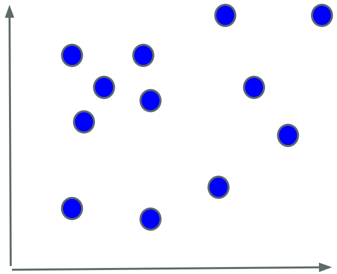
Linear Regression

By The LinAlg Squad

You have data!

-Goal: fit a mathematical model to a set of observed points



-want to find the linear equation that fits the points: f: x->y

in the form y=mx+b

-obviously we can easily fit two points but if the third is not on the same line then the line is not a very good predictor of future events

-all data points can be written in the form:

$$
\begin{aligned}
mx_1 + b &= y_1 \\
mx_2 + b &= y_2 \\
&\vdots \\
mx_n + b &= y_n
\end{aligned}
$$

Or in form Ax=b

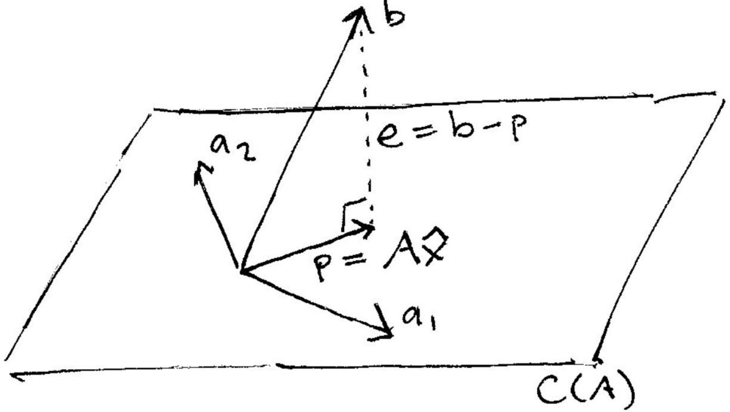
$$
\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_2 \end{bmatrix}
$$

1’s because b is “being multiplied by one”

Do matrix multiplication to do this

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_2 \end{bmatrix}$$

A x b



-a1 and a2 are the columns of A

-b is the observed values

-C(A) column space/our hoped for model lies on this space

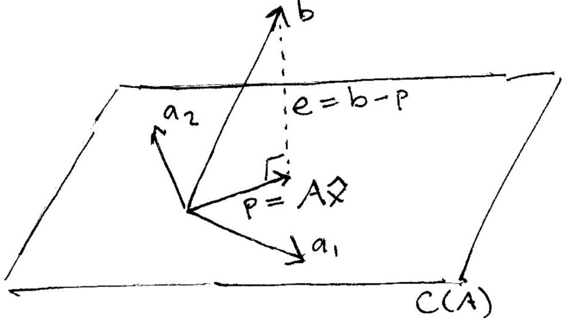
-idea is to find a new vector that lies on C(A) by projection (p)

-p is a combination of columns of A so $p = A\hat{x}$ where $\hat{x}$ is the predicted m and b

-e is the “error” between observed and projected, so want to minimize it so p and b should be perpendicular (geometry)

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_2 \end{bmatrix}$$

A x b



-to find the best line, we want to actually solve for x, ie m and b

$$A \begin{bmatrix} m \\ b \end{bmatrix} = \vec{y}$$

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \vec{y}$$

-multiply both sides by A^T

$$\begin{bmatrix} m \\ b \end{bmatrix} = (A^T A)^{-1} A^T \vec{y}$$

-solve for m and b by multiplying both sides by $(A^T A)^{-1}$. This will cancel out $A^T A$ on the left hand side.

Sometimes linear isn't the best! Quadratic is!

$$
\begin{aligned}
mx_1^2 + bx_1 + c &= y_1 \\
mx_2^2 + bx_2 + c &= y_2 \\
&\vdots \\
mx_n^2 + bx_n + c &= y_n
\end{aligned}
$$

$\xrightarrow{\text{Set up}}$

$$
\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & & \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}
$$

$\xrightarrow{\text{Solve!}}$

$$
A \begin{bmatrix} m \\ b \\ c \end{bmatrix} = \vec{y}
$$

$$
A^T A \begin{bmatrix} m \\ b \\ c \end{bmatrix} = A^T \vec{y}
$$

$$
\begin{bmatrix} m \\ b \\ c \end{bmatrix} = (A^T A)^{-1} A^T \vec{y}
$$

Code