^{-j\omega \frac{N-1}{2}} \left(\sum_{k=0}^{\frac{N-1}{2}} a(k)\cos(\omega k) \right)$$

where $a(0) = h(\frac{N-1}{2})$ and $a(k) = 2h(\frac{N-1}{2} - k), k = 1, 2, \ldots, \frac{N-1}{2}$

Type II FIR linear phase filter (N is even)

$$H(e^{j\omega}) = e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} b(k)\cos[\omega(k-1/2)] \right\}$$

where $b(k) = 2h(N/2-k), \qquad k = 1, 2, \ldots, N/2$

Generalised Linear Phase for FIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

Antisymmetric Filters

$$h(n) = -h(N-1-n), \qquad 0 \le n \le N-1$$

Type III FIR linear phase filter (N is odd)

$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}}\left\{\sum_{k=1}^{(N-1)/2} c(k)\sin(\omega k)\right\}$$

where $c(k) = 2h[(N-1)/2-k], \qquad k = 1, 2, \ldots, (N-1)/2$

Type IV FIR linear phase filter (N is even)

$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}}\left\{\sum_{k=1}^{N/2} d(k)\sin[\omega(k-\tfrac{1}{2})]\right\}$$

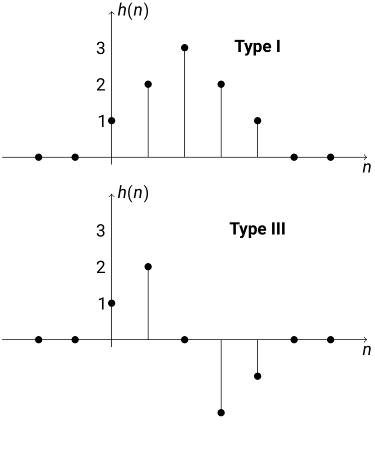
where $d(k) = 2h(N/2-k), \qquad k = 1, 2, \ldots, N/2$

Generalised Linear Phase for FIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

Type	Symmetry of $h(n)$	N	β	Form of $H_M(e^{j\omega})$	Constraints on $H_M(e^{j\omega}) \cdot e^{j\beta}$
I	$h(n) = h(N-1-n)$ symmetric	odd	0 π	$\sum_{n=0}^{\frac{N-1}{2}} a(n) \cdot \cos \omega n$	real
II	$h(n) = h(N-1-n)$ symmetric	even	0 π	$\sum_{n=1}^{\frac{N}{2}} b(n) \cdot \cos \omega(n - \frac{1}{2})$	real $H_M(e^{j\pi}) = 0$
III	$h(n) = -h(N-1-n)$ anti-symmetric	odd	$\pi/2$ $3\pi/2$	$\sum_{n=1}^{\frac{N-1}{2}} c(n) \cdot \sin \omega n$	Pure Imaginary $H_M(e^{j0}) = H_M(e^{j\pi}) = 0$
IV	$h(n) = -h(N-1-n)$ anti-symmetric	even	$\pi/2$ $3\pi/2$	$\sum_{n=1}^{\frac{N}{2}} d(n) \cdot \sin \omega(n - \frac{1}{2})$	Pure Imaginary $H_M(e^{j0}) = 0$

Generalised Linear Phase for FIR Filters
Example



Generalised Linear Phase for FIR Filters

Example

TECHNISCHE UNIVERSITÄT DARMSTADT

Example of a type I FIR filter. Given a filter with unit sample response

$$h(n) = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise,} \end{cases}$$

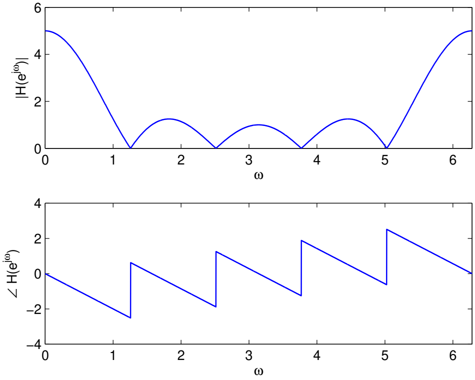
we can evaluate the frequency response to be

$$H(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = e^{-j2\omega} \frac{\sin\left(5\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} .$$

Note:

$$\sum_{n=0}^{N-1} q^n = \frac{1 - q^N}{1 - q} \quad \text{for } q \neq 1$$

Generalised Linear Phase for FIR Filters



Magnitude and phase of the frequency response

Generalised Linear Phase for FIR Filters

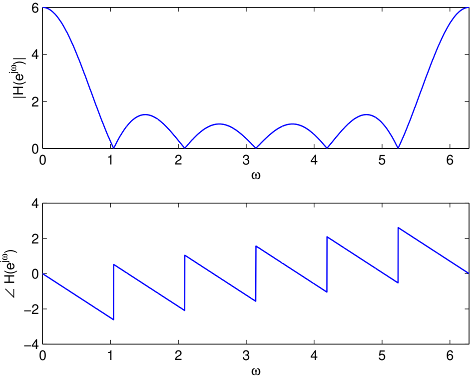
Example

TECHNISCHE UNIVERSITÄT DARMSTADT

Example of a type II FIR filter.
Consider an extension by one sample of the impulse response of the filter from the previous example. Then, the frequency response of the system is given by

$$H(e^{j\omega}) = \sum_{n=0}^{5} e^{-j\omega n} = \frac{1 - e^{-j\omega 6}}{1 - e^{-j\omega}} = e^{-j\omega \frac{5}{2}} \frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} .$$

Generalised Linear Phase for FIR Filters



Magnitude and phase of the frequency response

Generalised Linear Phase for FIR Filters

Example

Example of a type III FIR filter. Consider a filter with unit impulse response

$$h(n) = \delta(n) - \delta(n-2)$$

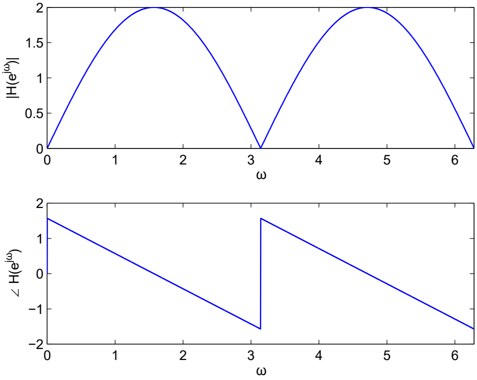
where $\delta(n)$ is Kronecker's delta defined as

$$\delta(n) = \begin{cases} 1 & , \quad n = 0 \\ 0 & , \quad n \neq 0 \end{cases} .$$

We can then evaluate the frequency response as

$$H(e^{j\omega}) = 1 - e^{-j2\omega} = j[2\sin(\omega)]e^{-j\omega} = 2e^{-j\left(\omega - \frac{\pi}{2}\right)} \cdot \sin(\omega).$$

Generalised Linear Phase for FIR Filters



Magnitude and phase of the frequency response

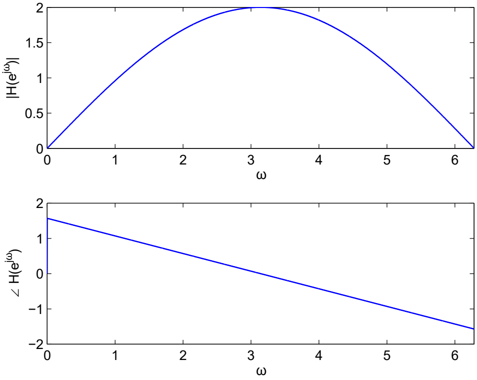
Example of a type IV FIR filter. Consider a filter with unit impulse response

$$h(n) = \delta(n) - \delta(n-1).$$

We can then evaluate the frequency response as

$$H(e^{j\omega}) = 1 - e^{-j\omega} = j\left[2\sin\left(\frac{\omega}{2}\right)\right]e^{-j\frac{\omega}{2}} = 2e^{-j\left(\frac{\omega}{2}-\frac{\pi}{2}\right)}\sin\left(\frac{\omega}{2}\right).$$

Generalised Linear Phase for FIR Filters

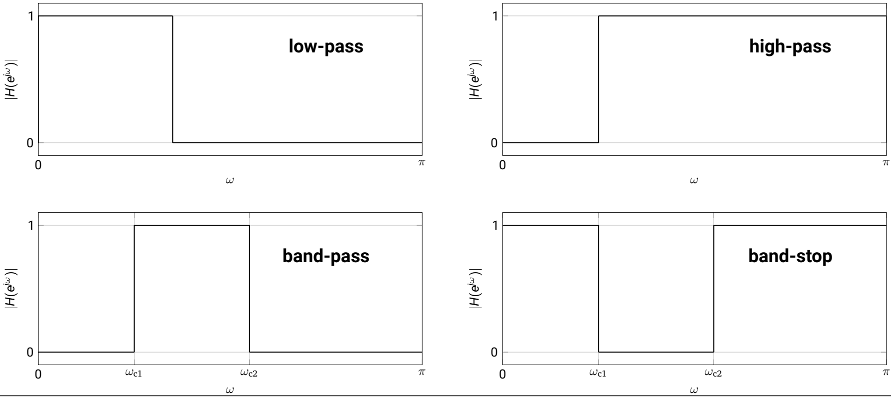


Magnitude and phase of the frequency response

Filter Specification

TECHNISCHE UNIVERSITÄT DARMSTADT

Since $h(n)$ is real, it follows that $H(e^{j\omega}) = H(e^{-j\omega})^*$
so $H(e^{j\omega})$ is completely specified for $0 \leq \omega < \pi$.



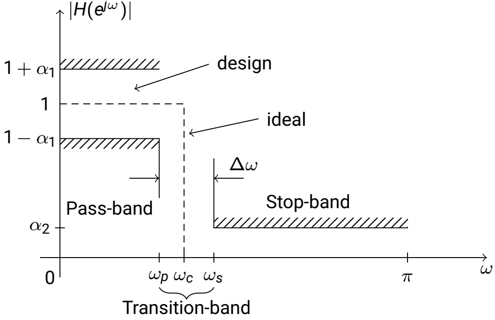
SPG

Filter Specification

Example: Specification of a low pass filter

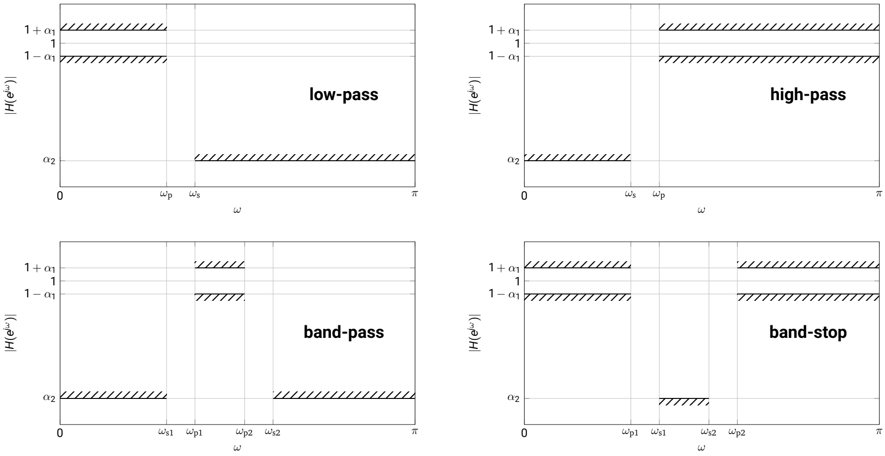
An ideal low pass filter has only a passband and a stopband, but in practice

- we also have a transition band
- and have to specify a tolerance in passband and stopband.



Filter Specification

Tolerance scheme for digital filter design



Properties of IIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

Restriction to the class of filters which:

- have a unit sample response $h(n)$ that is real, causal, and satisfies stability.
- possess a rational z-transform with $a_0 = 1$, i.e.,

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}}.$$

Infinito não é realizável

Properties of IIR Filters

Example

The filter with the impulse response

$$h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)\,,$$

with $H(z)$ given by

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}\,.$$

Can be realised by

$$y(n) = \frac{1}{2}y(n-1) + x(n)\,.$$

Properties of IIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

- **Stability**
  With an IIR filter, all the poles of $H(z)$ must lie inside the unit circle to ensure filter stability.
- **Linear phase**
  It is very difficult to control the phase of an IIR filter. As a result, we specify only the magnitude response for the IIR filter and accept the resulting phase characteristics.

Outline

TECHNISCHE UNIVERSITÄT DARMSTADT

1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. **Design of Finite Impulse Response Filter**
    - Design of FIR Filters by Windowing
    - Importance of the Window Shape and Length
    - Kaiser Window Filter Design
    - Optimal Filter Design and Implementation of FIR Filters
7. Design of Infinite Impulse Response Filters
8. Application: 2D Filters for Image Processing

Introduction

TECHNISCHE UNIVERSITÄT DARMSTADT

From the previously discussed four types of FIR filters is not every one suitable for a given design specification.

Type	Low pass	High pass	Band pass	Band stop
I				
II		Not suitable		Not suitable
III	Not suitable	Not suitable		Not suitable
IV	Not suitable			Not suitable

The two standard approaches for FIR filter design are:

- The window method.
- The optimal filter design method.

Design of FIR Filters by Windowing

TECHNISCHE UNIVERSITÄT DARMSTADT

Most idealised systems have non causal and infinite impulse responses.
$\Rightarrow$ The most straightforward approach is to truncate the ideal response.

If the desired (ideal) impulse response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \,,$$

the truncated impulse response would be

$$h(n) = \begin{cases} h_d(n), & 0 \leq n \leq N-1\,, \\ 0, & \text{elsewhere}\,. \end{cases}$$

Design of FIR Filters by Windowing

This can be expressed as

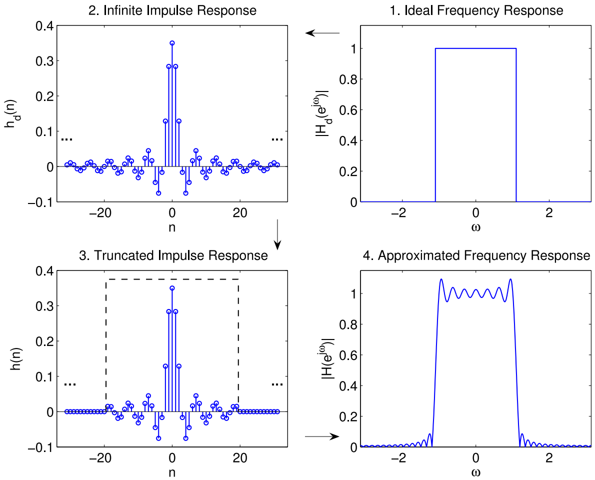
$$h(n) = h_d(n) \cdot w(n)\,,$$

and therefore

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta}) \cdot W(e^{j(\omega-\theta)})d\theta \\ &= H_d(e^{j\omega}) \circledast W(e^{j\omega})\,. \end{aligned}$$

If $h_d(n) \cdot w(n)$ is symmetric (or antisymmetric) then $h(n)$ is the unit sample response of a linear phase filter.

Design of FIR Filters by Windowing



TECHNISCHE UNIVERSITÄT DARMSTADT

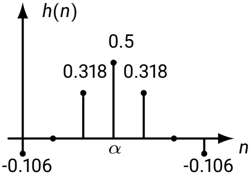
Example: Design a low pass filter

Ideal frequency response : $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & 0 \leq |\omega| < \omega_c\,, \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$

Ideal impulse response : $h_d(n) = \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}$

With a rectangular window, the resulting filter unit sample response is

$$h(n) = \begin{cases} \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}, & \text{for } 0 \leq n \leq N-1\,, \\ 0, & \text{otherwise}\,. \end{cases}$$



SPG

Importance of the Window Shape and Length

TECHNISCHE UNIVERSITÄT DARMSTADT

Remember:

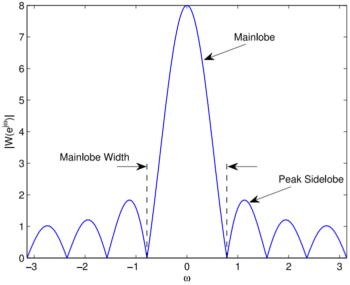
- In the time-domain : $h(n) = h_d(n) \cdot w(n)$,
- and in the frequency-domain :

$$H(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

Thus, $W(e^{j\omega})$ should approximate an impulse, which means that N would go to infinity !

Importance of the Window Shape and Length

- Increase N to decrease the "main-lobe" width
- Smooth the discontinuity of the window in the time-domain to reduce the "side-lobes" $\Rightarrow$ increased main-lobe width.

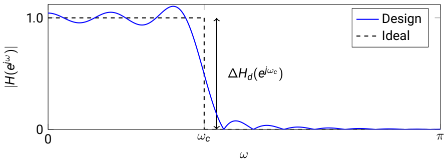


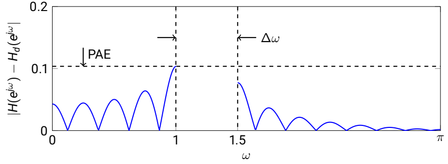
Magnitude of the Fourier transform of a rectangular window ($N = 8$)

$$
\begin{aligned}
W(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= e^{-j\omega \frac{N-1}{2}} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}
\end{aligned}
$$

Importance of the Window Shape and Length

Peak approximation error (PAE)



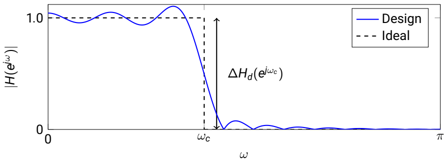


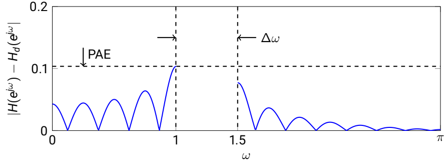
- Convolution with discontinuity leads to Gibbs phenomenon
- Resulting filter is an approximation of $H_d(e^{j\omega})$
- $\mathrm{PAE} = \gamma \cdot |\Delta H_d(e^{j\omega_c})|$
- Window has to be chosen such that all specifications are fulfilled, e.g. $\mathrm{PAE} \leq \min(\alpha_1, \alpha_2)$

Tolerances in pass- and stopband determine window shape!

Importance of the Window Shape and Length

Window length





- Distance between peak ripples is equal to main-lobe width
- Transition bandwidth slightly smaller than main-lobe width

Transition bandwidth determines window length!

Importance of the Window Shape and Length

TECHNISCHE UNIVERSITÄT DARMSTADT

Commonly used windows (1/2)

Rectangular

$$w(n) = \begin{cases} 1, & 0 \leq n \leq N-1\,, \\ 0, & \text{elsewhere.} \end{cases}$$

Bartlett

$$w(n) = \begin{cases} \dfrac{2n}{N-1}, & 0 \leq n \leq \dfrac{N-1}{2}\,, \\ 2 - \dfrac{2n}{N-1}, & \dfrac{N-1}{2} \leq n \leq N-1\,. \end{cases}$$

Importance of the Window Shape and Length

Commonly used windows (2/2)

Hanning

$$w(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right], \; 0 \leq n \leq N-1.$$

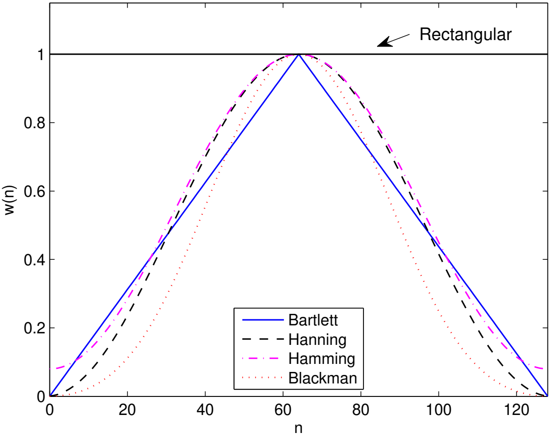
Hamming

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \; 0 \leq n \leq N-1.$$

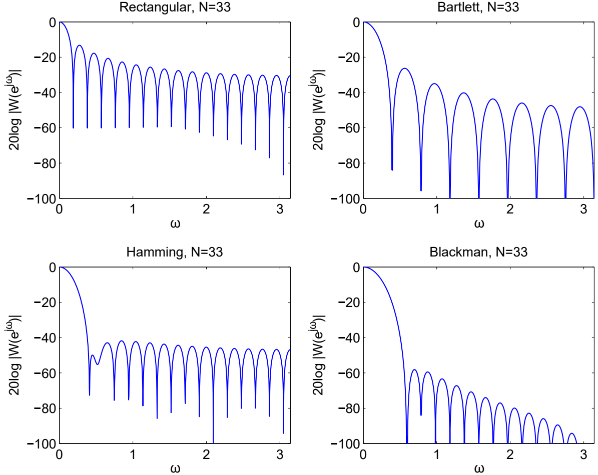
Blackman

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), \; 0 \leq n \leq N-1.$$

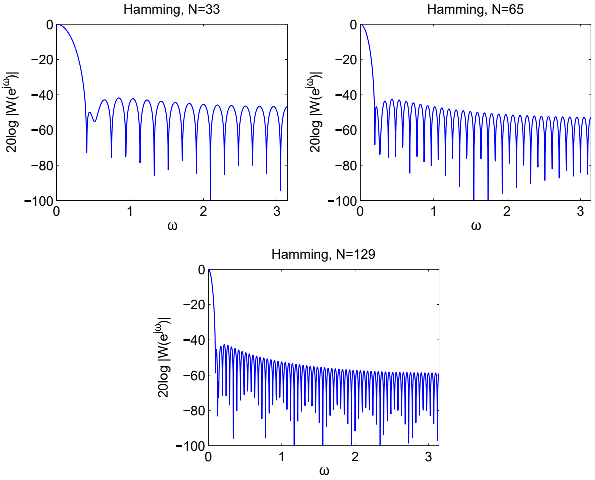
Importance of the Window Shape and Length



Importance of the Window Shape and Length



Importance of the Window Shape and Length



Importance of the Window Shape and Length

TECHNISCHE UNIVERSITÄT DARMSTADT

Summary:

- Window shape controls side-lobe as well as main-lobe behaviour.
- Window size controls main-lobe behaviour only.

Table: Properties of different windows

Window Type	Peak Amplitude of Side-Lobe (dB) (relative)	Approximate Width (rad) of Main-Lobe $\Delta\omega$	Peak Approximation Error (dB) $20 \log_{10} \gamma$
Rectangular	-13	$4\pi/N$	-21
Bartlett	-25	$8\pi/(N-1)$	-25
Hanning	-31	$8\pi/(N-1)$	-44
Hamming	-41	$8\pi/(N-1)$	-53
Blackman	-57	$12\pi/(N-1)$	-74

Importance of the Window Shape and Length

TECHNISCHE UNIVERSITÄT DARMSTADT

Example: Filter design problem
Design a low pass filter with a required stopband attenuation of
$20\log_{10}(\alpha_2) = -50$ *dB.*

Using the previous table, we have to

- determine all window types that satisfy the stopband constraint,
- choose the window type which results in the shortest window function.

Which one is that according to the above table? $\Rightarrow$ Hamming

Kaiser Window Filter Design

TECHNISCHE UNIVERSITÄT DARMSTADT

Largura do main-lobe -> transição do degrau -> demora mais pra cair de alto pra baixo

To achieve the best trade-off between main-lobe width and side-lobe level, we can formulate a general window function that can adapt to the specifications:

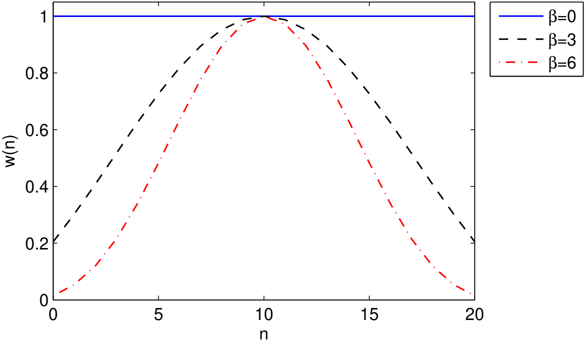
Kaiser window function

$$w(n) = \begin{cases} \dfrac{I_0[\beta(1 - \left[\frac{n-\alpha}{\alpha}\right]^2)^{\frac{1}{2}}]}{I_0(\beta)}, & 0 \leq n \leq N-1\,, \\ 0, & \text{elsewhere}\,, \end{cases}$$

where $\alpha = \frac{N-1}{2}$, and $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind.

- Thus, by varying N and β, we can trade side-lobe level for main-lobe width.

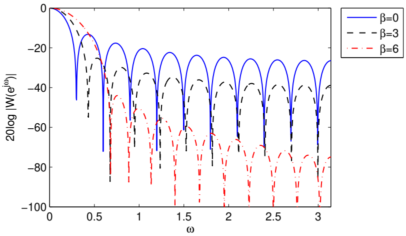
Kaiser Window Filter Design



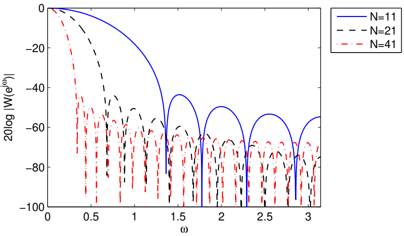
Kaiser windows for $\beta = 0$ (solid), 3 (dashed), and 6 (semi-dashed) and $N = 21$

Kaiser Window Filter Design

aumentar o beta, aumenta o main-lobe-width, mas diminui o peak do sidelob



Fourier transforms corresponding to windows with $N = 21$ for $\beta = 0$ (solid), 3 (dashed), and 6 (semi-dashed)



Fourier transforms of Kaiser windows with $\beta = 6$ and $N = 11$ (solid), 21 (dashed), and 41 (semi-dashed)

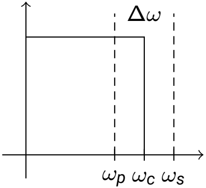
TECHNISCHE UNIVERSITÄT DARMSTADT

Design a filter with the Kaiser window

Assume that $\alpha_1 = \alpha_2$. The transition region width is therefore

$$\Delta\omega = \omega_s - \omega_p \,.$$

With $A = -20\log_{10}\alpha_1$



$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50, \\ 0, & A < 21. \end{cases}$$

and N must satisfy

try and error

$$N = \left\lceil \frac{A-8}{2.285\Delta\omega} + 1 \right\rceil \,.$$

SPG

Given the specifications $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $\alpha_1 = \alpha_2 = 0.001$ and $\Delta\omega = 0.2\pi$, find the unit sample response using Kaiser's method.

$$A = -20\log_{10}\alpha_1 = 60\,.$$

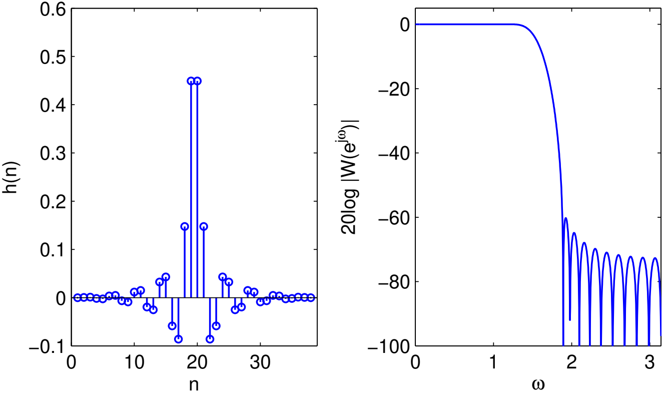
We find $\beta = 5.635,\ \ N = 38$ and

$$h(n) = \begin{cases} \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le N-1, \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha = \frac{N-1}{2} = \frac{37}{2} = 18.5$ and $\omega_c = (\omega_s + \omega_p)/2$. Since N is an even integer, the resulting linear phase system would be of type II.

Kaiser Window Filter Design

Example: Design a low pass filter



Kaiser Window Filter Design

Example: Design a high pass filter

TECHNISCHE UNIVERSITÄT DARMSTADT

We consider a high pass filter with

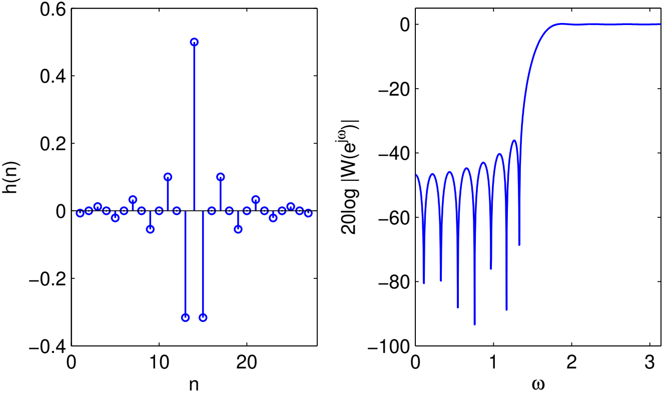
$$\begin{aligned} \omega_s &= 0.35\pi \\ \omega_p &= 0.5\pi \\ \alpha_1 &= 0.021 \end{aligned}$$

Again, using the formulae proposed by Kaiser, we get $\beta = 2.6$ and $N = 25$. The resulting peak error is $0.0213 > \alpha_1 = \alpha_2 = 0.021$ as specified. Thus, we increase N to 26. In this case, the filter would be of type II and is not appropriate for designing a high pass. Therefore, we use $N = 27$, which exceeds the required specifications.

Geralmente o N fica entre 20 e 50

Kaiser Window Filter Design

Example: Design a high pass filter



Optimal Filter Design

- Because of the Fourier approximation any filter design which uses the window method attempts to minimize the integrated squared error

$$\epsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega.$$

- Peak error always occurs at discontinuities of $H_d(e^{j\omega})$.

We get better results by minimizing the maximum error and distribute the approximation error equally across the frequency band.

Optimal Filter Design

TECHNISCHE UNIVERSITÄT DARMSTADT

Underlying Concept:

- The Parks-McClellan algorithm can be used to create an approximation of $H_d(e^{j\omega})$ that oscillates equally in all tolerance bands.
- It is based on the Alternation Theorem which states that $H(e^{j\omega}) = \sum_{n=0}^{L} \check{a}(n) \cos(\omega)^n$ is the best Chebyshev approximation to $H_d(e^{j\omega})$, given that $E(e^{j\omega})$ shows $L+2$ alternating frequencies.

As a result, we get *equi-ripple* filters that have a significantly lower order compared to the window method.

Optimal Filter Design

TECHNISCHE UNIVERSITÄT DARMSTADT

Property of Optimal Filters

Type I optimal filters have $\frac{N+3}{2}$ (with $L = \frac{N-1}{2}$) maxima in the passband and stopband. Thus, they are often called *equi-ripple* filters.

Remarks

- Designing an optimal filter requires many more computations than the window method.
- The length of the filter designed by the window method is 20% to 50% longer than the length of the optimal filter.
- The optimal filter design method is the most widely used FIR filter design method. [in MATLAB, the ’firpm’ function]

MATLAB Demo

TECHNISCHE UNIVERSITÄT DARMSTADT

The Filter Design and Analysis Toolbox in MATLAB

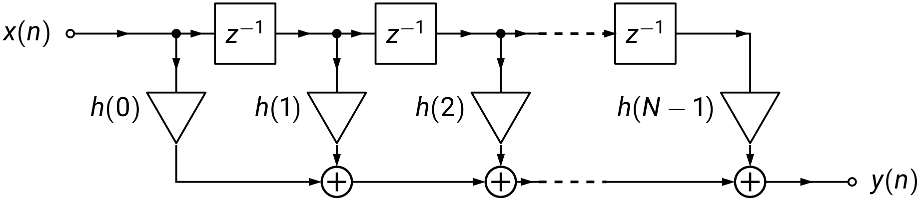
Implementation of FIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

The input-output relationship is given by a FIR filter through the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=0}^{N-1} h(k)x(n-k)\,,$$

represented in the signal flow graph.



Implementation of FIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

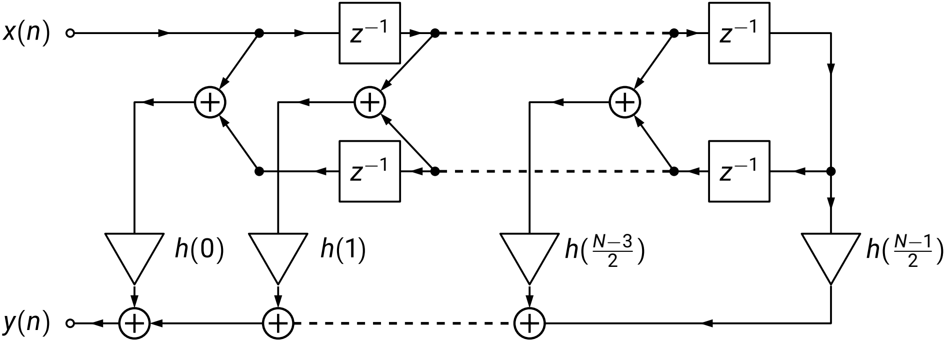
For type I Filters, we have $h(n) = h(N-1-n)$ and N is odd. This leads to

$$
\begin{aligned}
y(n) &= \sum_{k=0}^{N-1} h(k)x(n-k) \\
&= \sum_{k=0}^{\frac{N-1}{2}-1} h(k)x(n-k) + \sum_{k=\frac{N-1}{2}+1}^{N-1} h(k)x(n-k) \\
&\quad + h\left(\frac{N-1}{2}\right) x\left(n - \frac{N-1}{2}\right)
\end{aligned}
$$

Implementation of FIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

Efficient signal flow graph for the realisation of a type I FIR filter



Number of multiplications is halved!

Outline

TECHNISCHE UNIVERSITÄT DARMSTADT

1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. Design of Finite Impulse Response Filter
7. **Design of Infinite Impulse Response Filters**
   - Impulse Invariance Method
   - Bilinear Transformation
   - Implementation of IIR Filters
   - A Comparison of FIR and IIR Digital Filters
8. Application: 2D Filters for Image Processing

Introduction

TECHNISCHE UNIVERSITÄT DARMSTADT

Two different approaches to design an IIR filter :

- Design a digital filter from an analog or continuous-time filter.
- Design the filter directly.

Only the first approach is considered since it is much more useful.
Steps to design are

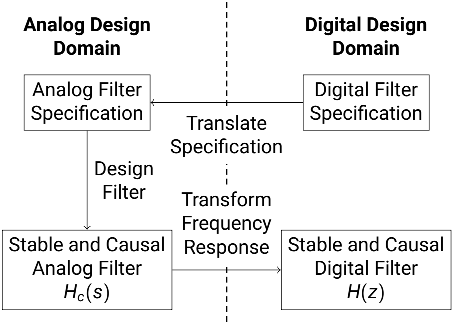
Step 1 : Specification of the digital filter.

Step 2 : Translation to a continuous-time filter specification.

Step 3 : Determination of the continuous-time filter system function $H_c(s)$.

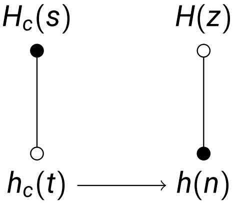
Step 4 : Transformation of $H_c(s)$ to a digital filter system function $H(z)$.

Introduction



- Translation of digital specifications and transformation of $H_c(s)$ to $H(z)$ have to be consistent.
- Causality and stability are desired for both $H_c(s)$ and $H(z)$.
- $H_c(j\Omega) = H_c(s)|_{s=j\Omega}$ should map $H(e^{j\omega}) = H(z = e^{j\omega})$.
- Different continuous-time prototype filters can be used.

Impulse Invariance Method



Steps:

1. From $H_c(s)$ and $H_c(j\Omega)$, we determine $h_c(t)$.
2. Obtain the digital sequence by $h(n) = T_d \cdot h_c(nT_d)$, where T_d is the design sampling interval.
3. $H(z)$ is obtained from $h(n)$.

Impulse Invariance Method

TECHNISCHE UNIVERSITÄT DARMSTADT

Example:

Continuous filter is described by $H_c(s) = \frac{1}{s - s_c}$.

The Inverse Laplace transform is $h_c(t) = e^{s_c t} \cdot u(t)$.

With $T_d = 1$ it follows that $h(n) = h_c(t)|_{t=nT_d} = e^{s_c n} \cdot u(n)$

which leads to $H(z) = \frac{1}{1 - e^{s_c} z^{-1}}$.

Impulse Invariance Method

TECHNISCHE UNIVERSITÄT DARMSTADT

From previous slides, we know

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right).$$

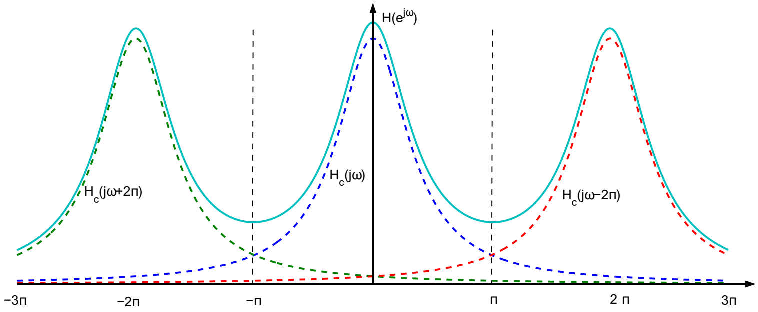
If there is no aliasing, then

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \quad \text{for} \quad -\pi \leq \omega \leq \pi\,.$$

Note that in practice $H_c(j\Omega)$ is not band-limited and therefore aliasing is present, which involves some difficulty in the design with the impulse invariance method.

We assume for implementation that $H_c(j\Omega)$ is band-limited.

Impulse Invariance Method

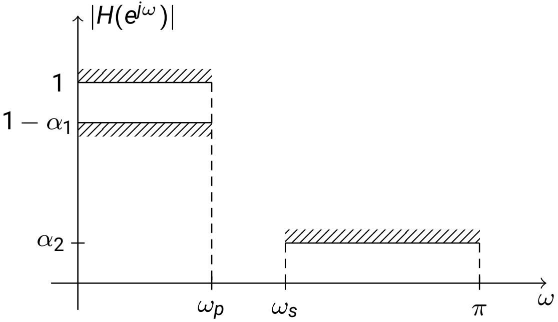


Impulse Invariance Method

Example

With: $\alpha_1 = 0.10875 \quad \alpha_2 = 0.17783$

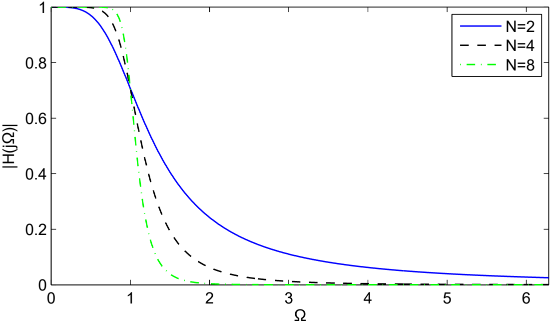
$\omega_p = 0.2\pi \quad \omega_s = 0.3\pi \quad T_d = 1$



Impulse Invariance Method

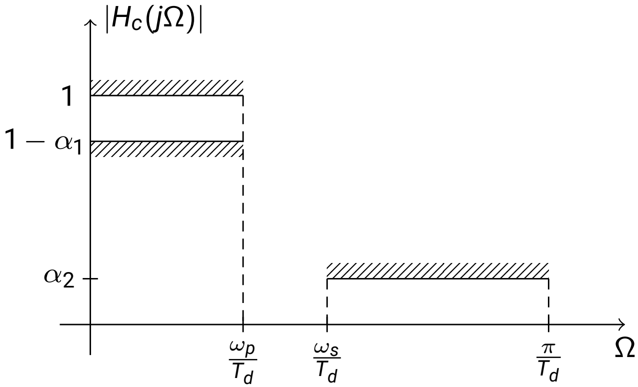
A Butterworth low-pass filter with an order N is defined as

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}} \,.$$



Impulse Invariance Method

Applying the relation $\Omega = \omega / T_d$ leads to the continuous-time filter specifications.



Impulse Invariance Method

The Butterworth filter should have the following specifications,

$$0.89125 \leq |H_c(j\Omega)| \leq 1\,, \quad 0 \leq |\Omega| \leq 0.2\pi\,,$$

and

$$|H_c(j\Omega)| \leq 0.17783\,, \quad 0.3\pi \leq |\Omega| \leq \pi\,.$$

Because $|H_c(j\Omega)|$ is a monotonic function of Ω, the specifications will be satisfied if

$$|H_c(j0.2\pi)| \geq 0.89125$$

and

$$|H_c(j0.3\pi)| \leq 0.17783.$$

Impulse Invariance Method

TECHNISCHE UNIVERSITÄT DARMSTADT

Now, we determine N and Ω_c, which are parameters of

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \, ,$$

to meet the specifications. Thus, we have

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad (\Omega_p = 0.2\pi)$$

and

$$1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 . \quad (\Omega_s = 0.3\pi)$$

Impulse Invariance Method

The given values lead to $N = 5.8858$ and $\Omega_c = 0.70474$. Rounding N up to 6, we find $\Omega_c = 0.702$. From

$$H_c(s)H_c(-s) = 1/[1 + (s/j\Omega_c)^{2N}],$$

we can determine the poles of $H_c(s)$ on the left half of the s-plane:

$$s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N+1)}, \quad k = 0, \ldots, N-1$$

We get

$$\begin{aligned} s_{1,2} &= -0.182 \pm j0.679 \\ s_{3,4} &= -0.497 \pm j0.497 \\ s_{5,6} &= -0.679 \pm j0.182 \end{aligned}$$

Impulse Invariance Method

From that, we obtain $H_c(s)$ as

$$H_c(s) = \frac{\Omega_c^N}{\Omega_c^N + s^N} = \frac{\Omega_c^N}{\prod_{k=0}^{N-1}(s - s_k)} \, .$$

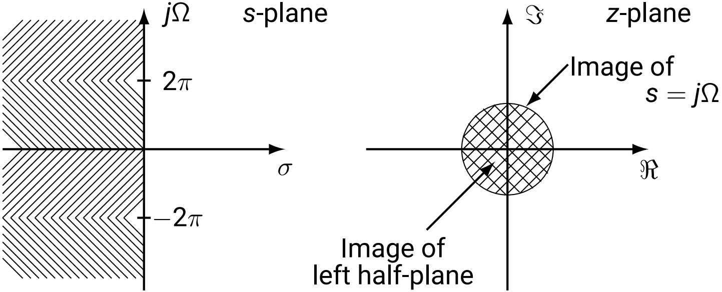
If we express $H_c(s)$ as a partial fraction expansion in $\frac{1}{s-s_k}$, we transform $\frac{1}{(s-s_k)}$ into $\frac{1}{(1-e^{s_k}z^{-1})}$ and get $H(z)$. This leads to

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}} \, .$$

Bilinear Transformation

TECHNISCHE UNIVERSITÄT DARMSTADT

- maps the imaginary axis of the s-plane to the unit circle.



$$z = \frac{\frac{2}{T_d} + s}{\frac{2}{T_d} - s} = \frac{1 + (T_d/2)s}{1 - (T_d/2)s} = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2}.$$

Bilinear Transformation

TECHNISCHE UNIVERSITÄT DARMSTADT

Now let

$$s = \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right],$$

so that

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right].$$

With $z = e^{j\omega}$ the transformation can be expressed as

$$s = \frac{2}{T_d}\cdot\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\frac{2}{T_d}\tan(\omega/2)$$

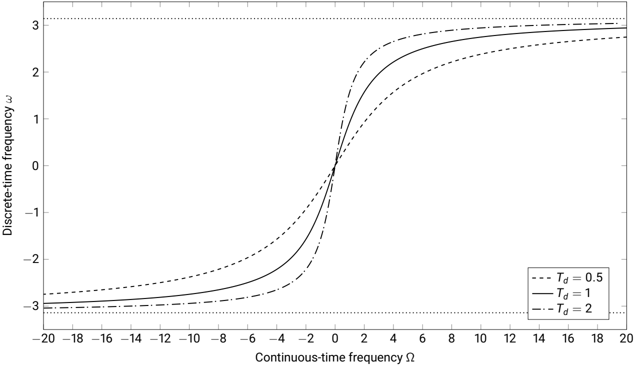
and

$$H\left(e^{j\omega}\right) = H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right).$$

Bilinear Transformation

From $s = j\frac{2}{T_d}\tan(\omega/2) = \sigma + j\Omega$, it follows that

$$\Omega = \frac{2}{T_d}\tan(\omega/2) \text{ and } \sigma = 0.$$



Bilinear Transformation

TECHNISCHE UNIVERSITÄT DARMSTADT

Remarks

- This transformation is the most used approach in practice. It calculates digital filters from known continuous-time filter responses.
- By taking into account the frequency deformation, $\Omega = \frac{2}{T_d} \tan \frac{\omega}{2}$, the detailed shape of $H_c(j\Omega)$ is not preserved.
- The delay time is also modified: $\tau_d = \tau_c[1 + (\omega_c T_d)/2]$. If the continuous-time filter has a constant delay time, the resulting digital filter will not have this property.
- The bilinear transformation has no aliasing problems.

Bilinear Transformation

TECHNISCHE UNIVERSITÄT DARMSTADT

Example

With: $\alpha_1 = 0.10875 \quad \alpha_2 = 0.17783$
$\omega_p = 0.2\pi \quad \omega_s = 0.3\pi \quad T_d = 1$

So we require that

$$0.89125 \leq |H_c(j\Omega)| \leq 1, \qquad 0 \leq |\Omega| \leq \frac{2}{T_d} \tan\left(\frac{0.2\pi}{2}\right),$$

$$|H_c(j\Omega)| \leq 0.17783, \qquad \frac{2}{T_d} \tan\left(\frac{0.3\pi}{2}\right) \leq |\Omega| < \infty.$$

Bilinear Transformation

Since a continuous-time Butterworth filter has monotonic magnitude response, we can equivalently require that

$$|H_c(j2\tan(0.1\pi))| \geq 0.89125$$

and

$$|H_c(j2\tan(0.15\pi))| \leq 0.17783.$$

The form of the magnitude-squared function for the Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}.$$

Bilinear Transformation

Solving for N and Ω_c with the equality sign, we obtain

$$1+\left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89125}\right)^2$$

and

$$1+\left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.17783}\right)^2$$

and solving for N gives

$$N=\frac{\log[((\frac{1}{0.17783})^2-1)/((\frac{1}{0.89125})^2-1)]}{2\log[\tan(0.15\pi)/\tan(0.1\pi)]}=5.30466.$$

We obtain $N=6$ and $\Omega_c=0.76622$.

Bilinear Transformation

For these values of Ω_c, N, the passband specifications are exceeded and stop band specifications are met exactly. As before, the system function of the continuous-time filter is obtained by assigning the left half-plane poles to $H_c(s)$ and the others to $H_c(-s)$.
Using the bilinear transformation, the resulting $H(z)$ will be different from the obtained one using the impulse invariance method.

Note: In practice, the bilinear transformation could lead to a slightly smaller order than the impulse invariance method.

Implementation of IIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

An IIR filter has an infinite extent and cannot be implemented by direct convolution in practice.

The output of the filter can be described by

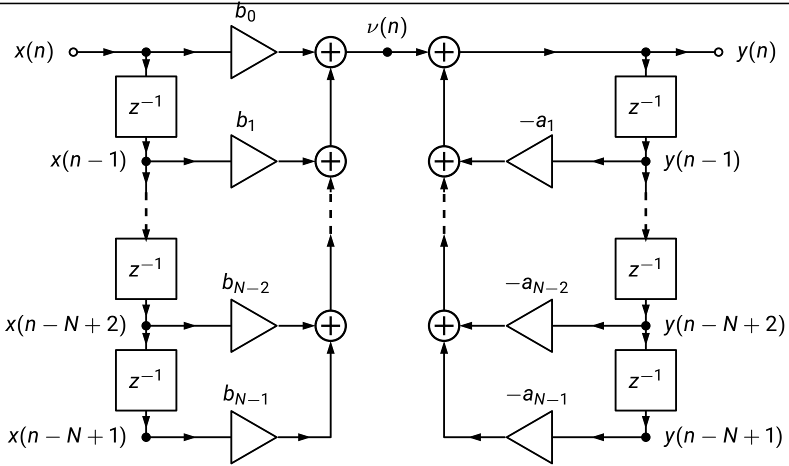
$$y(n) = -\sum_{k=1}^{M} a_k y(n-k) + \sum_{r=0}^{N-1} b_r x(n-r)\,.$$

Implementation of IIR Filters

Direct Form I

In the direct form I the filter $H(z)$ is implemented as a decomposition

$$H(z) = H_2(z)H_1(z) = \frac{1}{1 + \sum_{k=1}^{M} a_k z^{-k}} \left(\sum_{r=0}^{N-1} b_r z^{-r} \right).$$



Signal flow graph of a direct form I structure for an Mth-order system ($N-1=M$).

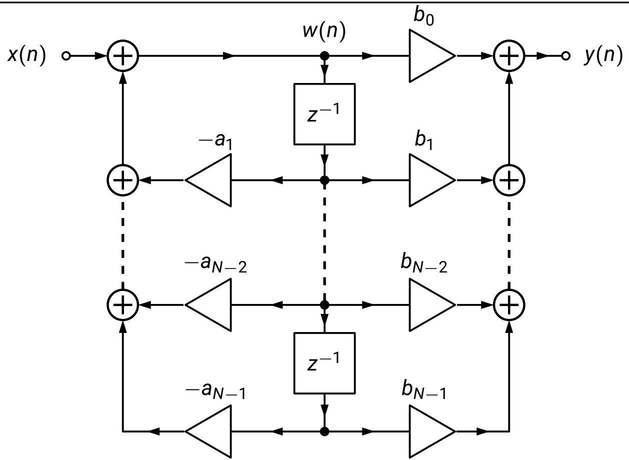
Implementation of IIR Filters

Direct Form II

In the direct form II we change the order of $H_1(z)$ and $H_2(z)$ with the advantage of reduction the number of storage elements by two.

$$H_1(z)H_2(z) = H(z) = H_2(z)H_1(z)$$

Implementation of IIR Filters



Signal flow graph of direct form II structure for an Mth-order system ($N-1=M$).

Implementation of IIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

Cascade form

The system function $H(z)$ can be expressed by

$$H(z) = \frac{\sum_{r=0}^{N-1} b_r z^{-r}}{1 + \sum_{k=1}^{M} a_k z^{-k}} = \prod_k H_k(z)$$

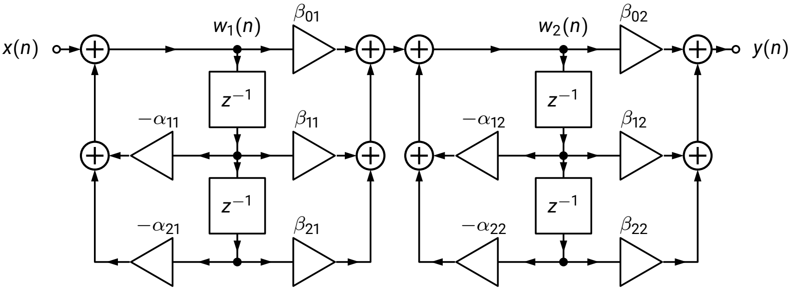
where $H_k(z)$ is given by :

$$H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

Implementation of IIR Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

- The cascade form is less sensitive to coefficient quantisation.
- In the cascade form, the change in one transmittance branch will affect the poles or zeroes only in the second order section where the branch transmittance is affected.



Cascade structure for a 4th-order system.

A Comparison of FIR and IIR Digital Filters

TECHNISCHE UNIVERSITÄT DARMSTADT

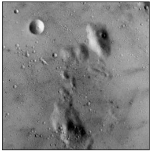
- A variety of frequency selective IIR filters can be designed using closed-form design formulae. For FIR filters closed-form design equations do not exist.
- Closed-form IIR design is limited to standard frequency selective filters and even these disregard the phase response.
- FIR filters can have precise linear phase and be designed for arbitrary frequency responses with almost the same effort.
- An IIR filter will meet a given amplitude response specification more efficiently than an FIR filter. As a consequence the complexity of IIR filters is lower.

Outline

1. Discrete-Time Signals and Systems
2. Digital Signal Processing of Continuous-Time Signals
3. Application: Ultrasound
4. Application: Photoplethysmography (PPG)
5. Filter Design
6. Design of Finite Impulse Response Filter
7. Design of Infinite Impulse Response Filters
8. **Application: 2D Filters for Image Processing**

2D Filters for Image Processing

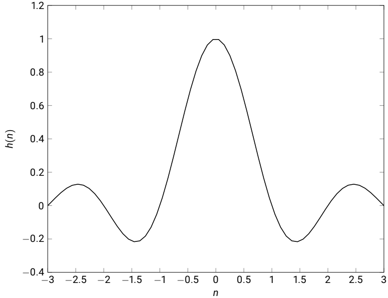
TECHNISCHE UNIVERSITÄT DARMSTADT

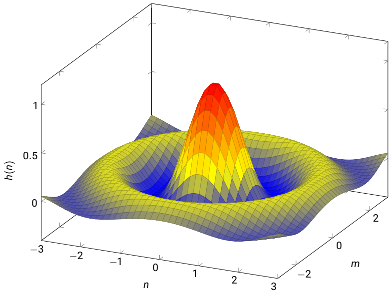


TECHNISCHE UNIVERSITÄT DARMSTADT

The theory of 1D filter design can easily be extended to 2D filter design.

Example: Low-Pass filters





1D sinc function:

$$h(n) = \frac{\sin(\pi n)}{\pi n}$$

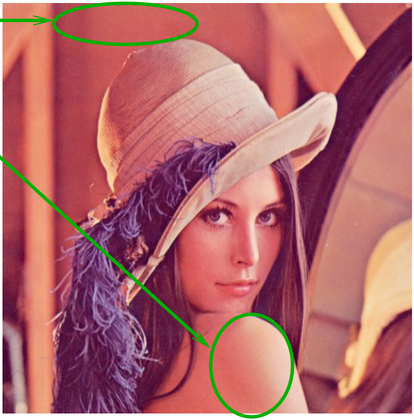
2D sinc function:

$$h(n,m) = \frac{\sin(\pi\sqrt{n^2+m^2})}{\pi\sqrt{n^2+m^2}}$$

SPG

Frequency concept in images

Low frequencies correspond to homogeneous areas in the image (e.g. surfaces)



Frequency concept in images

High frequencies correspond to 'active' areas in the image (e.g. edges, fine details, noise ...)

Example: Low Pass Filter

Noisy image

Low pass filtered image

High cutoff frequency: Only minor noise suppression, but the image details are well preserved

Example: Low Pass Filter

Noisy image

Low pass filtered image

Low cutoff frequency: Very good noise suppression, but the high frequency content in the original image is also affected

Example: Low Pass Filter

TECHNISCHE UNIVERSITÄT DARMSTADT

Noisy image

Low pass filtered image

Tradeoff: Choose cutoff frequency such that an acceptable noise suppression is achieved and important image details are still preserved.

Example: High Pass Filter

Original image

High pass filtered image

High pass filter: Preserves only the high frequency image content $\rightarrow$ HP filters are used for edge detection and object identification