Date

一. convex polyhedron cone

$V = N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ N lattice

$M = \mathrm{Hom}(N, \mathbb{Z})$ $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = V^*$

(rational cone) $\sigma = \{ r_1 v_1 + \cdots + r_s v_s \in V,\ r_i \geq 0 \}$ $v_i \in N$

$\sigma^\vee = \{ u \in V^*,\ \langle u, v \rangle \geq 0 \ \ \forall v \in \sigma \}$

(the dual cone)

Some conclusion:

Gorden's lemma

σ is a rational convex polyhedron cone

$S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup

pf: Take $u_1, \cdots, u_s \in \sigma^\vee \cap M$ that generate $\sigma^\vee$

$K = \{ \sum t_i u_i : 0 \leq t_i \leq 1 \}$

$K \cap M$ is finite

$\forall u \in \sigma^\vee \cap M$, write $u = \sum r_i u_i$ $(r_i \geq 0)$

$r_i = m_i + t_i$ s.t $m_i \in \mathbb{Z}_{\geq 0}$, $0 \leq t_i \leq 1$

$\Rightarrow u = \sum m_i u_i + u'$ where u_i and u' in $K \cap M$

$\Rightarrow K \cap M$ generates S_σ.

Prop: σ is rational, then $\sigma^\vee$ is rational

pf: ① If σ spans V, and $v_1, \cdots, v_s$ is the generators of σ. where $v_i \in N$

Let τ is a facet of σ, then τ

$\tau = \sigma \cap u_\tau^\perp$ for ~~some~~ a $u_\tau \in \sigma^\vee$ unique up to multiplication by a positive scalar

~~Take $v_{i_1}, \cdots, v_{i_{n-1}} \in \tau$ are independent~~

~~and $v_{i_n} \notin \tau$~~

~~Then there exist $u_0 \in M$ s.t~~

~~$\langle u_0, v_{i_1} \rangle = \cdots = \langle u_0, v_{i_{n-1}} \rangle = 0$~~

~~and $\langle u_0, v_{i_n} \rangle = 1$~~

Take $v_\tau \notin \tau$ $\quad \tilde{u}_\tau = \frac{u_\tau}{\langle u_\tau, v_\tau \rangle} \in \sigma^\vee$

then $\langle \tilde{u}_\tau, v_\tau \rangle = 1$

Take $v_{i_1}, \cdots, v_{i_{n-1}} \in \tau$ are independent

then $\{v_{i_1}, \cdots, v_{i_{n-1}}, v_\tau\}$ are independent

$\Rightarrow \tilde{u}_\tau \in M$

$\Rightarrow \tilde{u}_\tau \in M \cap \sigma^\vee$

Let S be a cone generated by $\tilde{u}_\tau$ where τ ranges over the facets of σ.

claim: $S = \sigma^\vee$

① $S \subseteq \sigma^\vee$ is obviously

② $\sigma^\vee \subseteq S$. If $\exists\, u \in \sigma^\vee$ but $u \notin S$

then $\exists\, v \in V$ s.t ~~$\langle v, u$~~ $\langle u, v \rangle < 0$

and $\langle u_\tau, v \rangle \geq 0$

② If $W = \mathbb{R}\cdot\sigma \neq V$

let $W^* = V^*/W^\perp$

then $\sigma^\vee$ is generated by the dual cone in W^* ~~and u~~ together with u and $-u$ as u ranges over the basis of $W^\perp$

(1) σ, τ are faces $\Rightarrow$ $\sigma \cap \tau$ is face

pf: $\bigcap (\sigma \cap u_i^\perp) = \sigma \cap (\Sigma u_i)^\perp$

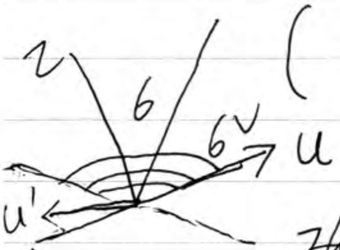
(2) face of a face is a face

pf: $\tau = \sigma \cap u^\perp \quad \gamma = \tau \cap (u')^\perp$

$u \in \cancel{(\sigma^\vee)}\ \sigma^\vee \qquad u' \in \tau^\vee$

$\exists$ large positive p

s.t. $u' + pu \in \sigma^\vee$ and $\gamma = \sigma \cap (u'+pu)^\perp$



$(\langle u'+pu, v\rangle = \langle u', v\rangle + p\langle u, v\rangle$

$\forall v \in \sigma \quad \cancel{p \geq -\frac{\langle u,v\rangle}{\langle u',v\rangle}} \quad \cancel{p \geq \min\langle u',v\rangle}$

~~Then~~ $\cancel{\gamma = \sigma \cap (u'+pu)^\perp}$

$\gamma = \sigma \cap u^\perp \cap (u')^\perp$

$\forall v \in \sigma \cap u^\perp \cap (u')^\perp$

$\langle u'+pu, v\rangle = 0 \Rightarrow v \in \sigma \cap (u'+pu)^\perp$

$\forall v \in (u'+pu)^\perp, \ v \in \sigma$

$\langle u'+pu, v\rangle = \langle u', v\rangle + p\langle u, v\rangle = 0$

(3) $\tau = \sigma \cap u^{\perp}$, $u \in \sigma^{\vee}$
then $\tau^{\vee} = \sigma^{\vee} + \mathbb{R}_{\geq 0}(-u)$

pf: It is enough to prove their dual are equal

$(\tau^{\vee})^{\vee} = \tau$

$(\sigma^{\vee} + \mathbb{R}_{\geq 0}(-u))^{\vee} = \{ v \in V \mid \langle u, v \rangle \geq 0 \ \forall u \in \sigma^{\vee} + \mathbb{R}_{\geq 0}(-u) \}$

① $v \in \sigma \cap (-u)^{\vee}$

$\Rightarrow v \in \sigma$ $\cancel{\langle v, u \rangle = 0}$ $-\langle u, v \rangle \geq 0$

$\Rightarrow \langle u, v \rangle = \langle u_0, v \rangle - r \langle u, v \rangle \geq 0$

$\Rightarrow v \in (\sigma^{\vee} + \mathbb{R}_{\geq 0}(-u))^{\vee}$

② $v \in (\sigma^{\vee} + \mathbb{R}_{\geq 0}(-u))^{\vee}$

$\cancel{\langle u_0, v \rangle - r \langle u, v \rangle \geq 0}$

$r = 0 \Rightarrow v \in \sigma$ $u_0 = 0 \Rightarrow v \in (-u)^{\vee}$

① + ② $\Rightarrow (\sigma^{\vee} + \mathbb{R}_{\geq 0}(-u))^{\vee} = \sigma \cap (-u)^{\vee}$

(4) σ is rational

$\tau < \sigma$ then $\tau = \sigma \cap u^{\perp}$ $u \in S_{\sigma} = \sigma^{\vee} \cap M$

and $S_{\tau} = S_{\sigma} + \mathbb{Z}_{\geq 0}(-u)$

pf: $\tau < \sigma \Rightarrow \tau = \sigma \cap u^{\perp}$
u is in the relative interior of $\sigma^{\vee} \cap \tau^{\perp}$

$\sigma^{\vee} \cap \tau^{\perp}$ rational $\Rightarrow u \in M$
Take $w \in S_{\tau}$ $\exists$ large positive integre p s.t. $w + pu \in \sigma^{\vee}_{\cap M}$

(6) $\gamma < \sigma$, $\tau < \sigma'$

then there is a u in $\sigma^\vee \cap (-\sigma')^\vee$

with
$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$$

pf: Let $\gamma = \sigma - \sigma' = \sigma + (-\sigma')$

Take u in the relative interior of $\gamma^\vee$

then $\gamma \cap u^\perp$ is the smaller face of γ

i.e
$$\gamma \cap u^\perp = \gamma \cap (-\gamma)$$
$$= \cancel{\sigma \cap} (\sigma - \sigma') \cap (\sigma' - \sigma)$$

① $u \in \gamma^\vee = (\sigma - \sigma')^\vee$
$$= \sigma^\vee \cap (-\sigma')^\vee$$

② "$\tau \subseteq \sigma \cap u^\perp$" and "$\tau \subseteq \sigma' \cap u^\perp$"

$\tau \subseteq \gamma \cap (-\gamma) \cancel{= \gamma \cap u^\perp} \subseteq u^\perp$

$\Rightarrow \tau \subseteq \sigma \cap u^\perp$

③ "$\sigma \cap u^\perp \subseteq \tau$"

$\forall\ v \in \sigma \cap u^\perp \subseteq \gamma \cap u^\perp \subseteq \sigma' - \sigma$

Then $\exists\ w' \in \sigma'$, $w \in \sigma$

$v = w' - w$

$\Rightarrow v + w \in \sigma' \cap \sigma = \tau$

$\Rightarrow v \in \tau$

~~④ "$\tau \subseteq \sigma' \cap u^\perp$"~~ ④ "$\sigma' \cap u^\perp \subseteq \tau$"

$\forall\ v \in \sigma' \cap u^\perp \subseteq \sigma - \sigma'$ $\exists\ w' \in \sigma'$, $w \in \sigma$

s.t $v + w' = w \in \sigma' \cap \sigma = \tau \Rightarrow v \in \tau$

σ, σ' rational

(7) $\tau = \sigma \cap \sigma'$

$\Rightarrow S_\tau = S_\sigma + S_{\sigma'}$

Pf: "$\supseteq$" $S_\tau \supseteq S_\sigma + S_{\sigma'}$ is obvious

"$\subseteq$" Take $u \in \sigma^\vee \cap (-\sigma')^\vee \cap M$

s.t $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$

Then $-u \in S_{\sigma'}$

and $S_\tau \subset S_\sigma + \mathbb{Z}_{\geq 0}(-u) \subset S_\sigma + S_{\sigma'}$

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二. Affine variety

1. Group ring $R[S]$ R: ring S: group

~~$f\in$~~ $R[S]=\{f \mid f: S\to R$ of finite support$\}$

① $R[S]$ as a free R-module

$f, g\in R[S]$ $f+g: x\mapsto f(x)+g(x)$

$\alpha f: x\mapsto \alpha\cdot f(x)$

~~basis: f_g: $f(g)=1$ ... 0 other~~

② $R[S]$ as a ring

$f, g\in R[S]$ $f\cdot g: x\mapsto \sum_{uv=x} f(u)g(v)$

③ if R commutative, $R[S]$ as a R-algebra

$S_6 = 6^{\vee}\cap M$ is a finitely generated semigroup

$\Rightarrow \mathbb{C}[S_6]$ is a commutative $\mathbb{C}$-algebra

Its basis: X^u, u varies over S_6 (as a complex vector space)

$X^u\cdot X^{u'} = X^{u+u'}$

the unit 1 is X^0

$\{u_i\}$ the generators for S_6

$\Rightarrow \{X^{u_i}\}$ the generator for $\mathbb{C}[S_6]$

2. Spec$[A]$ a complex affine variety

where A is a finitely generated commutative $\mathbb{C}$-algebra

finitely generated
commutative $\mathbb{C}$-algebra $\longrightarrow$ $A \cong \mathbb{C}[X_1, \cdots, X_m]/I$
A
(domain)

I ideal (I prime)

$\updownarrow V$

$\mathrm{Spec}(A) \stackrel{\Delta}{=} V(I)$ (irreducible)
$\subseteq \mathbb{C}^m$

maximal ideal of A $\longleftrightarrow$ $P \subseteq \mathrm{Spec}(A)$ P is closed point

$\mathrm{Specm}(A) = \{$all closed points$\}$

prime ideal of A $\longleftrightarrow$ $V \subseteq \mathrm{spec}(A)$ V subvariety of $V(I)$

homomorphism
$A \longrightarrow B$
$\mathrm{Spec}(A) \longleftarrow \mathrm{Spec}(B)$
morphism

① In particular
$A \to \mathbb{C}$
(closed point $\leftarrow \mathbb{C}$)

② $\forall f \in A$
localization homomorphism
$A \to A_f$
$\Rightarrow X_f = \mathrm{Spec}(A_f) \subseteq X = \mathrm{Spec}(A)$

$\mathbb{C}[S_6] \longrightarrow U_6 = \mathrm{spec}(\mathbb{C}[S_6])$
$\parallel$
A_6

prop: $\mathbb{C}[S_6]$ is integral

$S_6 \hookrightarrow \mathbb{Z}^n$

$\mathbb{C}[S_6] \hookrightarrow \mathbb{C}[X_1, \cdots, X_n, X_1^{-1}, \cdots, X_n^{-1}]$
$= (((\mathbb{C}[X_1, \cdots, X_n])_{X_1})_{X_2} \cdots)_{X_n}$

(1) example

① singular example

rank $(N) = 3$ N lattice

$\sigma = \mathbb{Z}_{\geq 0} v_1 + \mathbb{Z}_{\geq 0} v_2 + \mathbb{Z}_{\geq 0} v_3 + \mathbb{Z}_{\geq 0} v_4$

where $v_1 + v_3 = v_2 + v_4$

let $v_i = e_i$ $i = 1, 2, 3$ $v_4 = e_1 + e_3 - e_2$

$S_\sigma = \mathbb{Z}_{\geq 0} e_1^* + \mathbb{Z}_{\geq 0} e_3^* + \mathbb{Z}_{\geq 0}(e_1^* + e_2^*)$
$+ \mathbb{Z}_{\geq 0}(e_2^* + e_3^*)$

$\Rightarrow A_\sigma = \mathbb{C}[S_\sigma]$
$= \mathbb{C}[X_1, X_3, X_1X_2, X_2X_3]$
$= \mathbb{C}[W, X, Y, Z]/(WZ - XY)$

② $S_\sigma = \sum_{i=1}^{t} \mathbb{Z}_{\geq 0} u_i$

$\Rightarrow A_\sigma = \mathbb{C}[X^{u_1}, \cdots, X^{u_t}] = \mathbb{C}[Y_1, \cdots, Y_t]/I$

$I = (f)$, where $f = Y_1^{a_1} Y_2^{a_2} \cdots Y_t^{a_t}$
$- Y_1^{b_1} \cdots Y_t^{b_t}$

$a_1 u_1 + \cdots + a_t u_t = b_1 u_1 + \cdots + b_t u_t$

~~(2) $\tau < \sigma$ $U_\sigma = \mathrm{spec}\,\mathbb{C}[S_\sigma]$~~ & $U_\tau = \mathrm{spec}\,\mathbb{C}[S_\tau]$

③ $M = S_{\{0\}}$ $e_1^*, \cdots, e_n^*$ dual basis of M

$X_1 = X^{e_1^*} \in \mathbb{C}[M]$

$\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}]$

$U_{\{0\}} = \mathbb{C}^* \times \cdots \times \mathbb{C}^* = (\mathbb{C}^*)^n$

$\forall S_\sigma \subset M$ $\mathbb{C}[S_\sigma]$ is a subalgebra of $\mathbb{C}[M]$

④ σ with generators $e_1, \cdots, e_k$ $(1 \le k \le n)$

$S_\sigma = \mathbb{Z}_{\ge 0} e_1^* + \cdots + \mathbb{Z}_{\ge 0} e_k^*$
$\quad + \mathbb{Z} e_{k+1}^* + \cdots + \mathbb{Z} e_k^*$

~~U_σ~~ $A_\sigma = \mathbb{C}[X_1, X_2, \cdots X_k, X_{k+1}, X_{k+1}^{-1}, \cdots X_n, X_n^{-1}]$

$U_\sigma =$ ~~$\mathbb{C} \times \mathbb{C}$~~ $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$

Such U_σ is nonsingular

(2) $\tau < \sigma$

- If A, B $\mathbb{C}$-algebra
  $\varphi: A \to B$ homomorphism
  determins a morphism $\mathrm{spec}(B) \to \mathrm{Spec}(A)$

  $S_\sigma \to S_\tau$
- $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\tau]$

  $U_\tau = \mathrm{spec}(\mathbb{C}[S_\tau]) \longrightarrow U_\sigma = \mathrm{spec}(\mathbb{C}[S_\sigma])$

Lemma: $\tau < \sigma$, then $U_\tau \to U_\sigma$ embeds U_τ as a principle open subset of U_σ

pf: $\exists u \in S_\sigma$ with $\tau = \sigma \cap u^\perp$
and $S_\tau = S_\sigma + \mathbb{Z}_{\ge 0}(-u)$

$\Rightarrow$ each basis element for $\mathbb{C}[S_\tau]$ can be written in the form

$X^{w-pu} = X^w / (X^u)^p \quad w \in S_\sigma$

$\Rightarrow A_\tau = (A_\sigma)_{X^u} \qquad U_\tau = U_\sigma \setminus V(f)$
$\qquad f \in \mathbb{C}[S_\sigma]$

(3) $T_N = \mathrm{Spec}(\mathbb{C}[M])$

$t \in T_N \longleftrightarrow$ a ϕ map $M \to \mathbb{C}^*$

$x \in U_\sigma \longleftrightarrow$ a map $S_\sigma \to \mathbb{C}^*$

$T_N \times U_\sigma \to U_\sigma$

$(t, x) \mapsto t \cdot x : S_\sigma \to \mathbb{C}^*$

$u \mapsto t(u)x(u)$

the dual map $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$

$\chi^u \mapsto \chi^u \otimes \chi^u$

Ex: σ a cone in N, σ' is a cone in N'

$\sigma \times \sigma'$ is a cone in $N \oplus N'$

and construct a canonical isomorphism

$U_{\sigma\times\sigma'} = U_\sigma \times U_{\sigma'}$

sol: σ with generator $v_1 \cdots v_n$

σ' with genarator $w_1 \cdots w_m$

$\sigma \times \sigma' \cong \sum_{i=1}^{n} \mathbb{R}_{\geq 0} v_i \oplus 0 + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}\, 0 \oplus w_i$

$S_{\sigma\times\sigma'} = (\sigma \times \sigma')^\vee \cap \cancel{(N \oplus N')^*}$ $M \oplus M'$

$\cong$

$S_\sigma \times S_{\sigma'}$

$\mathbb{C}[S_\sigma \times S_{\sigma'}] = \mathbb{C}[S_\sigma] \times \mathbb{C}[S_{\sigma'}]$

三. fans and toric variety

Def: (fans)

Δ: a set of rational strongly convex polyhedral cones

satisfy: ① $\sigma \in \Delta$, $\tau < \sigma \Rightarrow \tau \in \sigma$

② $\sigma, \sigma' \in \Delta \Rightarrow \sigma \cap \sigma' < \sigma$, $\sigma \cap \sigma' < \sigma'$

assume: fans are finite

toric variety $X(\Delta)$ disjoint union of the affine toric variety U_σ.

Gluing lemma

Let $\{X_i\}$ be a family of scheme.

For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have induced structure

~~Let~~ ~~$\tau < \sigma$~~ Given σ_i, σ_j ~~be~~ are the faces of σ, then $\underbrace{\sigma_{ij} = \sigma_i \cap \sigma_j}_{\S 1.2(3)}$ is also a face of σ

$\sigma_i, \sigma_j \in \Delta$

$\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$ $\quad$ $\sigma_{ij} < \sigma_i$, $\sigma_{ij} < \sigma_j$

~~def → ℓ~~

Then $\mathcal{U}_{\sigma_{ij}} \subseteq \mathcal{U}_{\sigma_i}$ is an open set of $\mathcal{U}_{\sigma_i}$

$$\underline{\mathcal{O}_{\sigma_{ij}} = \mathcal{O}_{\sigma_i}\big|_{\mathcal{U}}}$$

We have $\mathcal{U}_{\sigma_{ij}} = \mathcal{U}_{\sigma_{ji}}$ and ~~$\varphi_{ij} : \mathcal{U}_{ij} \to \mathcal{U}_{ji}$~~

$$\varphi_{ij} : \mathcal{U}_{\sigma_{ij}} \to \mathcal{U}_{\sigma_{ji}} := \mathrm{id}$$

Obviously, (1) for each i, j $\varphi_{ji} = \varphi_{ij}^{-1}$

(2) $\varphi_{ij}(\mathcal{U}_{\sigma_{ij}} \cap \mathcal{U}_{\sigma_{ik}})$ $\quad \sigma_{ij} = \sigma_i \cap \sigma_j$

$\sigma_{ik} = \sigma_i \cap \sigma_k$

~~$\forall x \in$~~

$\sigma_i = \sigma \cap u_i^{\perp}$ $\quad \sigma_j = \sigma \cap u_j^{\perp}$ $\quad \sigma_k = \sigma \cap u_k^{\perp}$

$(u_i, u_j, u_k \in \sigma^{\vee})$

$$\sigma_{ij} = (\sigma \cap u_i^{\perp}) \cap (\sigma \cap u_j^{\perp})$$

$$= \sigma \cap (u_i + u_j)^{\perp}$$

$$\sigma_{ik} = \sigma \cap (u_i + u_k)^{\perp}$$

$$\sigma_{jk} = \sigma \cap (u_j + u_k)^{\perp}$$

$$S_{\sigma_i} = S_\sigma + \mathbb{Z}_{\geq 0}(-u_i) \quad (j, k)$$

$$S_{\sigma_{ij}} = S_\sigma + \mathbb{Z}_{\geq 0}(-(u_i + u_j)) \quad (ik, jk)$$

$$A_{\sigma_{ij}} = (A_{\sigma_i})_{\chi^{u_j}} \qquad \mathcal{U}_{\sigma_{ij}} = \mathcal{U}_{\sigma_i} - V(\chi^{u_j})$$

$$= \mathcal{U}_{\sigma_j} - V(\chi^{u_i})$$

$$\mathcal{U}_{\sigma_{ij}} \cap \mathcal{U}_{\sigma_{ik}} = (\mathcal{U}_{\sigma_i} - V(\chi^{u_j})) \cap (\mathcal{U}_{\sigma_i} - V(\chi^{u_k}))$$

$$= \cancel{\mathcal{U}_{\sigma_i} - (V(\chi^{u_j}) \cup V(\chi^{u_k}))}$$

$$= \mathcal{U}_\sigma \setminus V(\chi^{u_i}) \setminus V(\chi^{u_j}) \setminus V(\chi^{u_k})$$

$$\mathcal{U}_{\sigma_{ji}} \cap \mathcal{U}_{\sigma_{jk}} = \mathcal{U}_\sigma \setminus V(\chi^{u_j}) \setminus V(\chi^{u_i}) \setminus V(\chi^{u_k})$$

$$\varphi_{jk}\circ\varphi_{ij}(U_{ij}\cap U_{ik}) = \varphi_{jk}(U_{ji}\cap U_{jk})$$
$$= U_{kj}\cap U_{ki}$$
$$= \varphi_{ik}(U_{ij}\cap U_{ik})$$

~~According to Glueing Lemma~~

Given a fan Δ, and $\sigma_i, \sigma_j \in \Delta$.

Then $\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$ $\quad \cancel{U_\sigma}\ \sigma_{ij} < \sigma_i \quad \sigma_{ij} < \sigma_j$

$U_{\sigma_{ij}} \subseteq U_{\sigma_i}$ is an open set of U_{σ_i}

~~(1) φ_{ji} =~~ $\varphi_{ij} : U_{\sigma_{ij}} \to U_{\sigma_{ji}} := id$

(1) $\varphi_{ji} = \varphi_{ij}^{-1}$

(2) $\sigma_{ij} = \sigma_i \cap u_{ij}^{\perp}$ $\quad (u_{ij} \in S_{\sigma_i})$ $\quad u_{ij} \in \sigma_i^{\vee} \cap (-\sigma_j)^{\vee} \cap M$

$\quad = \sigma_j \cap u_{ij}^{\perp}$ $\quad (u_{ji} \in S_{\sigma_j})$

$\sigma_{ik} = \sigma_i \cap u_{ik}^{\perp}$ $\quad (u_{ik} \in S_{\sigma_i})$ $\quad u_{ik} \in$

$\quad = \sigma_k \cap u_{ki}^{\perp}$ $\quad (u_{ki} \in S_{\sigma_k})$

$\sigma_{jk} = \sigma_j \cap u_{jk}^{\perp}$ $\quad (u_{jk} \in S_{\sigma_j})$

$\quad = \sigma_k \cap u_{kj}^{\perp}$ $\quad (u_{kj} \in S_{\sigma_k})$

$A_{\sigma_{ij}} = (A_{\sigma_i})_{x^{u_{ij}}}$ $\quad U_{\sigma_{ij}} = U_{\sigma_i} \setminus V(x^{u_{ij}})$

$\quad = U_{\sigma_j} \setminus V(x^{u_{ji}})$

$U_{\sigma_{ik}} = \cancel{U_{\sigma_i}}\ U_{\sigma_i} \setminus V(x^{u_{ik}})$ $\quad U_{\sigma_{jk}} = U_{\sigma_j} \setminus V(x^{u_{jk}})$

$\quad = U_{\sigma_k} \setminus V(x^{u_{ki}})$ $\quad = U_{\sigma_k} \setminus V(x^{u_{kj}})$

$S_{\sigma_{ij}} = S_{\sigma_i} + S_{\sigma_j}$ $\quad \varphi(U_{\sigma_{ij}} \cap U_{\sigma_{ik}})$

$\quad = U_{\sigma_i} \setminus V(x^{u_{ij}}) \setminus V(x^{u_{ik}})$

(2) $\sigma_{ij} = \sigma_i \cap u_{ij}^{\perp} = \sigma_j \cap u_{ij}^{\perp}$

$u_{ij} \in \sigma_i^{\vee} \cap (-\sigma_j^{\vee}) \cap M$

$S_{\sigma_{ij}} = S_{\sigma_i} + \mathbb{Z}_{\geq 0}(-u_{ij}) = S_{\sigma_j} + \mathbb{Z}_{\geq 0}(u_{ij})$

$(-u_{ij} \in \sigma_j^{\vee})$

$A_{\sigma_{ij}} = (A_{\sigma_i})_{\chi^{u_{ij}}} = (A_{\sigma_j})_{\chi^{-u_{ij}}}$

$\Rightarrow U_{\sigma_{ij}} = U_{\sigma_i} \setminus V(\chi^{u_{ij}})$

$= U_{\sigma_j} \setminus V(\chi^{-u_{ij}})$

Since $V(\chi^{u_{ij}}) \cap V(\chi^{-u_{ij}}) = \phi$,

$U_{\sigma_{ij}} = U_{\sigma_i} \cap U_{\sigma_j}$

~~Then~~ Similarly, $U_{\sigma_{ik}} = U_{\sigma_i} \cap U_{\sigma_k}$

$U_{\sigma_{jk}} = U_{\sigma_j} \cap U_{\sigma_k}$

Then $\varphi_{ij}(U_{\sigma_{ij}} \cap U_{\sigma_{ik}}) = U_{\sigma_i} \cap U_{\sigma_j} \cap U_{\sigma_k}$

$= U_{\sigma_{ji}} \cap U_{\sigma_{jk}}$

and $\varphi_{jk} \circ \varphi_{ij}(U_{\sigma_{ij}} \cap U_{\sigma_{ik}}) = \varphi_{jk}(U_{\sigma_{ji}} \cap U_{\sigma_{jk}})$

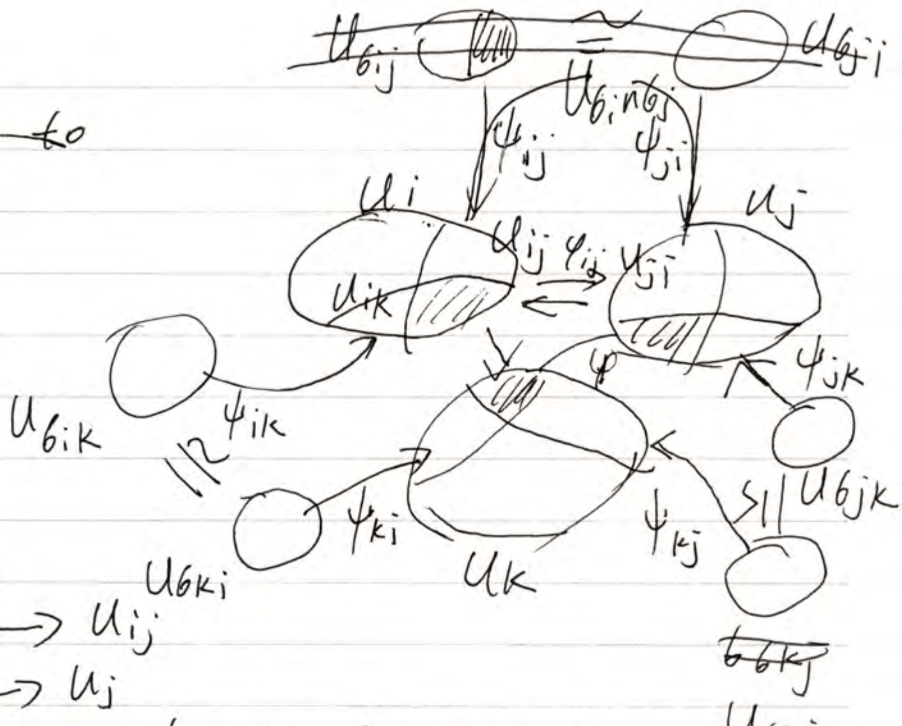
$= U_{\sigma_{kj}} \cap U_{\sigma_{ki}}$

$= \varphi_{ik}(U_{\sigma_{ij}} \cap U_{\sigma_{ik}})$

~~According to glueing lemma~~

~~We need to~~

~~①~~



Denote:

$\psi_{ij} : U_{\sigma_{ij}} \to U_{ij}$

$\varphi_{ij} : U_i \to U_j$

We have proved all the U_{ij} are the open subset of U_i

We need to confirm the following three conditions

① $\varphi_{ji} = \varphi_{ij}^{-1}$

$$\varphi_{ij}^{-1} = \cancel{\psi_{ji}}\ \psi_{ij} \circ \psi_{ji}^{-1} = \varphi_{ji}$$

② $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$

$$S_{\sigma_{ij}} = S_{\sigma_i} + \mathbb{Z}_{\geq 0}(-u_{ij})$$
$$\parallel$$
$$S_{\sigma_{ji}} = S_{\sigma_j} + \mathbb{Z}_{\geq 0}(-u_{ji})$$

$\varphi_{ij}(U_{\sigma_i \cap \sigma_j})$

$$A_{\sigma_i \cap \sigma_j} = (A_{\sigma_i})_{\chi^{u_{ij}}} \qquad U_{ij} = U_i \setminus V(\chi^{u_{ij}})$$
$$= (A_{\sigma_j})_{\chi^{u_{ji}}} \qquad U_{ji} = U_j \setminus V(\chi^{u_{ji}})$$

On the one hand

~~$U_{ijk} \cong$~~ $U_{ij} \cap U_{ik} =$ ~~$U_i \setminus V(\chi^{u_{ij}}) \setminus V(\chi^{u_{ik}})$~~

$$= U_{ij} \setminus V(\chi^{u_{ik}})$$

and $\left((A_{\sigma_i})_{\chi^{u_{ij}}}\right)_{\chi^{u_{ik}}}$

$$= (A_{\sigma_i})_{\chi^{u_{ij} + u_{ik}}}$$

Prop: $(A_f)_g = A_{fg}$

$\therefore$ $U_{ji} \cap U_{jk} = U_{ji} \setminus V(\chi^{jk})$

$= \underline{U_j \setminus V(\chi^{ji}) \setminus V(\chi^{jk})}$

$((A_{\sigma_j})_{\chi^{ji}})_{\chi^{jk}} = (A_{\sigma_j})_{\chi^{ji+jk}}$

On the other hand

$\sigma_i \cap (u_{ij} + u_{ik})^\perp$

$= \sigma_i \cap u_{ij}^\perp \cap u_{ik}^\perp$

$= \sigma_{ij} \cap \sigma_{ik}$

$= \sigma_i \cap \sigma_j \cap \sigma_k$

$\sigma_j \cap (u_{ji} + u_{jk})^\perp$

$= \sigma_j \cap u_{ji}^\perp \cap u_{jk}^\perp$

$= \sigma_{ji} \cap \sigma_{jk} = \sigma_i \cap \sigma_j \cap \sigma_k$

$\Rightarrow \sigma_i \cap (u_{ij} + u_{ik})^\perp$

$= \sigma_j \cap (u_{ji} + u_{jk})^\perp$

$\Rightarrow A_{\sigma_i \cap \sigma_j \cap \sigma_k} = (A_{\sigma_i})_{\chi^{u_{ij}+u_{ik}}} = (A_{\sigma_j})_{\chi^{u_{ji}+u_{jk}}}$

Thus $\varphi_{ij}(U_{ij} \cap U_{ik}) = \psi_{ji} \circ \psi_{ij}^{-1}(U_{ij} \cap U_{ik})$

(preimage $\quad = \psi_{ji}(U_{\sigma_i \cap \sigma_j \cap \sigma_k})$

$= U_{ji} \cap U_{jk}$

Then

$\varphi_{jk} \circ \varphi_{ij}(U_{ij} \cap U_{ik})$

$= \varphi_{jk}(U_{ji} \cap U_{jk}) = U_{ki} \cap U_{kj}$

$= \varphi_{ik}$

(consistency)

Example

1. ~~When $N=\mathbb{Z}$~~ In dimension one, ~~wher~~ with $N=\mathbb{Z}$

All possible cones are σ_+, σ_-, σ_0

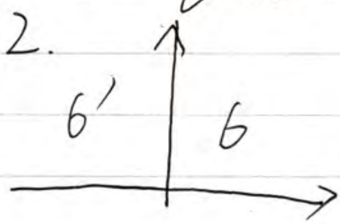
① $\Delta=\{\sigma_+,\sigma_-,\sigma_0\}$ $\quad X(\Delta)=\mathbb{P}^1$

② $\Delta=\{\sigma_+,\sigma_0\}$ corresponding $X(\Delta)=\mathbb{C}$

③ $\Delta=\{\sigma_-,\sigma_0\}$ $\quad X(\Delta)=\mathbb{C}$

④ $\Delta=\{\sigma_0\}$ $\quad X(\Delta)=\mathbb{C}^*$

2.



In dimension two

$\Delta=\{\sigma,\sigma',\tau,\{0\}\}$

~~$U_\sigma=\mathbb{C}$~~ $A_\sigma=\mathbb{C}[X,Y]$

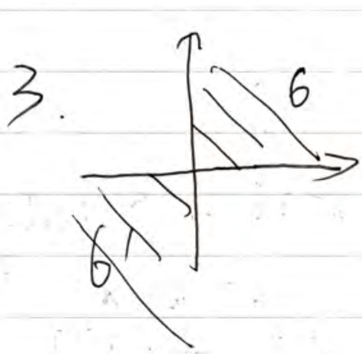
$A_{\sigma'}=\mathbb{C}[X^{-1},Y]$

$U_\sigma=\mathbb{C}\times\mathbb{C}$ $\quad U_{\sigma'}=\mathbb{C}\times\mathbb{C}$

the ~~cross~~ product of $\mathbb{C}$ and $\mathbb{C}$

$\Rightarrow X(\Delta)=\mathbb{P}^1\times\mathbb{C}$

3.



$\mathbb{C}[S_\sigma]=\mathbb{C}[X,Y]$

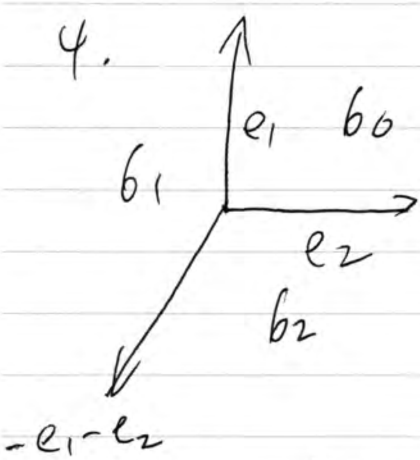
$U_\sigma=\mathbb{C}\times\mathbb{C}$

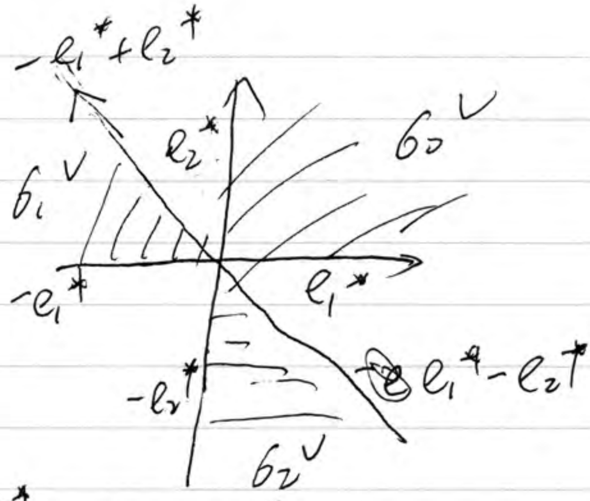
$\mathbb{C}[S_{\sigma'}]=\mathbb{C}[X^{-1},Y^{-1}]$

$U_{\sigma'}=\mathbb{C}\times\mathbb{C}$

$X(\Delta)=\mathbb{P}^1\times\mathbb{P}^1$

4.

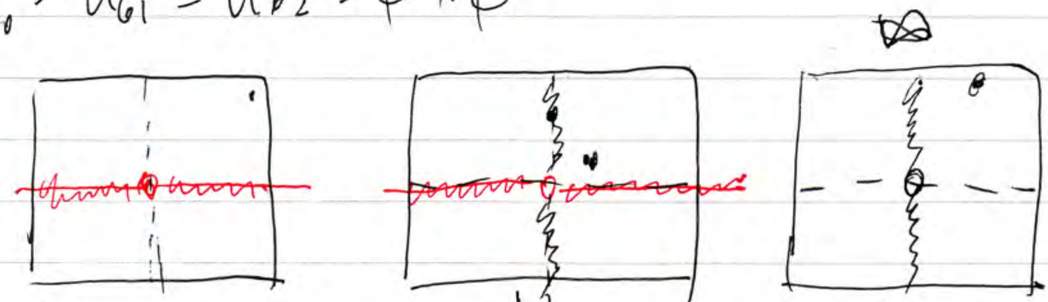




denote $\chi^{e_1^*} = X$, $\chi^{e_2^*} = Y$

$$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X, Y]$$
$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X^{-1}, X^{-1}Y] \cong \mathbb{C}[X, Y]$$
$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[XY^{-1}, Y^{-1}]$$

$$U_{\sigma_0} = U_{\sigma_1} = U_{\sigma_2} = \mathbb{C} \times \mathbb{C}$$

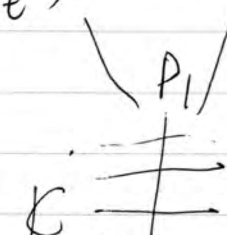


U_{σ_0} $\quad$ U_{σ_1} $\quad$ U_{σ_2}

(u, v) $\quad$ (w, z) $\quad$ (s, t)

$= (\frac{1}{w}, \frac{z}{w})$ $\quad$ $= (\frac{s}{t}, \frac{1}{t})$ $\quad$ $= (\frac{w}{z}, \frac{1}{z})$

$= (\frac{t}{s}, \frac{1}{s})$



$$\mathbb{C}[S_{\sigma_0 \cap \sigma_1}] = \mathbb{C}[X, X^{-1}, Y]$$

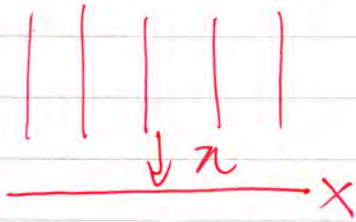
$$\mathbb{C}[X, Y] \longmapsto \mathbb{C}[X, Y]_X \underset{id}{\cong} \mathbb{C}[X^{-1}, X^{-1}Y]_{X^{-1}} \longleftarrow \mathbb{C}[X^{-1}, X^{-1}Y]$$

$X \longmapsto X$ $\qquad$ $X^{-1} \longleftarrow X^{-1}$

$Y \longmapsto Y$ $\qquad$ $X^{-1}Y \longleftarrow X^{-1}Y$

(u, v) $\qquad$ $u = \frac{1}{w}$ $\quad$ $v = \frac{z}{w}$ $\qquad$ (w, z)

The line bundle



- $\pi^{-1}(x) \simeq \mathbb{C}$
- $\forall x \in X$, $\exists$ open neighborhood $U \ni x$

  s.t. $\pi^{-1}(U) \underset{\varphi_U}{\simeq} U \times \mathbb{C}$
- on $U \cap V$

  $\varphi_V \circ \varphi_U^{-1} : (U \cap V) \times \mathbb{C} \to (U \cap V) \times \mathbb{C}$

  is fibrewise linear $(x, v) \to (x, (\varphi_{UV}(x))(v))$

  ($\varphi_{UV} : U \cap V \to GL(\mathbb{C})$ is <u>algebraic</u>
  morphism)

$L \xrightarrow{\pi} X$ is called an algebraic line bundle on X.

- X complex mainfold, L is an holomorphic line bundle
- X is algebraic scheme, $E \xrightarrow{\pi} X$

  is called an algebraic vector bundle

  if $\pi^{-1}(x) \simeq \mathbb{C}^r$

  $\varphi_{UV} : \cancel{U \times V}\ U \cap V \to GL_r(\mathbb{C})$

  is algebraic morphism

Ex: $v_0, v_1, \cdots, v_n$ generate a lattice N of rank n, with $v_0 + v_1 + \cdots + v_n = 0$. Let Δ be the fan whose cones are generated by any proper subset of the vectors $v_0, \cdots, v_n$. Construct $X(\Delta) \cong \mathbb{P}^n$

Sol: Take $v_1, \cdots, v_n$ to be the standard basis $e_1, \cdots, e_n$ for $N = \mathbb{Z}^n$, with

$v_0 = -e_1 - e_2 \cdots - e_n$

$\sigma_0 = \mathrm{Cone}\{e_1, \cdots, e_n\}$

$\sigma_i = \mathrm{Cone}\{e_1, \cdots, \hat{e_i}, \cdots, e_n, -e_1 - e_2 \cdots - e_n\}$

$S_{\sigma_0} = \mathrm{Cone}\{e_1^*, \cdots, e_n^*\}$

$S_{\sigma_i} = \mathrm{Cone}\{e_1^* - e_i^*, \cdots, -e_i^*, \cdots e_n^* - e_i^*\}$

$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2, \cdots, X_n]$

$\mathbb{C}[S_{\sigma_i}] = \mathbb{C}[X_1 X_i^{-1}, \cdots, X_i^{-1}, \cdots X_n X_i^{-1}]$

~~We~~ $U_{\sigma_i} = \mathrm{spec}(\mathbb{C}[S_\sigma])$

Let $(u_{i1}, \cdots, u_{in})$ be the coordinate of U_{σ_i}

We have $(u_{01}, \cdots u_{0n})$

$$= \left(\frac{u_{i1}}{u_{ii}}, \cdots \underset{\substack{\uparrow \\ i\text{-th}}}{\frac{1}{u_{ii}}}, \cdots, \frac{u_{in}}{u_{ii}}\right)$$

$(u_{i1}, \cdots, u_{in})$

$$= \left(\frac{u_{01}}{u_{0i}}, \cdots, \underset{\substack{\uparrow \\ i\text{-th}}}{\frac{1}{u_{0i}}}, \cdots \frac{u_{0n}}{u_{0i}}\right) = \left(\frac{u_{j1}}{u_{ji}}, \cdots \underset{\substack{\uparrow \\ j\text{-th}}}{\frac{1}{u_{ji}}}, \cdots \underset{\substack{\uparrow \\ i\text{-th}}}{\frac{u_{jj}}{u_{ji}}}, \cdots \frac{u_{jn}}{u_{ji}}\right)$$

Toric variety from polytope

Def: Convex polytope K in a finite dimension vector space E is the convex hull of a finite set of points

$$K = \mathrm{Conv}(v_0, \cdots, v_n)$$

Def: (face)

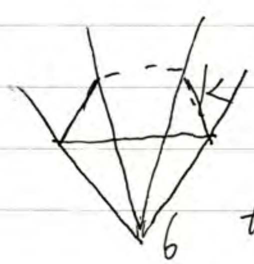
A face ~~of~~ F of K is

$$F = \{v \in K : \langle u, v \rangle = r\}$$

where $u \in E^*$ with $\langle u, v \rangle \geq r$ for all v in K

Assume: K is in n-dimension, K contains the origin in ~~ith~~ its interior $0 \in K$

Let σ be the cone over $K \times 1$ in the vector~~a~~ space $E \times R$



the faces of cone σ $\longleftrightarrow$ the faces of K

the ~~cone~~ face $\{0\}$ $\longleftrightarrow$ empty set

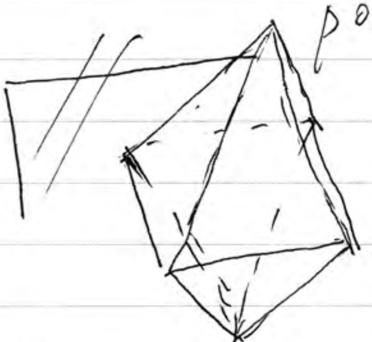
For each $\tau < \sigma$, $\tau = H \cap \sigma$,
Let $H' = H \cap \{E \times 1\}$, then H' is a face of K

For each $\tau^{\circ} < K$, $\tau^{\circ} = H \cap \sigma$
Let H' be a hyper~~sp~~plane in $E \times R$ containing H and $\{0\}$

Anologue to dual cone

The ~~pa~~ polor set of K is defined to be $K^o = \{u \in E^*, \langle u, v\rangle \geq -1, \forall v \in K\}$

let P ~~is~~ be a polytope with vertices $\{\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1$



$P^o = (\pm 1, \pm 1, \pm 1)$

prop: ① $(K^o)^o = K$

② If $F < K$, then $F^* = \{u \in K^o, \langle u, v\rangle = -1, \forall v \in F\}$ is a face of K^o

$F \mapsto F^*$ is one to one, ~~order-rever~~ order-reversing

~~dim~~ with $\dim(F) + \dim(F^*) = \dim(E) - 1$

③ K rational $\Rightarrow K^o$ rational

· Subdivision of the boundary of K

Two method of getting a fan from a ~~pt~~ polytope

· rational convex polytope

$K \subseteq N_{\mathbb{R}}$

determines a fan Δ whose cones over the properfaces of K (like)

· P is a rational in the dual space $M_{\mathbb{R}}$

Assume: P is n-dimension

but it is not necessary that $\{0\} \in P$

For each $Q < P$, we define a cone σ_Q

as $\sigma_Q = \{v \in N_{\mathbb{R}} : \langle u, v\rangle \leq \langle u', v\rangle$ for all $u \in Q$. and $u' \in P\}$

prop. ① $\sigma_Q^{\vee}$ is spanned by $u' - u$ where u' and u vary among $\mathrm{ver}(P)$ and $\mathrm{ver}(Q)$

~~and ② $\{\sigma_Q : Q$ vary from th~~ among the f

② $\{\sigma_Q ;\ Q$ vary among the faces of $P\}$ is the fan, denoted by Δ_P

③ a. $\forall Q, Q' < P$

$\sigma_Q \cap \sigma_{Q'} \cancel{= \sigma_{Q\cap Q'}} = \sigma_\tau \quad \tau = \cap\{\tau' < P \mid Q \le \tau'$ and $Q' \le \tau'\}$

~~$\forall v \in \sigma_Q \cap \sigma_{Q'}$~~

~~$\forall \langle u, v\rangle \le \langle u', v\rangle$~~ $\forall u \in Q, u' \in P$

$\langle u'', v\rangle \le \langle u', v\rangle \quad \forall u'' \in Q', u' \in P$

$\Rightarrow \langle u, v\rangle \le \langle u', v\rangle \quad \forall u \in Q \cap Q'$

$\exists v \in \sigma_{Q\cap Q'}, v \notin \sigma_Q \cap \sigma_{Q'} \quad u' \in P$

$\langle u, v\rangle \not\le \langle u', v\rangle \quad u \in Q \cap Q', u' \in P$

assume $v \notin \sigma_Q$ $\exists u_1 \in Q, u_2 \in P$ s.t

$\langle u_1, v\rangle \not\le \langle u_2, v\rangle$

~~$\exists u_3 \in$ $u_1 = \sum \lambda_i v_i$~~

τ ~~is ex~~ exists, since $Q < P$, $Q' < P$

and $\tau \not< P$

$\forall v \in \sigma_Q \cap \sigma_{Q'}$

$\langle u, v\rangle \le \langle u', v\rangle \quad \forall u \in Q, u' \in P$

$\langle u'', v\rangle \le \langle u', v\rangle \quad \forall u'' \in Q', u' \in P$

$\forall u \in Q, u'' \in Q'$

we have $\langle u, v\rangle \le \langle u'', v\rangle$

and $\langle u, v\rangle \ge \langle u'', v\rangle$

So $\langle u, v\rangle = \langle u'', v\rangle$

$P \subseteq M_{\mathbb{R}}$

prop: If P contains the origin as an interior point, then Δ_P consists of cones over the faces of the polar polytope P°

pf: (1) $\sigma_Q = \cancel{Q^*}$ the cone over the dual face Q^* of P° $\cancel{\sigma_{Q^*}}$ $C(Q^*)$

~~$P \subseteq M_{\mathbb{R}}$~~ ~~$\forall v \in Q^*$~~

If $v \in Q^*$, we have $\langle u, v \rangle = -1$ $\forall u \in Q$ and $\langle u, v \rangle \geq -1$ $\forall u \in P$

Then $\forall v \in C(Q^*)$ $\exists v' \in Q^*$ s.t $v = r v'$ $(r \geq 0)$

Thus $\langle u, v \rangle = -r \leq \langle u', v \rangle$ $\forall u \in Q,\ u' \in P$

that is, $v \in \sigma_Q$

$\forall v \in \sigma_Q$, we have $\langle u, v \rangle \leq \langle u', v \rangle$ $\forall u \in Q, u' \in P$

~~Then~~ ~~If $u \neq u'$~~ $\forall u, u' \in Q$
If we have $\langle u, v \rangle \leq \langle u', v \rangle \leq \langle u, v \rangle$

~~Let $r = |\langle u, v \rangle| (\forall u \in Q)$~~

~~and $v' = \frac{v}{r}$~~ Let $r = -\langle u, v \rangle$ $(\forall u \in Q)$

Since $0 \in P$, so $r = -\langle u, v \rangle \geq 0$

Let $v' = \frac{v}{r}$, then $v' \in Q^*$

(2) Δ_P is a fan

$\Delta_{mP+u} = \Delta_P$ ($\forall m \geq 0$, $u \in M$)

Then for any P can be changed to one containing the origin as an interior point by translating and expansion

($\exists m \geq 0$, $u \in M$, s.t $\{0\} \in$ interior of $mP+u$)

Since $C((mP+u)^\circ) = \Delta_{mP+u}$ is a fan,

so Δ_P is a fan

Δ Properties of affine toric varieties

- points of affine varieties

Let $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, the following ways to describe the points of U_σ is equivalent

(a) $p \in U_\sigma$

(b) Maximum ideals $m \subseteq \mathbb{C}[S_\sigma]$

(c) $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S_\sigma] \to \mathbb{C}$

(d) Semigroup homomoiphisms $S_\sigma \to \mathbb{C}$

pf: the correspondence between (a), (b), (c) is standard

- the action of T_N on U_σ

~~$\forall t \in T_N$ correspond to a semigroup homomorphism $t: M \to \mathbb{C}^*$~~

$\forall x \in U_\sigma$ correspond to $x: S_\sigma \to \mathbb{C}$

$$T_N \times U_\sigma \to U_\sigma$$
$$(t, x) \mapsto t\cdot x : S_\sigma \to \mathbb{C}$$
$$u \mapsto t(u)\, x(u)$$

$x_\sigma \in U_\sigma$ is defined by

$$x_\sigma : S_\sigma \to \mathbb{C}$$
$$u \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

σ spans $N_\mathbb{R} \Leftrightarrow \sigma^\perp = \{0\}$

$\Leftrightarrow x_\sigma$ is a fixed point of torus action

"$\Rightarrow$" $x : u \mapsto x(u)$

$t\cdot x : u \mapsto t(u)\, x(u)$

$u \in \sigma^\perp = \{0\}$ ~~$u \in \sigma^\perp$~~ $t \cdot x(u) = 1$

othewise $t\cdot x(u) = 0$

"$\Leftarrow$" $u \in \sigma^\perp$ $t \cdot x(u) = t(u) = x(u) = 1$

otherwise $t \cdot x(u) = 0$

$\forall u \in \sigma^\perp$ $\cancel{t \cdot x}$ $t(u) = 1$ $\forall t$

$\Rightarrow \cancel{u} \ \sigma^\perp = \{0\}$

$(\mathbb{C}^*)^n = T_N = \mathrm{Hom}(M, \mathbb{C}^*)$

$t : M \to \mathbb{C}^*$

$(a_1, \cdots a_n) \to t_1^{a_1} \cdots t_n^{a_n}$

△ Smooth points of Affine variety

X is irreducible

$\mathcal{O}_{X,p} = (\mathbb{C}[X])_S$

$S = \{g \in \mathbb{C}[X] \mid g(p) \neq 0\}$

- the local ring of X at p is

$$\mathcal{O}_{X,p} = \{f/g \in k(X) \mid f, g \in k[X] \text{ and } g(p) \neq 0\}$$

the maximal ideal of $\mathcal{O}_{X,p}$

$$m_{X,p} = \{\phi \in \mathcal{O}_{X,p} \mid \phi(p) = 0\}$$

($m_{X,p}$ is unique maximal ideal)

- The Zariski tangent space of X at p is defined to be

$$T_p(X) = \mathrm{Hom}_{\mathbb{C}}(m_{X,p}/m_{V,p}^2, \mathbb{C})$$

~~ad~~ and the cotangent space

$$T_p(X)^* = m_{X,p}/m_{X,p}^2$$