Stochastic Local Volatility Models: Theory and Implementation

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Stochastic Local Volatility Models: Theory and Implementation

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Plan of the presentation

1) Hedging and volatility

2) Review of volatility models

3) Local volatility models with jumps and stochastic volatility

4) Calibration using Kolmogorov equations

5) PDE based methods in one dimension

6) PDE based methods in two dimensions

7) Illustrations

References

Some theoretical and practical details for my presentation can be found in:

1) Sepp, A. (2007) Affine Models in Mathematical Finance: an Analytical Approach, PhD thesis, University of Tartu
`http://math.ut.ee/~spartak/papers/seppthesis.pdf`

2) Sepp, A. (2012) An Approximate Distribution of Delta-Hedging Errors in a Jump-Diffusion Model with Discrete Trading and Transaction Costs, Quantitative Finance, 12(7), 1119-1141
`http://ssrn.com/abstract=1360472`

Volatility modelling

What is important for a competitive pricing and hedging model?

1) Consistency with the observed market dynamics implies stable model parameters and hedges

2) Consistency with vanilla option prices ensures that the model fits the risk-neutral distribution implied by these prices and that the model gamma is not much different from the market implied gamma

Hedging and volatility I

Let us consider a vanilla option and for simplicity assume zero interest rates and dividends

We can find the option value $U^{(\sigma)}(t,S)$ by solving the celebrated Black-Scholes-Merton (1973) partial differential equation (PDE) with log-normal volatility parameter σ:

$$U_t^{(\sigma)} + \frac{1}{2}\sigma^2 S^2 U_{SS}^{(\sigma)} = 0, \tag{1}$$

where t is current time and S is the spot price

To delta-hedge a short-position in this option we consider the following volatilities:
σ_i - implied volatility for computing option value $U^{(\sigma_i)}(t,S)$
σ_h - hedging volatility for computing option delta $U_S^{(\sigma_h)}(t,S)$

Note that σ_h might also include the vega adjustment, which is change in σ_i given change in the spot, so in general $\sigma_i \neq \sigma_h$

Hedging and volatility II

The delta-hedged position at time t_n:

$$\Pi(t_n) = S(t_n)\Delta^{(\sigma_h)}(t_n, S(t_n)) - U^{(\sigma_i)}(t_n, S(t_n))$$

One period profit-and-loss realized over time period δt_n is:

$$\begin{aligned}\delta\Pi(t_n) = &\left(S(t_{n-1}) + \delta S(t_n)\right)\Delta^{(\sigma_h)}(t_{n-1}, S(t_{n-1})) \\ &- U^{(\sigma_i)}(t_{n-1} + \delta t_n, S(t_{n-1}) + \delta S(t_n)) - \Pi(t_{n-1})\end{aligned}$$

where $\delta S(t_n) = S(t_n) - S(t_{n-1})$ and $\delta t_n = t_n - t_{n-1}$

Hedging and volatility III Consider Taylor series of $U^{(\sigma_i)}(t_{n-1} + \delta t_n, S(t_{n-1}) + \delta S(t_n))$:

$$U^{(\sigma_i)}(t_{n-1} + \delta t_n, S(t_{n-1}) + \delta S(t_n)) \approx U^{(\sigma_i)}(t_{n-1}, S(t_{n-1}))$$
$$+ \delta t_n \Theta^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) + \delta S(t_n) \Delta^{(r)}(t_{n-1}, S(t_{n-1})) + \frac{1}{2} (\delta S(t_n))^2 \Gamma^{(r)}(t_{n-1}$$

where $\Theta^{(\sigma_i)}(t_{n-1}, S(t_{n-1}))$ is the option theta, and $\Delta^{(r)}(t_{n-1}, S(t_{n-1}))$ and $\Gamma^{(r)}(t_{n-1}, S(t_{n-1}))$ are realized delta and gamma:

$$\begin{aligned}
\Theta^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) &= \frac{\partial}{\partial t} U^{(\sigma_i)}(t, S(t_{n-1})) \,|_{t=t_{n-1}} \\
\Delta^{(r)}(t_{n-1}, S(t_{n-1})) &= \frac{\partial}{\partial S} U^{(\sigma_i)}(t_{n-1}, S) \,|_{S=S(t_n)} \\
&\approx \frac{U^{(\sigma_i)}(t_{n-1}, S(t_n)) - U^{(\sigma_i)}(t_{n-1}, S(t_{n-1}))}{S(t_n) - S(t_{n-1})} \\
\Gamma^{(r)}(t_{n-1}, S(t_{n-1})) &= \frac{\partial}{\partial S} \Delta^{(r)}(t_{n-1}, S) \,|_{S=S(t_n)} \\
&\approx \frac{\Delta^{(r)}(t_{n-1}, S(t_n)) - \Delta^{(r)}(t_{n-1}, S(t_{n-1}))}{S(t_n) - S(t_{n-1})}
\end{aligned}$$

Hedging and volatility IV

One period realized profit-and-loss is approximated by:

$$\delta\Pi(t_n) = \left[\Delta^{(\sigma_h)}(t_{n-1}, S(t_{n-1})) - \Delta^{(r)}(t_{n-1}, S(t_n))\right] \delta S(t_n)$$
$$- \delta t_n \Theta^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) - \frac{1}{2} (\delta S(t_n))^2 \, \Gamma^{(r)}(t_{n-1}, S(t_{n-1}))$$

Note that option theta satisfies the BSM PDE:

$$\Theta^{(\sigma_i)}(t, S) = -\frac{1}{2}\sigma_i^2 S^2 \Gamma^{(\sigma_i)}(t, S)$$

If the delta-hedging is rebalanced at discrete times t_n, the realized profit-and-loss, P&L, $\Pi(T)$ at the option maturity is

$$\Pi(T) = \sum_n \left[\Delta^{(\sigma_h)}(t_{n-1}, S(t_{n-1})) - \Delta^{(r)}(t_{n-1}, S(t_{n-1}))\right] \delta S(t_n)$$
$$+ \frac{1}{2} \sum_n \left[\sigma_i^2 \Gamma^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) - V(t_n) \Gamma^{(r)}(t_{n-1}, S(t_{n-1}))\right] S^2(t_{n-1}) \delta t_n$$

where $V(t_n)$ is the realized variance

$$V(t_n) = \frac{1}{\delta t_n} \left(\frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}\right)^2$$

Hedging and volatility V. Modelling objective A:
Find a pricing and hedging model that
1) predicts the correct delta: $\Delta^{(\sigma_h)}(t_{n-1}, S(t_{n-1})) \approx \Delta^{(r)}(t_{n-1}, S(t_{n-1}))$, then the P&L will be independent from the realized price path (note that $\Delta^{(\sigma_h)}(t_{n-1}, S(t_{n-1}))$ is implied by the hedging model while $\Delta^{(r)}(t_{n-1}, S(t_{n-1}))$ represents actual change in option price observed in the market given change in the spot and associated change in implied volatility σ_i)

2) predicts the correct gamma: $\Gamma^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) \approx \Gamma^{(r)}(t_{n-1}, S(t_{n-1}))$ (note that $\Gamma^{(\sigma_i)}(t_{n-1}, S(t_{n-1}))$ is also implied by the hedging model while $\Gamma^{(r)}(t_{n-1}, S(t_{n-1}))$ is the realized change in delta given change in spot and associated change in implied volatility σ_i)

If the model satisfies **1)** and **2)**, realized P&L greatly simplifies to

$$\Pi(T) = \frac{1}{2} \sum_n \left(\sigma_i^2 - V(t_n)\right) \Gamma^{(\sigma_i)}(t_{n-1}, S(t_{n-1})) S^2(t_{n-1}) \delta t_n$$

The "edge" comes from the spread between the implied volatility squared and the realized variance: $\sigma_i^2 - V(t_n)$

Volatility modelling VI. Modelling objective B

If the option can be sold at implied variance σ_i^2 that is (much) higher than the realized variance $\frac{1}{T}\int_0^T V(t')dt'$ then $\Pi(T)$ is expected to be positive because $\Gamma^{(\sigma_i)}(t, S(t))$ is positive for vanilla options with convex pay-offs (see Sepp (2011) for approximate distribution of $\Pi(T)$)

To "materialize" the "edge" as the P&L, it is important to have a model that is consistent with **1)** and **2)**

For exotic options, considerations might be more complicated as $\Gamma^{(\sigma_i)}(t, S(t))$ and $\Gamma^{(r)}(t, S(t))$ might change the sign and generally higher order risks need to be hedged

Still, implications **1)** and **2)** are important for a competitive pricing and hedging model. To conclude:
1) Consistency with the observed market dynamics implies stable model parameters and hedge ratios
2) Consistency with vanilla option prices ensures that the model fits the price distribution implied by these prices and that the model gamma is not much different from the market implied gamma

Overview I. Black-Scholes-Merton

I will start with a brief review of volatility models

Black-Scholes-Merton (1973) considered log-normal model:

$$dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t),\ S(0) = S, \tag{2}$$

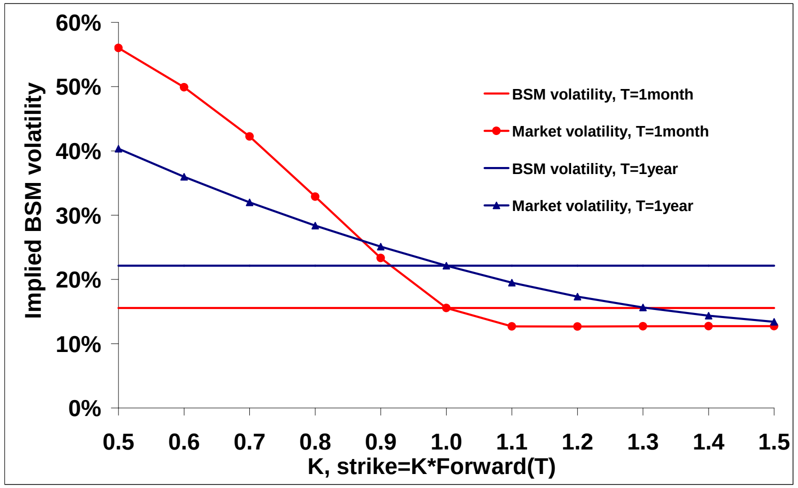
where $W(t)$ is the Brownian motion and $\mu(t)$ is the risk-neutral drift (typically, $\mu(t) = r(t) - d(t)$, where $r(t)$ and $d(t)$ are deterministic instantaneous interest and dividend rates)

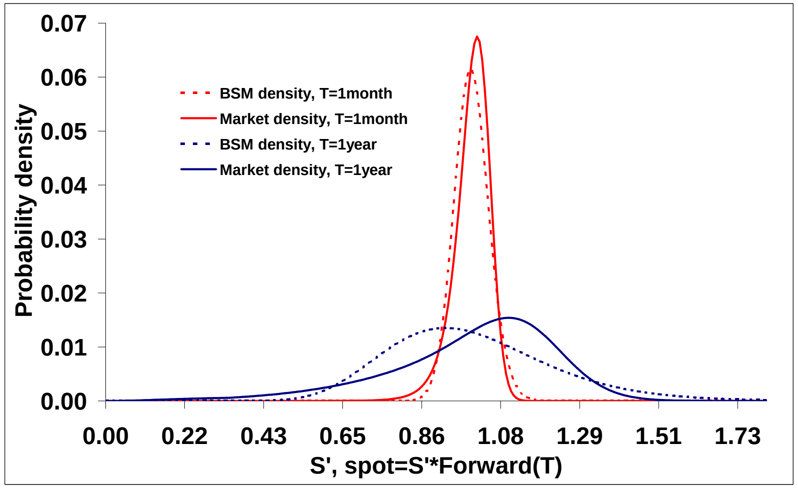
The constant volatility σ is independent from both the spot price and any extra factors

The model is not realistic in the presence of the skew observed in the equity markets (out-of-the money puts are relatively expensive than out-of-the money calls)

In practice, the BSM model is as a "static" tool: price quotation, implied volatility parametrization, benchmarking

Illustration for options on the S&P 500





Left: Market implied and BSM volatility at ATM strike for T =1month an T =1year
Right: Corresponding market and BSM implied probability density

Market implied distribution is more skewed to the left

Overview II. Bachelier model, A

Bachelier (1900) proposed a normal dynamics:

$$dS(t) = \mu(t)S(0)dt + \sigma_{\mathrm{normal}}S(0)dW(t), \;\; S(0) = S, \tag{3}$$

The implied local log-normal volatility $\sigma_{\mathrm{loc}}(S(t))$ (approximately):

$$\sigma_{\mathrm{loc}}(S(t)) = \sigma_{\mathrm{normal}}\frac{S(0)}{S(t)}$$

The Bachelier model produces the leverage effect: for small price the volatility increases and vice versa

The non-zero probability of negative prices: the default event in the financial terms (modelled as the first time S(t) hits zero)

Overview II. Bachelier model, B

From the modelling point of view, the Bachelier model is more realistic than that of Black-Scholes-Merton

However, the model is little accepted in practice: people prefer to think in relative terms rather than in absolute

In Bachelier model, we cannot compare risks of stock A with the volatility of $100,000\%$ and stock B with the volatility of 10% without looking at their spot prices

Now: stock A is the FTSE100 index with $S(0) = 5,700$ and stock B is a penny share with $S(0) = 0.1$

The equivalent log-normal volatility of A is 17.54% and that of B is 100.00%

Overview III. Jump-diffusion models

Merton (1976) introduced log-normal dynamics with normal jumps:

$$dS(t) = \mu(t)S(t_-)dt + \sigma S(t_-)dW(t) + \Big((e^J - 1)dN(t) - \lambda\nu dt\Big) S(t_-) \tag{4}$$

where $N(t)$ is a Poisson process with intensity λ, J is jump amplitude with PDF $\varpi(J) = \mathbf{n}(\eta, \delta)$, and ν is the compensator:

$$\nu = \int_{-\infty}^{\infty} e^{J'} \varpi(J') dJ' - 1 = e^{\eta + \frac{1}{2}\delta^2} - 1$$

Given jump in $N(t)$, jump in $S(t)$ is: $\Delta S(t) = S(t_-)(e^J - 1) \approx S(t_-)J$

Since the possibility of large jumps makes out-of-the-money puts more expensive, jump-diffusions introduce the implied volatility skew

However, for longer periods (more than one year) the implied distribution resembles to the Gaussian (by the central limit theorem)

Jump models have been extended to general Levy processes (Lewis (2001), Levendorskii-Boyarchenko (2002), Carr *et al* (2003), Cont-Tankov (2004), ...)

Overview IV. Stochastic volatility models Hull-White (1988):

$$dS(t) = \mu(t)S(t)dt + V(t)S(t)dW^{(1)}(t)$$
$$dV(t) = \kappa(\theta - V(t))dt + \epsilon V(t)dW^{(2)}(t)$$

where $< dW^{(1)}(t), dW^{(2)}(t) >= \rho dt$
Scott (1987):

$$dS(t) = \mu(t)S(t)dt + e^{V(t)}S(t)dW^{(1)}(t)$$
$$dV(t) = \kappa(\theta - V(t))dt + \epsilon dW^{(2)}(t)$$

Stein-Stein (1991):

$$dS(t) = \mu(t)S(t)dt + V(t)S(t)dW^{(1)}(t)$$
$$dV(t) = \kappa(\theta - V(t))dt + \epsilon dW^{(2)}(t)$$

Heston (1993):

$$dS(t) = \mu(t)S(t)dt + \sqrt{V(t)}S(t)dW^{(1)}(t)$$
$$dV(t) = \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW^{(2)}(t)$$

Jumps can be included (Bates (1996), Duffie *et al* (2000), ...)

Overview V. Parametric local volatility

Cox (1975) proposes the constant elasticity of the variance (CEV) model:

$$dS(t) = \mu(t)S(S)dt + \sigma_{\text{cev}}\left(S^{\beta-1}(t)\right)S(t)dW(t) \tag{5}$$

Rubinstein (1983) proposes the dis-placed diffusion model

$$dS(t) = \mu(t)S(t)dt + \sigma_{\text{dis}}\left(\beta + \gamma\frac{S(0)}{S(t)}\right)S(t)dW(t) \tag{6}$$

Both the CEV with $\beta < 1$ (typically, $\beta \in [-5, -3]$ for single stocks and indexes) and dis-placed diffusions exhibit the leverage effect and the default possibility (from my experience, the dis-placed diffusion model is easier to deal with)

Ingersoll (1996), Zuhlsdorff (1999) and Lipton (2000) consider the hyperbolic volatility:

$$dS(t) = \mu(t)S(S)dt + \sigma_{\text{hyp}}\left(\alpha\frac{S(t)}{S(0)} + \beta + \gamma\frac{S(0)}{S(t)}\right)S(t)dW(t) \tag{7}$$

Overview VI. Non-parametric local volatility

In general, parameters of parametric local volatility models need to be calibrated to observed market prices in least squares sense

Derman-Kani (1994), Rubinstein (1994), Dupire (1994) consider the one-dimensional diffusion model

$$dS(t) = \mu(t)S(t)dt + \sigma_{(\text{loc})}(t, S(t))S(t)dW(t), \tag{8}$$

with local volatility $\sigma_{(\text{loc})}(t, S(t))$ implied from observed market prices of vanilla options

Andersen-Andreasen (2000) and Carr *et al* (2004) introduce the local volatility with jumps

JP Morgan (1999) and Blacher (2001) consider the local volatility model with stochastic variance driven by the Heston model

Lipton (2002) introduces the local stochastic volatility with jumps

Let me summarise the two main categories of the models used in practice for exotic options

Overview VII. Volatility models

1) **Non-parametric local volatility models** (Dupire (1994), Derman-Kani (1994), Rubinstein (1994))
"+" are consistent with today's market prices by construction
"-" but they tend to be poor in replicating the market dynamics of spot and volatility (implied volatility tends to move too much given a change in the spot, no mean-reversion effect)
"-" especially, it is impossible to tune-up the volatility of the implied volatility as there is simply no parameter for that!

2) **Parametric stochastic volatility models** (Heston (1993))
"+" tend to be more in line with the market dynamics
"+" equipped to model the term-structure (by mean-reversion parameters) and the volatility of the variance (by vol-of-vol parameters)
"-" need a least-squares calibration to today's option prices
"-" unfortunately, any change in any of the parameters of the mean-reversion or the vol-of-vol requires re-calibration of other parameters

3) **Stochastic local volatility models** (JP Morgan (1999), Blacher (2001), Lipton (2002))
aim to include "+"'s and cross "-"'s of the first two models

Overview VIII. Model specification

1) Global factors

Select factors relevant for product risk: stochastic volatility, stochastic interest rate, jumps, default risk, etc

(Guess) Estimate or calibrate model parameters for the dynamics of these factors using either historical or market data

Parameters are updated infrequently

2) Local factors

Specify local factors such as local volatility or local drift (for quantos)

Parameters of local factors are updated frequently (on the run)

Stochastic local volatility models

Stochastic local volatility models allow:

1) Specify the dynamics for the instantaneous volatility

2) Fit the risk-neutral distributions implied by market prices of vanilla options by specifying local volatility as function of the spot price

However, the model calibration requires robust numerical methods for solving two-dimensional PDE-s

In general, a successful implementation of the model involves a decent amount of both theoretical and practical exercises

Calibration of parametric models I. Forward-backward equations
I apply the methods for stochastic processes and partial differential equations dating back to **Kolmogorov foundational work** (1931)

Consider a 1-d problem with stochastic factor $S(t)$ for a product that depends only on terminal value of $S(T)$
PV function $U(t, S; T, K)$, $0 \leq t \leq T$, solves the **backward equation** as function of (t, S):

$$U_t(t, S) + \mathcal{L}U(t, S) = -c(t, S),\ U(T, S) = u(S) \tag{9}$$

$u(S)$ and $c(t, S)$ are pay-off and coupon functions, respectively
$\mathcal{L}$ is the **infinitesimal operator** corresponding to the dynamics of $S(t)$ (includes volatility, drift, discounting, jumps)

Transition probability density function, $G(0, S_0; t, S')$, aka **Green's function** in mathematical physics, solves the **forward equation** as function of (t, S'):

$$G_t(t, S') - \mathcal{L}^{\dagger}G(t, S') = 0,\ G(0, S') = \delta(S' - S_0) \tag{10}$$

where $\delta()$ is Dirac delta function and $\mathcal{L}^{\dagger}$ is **operator adjoint to** $\mathcal{L}$

Calibration of parametric models II. Valuation formula

Multiply backward equation by $G(0, S_0; t, S)$ and integrate over (t, S):

$$
\begin{aligned}
&- \int_0^T \int_{-\infty}^{\infty} G(0, S_0; t', S') c(t', S') dS' dt' \\
&\equiv \int_0^T \int_{-\infty}^{\infty} \left[G(0, S_0; t', S') U_t(t', S') + G(0, S_0; t', S') \mathcal{L} U(t', S') \right] dS' dt' \\
&= \int_{-\infty}^{\infty} \left[G(0, S_0; T, S') U(T, S') - G(0, S_0; 0, S') U(0, S') \right. \\
&\left. - \int_0^T G_t(0, S_0; t', S') U(t', S') dt' \right] dS' + \int_{-\infty}^{\infty} \int_0^T G(0, S_0; t', S') \mathcal{L} U(t', S') dt' dS' \\
&= \int_{-\infty}^{\infty} G(0, S_0; T, S') u(S') dS' - U(0, S_0) \\
&- \int_0^T \int_{-\infty}^{\infty} \left[\left\{ \mathcal{L}^{\dagger} G(0, S_0; t', S') \right\} U(t', S') - G(0, S_0; t', S') \left\{ \mathcal{L} U(t', S') \right\} \right] dS' dt'
\end{aligned}
$$

where, in second line, the first term is integrated by parts, and, in the last line, terminal condition for $U(T, S)$ and initial condition for $G(0, S_0; 0, S)$ are applied

Calibration of parametric models III. Consistency
Consistency condition for adjoint operators $\mathcal{L}$ and $\mathcal{L}^\dagger$ (can be checked by integration by parts):

$$\int_0^T \int_{-\infty}^{\infty} \left[\left\{ \mathcal{L}^\dagger G(0, S_0; t', S') \right\} U(t', S') - G(0, S_0; t', S') \left\{ \mathcal{L} U(t', S') \right\} \right] dS' dt' = 0 \tag{11}$$

Then the PV can be computed by convolution:

$$\begin{aligned} U(0, S_0; T, K) = & \int_{-\infty}^{\infty} G(0, S_0; T, S') u(S') dS' \\ & + \int_0^T \int_{-\infty}^{\infty} G(0, S_0; t', S') c(t', S') dS' dt' \end{aligned} \tag{12}$$

This formula is known as **Duhamel's principle** (19th century) in mathematical physics or **Feynman-Kac formula** (1949) in probability

Calibration of parametric models IV. Implications

Typically, for a parametric model, model parameters are assumed to be piece-wise constant in time with jumps at times $\{T_n\}$, where $\{T_n\}$ is set of maturity times of listed options

Implications for calibration by bootstrap:

1) Given calibrated set of piece-wise constant model parameters at time T_{m-1} and known values of $G(T_{m-1}, S)$, make a guess for parameters at time T_m and compute $G(T_m, S)$

2) Apply Duhamel's formula (12) to compute PV-s of specified vanilla instruments

3) By changing parameters for time T_m, minimize the sum of squared differences between model and market prices

4)After convergence, store $G(T_m, S)$ and go to the next time slice

Calibration of parametric models V. Numerical schemes

Consistency condition (11) for adjoint operators $\mathcal{L}$ and $\mathcal{L}^\dagger$ is based on theoretical arguments:

$$\int_0^T \int_{-\infty}^{\infty} \left[\left\{ \mathcal{L}^\dagger G(0, S_0; t', S') \right\} U(t', S') - G(0, S_0; t', S') \left\{ \mathcal{L} U(t', S') \right\} \right] dS' dt' = 0$$

It is not necessarily true for discrete numerical schemes! (see Lipton (2007) for a consistent scheme)

Implications for numerical schemes:

1) For adjoint operators, if M is spacial discretisation of $\mathcal{L}$ then M^T should be spacial discretisation of $\mathcal{L}^\dagger$

2) Both forward and backward equations should be solved using the same schemes

3) It is enough to develop either forward or backward scheme

4) $\mathcal{L}$ and $\mathcal{L}^\dagger$ have the same values of the operator norm, so the convergence of the forward scheme implies convergence of the backward scheme and vice versa

Local volatility calibration I. Basic facts

Let $G(t,S;T,S')$ be the risk-neutral probability density at time T and state S' given that $S(t)=S$:

$$G(t,S;T,S') = \mathbb{P}\left[S(T)\in dS' \mid S(t)=S\right] dS' \tag{13}$$

Assume that the discount rate is deterministic

By applying the risk-neutral valuation and **Duhamel's formula** (12), un-discounted call value $C(T,K)$ solves:

$$\begin{aligned} C(T,K) &= \int_0^\infty (S'-K)^+ G(t,S;T,S')dS' \\ &= \int_K^\infty (S'-K) G(t,S;T,S')dS' \end{aligned} \tag{14}$$

By basic calculus:

$$\begin{aligned} C_K(T,K) &= -\int_K^\infty G(t,S;T,S')dS' \\ C_{KK}(T,K) &= G(t,S;T,K) \end{aligned} \tag{15}$$

Local volatility calibration II. Breeden-Litzenberger formula

Consider a one-dimensional diffusion:

$$dS(t) = \mu(t)S(t)dt + \sigma_{(\text{loc,dif})}(t, S(t))S(t)dW(t), \;\; S(0) = S \qquad (16)$$

where $\sigma_{(\text{loc,dif})}$ is the local volatility of the diffusion

Note that if call prices are given for all strikes and maturities, $\{C^{(\text{market})}(K,T)\}$, then by (15) the risk-neutral distribution satisfies (**Breeden-Litzenberger** (1978)):

$$G^{(\text{market})}(T,K) = C^{(\text{market})}_{KK}(T,K)$$

Calibration objective: how we should specify $\sigma_{(\text{loc,dif})}(T,S')$ so that the implied risk-neutral distribution $G(t,S;T,S')$ is consistent with $G^{(\text{market})}(T,K)$

Local volatility calibration III. Kolmogorov equation

The risk-neutral probability density $G(t,S;T,S')$ solves the **forward Kolmogorov** equation:

$$G_T - \frac{1}{2}\Big((\sigma_{\text{(loc,dif)}}(T,S')S')^2 G\Big)_{S'S'} + \Big(\mu(T)S'G\Big)_{S'} = 0$$
$$G(t,S;t,S') = \delta(S'-S)$$

Multiply by $(S'-K)^+$ and integrate over S'

Initial condition:

$$\int_0^\infty (S'-K)^+\delta(S'-S)dS' = (S-K)^+$$

Time derivative:

$$\int_0^\infty (S'-K)^+ G_T dS' = C_T(T,K)$$

Diffusion term:

$$\int_0^\infty (S'-K)^+ \Big((\sigma_{\text{(loc,dif)}}(T,S')S')^2 G\Big)_{S'S'} dS' = \sigma^2_{\text{(loc,dif)}}(T,K)K^2 G(t,S;T,K)$$
$$= \sigma^2_{\text{(loc,dif)}}(T,K)K^2 C_{KK}(T,K)$$

Drift term:

$$\int_0^\infty (S'-K)^+ \Big(\mu(T)S'G\Big)_{S'} dS' = \mu(T)KC_K(T,K) - \mu(T)C(T,K)$$

As a result, we obtain the forward equation for call option prices as function of the "forward" arguments T and K:

$$C_T - \frac{1}{2}\sigma^2_{\text{(loc,dif)}}(T,K)K^2 C_{KK} + \mu(T)KC_K - \mu(T)C = 0$$
$$C(t,K) = (S(t)-K)^+$$

Inverting the PDE in terms of the $\sigma^2_{\text{(loc,dif)}}$, we obtain **Dupire equation** (1994) for local volatility:

$$\sigma^2_{\text{(loc,dif)}}(T,K) = \frac{C_T(T,K) + \mu(T)KC_K(T,K) - \mu(T)C(T,K)}{\frac{1}{2}K^2 C_{KK}(T,K)} \qquad (17)$$

Jump-diffusion model I

Andersen-Andreasen (2000) consider a Merton jump-diffusion extended to local volatility:

$$
\begin{aligned}
dS(t) =& \mu(t)S(t_-)dt + \sigma_{(\text{loc,jd})}(t, S(t_-))S(t_-)dW(t) \\
&+ \Big((e^J - 1)dN(t) - \lambda\nu dt\Big) S(t_-), \;\; S(0) = S
\end{aligned}
\tag{18}
$$

where $\sigma_{(\text{loc,jd})}$ is the local volatility corresponding to the jump-diffusion

Jump-diffusion model II Kolmogorov equation
Now $G(t,S;T,S')$ solves the forward **Kolmogorov equation**:

$$G_T - \frac{1}{2}\Big((\sigma_{(\text{loc,jd})}(T,S')S')^2 G\Big)_{S'S'} + \Big(\mu(T)S'G\Big)_{S'} - \lambda I(S') = 0$$
$$G(t,S;t,S') = \delta(S'-S)$$
$$I(S') = \int_{-\infty}^{\infty} G(S'e^{-J'})e^{-J'}\varpi(J')dJ' + (\nu S'G)_{S'} - G$$

Proof: for the backward equation the jump term is given by:

$$I^\star(S') = \int_{-\infty}^{\infty} U(S'e^{J'})\varpi(J')dJ' - \nu S'U_{S'} - U$$

Multiply by $G(t,S;T,S')$ and integrate over S':

$$\int_{-\infty}^{\infty} G(t,S;T,S')I^\star(S')dS' = \int_{-\infty}^{\infty} G(t,S;T,S')U(S'e^{J'})\varpi(J')dJ'dS'$$
$$-\int_{-\infty}^{\infty} \nu G(S')S'U_{S'}dS' - \int_{-\infty}^{\infty} GU dS'$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(t,S;T,S'e^{-J'})e^{-J'}\varpi(J')dJ' + (\nu S'G)_{S'} - G\right]U(S')dS'$$

by making substitution $S'e^{J'} \to S'$ and integrating by parts

Jump-diffusion model IV

Following the same steps consider:

$$
\begin{aligned}
I^{\dagger}(K) &= \int_0^{\infty} (S' - K)^+ I(S')dS' \\
&= \int_0^{\infty} (S' - K)^+ \int_{-\infty}^{\infty} G(S'e^{-J'})e^{-J'}\varpi(J')dJ'dS' + \nu K C_K - (\nu + 1)C \\
&= \int_{-\infty}^{\infty} C(Ke^{-J'})e^{J'}\varpi(J')dJ' + \nu K C_K - (\nu + 1)C
\end{aligned}
$$

As a result, obtain the **forward equation for call option prices**:

$$
C_T - \frac{1}{2}\sigma^2_{(\mathrm{loc,jd})}(T,K)K^2 C_{KK} + \mu(T)KC_K - \mu(T)C - \lambda I^{\dagger}(K) = 0
$$
$$
C(t,K) = (S(t) - K)^+
$$

Inverting $\sigma^2_{(\mathrm{loc,jd})}$ yields **Andersen-Andreasen equation** (2000):

$$
\begin{aligned}
\sigma^2_{(\mathrm{loc,jd})}(T,K) &= \frac{C_T(T,K) + \mu(T)KC_K(T,K) - \mu(T)C(T,K) - \lambda I^{\dagger}(K)}{\frac{1}{2}K^2 C_{KK}(T,K)} \\
&= \sigma^2_{(\mathrm{loc})}(T,K) - \frac{\lambda I^{\dagger}(K)}{\frac{1}{2}K^2 C_{KK}(T,K)}
\end{aligned}
$$

Stochastic local volatility I

Consider 2-d stochastic local volatility diffusion:

$$dS(t) = \mu(t)S(t)dt + \sigma_{(\text{loc,sv})}(t, S(t))\vartheta(t, Y(t))S(t)dW^{(1)}(t), \ S(0) = S,$$
$$dY(t) = \theta(Y(t))dt + \epsilon(Y(t))dW^{(2)}(t), \ Y(0) = Y \tag{19}$$

where $\sigma_{(\text{loc,sv})}$ is the local volatility of the diffusion with stochastic volatility and $< dW^{(1)}(t), dW^{(2)}(t) >= \rho dt$

$\vartheta(t, Y(t))$ is the mapping function of $Y(t)$ into the spot dynamics

Stochastic local volatility II. Calibration

Conventional approach (for example, Ren-Madan-Qian (2007)) is to use Gyöngy (1986) mimicking theorem yielding:

$$\sigma^2_{(\mathsf{loc,sv})}(T,K)\mathbb{E}\left[\vartheta^2(T,Y(T))\,|\,S(T)=K\right]=\sigma^2_{(\mathsf{loc,dif})}(T,K)$$

Gyöngy theorem is purely probabilistic analysis, we follow original Lipton's (2002) PDE-based approach applied directly in the context of stochastic local volatility calibration

The risk-neutral probability density $G(t,S,Y;T,S',Y')$ solves the forward **Kolmogorov equation**:

$$\begin{aligned} G_T &- \frac{1}{2}\Big((\sigma_{(\mathsf{loc,sv})}(T,S')\vartheta(T,Y')S')^2 G\Big)_{S'S'} + \Big(\mu(T)S'G\Big)_{S'} \\ &- \frac{1}{2}\Big(\epsilon^2(Y')G\Big)_{Y'Y'} + \Big(\theta(Y')G\Big)_{Y'} \\ &- \Big(\rho\sigma_{(\mathsf{loc,sv})}(T,S')\vartheta(T,Y')\epsilon(Y')S'G\Big)_{S'Y'} = 0 \\ G(t,S,&Y;t,S',Y') = \delta(S'-S)\delta(Y'-Y) \end{aligned}$$

Stochastic local volatility III. Calibration

Consider the unconditional density:

$$U(t,S;t,S') = \int_{-\infty}^{\infty} G(t,S,Y;T,S',Y')dY'$$

The diffusion term becomes:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Big((\sigma_{\text{(loc,sv)}}(T,S')\vartheta(T,Y')S')^2 G\Big)_{S'S'} dY' \\
&= \left(\left[\int_{-\infty}^{\infty} \vartheta^2(T,Y')GdY'\right](\sigma_{\text{(loc,sv)}}(T,S')S')^2\right)_{S'S'} \\
&= \Big(V(T,S')(\sigma_{\text{(loc,sv)}}(T,S')S')^2 U(T,S')\Big)_{S'S'}
\end{aligned}$$

where $V(T,S')$ is **the conditional variance**:

$$\begin{aligned}
V(T,S') &= \frac{\int_{-\infty}^{\infty} \vartheta^2(T,Y')G(t,S,Y;T,S',Y')dY'}{U(t,S;T,S')} \\
&= \frac{\int_{-\infty}^{\infty} \vartheta^2(T,Y')G(t,S,Y;T,S',Y')dY'}{\int_{-\infty}^{\infty} G(t,S,Y;T,S',Y')dY'}
\end{aligned}$$

Stochastic local volatility IV. Calibration

Remaining terms are trivial, yielding:

$$U_T - \frac{1}{2}\Big(V(T,S')(\sigma_{(\text{loc,sv})}(T,S')S')^2 U\Big)_{S'S'} + \Big(\mu(T)S'U\Big)_{S'} = 0$$
$$U(t,S;t,S') = \delta(S'-S)$$

PDE for call prices:

$$C_T - \frac{1}{2}(V(T,K)(\sigma_{(\text{loc,sv})}(T,K)K)^2 C_{KK} + \mu(T)KC_K - \mu(T)C = 0$$

The model is consistent with the **Dupire local volatility** if

$$V(T,K)\sigma^2_{(\text{loc,sv})}(T,K) = \sigma^2_{(\text{loc,dif})}(T,K)$$

As a result, the **stochastic local volatility** is specified by:

$$\sigma^2_{(\text{loc,sv})}(T,K) = \frac{\sigma^2_{(\text{loc,dif})}(T,K)}{V(T,K)}$$

Stochastic local volatility with jumps I

Consider a two-dimensional stochastic local volatility jump-diffusion:

$$\begin{aligned} dS(t) =&\mu(t)S(t_-)dt + \sigma_{(\text{loc,svj})}(t_-, S(t_-))\vartheta(t_-, Y(t_-))S(t_-)dW^{(1)}(t) \\ &+ \Big((e^J - 1)dN(t) - \lambda\nu dt\Big) S(t_-),\ S(0) = S, \\ dY(t) =&\theta(Y(t))dt + \epsilon(Y(t))dW^{(2)}(t) + \Upsilon dN(t),\ Y(0) = Y \end{aligned} \tag{20}$$

where $\sigma_{(\text{loc, svj})}$ is the local volatility of the diffusion with stochastic volatility and jumps

Υ is the amplitude of the jump in $Y(t)$ with PDF $\varsigma(\Upsilon)$

Jumps are simultaneous in $S(t)$ and $Y(t)$

Stochastic local volatility with jumps II

The risk-neutral probability density $G(t, S, Y; T, S', Y')$ solves the forward **Kolmogorov equation**:

$$
\begin{aligned}
&G_T - \frac{1}{2}\Big((\sigma_{(\text{loc,svj})}(T, S')\vartheta(T, Y')S')^2 G\Big)_{S'S'} + \Big(\mu(T)S'G\Big)_{S'} \\
&\quad - \frac{1}{2}\Big(\epsilon^2(Y')G\Big)_{Y'Y'} + \Big(\theta(Y')G\Big)_{Y'} \\
&\quad - \Big(\rho\sigma_{(\text{loc,svj})}(T, S')\vartheta(T, Y')\epsilon(Y')S'G\Big)_{S'Y'} - \lambda I(S') = 0 \\
&\qquad I(S') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(S'e^{-J}, Y' - \Upsilon)e^{-J}\varpi(J)\varsigma(\Upsilon)\,dJ\,d\Upsilon + (\nu S'G)_{S'} - G \\
&G(t, S, Y; t, S', Y') = \delta(S' - S)\delta(Y' - Y)
\end{aligned}
$$

Stochastic local volatility with jumps III

PDE for call prices:

$$C_T - \frac{1}{2}(V(T,K)(\sigma_{\text{(loc,svj)}}(T,K)K)^2 C_{KK} + \mu(T)KC_K - \mu(T)C - \lambda I^\dagger(K) = 0$$

where $V(T,K)$ is **the unconditional variance**

Inverting $\sigma^2_{\text{(loc,svj)}}(T,K)$, yields **Lipton equation** (2002):

$$\begin{aligned}
\sigma^2_{\text{(loc,svj)}}(T,K) &= \frac{C_T(T,K) + \mu(T)KC_K(T,K) - \mu(T)C(T,K) - \lambda I^\dagger(K)}{\frac{1}{2}V(T,K)K^2C_{KK}(T,K)} \\
&= \frac{1}{V(T,K)}\left(\sigma^2_{\text{(loc,dif)}}(T,K) - \frac{\lambda I^\dagger(K)}{\frac{1}{2}K^2C_{KK}(T,K)}\right) \\
&= \frac{\sigma^2_{\text{(loc,jd)}}(T,K)}{V(T,K)}
\end{aligned}$$

PDE based methods

Non-parametric local stochastic volatility can only be implemented using numerical methods for partial integro-differential equations

These methods should be flexible to handle jumps in one and two dimensions and the correlation term for two dimensions

I will review some of the available methods

For the backward problem, $\mathcal{L}$ is the infinitesimal operator corresponding to model dynamics:

$$\mathcal{L} = \mathcal{D} + \lambda \mathcal{J}$$

1-d Problem
Here $\mathcal{D}$ is the diffusion-convection operator:

$$\mathcal{D}U(t,S) \equiv \frac{1}{2}\sigma^2(t,S)S^2U_{SS} + \mu(t)SU_S + \lambda U$$

$\mathcal{J}$ is the integral operator:

$$\mathcal{J}U(t,S) \equiv \int_{-\infty}^{\infty} U(t, Se^{J'})\varpi(J')dJ'$$

For the forward problem, we consider the operator $\mathcal{L}^\dagger$ adjoint to $\mathcal{L}$:

$$\mathcal{L}^\dagger = \mathcal{D}^\dagger + \lambda\mathcal{J}^\dagger$$

For both problems we denote the discretized diffusion and integral operators by $\mathbf{D}^n$ and $\mathbf{J}$, respectively

The diffusion operator is space and time dependent (denoted by ^{n})

Care must be taken for discretization of the operator for the mean-reverting process!

1-d Problem for diffusion problem I

N - number of points in the space grid
$\mathbf{D}^n$ - discrete diffusion matrix at time t_n with dimension $N \times N$
U^n - the solution vector at time t_n with dimension $N \times 1$
$\mathbf{I}$ - the unit matrix with dimension $N \times N$

Time-stepping methods of the choice:

Explicit method:

$$U^{n+1} = \left(\mathbf{I} + \mathbf{D}^{n+1}\right) U^n$$

Implicit method:

$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right) U^{n+1} = U^n$$

θ method:

$$\left(\mathbf{I} - \theta \mathbf{D}^{n+1}\right) U^{n+1} = \left(\mathbf{I} + \theta \mathbf{D}^{n+1}\right) U^n$$

with $\theta = \frac{1}{2}$ corresponding to Crank-Nicolson scheme

1-d Problem for diffusion problem II

The explicit method requires very fined grids and is highly unstable - it should be avoided

The Crank-Nicolson scheme is unconditionally stable and is of order $(\delta S)^2$ and $(\delta t)^2$, but might lead to negative probabilities for the forward equation

The implicit method is unconditionally stable and does not lead to negative densities, but it is of order (δt) and becomes less accurate for longer maturities

My favourite is the predictor-corrector scheme:

$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right) \widetilde{U} = U^n$$
$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right) U^{n+1} = U^n + \frac{1}{2}\mathbf{D}^{n+1}(U^n - \widetilde{U})$$

which is of order $(\delta S)^2$ and $(\delta t)^2$, and does not lead to negative probabilities

Numerics for jump-diffusions I

Let me consider the forward equation with additive jumps (multiplicative jumps can be handled in the log-space):

$$\mathcal{J}G(x) = \int_{-\infty}^{\infty} G(x-j)\varpi(J')dJ'$$

Let me consider discrete negative jumps with size $-\eta$, $\eta > 0$:

$$\mathcal{J}G(x) = G(x+\eta)$$

Discretization is a simple linear interpolation:

$$\mathcal{J}G_i = wG_k + (1-w)G_{k+1}$$

where $k = \min\{k : x_{k+1} \geq x_i + \nu\}$ and $w = \frac{x_{k+1}-(x_i+\nu)}{x_{k+1}-x_k}$

Numerics for jump-diffusions II

In the matrix form, for uniform spacial grid:

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & ... & w & 1-w & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & w & 1-w & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & w & 1-w \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & 1 \\ ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

$\mathbf{J}_{(p,N)} = 1$, where $p = \min\{p : x_p + \nu \geq x_N\}$

In general, for two-sided jumps, $\mathbf{J}$ is a full matrix:
The implicit method for the integral term leads to $O(N^3)$ complexity and is not practical
The explicit method leads to $O(N^2)$ complexity but needs extra care for stability

For exponential jumps, Lipton (2003) develops recursive scheme with $O(N)$ complexity

Numerics for jump-diffusions III

In case of discrete jumps, we have two-banded matrix with p rows that have non-zero elements

Consider solving a linear system:

$$(\mathbf{I} - \beta \mathbf{J})X = R$$

where $\beta = \lambda \delta t$, $0 < \beta < 1$

We can solve this system by back-substitution with cost of $O(N)$ operations

Numerics for jump-diffusions IV

Consider θ and θ_J schemes for the diffusion and the jump parts:

$$\left(\mathbf{I} - \theta\mathbf{D}^{n+1} - \theta_J\lambda_{n+1}\mathbf{J}\right)U^{n+1} = \left(\mathbf{I} + \theta\mathbf{D}^{n+1} + \theta_J\lambda_{n+1}\mathbf{J}\right)U^n$$

where $\lambda_{n+1} = (t_{n+1} - t_n)\lambda(t_{n+1})$

The accuracy with $\theta = \theta_J = \frac{1}{2}$ is supposed to be $O(dx^2) + O(dt^2)$

However, the matrix in the lhs will be full: the cost to invert is $O(N^3)$

Implicit-explicit method:

$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right)U^{n+1} = \left(\mathbf{I} + \lambda_{n+1}\mathbf{J}\right)U^n$$

with accuracy of $O(dx^2) + O(dt)$

θ-**explicit method**:

$$\left(\mathbf{I} - \theta\mathbf{D}^{n+1}\right)U^{n+1} = \left(\mathbf{I} + \theta\mathbf{D}^{n+1} + \lambda_{n+1}\mathbf{J}\right)U^n$$

with accuracy of $O(dx^2) + O(dt)$

Numerics for jump-diffusions V. Andersen-Andreasen (2000) scheme

Make the first half step with $\theta = 1$ and $\theta_J = 0$ and the second half step with $\theta = 1$ and $\theta_J = 1$:

$$\left(\mathbf{I} - \frac{1}{2}\mathbf{D}^{n+1}\right)\widetilde{U} = \left(\mathbf{I} + \frac{1}{2}\lambda_{n+1}\mathbf{J}\right)U^n$$
$$\left(\mathbf{I} - \frac{1}{2}\lambda_{n+1}\mathbf{J}\right)U^{n+1} = \left(\mathbf{I} + \frac{1}{2}\mathbf{D}^{n+1}\right)\widetilde{U}$$

with accuracy of $O(dx^2) + O(dt^2)$

When $\mathbf{J}$ is full, the second equation can be solved by the application of the discrete Fourier transform (DFT) with the cost of $O(N \log N)$

For discrete jumps, the second equation solved with cost of $O(N)$ operations

Numerics for jump-diffusions VI. d'Halluin et al (2005) scheme

Consider fixed point iterations:

Set $V^1 = U^n$

Iterate for $p = 1, .., \overline{p}$ ($\overline{p} = 2$ is good enough):

$$\left(\mathbf{I} - \theta \mathbf{D}^{n+1}\right) V^{p+1} = \left(\mathbf{I} + \theta \mathbf{D}^{n+1}\right) U^n + \lambda_{n+1} \mathbf{J} V^p,$$

Set $U^{n+1} = V^{\overline{p}}$

The accuracy is $O(dx^2) + O(dt)$

Numerics for jump-diffusions VII

My favourite is the implicit scheme with predictor-corrector applied twice:

$$\widetilde{U}^{(0)} = \left(\mathbf{I} + \mathbf{D}^{n+1} + \lambda_{n+1}\mathbf{J}\right) U^n$$
$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right) \widetilde{U} = \widetilde{U}^{(0)}$$
$$\left(\mathbf{I} - \mathbf{D}^{n+1}\right) U^{n+1} = \widetilde{U}^{(0)} + \frac{1}{2}(\mathbf{D}^{n+1} + \lambda_{n+1}\mathbf{J})(\widetilde{U} - U^n)$$

The accuracy is $O(dx^2) + O(dt^2)$

Numerical Methods for Two Dimensional Problem I

We consider the forward problem for $U(t, x_1, x_2)$:

$$\begin{aligned} &U_T - \mathcal{M}U = 0 \\ &U(0, x_1, x_2) = \delta(x_1 - x_1(0))\delta(x_2 - x_2(0)) \end{aligned} \tag{21}$$

where

$$\mathcal{M} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{C} + \lambda\mathcal{J}$$

$\mathcal{D}_1$ and $\mathcal{D}_2$ are 1-d diffusion-convection operators in x_1 and x_2 directions, respectively
$\mathcal{C}$ is the correlation operator
$\mathcal{J}$ is the integral operator for joint jumps in x_1 and x_2

Let $\mathbf{D}_1$ and $\mathbf{D}_2$ denote the discretized 1-d diffusion-convection operators in x_1 and x_2 directions, respectively

$\mathbf{C}$ and $\mathbf{J}$ are the discretized correlation and integral operator, respectively

Douglas-Rachford (1956) scheme

Make a predictor and apply two orthogonal corrector steps:

$$
\begin{aligned}
\widetilde{U}^{(0)} &= (\mathbf{I} + \mathbf{C} + \lambda\mathbf{J} + \mathbf{D}_1 + \mathbf{D}_2)U^n \\
(\mathbf{I} - \theta\mathbf{D}_1)\widetilde{U} &= \widetilde{U}^{(0)} - \theta\mathbf{D}_1 U^n \\
(\mathbf{I} - \theta\mathbf{D}_2)U^{n+1} &= \widetilde{U} - \theta\mathbf{D}_2 U^n
\end{aligned}
$$

In the second line, for each fixed index j we apply the diffusion operator in x_1 direction; and solve the tridiagonal system of equations to get the auxiliary solution $\widetilde{U}(\cdot, x_2(j))$

In the third line, keeping i fixed, we apply the diffusion step in x_2 direction and solve the system of tridiagonal equations to get the solution $U^{n+1}(x_1(i), \cdot)$ at time t_{n+1}

Complexity is $O(N_1 N_2)$ per time step

Craig-Sneyd (1988) scheme

Start as with Douglas-Rachford scheme make a second predictor and again apply two orthogonal corrector steps:

$$\widetilde{U}^{(0)} = (\mathbf{I} + \mathbf{C} + \lambda\mathbf{J} + \mathbf{D}_1 + \mathbf{D}_2)U^n$$
$$(\mathbf{I} - \theta\mathbf{D}_1)\widetilde{U}^{(1)} = \widetilde{U}^{(0)} - \theta\mathbf{D}_1 U^n$$
$$(\mathbf{I} - \theta\mathbf{D}_2)\widetilde{U}^{(2)} = \widetilde{U}^{(1)} - \theta\mathbf{D}_2 U^n$$
$$\widetilde{U}^{(3)} = \widetilde{U}^{(0)} + \frac{1}{2}(\mathbf{C} + \lambda\mathbf{J})(\widetilde{U}^{(2)} - U^n)$$
$$(\mathbf{I} - \theta\mathbf{D}_1)\widetilde{U}^{(4)} = \widetilde{U}^{(3)} - \theta\mathbf{D}_1 U^n$$
$$(\mathbf{I} - \theta\mathbf{D}_2)U^{n+1} = \widetilde{U}^{(4)} - \theta\mathbf{D}_2 U^n$$

Hundsdorfer-Verwer (2003) scheme (in't Hout-Foulon (2008))

Similar to Craig-Sneyd scheme with predictor including $\mathbf{D}_1$ and $\mathbf{D}_2$ and the second corrector applied on $\widetilde{U}^{(2)}$:

$$
\begin{aligned}
&\widetilde{U}^{(0)} = (\mathbf{I} + \mathbf{C} + \lambda\mathbf{J} + \mathbf{D}_1 + \mathbf{D}_2)U^n \\
&(\mathbf{I} - \theta\mathbf{D}_1)\widetilde{U}^{(1)} = \widetilde{U}^{(0)} - \theta\mathbf{D}_1 U^n \\
&(\mathbf{I} - \theta\mathbf{D}_2)\widetilde{U}^{(2)} = \widetilde{U}^{(1)} - \theta\mathbf{D}_2 U^n \\
&\widetilde{U}^{(3)} = \widetilde{U}^{(0)} + \frac{1}{2}(\mathbf{C} + \lambda\mathbf{J} + \mathbf{D}_1 + \mathbf{D}_2)\left(\widetilde{U}^{(2)} - U^n\right) \\
&(\mathbf{I} - \theta\mathbf{D}_1)\widetilde{U}^{(4)} = \widetilde{U}^{(3)} - \theta\mathbf{D}_1 \widetilde{U}^{(2)} \\
&(\mathbf{I} - \theta\mathbf{D}_2)U^{n+1} = \widetilde{U}^{(4)} - \theta\mathbf{D}_2 \widetilde{U}^{(2)}
\end{aligned}
$$

Discretisation of integral term

Direct methods are infeasible because of $O(N_1^2 N_2^2)$ complexity

DFT method (Clift-Forsyth (2008)) has $O(N_1 N_2 \log N_1 N_2)$ complexity but suffers from problems associated with the DFT

Explicit methods with $O(N_1 N_2)$ complexity are available for discrete and exponential jumps (Lipton-Sepp (2011))

The simplest case is if jumps are discrete with sizes η_1 and η_2:

$$\mathcal{J}G = G(x_1 - \eta_1, x_2 - \eta_2)$$

This term is approximated by bi-linear interpolation with the second order accuracy leading to the $O(N_1 N_2)$ complexity

Illustration I. Model specification
We specify a particular version of the LSV dynamics (20):

$$\begin{aligned}
dS(t) &= \mu(t)S(t_-)dt + \sigma_{(\text{loc,svj})}(t_-, S(t_-))\vartheta(t_-, Y(t_-))S(t_-)dW^{(1)}(t) \\
&+ \Big((e^{-\nu} - 1)dN(t) + \lambda\nu dt\Big) S(t_-),\ S(0) = S \\
dY(t) &= -\kappa Y(t)dt + \epsilon dW^{(2)}(t) + \eta dN(t),\ Y(0) = 0
\end{aligned} \tag{22}$$

where $dW^{(1)}(t)dW^{(2)}(t) = \rho dt$
Convenient choice of $\vartheta(t, Y(t))$:

$$\vartheta(t, Y(t)) = e^{Y(t) - \mathbb{V}[Y(t)]}$$

where $\mathbb{V}[Y(t)]$ is the variance of $Y(t)$. Then if $\rho = 0$:

$$\mathbb{E}\left[\vartheta^2(t, Y(t))\,|\,Y(0) = 0\right] = 1$$

Mapping $\vartheta(t, Y(t))$ introduces the "volatility-of-volatility" effect without affecting the local volatility close to the spot
Simultaneous jumps in $S(t)$ and $Y(t)$ are discrete with magnitudes $-\nu < 0$ and $\eta > 0$ (note that jump variance will decrease the model implied correlation between spot and volatility)

Illustration II. Model calibration

1) Specify time and space grids

2) Compute the local volatility using Dupire equation (17) on the specified grid (interpolating implied volatilities in strikes and maturities)

3) Calibrate local stochastic volatility on the specified grid

4) Use backward PDE methods or Monte-Carlo simulations of the local stochastic volatility model to compute values of exotic options

Illustration III. Local volatility calibration A
At initial time $t_0 = 0$
i) initialize $G^0(x_1(i), x_2(j)) = \mathbf{1}_{x_1(i)=x_1(t_0)}\mathbf{1}_{x_2(j)=x_2(t_0)}$

ii) Set the conditional variance $V(t_1, x_1(i)) = 1$ so that

$$\sigma^2_{\text{(loc,sv)}}(t_1, x_1(i)) = \sigma^2_{\text{(loc,dif)}}(t_1, x_1(i))$$

iii) Compute $G^1(x_1(i), x_2(j))$ by solving the forward PDE

iv) Compute $V(t_1, x_1(i))$ by

$$V(t_1, x_1(i)) = \frac{\sum_i \sum_j \vartheta^2(t_1, x_2(j)) G^1(x_1(i), x_2(j))}{\sum_j G^1(x_1(i), x_2(j))}$$

and set

$$\sigma^2_{\text{(loc,sv)}}(t_1, x_1(i)) = \frac{\sigma^2_{\text{(loc,dif)}}(t_1, x_1(i))}{V(t_1, x_1(i))}$$

G) Repeat **C)** and **D)** and go to the next time step

Illustration III. Local volatility calibration B
At time t_n given $G^{n-1}(x_1(i), x_2(j))$ and $V(t_{n-1}, x_1(i))$
i) Compute the local variance by:

$$\sigma^2_{\text{(loc,sv)}}(t_n, x_1(i)) = \frac{\sigma^2_{\text{(loc,dif)}}(t_n, x_1(i))}{V(t_{n-1}, x_1(i))}$$

ii) Compute $G^n(x_1(i), x_2(j))$ by solving the forward PDE

iii) Compute $V(t_n, x_1(i))$ by

$$V(t_n, x_1(i)) = \frac{\sum_i \sum_j \vartheta^2(t_n, x_2(j)) G^n(x_1(i), x_2(j))}{\sum_j G^n(x_1(i), x_2(j))}$$

and set

$$\sigma^2_{\text{(loc,sv)}}(t_n, x_1(i)) = \frac{\sigma^2_{\text{(loc,dif)}}(t_n, x_1(i))}{V(t_n, x_1(i))}$$

D) Repeat **B)** and **C)** and go to the next time step

For stochastic volatility with jumps we use $\sigma^2_{\text{(loc,jd)}}$

Illustration III. Local volatility calibration C

When we use predictor-corrector schemes:

i) Compute the predictor step using local stochastic volatility from previous time step

ii) Update local stochastic volatility and compute the corrector step with new volatility

iii) After the corrector step, compute new local stochastic volatility and go to the next step

Illustration IV. Discrete dividends

At ex-dividend time t_n:

$$S(t_n+) = S(t_n-) - D_n$$

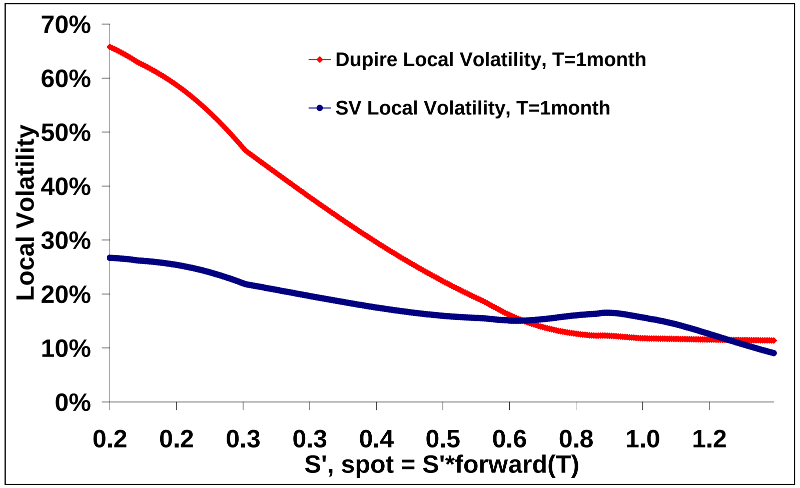
where D_n is the cash dividend at time t_n

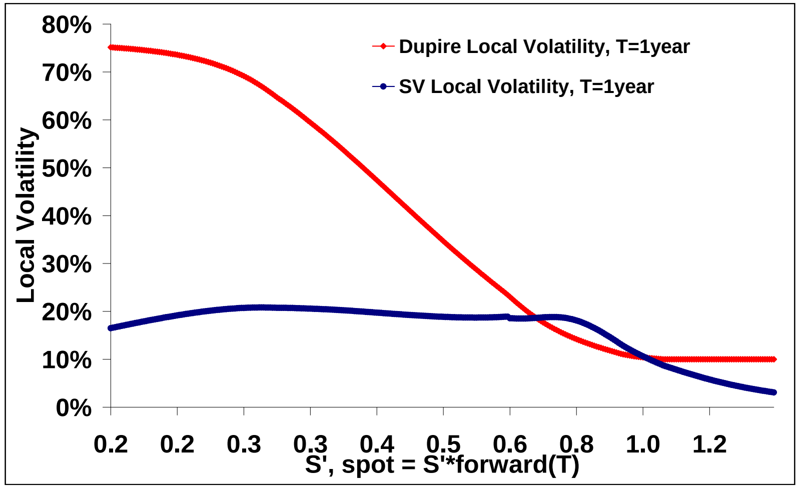
Note that this corresponds to the negative jump in $S(t)$ with a constant magnitude D_n at deterministic time t_n

Therefore the developed method for discrete negative jumps are readily applied for discrete dividends

It is important that $\sigma^2_{\text{(loc,dif)}}$ is consistent with the discrete dividends

Illustration V. Local volatilities for options on the S&P 500

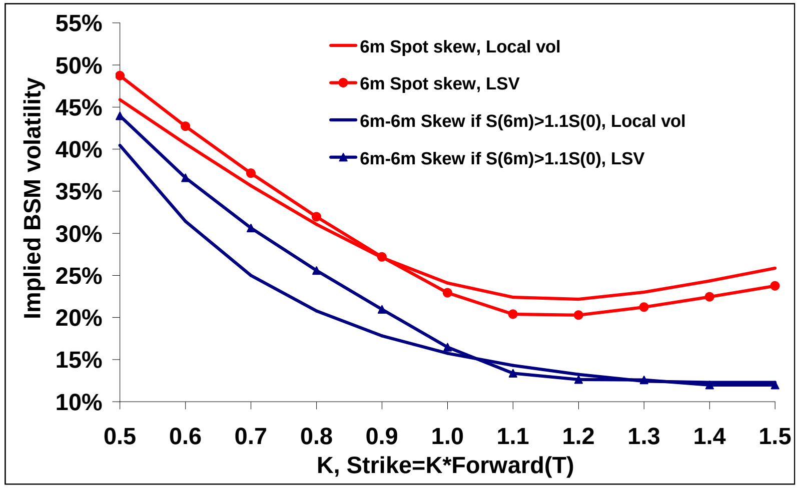


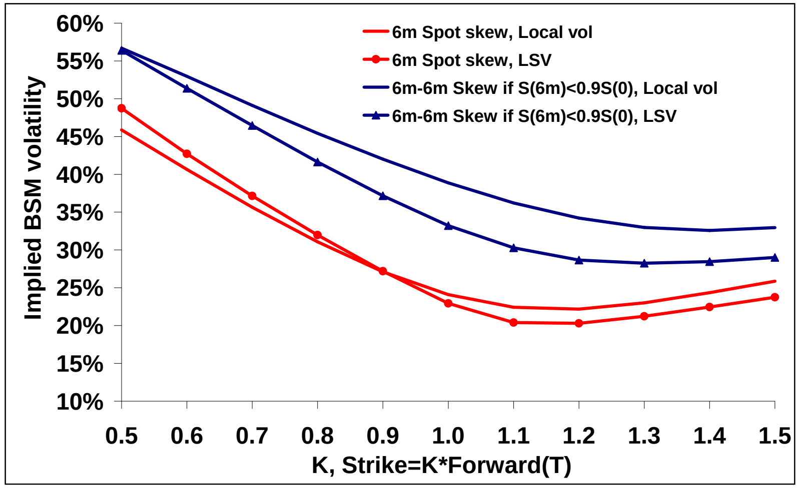


Left: Dupire local volatility and LSV local volatilities at T=1 month
Right: ... at T=1 year

LSV local volatility is an adjustment to the skew implied by the stochastic volatility part and is mostly flat

Illustration VI. Forward volatilities for options on the S&P 500





6m-6m skew: the implied volatility for a vanilla call option starting in 6 month and maturing in one year from now with strike fixed in 6 months: $K = S(T = 6m)$

Left: 6m-6m skew if $S(T = 6m) > 1.1S(0)$: the implied volatility for the forward-start option conditional that $S(T = 6m) > 1.1S(0)$

Right: 6m-6m skew if $S(T = 6m) < 0.9S(0)$: the implied volatility for the forward-start option conditional that $S(T = 6m) < 0.9S(0)$

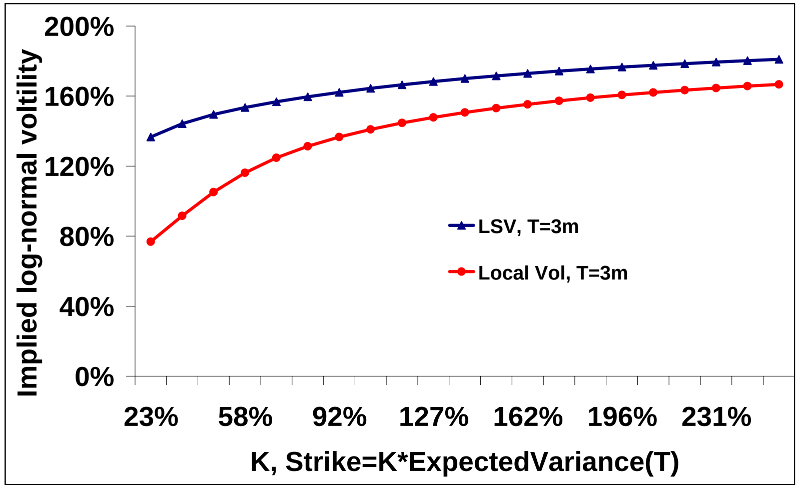
Illustration VII. Implications

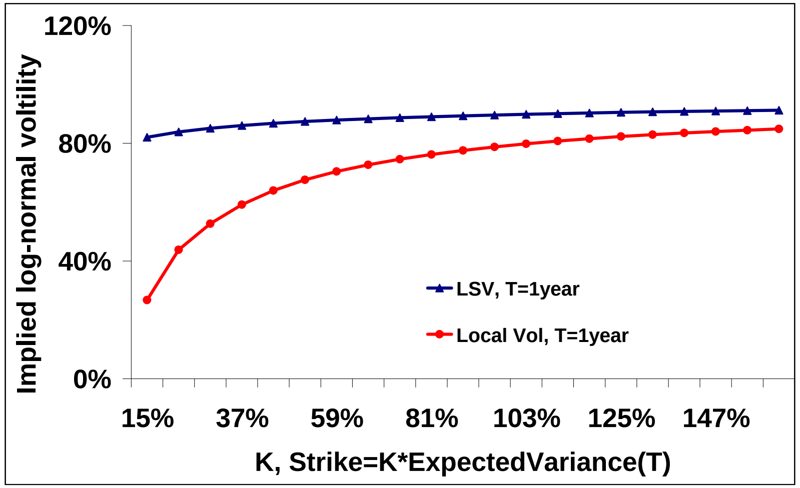
Conditional that spot moves up, the LSV model preserves the shape of the volatility skew while the pure local volatility does not

Conditional that spot moves down, the pure local volatility implies that the ATM volatility goes too high and the skew flattens unlike the LSV model

The LSV is more closer to the actual dynamics!

Illustration VIII. Implied volatilities of options on the daily realized variance of the S&P 500





Left: Implied log-normal volatilities on options on the daily realized variance of the S&P 500 with maturity T=3 month
Right: ... at T=1 year

LSV local volatility implies higher volatility of the realized variance ad the vol-of-vol parameter can be "tuned-up" to match the market

Illustration IX. Options on the VIX

Augment the model dynamics (22) with the third variable for the realized quadratic variance of $S(t)$:

$$dI(t) = \Big(\sigma_{(\text{loc,svj})}(t_-, S(t_-))\vartheta(t_-, Y(t_-))\Big)^2 dt + \nu^2 dN(t),\ \ I(0) = 0$$

Consider the expected variance realized over period $[T, T + T_{\text{vix}}]$:

$$\widetilde{I}(T, T + T_{\text{vix}}) = \frac{1}{T_{\text{vix}}}\mathbb{E}\left[\int_T^{T+T_{\text{vix}}} dI(t')\right]$$

where $T_{\text{vix}} = 30/365.25$ is the annualized tenor of the VIX

A call option on the VIX maturing at T is computed by:

$$C(T, K) = \mathbb{E}\left[\Big(\widetilde{I}(T, T + T_{\text{vix}}) - K\Big)^+ \mid \widetilde{I}(T, T) = 0\right]$$

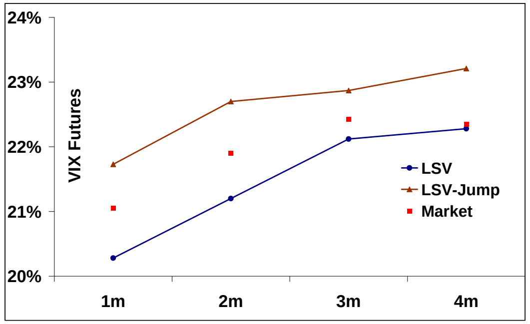
Valuation:

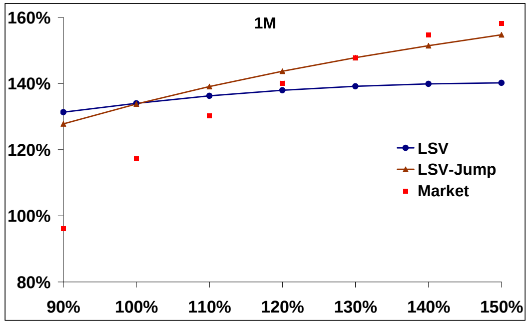
i) Solve 2-d (!) backward problem for $\widetilde{I}(T, T + T_{\text{vix}})$ as function of (S, Y)

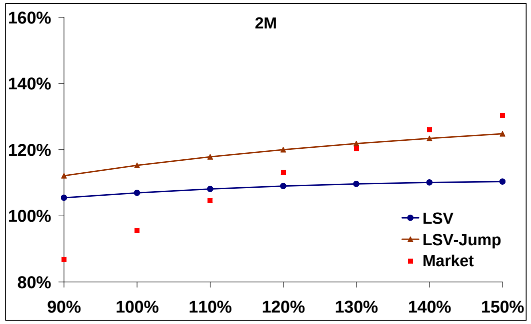
ii) Compute $G(t, S, Y; T, S', Y')$ and evaluate $C(T, K)$ by convolution

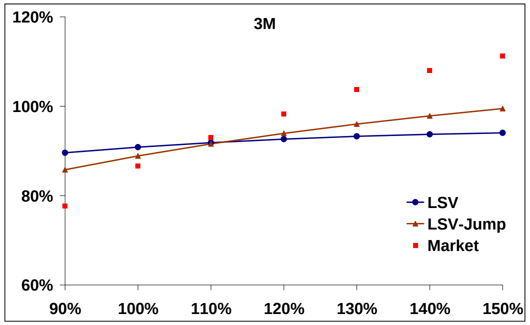
iii) Can price call with different strikes at a time

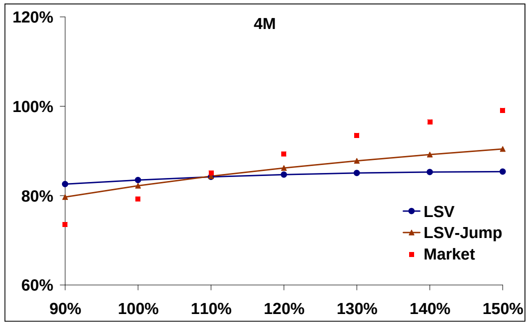
VIX Implied volatilities











Conclusion: consistency with options on the SPX does not lead to consistency with options on the VIX; Need extra flexibility for jumps in volatility (time and/or space dependency)

Conclusions

I have presented theoretical and practical grounds for stochastic local volatility models and highlighted details of model implementation

I am grateful to members of the quantitative group at Bank of America Merrill Lynch for their help and discussions during work on this project

Thank you for your attention!

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