

Verified Construction of Fair Voting Rules

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Abstract

Voting rules aggregate multiple individual preferences in order to make a collective decision. Commonly, these mechanisms are expected to respect a multitude of different notions of fairness and reliability, which must be carefully balanced to avoid inconsistencies.

This article contains a formalisation of a framework for the construction of such fair voting rules using composable modules [1, 2]. The framework is a formal and systematic approach for the flexible and verified construction of voting rules from individual composable modules to respect such social-choice properties by construction. Formal composition rules guarantee resulting social-choice properties from properties of the individual components which are of generic nature to be reused for various voting rules. We provide proofs for a selected set of structures and composition rules. The approach can be readily extended in order to support more voting rules, e.g., from the literature by extending the sets of modules and composition rules.

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Chapter 1

Social-Choice Types

1.1 Auxiliary Lemmas

```
theory Auxiliary-Lemmas
  imports Main
begin
```

Summation function is invariant under application of a (bijective) permutation on the elements.

```
lemma sum-comp:
  fixes
     $f :: 'x \Rightarrow ('z :: \text{comm-monoid-add})$  and
     $g :: 'y \Rightarrow 'x$  and
     $X :: 'x \text{ set}$  and
     $Y :: 'y \text{ set}$ 
  assumes bij-betw  $g$   $Y$   $X$ 
  shows  $(\sum x \in X. f\ x) = (\sum x \in Y. (f \circ g)\ x)$ 
  <proof>
```

The inversion of a composition of injective functions is equivalent to the composition of the two individual inverted functions.

```
lemma the-inv-comp:
  fixes
     $X :: 'x \text{ set}$  and
     $Y :: 'y \text{ set}$  and
     $Z :: 'z \text{ set}$  and
     $f :: 'y \Rightarrow 'x$  and
     $g :: 'z \Rightarrow 'y$  and
     $x :: 'x$ 
  assumes
    bij-betw  $f$   $Y$   $X$  and
    bij-betw  $g$   $Z$   $Y$  and
     $x \in X$ 
  shows the-inv-into  $Z$   $(f \circ g)\ x = ((\text{the-inv-into } Z\ g) \circ (\text{the-inv-into } Y\ f))\ x$ 
  <proof>
```

end

1.2 Preference Relation

```
theory Preference-Relation
  imports Main
begin
```

The very core of the composable modules voting framework: types and functions, derivations, lemmas, operations on preference relations, etc.

1.2.1 Definition

Each voter expresses pairwise relations between all alternatives, thereby inducing a linear order.

```
type-synonym 'a Preference-Relation = 'a rel
```

```
type-synonym 'a Vote = 'a set  $\times$  'a Preference-Relation
```

```
fun is-less-preferred-than :: 'a  $\Rightarrow$  'a Preference-Relation  $\Rightarrow$  'a  $\Rightarrow$  bool
  (-  $\preceq$  - [50, 1000, 51] 50) where
  a  $\preceq_r$  b = ((a, b)  $\in$  r)
```

```
fun alts- $\mathcal{V}$  :: 'a Vote  $\Rightarrow$  'a set where
  alts- $\mathcal{V}$  V = fst V
```

```
fun pref- $\mathcal{V}$  :: 'a Vote  $\Rightarrow$  'a Preference-Relation where
  pref- $\mathcal{V}$  V = snd V
```

```
lemma lin-imp-antisym:
  fixes
    A :: 'a set and
    r :: 'a Preference-Relation
  assumes linear-order-on A r
  shows antisym r
  <proof>
```

```
lemma lin-imp-trans:
  fixes
    A :: 'a set and
    r :: 'a Preference-Relation
  assumes linear-order-on A r
  shows trans r
```

$\langle \text{proof} \rangle$

1.2.2 Ranking

fun *rank* :: 'a *Preference-Relation* \Rightarrow 'a \Rightarrow nat **where**
 rank *r* *a* = card (above *r* *a*)

lemma *rank-gt-zero*:

fixes

r :: 'a *Preference-Relation* **and**

a :: 'a

assumes

refl: *a* \preceq_r *a* **and**

fin: finite *r*

shows *rank* *r* *a* \geq 1

$\langle \text{proof} \rangle$

1.2.3 Limited Preference

definition *limited* :: 'a set \Rightarrow 'a *Preference-Relation* \Rightarrow bool **where**
 limited *A* *r* \equiv *r* \subseteq *A* \times *A*

lemma *limited-dest*:

fixes

A :: 'a set **and**

r :: 'a *Preference-Relation* **and**

a *b* :: 'a

assumes

a \preceq_r *b* **and**

limited *A* *r*

shows *a* \in *A* \wedge *b* \in *A*

$\langle \text{proof} \rangle$

fun *limit* :: 'a set \Rightarrow 'a *Preference-Relation* \Rightarrow 'a *Preference-Relation* **where**
 limit *A* *r* = {(*a*, *b*) \in *r*. *a* \in *A* \wedge *b* \in *A*}

definition *connex* :: 'a set \Rightarrow 'a *Preference-Relation* \Rightarrow bool **where**
 connex *A* *r* \equiv *limited* *A* *r* \wedge (\forall *a* \in *A*. \forall *b* \in *A*. *a* \preceq_r *b* \vee *b* \preceq_r *a*)

lemma *connex-imp-refl*:

fixes

A :: 'a set **and**

r :: 'a *Preference-Relation*

assumes *connex* *A* *r*

shows *refl-on* *A* *r*

$\langle \text{proof} \rangle$

lemma *lin-ord-imp-connex*:

fixes

A :: 'a set **and**

$r :: 'a \text{ Preference-Relation}$
assumes $\text{linear-order-on } A \ r$
shows $\text{connex } A \ r$
 $\langle \text{proof} \rangle$

lemma $\text{connex-antsym-and-trans-imp-lin-ord:}$
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
assumes
 $\text{connex-r: } \text{connex } A \ r$ **and**
 $\text{antisym-r: } \text{antisym } r$ **and**
 $\text{trans-r: } \text{trans } r$
shows $\text{linear-order-on } A \ r$
 $\langle \text{proof} \rangle$

lemma limit-to-limits:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
shows $\text{limited } A \ (\text{limit } A \ r)$
 $\langle \text{proof} \rangle$

lemma $\text{limit-presv-connex:}$
fixes
 $A \ B :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
assumes
 $\text{connex: } \text{connex } B \ r$ **and**
 $\text{subset: } A \subseteq B$
shows $\text{connex } A \ (\text{limit } A \ r)$
 $\langle \text{proof} \rangle$

lemma $\text{limit-presv-antisym:}$
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
assumes $\text{antisym } r$
shows $\text{antisym } (\text{limit } A \ r)$
 $\langle \text{proof} \rangle$

lemma $\text{limit-presv-trans:}$
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
assumes $\text{trans } r$
shows $\text{trans } (\text{limit } A \ r)$
 $\langle \text{proof} \rangle$

lemma *limit-presv-lin-ord*:
fixes
 $A\ B :: 'a\ set$ **and**
 $r :: 'a\ Preference-Relation$
assumes
 $linear-order-on\ B\ r$ **and**
 $A \subseteq B$
shows $linear-order-on\ A\ (limit\ A\ r)$
 $\langle proof \rangle$

lemma *limit-presv-prefs*:
fixes
 $A :: 'a\ set$ **and**
 $r :: 'a\ Preference-Relation$ **and**
 $a\ b :: 'a$
assumes
 $a \preceq_r b$ **and**
 $a \in A$ **and**
 $b \in A$
shows $let\ s = limit\ A\ r\ in\ a \preceq_s b$
 $\langle proof \rangle$

lemma *limit-rel-presv-prefs*:
fixes
 $A :: 'a\ set$ **and**
 $r :: 'a\ Preference-Relation$ **and**
 $a\ b :: 'a$
assumes $(a, b) \in limit\ A\ r$
shows $a \preceq_r b$
 $\langle proof \rangle$

lemma *limit-trans*:
fixes
 $A\ B :: 'a\ set$ **and**
 $r :: 'a\ Preference-Relation$
assumes $A \subseteq B$
shows $limit\ A\ r = limit\ A\ (limit\ B\ r)$
 $\langle proof \rangle$

lemma *lin-ord-not-empty*:
fixes $r :: 'a\ Preference-Relation$
assumes $r \neq \{\}$
shows $\neg linear-order-on\ \{\}\ r$
 $\langle proof \rangle$

lemma *lin-ord-singleton*:
fixes $a :: 'a$
shows $\forall\ r.\ linear-order-on\ \{a\}\ r \longrightarrow r = \{(a, a)\}$
 $\langle proof \rangle$

1.2.4 Auxiliary Lemmas

lemma *above-trans*:

fixes

$r :: 'a \text{ Preference-Relation}$ **and**
 $a \ b :: 'a$

assumes

$\text{trans } r$ **and**
 $(a, b) \in r$

shows $\text{above } r \ b \subseteq \text{above } r \ a$
<proof>

lemma *above-refl*:

fixes

$A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$

assumes

$\text{refl-on } A \ r$ **and**
 $a \in A$

shows $a \in \text{above } r \ a$
<proof>

lemma *above-subset-geq-one*:

fixes

$A :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$

assumes

$\text{linear-order-on } A \ r$ **and**
 $\text{linear-order-on } A \ r'$ **and**
 $\text{above } r \ a \subseteq \text{above } r' \ a$ **and**
 $\text{above } r' \ a = \{a\}$

shows $\text{above } r \ a = \{a\}$
<proof>

lemma *above-connex*:

fixes

$A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$

assumes

$\text{connex } A \ r$ **and**
 $a \in A$

shows $a \in \text{above } r \ a$
<proof>

lemma *pref-imp-in-above*:

fixes

$r :: 'a \text{ Preference-Relation}$ **and**

$a \ b :: 'a$
shows $(a \preceq_r b) = (b \in \text{above } r \ a)$
 $\langle \text{proof} \rangle$

lemma *limit-presv-above*:

fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a \ b :: 'a$
assumes
 $b \in \text{above } r \ a$ **and**
 $a \in A$ **and**
 $b \in A$
shows $b \in \text{above } (\text{limit } A \ r) \ a$
 $\langle \text{proof} \rangle$

lemma *limit-rel-presv-above*:

fixes
 $A \ B :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a \ b :: 'a$
assumes $b \in \text{above } (\text{limit } B \ r) \ a$
shows $b \in \text{above } r \ a$
 $\langle \text{proof} \rangle$

lemma *above-one*:

fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$
assumes
 $\text{lin-ord-}r$: $\text{linear-order-on } A \ r$ **and**
 $\text{fin-}A$: $\text{finite } A$ **and**
 $\text{non-empty-}A$: $A \neq \{\}$
shows $\exists \ a \in A. \text{above } r \ a = \{a\} \wedge (\forall \ a' \in A. \text{above } r \ a' = \{a'\} \longrightarrow a' = a)$
 $\langle \text{proof} \rangle$

lemma *above-one-eq*:

fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a \ b :: 'a$
assumes
 lin-ord : $\text{linear-order-on } A \ r$ **and**
 $\text{fin-}A$: $\text{finite } A$ **and**
 $\text{not-empty-}A$: $A \neq \{\}$ **and**
 $\text{above-}a$: $\text{above } r \ a = \{a\}$ **and**
 $\text{above-}b$: $\text{above } r \ b = \{b\}$
shows $a = b$
 $\langle \text{proof} \rangle$

lemma *above-one-imp-rank-one*:
fixes
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes $\text{above } r \ a = \{a\}$
shows $\text{rank } r \ a = 1$
 $\langle \text{proof} \rangle$

lemma *rank-one-imp-above-one*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes
 $\text{lin-ord: linear-order-on } A \ r$ **and**
 $\text{rank-one: rank } r \ a = 1$
shows $\text{above } r \ a = \{a\}$
 $\langle \text{proof} \rangle$

theorem *above-rank*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes $\text{linear-order-on } A \ r$
shows $(\text{above } r \ a = \{a\}) = (\text{rank } r \ a = 1)$
 $\langle \text{proof} \rangle$

lemma *rank-unique*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a \ b :: 'a$
assumes
 $\text{lin-ord: linear-order-on } A \ r$ **and**
 $\text{fin-A: finite } A$ **and**
 $\text{a-in-A: } a \in A$ **and**
 $\text{b-in-A: } b \in A$ **and**
 $\text{a-neq-b: } a \neq b$
shows $\text{rank } r \ a \neq \text{rank } r \ b$
 $\langle \text{proof} \rangle$

lemma *above-presv-limit*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
shows $\text{above } (\text{limit } A \ r) \ a \subseteq A$

$\langle \text{proof} \rangle$

1.2.5 Lifting Property

definition *equiv-rel-except-a* :: 'a set \Rightarrow 'a Preference-Relation \Rightarrow 'a Preference-Relation \Rightarrow bool **where**
equiv-rel-except-a A r r' a \equiv
 linear-order-on A r \wedge linear-order-on A r' \wedge a \in A \wedge
 $(\forall a' \in A - \{a\}. \forall b' \in A - \{a\}. (a' \preceq_r b') = (a' \preceq_{r'} b'))$

definition *lifted* :: 'a set \Rightarrow 'a Preference-Relation \Rightarrow 'a Preference-Relation \Rightarrow bool **where**
lifted A r r' a \equiv
equiv-rel-except-a A r r' a \wedge $(\exists a' \in A - \{a\}. a \preceq_r a' \wedge a' \preceq_{r'} a)$

lemma *trivial-equiv-rel*:

fixes

A :: 'a set **and**

r :: 'a Preference-Relation

assumes linear-order-on A r

shows $\forall a \in A. \text{equiv-rel-except-a } A \ r \ r \ a$

$\langle \text{proof} \rangle$

lemma *lifted-imp-equiv-rel-except-a*:

fixes

A :: 'a set **and**

r r' :: 'a Preference-Relation **and**

a :: 'a

assumes *lifted* A r r' a

shows *equiv-rel-except-a* A r r' a

$\langle \text{proof} \rangle$

lemma *lifted-imp-switched*:

fixes

A :: 'a set **and**

r r' :: 'a Preference-Relation **and**

a :: 'a

assumes *lifted* A r r' a

shows $\forall a' \in A - \{a\}. \neg (a' \preceq_r a \wedge a \preceq_{r'} a')$

$\langle \text{proof} \rangle$

lemma *lifted-mono*:

fixes

A :: 'a set **and**

r r' :: 'a Preference-Relation **and**

a a' :: 'a

assumes

lifted: *lifted* A r r' a **and**

a'-pref-a: $a' \preceq_r a$

shows $a' \preceq_{r'} a$
 $\langle proof \rangle$

lemma *lifted-above-subset*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes *lifted* $A \ r \ r' \ a$
shows $\text{above } r' \ a \subseteq \text{above } r \ a$
 $\langle proof \rangle$

lemma *lifted-above-mono*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a \ a' :: 'a$
assumes
lifted-a: *lifted* $A \ r \ r' \ a$ **and**
a'-in-A-sub-a: $a' \in A - \{a\}$
shows $\text{above } r \ a' \subseteq \text{above } r' \ a' \cup \{a\}$
 $\langle proof \rangle$

lemma *limit-lifted-imp-eq-or-lifted*:
fixes
 $A \ A' :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes
lifted: *lifted* $A' \ r \ r' \ a$ **and**
subset: $A \subseteq A'$
shows $\text{limit } A \ r = \text{limit } A \ r' \vee \text{lifted } A \ (\text{limit } A \ r) \ (\text{limit } A \ r') \ a$
 $\langle proof \rangle$

lemma *negl-diff-imp-eq-limit*:
fixes
 $A \ A' :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes
change: *equiv-rel-except-a* $A' \ r \ r' \ a$ **and**
subset: $A \subseteq A'$ **and**
not-in-A: $a \notin A$
shows $\text{limit } A \ r = \text{limit } A \ r'$
 $\langle proof \rangle$

theorem *lifted-above-winner-alts*:
fixes
 $A :: 'a \text{ set}$ **and**

$r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a \ a' :: 'a$
assumes
 $lifted\text{-}a$: $lifted \ A \ r \ r' \ a$ **and**
 $a'\text{-above-}a'$: $above \ r \ a' = \{a'\}$ **and**
 $fin\text{-}A$: $finite \ A$
shows $above \ r' \ a' = \{a'\} \vee above \ r' \ a = \{a\}$
 $\langle proof \rangle$

theorem $lifted\text{-}above\text{-}winner\text{-}single$:
fixes
 $A :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a :: 'a$
assumes
 $lifted \ A \ r \ r' \ a$ **and**
 $above \ r \ a = \{a\}$ **and**
 $finite \ A$
shows $above \ r' \ a = \{a\}$
 $\langle proof \rangle$

theorem $lifted\text{-}above\text{-}winner\text{-}other$:
fixes
 $A :: 'a \text{ set}$ **and**
 $r \ r' :: 'a \text{ Preference-Relation}$ **and**
 $a \ a' :: 'a$
assumes
 $lifted\text{-}a$: $lifted \ A \ r \ r' \ a$ **and**
 $a'\text{-above-}a'$: $above \ r' \ a' = \{a'\}$ **and**
 $fin\text{-}A$: $finite \ A$ **and**
 $a\text{-not-}a'$: $a \neq a'$
shows $above \ r \ a' = \{a'\}$
 $\langle proof \rangle$

end

1.3 Norm

theory $Norm$
imports $HOL\text{-}Library.Extended\text{-}Real$
 $HOL\text{-}Combinatorics.List\text{-}Permutation$
 $Auxiliary\text{-}Lemmas$
begin

A norm on R to n is a mapping $N: R \mapsto n$ on R that has the following properties for all mappings u (and v) in R to n :

- positive scalability: $N(a * u) = |a| * N(u)$ for all a in R .
- positive semidefiniteness: $N(u) \geq 0$ with $N(u) = 0$ if and only if $u = (0, 0, \dots, 0)$.
- triangle inequality: $N(u + v) \leq N(u) + N(v)$.

1.3.1 Definition

type-synonym $Norm = \text{ereal list} \Rightarrow \text{ereal}$

definition $norm :: Norm \Rightarrow bool$ **where**

$norm\ n \equiv \forall\ x :: \text{ereal list}. n\ x \geq 0 \wedge (\forall\ i < \text{length}\ x. x[i] = 0 \longrightarrow n\ x = 0)$

1.3.2 Auxiliary Lemmas

lemma *sum-over-image-of-bijection*:

fixes

$A :: 'a\ \text{set}$ **and**

$A' :: 'b\ \text{set}$ **and**

$f :: 'a \Rightarrow 'b$ **and**

$g :: 'a \Rightarrow \text{ereal}$

assumes *bij-betw* $f\ A\ A'$

shows $(\sum\ a \in A. g\ a) = (\sum\ a' \in A'. g\ (\text{the-inv-into}\ A\ f\ a'))$

<proof>

1.3.3 Common Norms

fun *l-one* $:: Norm$ **where**

$l-one\ x = (\sum\ i < \text{length}\ x. |x[i]|)$

1.3.4 Properties

definition *symmetry* $:: Norm \Rightarrow bool$ **where**

$symmetry\ n \equiv \forall\ x\ y. x <^{\sim\sim} y \longrightarrow n\ x = n\ y$

1.3.5 Theorems

theorem *l-one-is-sym*: $symmetry\ l-one$

<proof>

end

1.4 Electoral Result

```
theory Result
  imports Main
begin
```

An electoral result is the principal result type of the composable modules voting framework, as it is a generalization of the set of winning alternatives from social choice functions. Electoral results are selections of the received (possibly empty) set of alternatives into the three disjoint groups of elected, rejected and deferred alternatives. Any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives.

1.4.1 Auxiliary Functions

```
type-synonym 'r Result = 'r set * 'r set * 'r set
```

A partition of a set A are pairwise disjoint sets that "set equals partition" A. For this specific predicate, we have three disjoint sets in a three-tuple.

```
fun disjoint3 :: 'r Result  $\Rightarrow$  bool where
```

```
  disjoint3 (e, r, d) =
    ((e  $\cap$  r = {})  $\wedge$ 
     (e  $\cap$  d = {})  $\wedge$ 
     (r  $\cap$  d = {}))
```

```
fun set-equals-partition :: 'r set  $\Rightarrow$  'r Result  $\Rightarrow$  bool where
```

```
  set-equals-partition X (e, r, d) = (e  $\cup$  r  $\cup$  d = X)
```

1.4.2 Definition

A result generally is related to the alternative set A (of type 'a). A result should be well-formed on the alternatives. Also it should be possible to limit a well-formed result to a subset of the alternatives.

Specific result types like social choice results (sets of alternatives) can be realized via sublocales of the result locale.

```
locale result =
```

```
  fixes
```

```
    well-formed :: 'a set  $\Rightarrow$  ('r Result)  $\Rightarrow$  bool and
```

```
    limit :: 'a set  $\Rightarrow$  'r set  $\Rightarrow$  'r set
```

```
assumes  $\forall$  (A :: 'a set) (r :: 'r Result).
```

```
  (set-equals-partition (limit A UNIV) r  $\wedge$  disjoint3 r)  $\longrightarrow$  well-formed A r
```

These three functions return the elect, reject, or defer set of a result.

```
fun (in result) limitR :: 'a set  $\Rightarrow$  'r Result  $\Rightarrow$  'r Result where
```

```
  limitR A (e, r, d) = (limit A e, limit A r, limit A d)
```

abbreviation *elect-r* :: 'r Result \Rightarrow 'r set **where**
elect-r r \equiv fst r

abbreviation *reject-r* :: 'r Result \Rightarrow 'r set **where**
reject-r r \equiv fst (snd r)

abbreviation *defer-r* :: 'r Result \Rightarrow 'r set **where**
defer-r r \equiv snd (snd r)

end

1.5 Preference Profile

theory *Profile*
imports *Preference-Relation*
Auxiliary-Lemmas
HOL-Library.Extended-Nat
HOL-Combinatorics.Permutations
begin

Preference profiles denote the decisions made by the individual voters on the eligible alternatives. They are represented in the form of one preference relation (e.g., selected on a ballot) per voter, collectively captured in a mapping of voters onto their respective preference relations. If there are finitely many voters, they can be enumerated and the mapping can be interpreted as a list of preference relations. Unlike the common preference profiles in the social-choice sense, the profiles described here consider only the (sub-)set of alternatives that are received.

1.5.1 Definition

A profile contains one ballot for each voter. An election consists of a set of participating voters, a set of eligible alternatives, and a corresponding profile.

type-synonym ('a, 'v) *Profile* = 'v \Rightarrow ('a *Preference-Relation*)

type-synonym ('a, 'v) *Election* = 'a set \times 'v set \times ('a, 'v) *Profile*

fun *alternatives- \mathcal{E}* :: ('a, 'v) *Election* \Rightarrow 'a set **where**
alternatives- \mathcal{E} E = fst E

fun *voters- \mathcal{E}* :: ('a, 'v) *Election* \Rightarrow 'v set **where**

$\text{voters-}\mathcal{E} \ E = \text{fst} (\text{snd } E)$

fun $\text{profile-}\mathcal{E} :: ('a, 'v) \text{ Election} \Rightarrow ('a, 'v) \text{ Profile} \text{ where}$
 $\text{profile-}\mathcal{E} \ E = \text{snd} (\text{snd } E)$

fun $\text{election-equality} :: ('a, 'v) \text{ Election} \Rightarrow ('a, 'v) \text{ Election} \Rightarrow \text{bool} \text{ where}$
 $\text{election-equality} (A, V, p) (A', V', p') =$
 $(A = A' \wedge V = V' \wedge (\forall v \in V. p \ v = p' \ v))$

A profile on a set of alternatives A and a voter set V consists of ballots that are linear orders on A for all voters in V. A finite profile is one with finitely many alternatives and voters.

definition $\text{profile} :: 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow \text{bool} \text{ where}$
 $\text{profile } V \ A \ p \equiv \forall v \in V. \text{linear-order-on } A \ (p \ v)$

abbreviation $\text{finite-profile} :: 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow \text{bool} \text{ where}$
 $\text{finite-profile } V \ A \ p \equiv \text{finite } A \wedge \text{finite } V \wedge \text{profile } V \ A \ p$

abbreviation $\text{finite-election} :: ('a, 'v) \text{ Election} \Rightarrow \text{bool} \text{ where}$
 $\text{finite-election } E \equiv \text{finite-profile } (\text{voters-}\mathcal{E} \ E) (\text{alternatives-}\mathcal{E} \ E) (\text{profile-}\mathcal{E} \ E)$

definition $\text{finite-elections-}\mathcal{V} :: ('a, 'v) \text{ Election set} \text{ where}$
 $\text{finite-elections-}\mathcal{V} \equiv \{E :: ('a, 'v) \text{ Election}. \text{finite } (\text{voters-}\mathcal{E} \ E)\}$

definition $\text{finite-elections} :: ('a, 'v) \text{ Election set} \text{ where}$
 $\text{finite-elections} \equiv \{E :: ('a, 'v) \text{ Election}. \text{finite-election } E\}$

definition $\text{well-formed-elections} :: ('a, 'v) \text{ Election set} \text{ where}$
 $\text{well-formed-elections} \equiv \{E. \text{profile } (\text{voters-}\mathcal{E} \ E) (\text{alternatives-}\mathcal{E} \ E) (\text{profile-}\mathcal{E} \ E)\}$

— This function subsumes elections with fixed alternatives, finite voters, and a default value for the profile value on non-voters.

fun $\text{elections-}\mathcal{A} :: 'a \text{ set} \Rightarrow ('a, 'v) \text{ Election set} \text{ where}$
 $\text{elections-}\mathcal{A} \ A =$
 $\text{well-formed-elections}$
 $\cap \{E. \text{alternatives-}\mathcal{E} \ E = A \wedge \text{finite } (\text{voters-}\mathcal{E} \ E)$
 $\wedge (\forall v. v \notin \text{voters-}\mathcal{E} \ E \longrightarrow \text{profile-}\mathcal{E} \ E \ v = \{\})\}$

— Here, we count the occurrences of a ballot in an election, i.e., how many voters specifically chose that exact ballot.

fun $\text{vote-count} :: 'a \text{ Preference-Relation} \Rightarrow ('a, 'v) \text{ Election} \Rightarrow \text{nat} \text{ where}$
 $\text{vote-count } p \ E = \text{card } \{v \in (\text{voters-}\mathcal{E} \ E). (\text{profile-}\mathcal{E} \ E) \ v = p\}$

1.5.2 Vote Count

lemma $\text{vote-count-sum}:$
fixes $E :: ('a, 'v) \text{ Election}$
assumes
 $\text{finite } (\text{voters-}\mathcal{E} \ E) \text{ and}$

$\text{finite } (UNIV :: ('a \times 'a) \text{ set})$
shows $(\sum p \in UNIV. \text{vote-count } p \ E) = \text{card } (\text{voters-}\mathcal{E} \ E)$
 $\langle \text{proof} \rangle$

1.5.3 Voter Permutations

A common action of interest on elections is renaming the voters, e.g., when talking about anonymity.

fun $\text{rename} :: ('v \Rightarrow 'v) \Rightarrow ('a, 'v) \text{ Election} \Rightarrow ('a, 'v) \text{ Election}$ **where**
 $\text{rename } \pi \ (A, V, p) = (A, \pi \text{ ` } V, p \circ (\text{the-inv } \pi))$

lemma rename-sound :

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$ **and**

$\pi :: 'v \Rightarrow 'v$

assumes

$\text{prof}: \text{profile } V \ A \ p$ **and**

$\text{renamed}: (A, V', q) = \text{rename } \pi \ (A, V, p)$ **and**

$\text{bij-perm}: \text{bij } \pi$

shows $\text{profile } V' \ A \ q$

$\langle \text{proof} \rangle$

lemma rename-prof :

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$ **and**

$\pi :: 'v \Rightarrow 'v$

assumes

$\text{profile } V \ A \ p$ **and**

$(A, V', q) = \text{rename } \pi \ (A, V, p)$ **and**

$\text{bij } \pi$

shows $\text{profile } V' \ A \ q$

$\langle \text{proof} \rangle$

lemma rename-finite :

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$ **and**

$\pi :: 'v \Rightarrow 'v$

assumes

$\text{finite } V$ **and**

$(A, V', q) = \text{rename } \pi \ (A, V, p)$ **and**

$\text{bij } \pi$

shows $\text{finite } V'$

$\langle \text{proof} \rangle$


```

lemma rename-inv:
  fixes
     $\pi :: 'v \Rightarrow 'v$  and
     $A :: 'a \text{ set}$  and
     $V :: 'v \text{ set}$  and
     $p :: ('a, 'v) \text{ Profile}$ 
  assumes bij  $\pi$ 
  shows  $\text{rename } \pi (\text{rename } (\text{the-inv } \pi) (A, V, p)) = (A, V, p)$ 
<proof>

```

```

lemma rename-inj:
  fixes  $\pi :: 'v \Rightarrow 'v$ 
  assumes bij  $\pi$ 
  shows inj ( $\text{rename } \pi$ )
<proof>

```

```

lemma rename-surj:
  fixes  $\pi :: 'v \Rightarrow 'v$ 
  assumes bij  $\pi$ 
  shows
     $\text{rename } \pi \text{ ' well-formed-elections} = \text{well-formed-elections}$  and
     $\text{rename } \pi \text{ ' finite-elections} = \text{finite-elections}$ 
<proof>

```

1.5.4 List Representation

A profile on a voter set that has a natural order can be viewed as a list of ballots.

```

fun to-list ::  $'v :: \text{linorder set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow$ 
   $(('a \text{ Preference-Relation}) \text{ list})$  where
  to-list  $V \ p = (\text{if } (\text{finite } V)$ 
     $\text{then } (\text{map } p (\text{sorted-list-of-set } V))$ 
     $\text{else } [])$ 

```

```

lemma map-helper:
  fixes
     $f :: 'x \Rightarrow 'y \Rightarrow 'z$  and
     $g :: 'x \Rightarrow 'x$  and
     $h :: 'y \Rightarrow 'y$  and
     $l :: 'x \text{ list}$  and
     $l' :: 'y \text{ list}$ 
  shows  $\text{map2 } f (\text{map } g \ l) (\text{map } h \ l') = \text{map2 } (\lambda \ x \ y. f \ (g \ x) \ (h \ y)) \ l \ l'$ 
<proof>

```

```

lemma to-list-simp:
  fixes
     $i :: \text{nat}$  and
     $V :: 'v :: \text{linorder set}$  and

```

```

    p :: ('a, 'v) Profile
  assumes i < card V
  shows (to-list V p)!i = p ((sorted-list-of-set V)!i)
  <proof>

lemma to-list-comp:
  fixes
    V :: 'v :: linorder set and
    p :: ('a, 'v) Profile and
    f :: 'a rel  $\Rightarrow$  'a rel
  shows to-list V (f  $\circ$  p) = map f (to-list V p)
  <proof>

lemma set-card-upper-bound:
  fixes
    i :: nat and
    V :: nat set
  assumes
    fin-V: finite V and
    bound-v:  $\forall v \in V. v < i$ 
  shows card V  $\leq$  i
  <proof>

lemma sorted-list-of-set-nth-equals-card:
  fixes
    V :: 'v :: linorder set and
    x :: 'v
  assumes
    fin-V: finite V and
    x-V: x  $\in$  V
  shows sorted-list-of-set V!(card {v  $\in$  V. v < x}) = x
  <proof>

lemma to-list-permutes-under-bij:
  fixes
     $\pi$  :: 'v :: linorder  $\Rightarrow$  'v and
    V :: 'v set and
    p :: ('a, 'v) Profile
  assumes bij  $\pi$ 
  shows
    let  $\varphi = \lambda i. \text{card } \{v \in \pi \text{ ` } V. v < \pi ((\text{sorted-list-of-set } V)!i)\}$ 
    in (to-list V p) = permute-list  $\varphi$  (to-list ( $\pi$  ` V) ( $\lambda x. p (\text{the-inv } \pi x)$ ))
  <proof>

```

1.5.5 Preference Counts

The win count for an alternative a with respect to a finite voter set V in a profile p is the amount of ballots from V in p that rank alternative a in first position. If the voter set is infinite, counting is not generally possible.

fun win-count :: 'v set \Rightarrow ('a, 'v) Profile \Rightarrow 'a \Rightarrow enat **where**
 win-count V p a = (if (finite V)
 then card {v \in V. above (p v) a = {a}} else ∞)

fun prefer-count :: 'v set \Rightarrow ('a, 'v) Profile \Rightarrow 'a \Rightarrow 'a \Rightarrow enat **where**
 prefer-count V p x y = (if (finite V)
 then card {v \in V. (let r = (p v) in (y \preceq_r x))} else ∞)

lemma pref-count-voter-set-card:

fixes

V :: 'v set **and**

p :: ('a, 'v) Profile **and**

a b :: 'a

assumes finite V

shows prefer-count V p a b \leq card V

\langle proof \rangle

lemma set-compr:

fixes

A :: 'a set **and**

f :: 'a \Rightarrow 'a set

shows {f x | x. x \in A} = f ' A

\langle proof \rangle

lemma pref-count-set-compr:

fixes

A :: 'a set **and**

V :: 'v set **and**

p :: ('a, 'v) Profile **and**

a :: 'a

shows {prefer-count V p a a' | a'. a' \in A - {a}} =
 (prefer-count V p a) ' (A - {a})

\langle proof \rangle

lemma pref-count:

fixes

A :: 'a set **and**

V :: 'v set **and**

p :: ('a, 'v) Profile **and**

a b :: 'a

assumes

prof: profile V A p **and**

fn: finite V **and**

a-in-A: a \in A **and**

b-in-A: b \in A **and**

neg: a \neq b

shows prefer-count V p a b = card V - (prefer-count V p b a)

\langle proof \rangle

lemma *pref-count-sym*:

fixes

$p :: ('a, 'v) \text{ Profile}$ **and**

$V :: 'v \text{ set}$ **and**

$a \ b \ c :: 'a$

assumes

pref-count-ineq: $\text{prefer-count } V \ p \ a \ c \geq \text{prefer-count } V \ p \ c \ b$ **and**

prof: $\text{profile } V \ A \ p$ **and**

a-in-A: $a \in A$ **and**

b-in-A: $b \in A$ **and**

c-in-A: $c \in A$ **and**

a-neg-c: $a \neq c$ **and**

c-neg-b: $c \neq b$

shows $\text{prefer-count } V \ p \ b \ c \geq \text{prefer-count } V \ p \ c \ a$

<proof>

lemma *empty-prof-imp-zero-pref-count*:

fixes

$p :: ('a, 'v) \text{ Profile}$ **and**

$V :: 'v \text{ set}$ **and**

$a \ b :: 'a$

assumes $V = \{\}$

shows $\text{prefer-count } V \ p \ a \ b = 0$

<proof>

fun *wins* :: $'v \text{ set} \Rightarrow 'a \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'a \Rightarrow \text{bool}$ **where**

wins $V \ a \ p \ b =$

$(\text{prefer-count } V \ p \ a \ b > \text{prefer-count } V \ p \ b \ a)$

lemma *wins-inf-voters*:

fixes

$p :: ('a, 'v) \text{ Profile}$ **and**

$a \ b :: 'a$ **and**

$V :: 'v \text{ set}$

assumes *infinite* V

shows $\neg \text{wins } V \ b \ p \ a$

<proof>

Having alternative a win against b implies that b does not win against a .

lemma *wins-antisym*:

fixes

$p :: ('a, 'v) \text{ Profile}$ **and**

$a \ b :: 'a$ **and**

$V :: 'v \text{ set}$

assumes *wins* $V \ a \ p \ b$ — This already implies that V is finite.

shows $\neg \text{wins } V \ b \ p \ a$

<proof>

lemma *wins-irreflex*:

```

fixes
   $p :: ('a, 'v) \text{ Profile}$  and
   $a :: 'a$  and
   $V :: 'v \text{ set}$ 
shows  $\neg \text{wins } V \ a \ p \ a$ 
 $\langle \text{proof} \rangle$ 

```

1.5.6 Condorcet Winner

```

fun condorcet-winner ::  $'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'a \Rightarrow \text{bool}$  where
  condorcet-winner  $V \ A \ p \ a =$ 
    ( $\text{finite-profile } V \ A \ p \wedge a \in A \wedge (\forall x \in A - \{a\}. \text{wins } V \ a \ p \ x)$ )

```

lemma *cond-winner-unique-eq*:

```

fixes
   $V :: 'v \text{ set}$  and
   $A :: 'a \text{ set}$  and
   $p :: ('a, 'v) \text{ Profile}$  and
   $a \ b :: 'a$ 
assumes
  condorcet-winner  $V \ A \ p \ a$  and
  condorcet-winner  $V \ A \ p \ b$ 
shows  $b = a$ 
 $\langle \text{proof} \rangle$ 

```

lemma *cond-winner-unique*:

```

fixes
   $A :: 'a \text{ set}$  and
   $p :: ('a, 'v) \text{ Profile}$  and
   $a :: 'a$ 
assumes condorcet-winner  $V \ A \ p \ a$ 
shows  $\{a' \in A. \text{condorcet-winner } V \ A \ p \ a'\} = \{a\}$ 
 $\langle \text{proof} \rangle$ 

```

lemma *cond-winner-unique'*:

```

fixes
   $V :: 'v \text{ set}$  and
   $A :: 'a \text{ set}$  and
   $p :: ('a, 'v) \text{ Profile}$  and
   $a \ b :: 'a$ 
assumes
  condorcet-winner  $V \ A \ p \ a$  and
   $b \neq a$ 
shows  $\neg \text{condorcet-winner } V \ A \ p \ b$ 
 $\langle \text{proof} \rangle$ 

```

1.5.7 Limited Profile

This function restricts a profile p to a set A of alternatives and a set V of voters s.t. voters outside of V do not have any preferences or do not cast a vote. This keeps all of A 's preferences.

fun *limit-profile* :: 'a set \Rightarrow ('a, 'v) Profile \Rightarrow ('a, 'v) Profile **where**
limit-profile A p = (λ v. *limit* A (p v))

lemma *limit-prof-trans*:

fixes

A B C :: 'a set **and**

p :: ('a, 'v) Profile

assumes

B \subseteq A **and**

C \subseteq B

shows *limit-profile* C p = *limit-profile* C (*limit-profile* B p)

<proof>

lemma *limit-profile-sound*:

fixes

A B :: 'a set **and**

V :: 'v set **and**

p :: ('a, 'v) Profile

assumes

profile V B p **and**

A \subseteq B

shows *profile* V A (*limit-profile* A p)

<proof>

1.5.8 Lifting Property

definition *equiv-prof-except-a* :: 'v set \Rightarrow 'a set \Rightarrow ('a, 'v) Profile \Rightarrow
('a, 'v) Profile \Rightarrow 'a \Rightarrow bool **where**
equiv-prof-except-a V A p p' a \equiv
profile V A p \wedge *profile* V A p' \wedge a \in A \wedge
(\forall v \in V. *equiv-rel-except-a* A (p v) (p' v) a)

An alternative gets lifted from one profile to another iff its ranking increases in at least one ballot, and nothing else changes.

definition *lifted* :: 'v set \Rightarrow 'a set \Rightarrow ('a, 'v) Profile \Rightarrow
('a, 'v) Profile \Rightarrow 'a \Rightarrow bool **where**

lifted V A p p' a \equiv

finite-profile V A p \wedge *finite-profile* V A p' \wedge a \in A

\wedge (\forall v \in V. \neg *Preference-Relation.lifted* A (p v) (p' v) a \longrightarrow (p v) = (p' v))

\wedge (\exists v \in V. *Preference-Relation.lifted* A (p v) (p' v) a)

lemma *lifted-imp-equiv-prof-except-a*:

fixes

A :: 'a set **and**

```

    V :: 'v set and
    p p' :: ('a, 'v) Profile and
    a :: 'a
    assumes lifted V A p p' a
    shows equiv-prof-except-a V A p p' a
  <proof>

```

lemma *negl-diff-imp-eq-limit-prof*:

```

  fixes
    A A' :: 'a set and
    V :: 'v set and
    p p' :: ('a, 'v) Profile and
    a :: 'a
  assumes
    change: equiv-prof-except-a V A' p q a and
    subset: A ⊆ A' and
    not-in-A: a ∉ A
  shows ∀ v ∈ V. (limit-profile A p) v = (limit-profile A q) v
  — With the current definitions of equiv-prof-except-a and limit-prof, we can only
  conclude that the limited profiles coincide on the given voter set, since limit-prof
  may change the profiles everywhere, while equiv-prof-except-a only makes state-
  ments about the voter set.
  <proof>

```

lemma *limit-prof-eq-or-lifted*:

```

  fixes
    A A' :: 'a set and
    V :: 'v set and
    p p' :: ('a, 'v) Profile and
    a :: 'a
  assumes
    lifted-a: lifted V A' p p' a and
    subset: A ⊆ A'
  shows (∀ v ∈ V. limit-profile A p v = limit-profile A p' v)
    ∨ lifted V A (limit-profile A p) (limit-profile A p') a
  <proof>

```

end

1.6 Social Choice Result

theory *Social-Choice-Result*

imports *Result*

begin

1.6.1 Definition

A social choice result contains three sets of alternatives: elected, rejected, and deferred alternatives.

fun *well-formed-SCF* :: 'a set \Rightarrow 'a Result \Rightarrow bool **where**
 well-formed-SCF A res = (disjoint3 res \wedge set-equals-partition A res)

fun *limit-SCF* :: 'a set \Rightarrow 'a set \Rightarrow 'a set **where**
 limit-SCF A r = A \cap r

1.6.2 Auxiliary Lemmas

lemma *result-imp-rej*:
 fixes A e r d :: 'a set
 assumes *well-formed-SCF* A (e, r, d)
 shows A - (e \cup d) = r
 \langle proof \rangle

lemma *result-count*:
 fixes A e r d :: 'a set
 assumes
 wf-result: *well-formed-SCF* A (e, r, d) **and**
 fin-A: finite A
 shows card A = card e + card r + card d
 \langle proof \rangle

lemma *defer-subset*:
 fixes
 A :: 'a set **and**
 r :: 'a Result
 assumes *well-formed-SCF* A r
 shows defer-r r \subseteq A
 \langle proof \rangle

lemma *elect-subset*:
 fixes
 A :: 'a set **and**
 r :: 'a Result
 assumes *well-formed-SCF* A r
 shows elect-r r \subseteq A
 \langle proof \rangle

lemma *reject-subset*:
 fixes
 A :: 'a set **and**
 r :: 'a Result
 assumes *well-formed-SCF* A r
 shows reject-r r \subseteq A
 \langle proof \rangle

end

1.7 Social Welfare Result

```
theory Social-Welfare-Result
  imports Result
           Preference-Relation
begin
```

A social welfare result contains three sets of relations: elected, rejected, and deferred. A well-formed social welfare result consists only of linear orders on the alternatives.

```
fun well-formed-SWF :: 'a set  $\Rightarrow$  ('a Preference-Relation) Result  $\Rightarrow$  bool where
  well-formed-SWF A res = (disjoint3 res  $\wedge$ 
                           set-equals-partition {r. linear-order-on A r} res)
```

```
fun limit-SWF :: 'a set  $\Rightarrow$  ('a Preference-Relation) set  $\Rightarrow$ 
  ('a Preference-Relation) set where
  limit-SWF A res = {limit A r | r. r  $\in$  res  $\wedge$  linear-order-on A (limit A r)}
```

end

1.8 Electoral Result Types

```
theory Result-Interpretations
  imports Social-Choice-Result
           Social-Welfare-Result
           Collections.Locale-Code
begin
```

Interpretations of the result locale are placed inside a Locale-Code block in order to enable code generation of later definitions in the locale. Those definitions need to be added via a Locale-Code block as well.

$\langle ML \rangle$

Results from social choice functions (*SCFs*), for the purpose of composability and modularity given as three sets of (potentially tied) alternatives. See `Social_Choice_Result.thy` for details.

```
global-interpretation SCF-result: result well-formed-SCF limit-SCF
 $\langle proof \rangle$ 
```

Results from committee functions, for the purpose of composability and

modularity given as three sets of (potentially tied) sets of alternatives or committees. *[[Not actually used yet.]]*

```
global-interpretation committee-result: result
   $\lambda A r. \text{set-equals-partition } (\text{Pow } A) r \wedge \text{disjoint3 } r$ 
   $\lambda A rs. \{r \cap A \mid r. r \in rs\}$ 
 $\langle \text{proof} \rangle$ 
```

Results from social welfare functions (*SWFs*), for the purpose of composability and modularity given as three sets of (potentially tied) linear orders over the alternatives. See `Social_Welfare_Result.thy` for details.

```
global-interpretation SWF-result: result well-formed-SWF limit-SWF
 $\langle \text{proof} \rangle$ 
```

$\langle ML \rangle$

end

1.9 Symmetry Properties of Functions

```
theory Symmetry-Of-Functions
  imports HOL-Algebra.Group-Action
           HOL-Algebra.Generated-Groups
begin
```

1.9.1 Functions

```
type-synonym ('x, 'y) binary-fun = 'x  $\Rightarrow$  'y  $\Rightarrow$  'y

fun extensional-continuation :: ('x  $\Rightarrow$  'y)  $\Rightarrow$  'x set  $\Rightarrow$  ('x  $\Rightarrow$  'y) where
  extensional-continuation f S = ( $\lambda x. \text{if } (x \in S) \text{ then } (f x) \text{ else undefined}$ )

fun preimg :: ('x  $\Rightarrow$  'y)  $\Rightarrow$  'x set  $\Rightarrow$  'y  $\Rightarrow$  'x set where
  preimg f S y = {x  $\in$  S. f x = y}
```

1.9.2 Relations for Symmetry Constructions

```
fun restricted-rel :: 'x rel  $\Rightarrow$  'x set  $\Rightarrow$  'x set  $\Rightarrow$  'x rel where
  restricted-rel r S S' = r  $\cap$  (S  $\times$  S')

fun closed-restricted-rel :: 'x rel  $\Rightarrow$  'x set  $\Rightarrow$  'x set  $\Rightarrow$  bool where
  closed-restricted-rel r S T = ((restricted-rel r T S) “ T  $\subseteq$  T)

fun action-induced-rel :: 'x set  $\Rightarrow$  'y set  $\Rightarrow$  ('x, 'y) binary-fun  $\Rightarrow$  'y rel where
  action-induced-rel S T  $\varphi$  = {(y, y'). y  $\in$  T  $\wedge$  ( $\exists x \in S. \varphi x y = y'$ )}

fun product :: 'x rel  $\Rightarrow$  ('x * 'x) rel where
```

$product\ r = \{(p, p'). (fst\ p, fst\ p') \in r \wedge (snd\ p, snd\ p') \in r\}$

fun *equivariance* :: 'x set \Rightarrow 'y set \Rightarrow ('x, 'y) binary-fun \Rightarrow ('y * 'y) rel **where**
equivariance S T $\varphi =$
 $\{((u, v), (x, y)). (u, v) \in T \times T \wedge (\exists\ z \in S. x = \varphi\ z\ u \wedge y = \varphi\ z\ v)\}$

fun *singleton-set-system* :: 'x set \Rightarrow 'x set set **where**
singleton-set-system S = $\{\{x\} \mid x. x \in S\}$

fun *set-action* :: ('x, 'r) binary-fun \Rightarrow ('x, 'r set) binary-fun **where**
set-action $\psi\ x = image\ (\psi\ x)$

1.9.3 Invariance and Equivariance

Invariance and equivariance are symmetry properties of functions: Invariance means that related preimages have identical images and equivariance denotes consistent changes.

datatype ('x, 'y) symmetry =
Invariance 'x rel |
Equivariance 'x set (('x \Rightarrow 'x) \times ('y \Rightarrow 'y)) set

fun *is-symmetry* :: ('x \Rightarrow 'y) \Rightarrow ('x, 'y) symmetry \Rightarrow bool **where**
is-symmetry f (*Invariance* r) = $(\forall\ x. \forall\ y. (x, y) \in r \longrightarrow f\ x = f\ y) \mid$
is-symmetry f (*Equivariance* s τ) = $(\forall\ (\varphi, \psi) \in \tau. \forall\ x \in s. f\ (\varphi\ x) = \psi\ (f\ x))$

definition *action-induced-equivariance* :: 'z set \Rightarrow 'x set \Rightarrow ('z, 'x) binary-fun \Rightarrow
('z, 'y) binary-fun \Rightarrow ('x, 'y) symmetry **where**
action-induced-equivariance T S $\varphi\ \psi \equiv Equivariance\ S\ \{(\varphi\ z, \psi\ z) \mid z. z \in T\}$

1.9.4 Auxiliary Lemmas

lemma *un-left-inv-singleton-set-system*: $\bigcup \circ singleton-set-system = id$
<proof>

lemma *preimg-comp*:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $g :: 'x \Rightarrow 'x$ **and**
 $S :: 'x\ set$ **and**
 $x :: 'y$
shows *preimg* f ($g\ ' S$) x = $g\ ' preimg\ (f \circ g)\ S\ x$
<proof>

1.9.5 Rewrite Rules

theorem *rewrite-invar-as-equivar*:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $S :: 'x\ set$ **and**

$T :: 'z \text{ set}$ **and**
 $\varphi :: ('z, 'x) \text{ binary-fun}$
shows $\text{is-symmetry } f \text{ (Invariance (action-induced-rel } T \ S \ \varphi)) =$
 $\text{is-symmetry } f \text{ (action-induced-equivariance } T \ S \ \varphi \ (\lambda g. \text{id}))$
 $\langle \text{proof} \rangle$

lemma *rewrite-invar-ind-by-act*:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $S :: 'z \text{ set}$ **and**
 $T :: 'x \text{ set}$ **and**
 $\varphi :: ('z, 'x) \text{ binary-fun}$
shows $\text{is-symmetry } f \text{ (Invariance (action-induced-rel } S \ T \ \varphi)) =$
 $(\forall x \in S. \forall y \in T. f \ y = f \ (\varphi \ x \ y))$
 $\langle \text{proof} \rangle$

lemma *rewrite-equivariance*:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $S :: 'z \text{ set}$ **and**
 $T :: 'x \text{ set}$ **and**
 $\varphi :: ('z, 'x) \text{ binary-fun}$ **and**
 $\psi :: ('z, 'y) \text{ binary-fun}$
shows $\text{is-symmetry } f \text{ (action-induced-equivariance } S \ T \ \varphi \ \psi) =$
 $(\forall x \in S. \forall y \in T. f \ (\varphi \ x \ y) = \psi \ x \ (f \ y))$
 $\langle \text{proof} \rangle$

lemma *rewrite-group-action-img*:
fixes
 $m :: 'x \text{ monoid}$ **and**
 $S \ T :: 'y \text{ set}$ **and**
 $\varphi :: ('x, 'y) \text{ binary-fun}$ **and**
 $x \ y :: 'x$
assumes
 $T \subseteq S$ **and**
 $x \in \text{carrier } m$ **and**
 $y \in \text{carrier } m$ **and**
 $\text{group-action } m \ S \ \varphi$
shows $\varphi \ (x \otimes_m y) \ ' T = \varphi \ x \ ' \varphi \ y \ ' T$
 $\langle \text{proof} \rangle$

lemma *rewrite-carrier*: $\text{carrier } (\text{BijGroup } \text{UNIV}) = \{f. \text{bij } f\}$
 $\langle \text{proof} \rangle$

lemma *universal-set-carrier-imp-bij-group*:
fixes $f :: 'a \Rightarrow 'a$
assumes $f \in \text{carrier } (\text{BijGroup } \text{UNIV})$
shows $\text{bij } f$
 $\langle \text{proof} \rangle$

lemma *rewrite-sym-group*:

fixes

$f\ g :: 'a \Rightarrow 'a$ **and**

$S :: 'a\ \text{set}$

assumes

$f \in \text{carrier}\ (BijGroup\ S)$ **and**

$g \in \text{carrier}\ (BijGroup\ S)$

shows

rewrite-mult: $f \otimes_{BijGroup\ S} g = \text{extensional-continuation}\ (f \circ g)\ S$ **and**

rewrite-mult-univ: $S = UNIV \longrightarrow f \otimes_{BijGroup\ S} g = f \circ g$

<proof>

lemma *simp-extensional-univ*:

fixes $f :: 'a \Rightarrow 'b$

shows *extensional-continuation* $f\ UNIV = f$

<proof>

lemma *extensional-continuation-subset*:

fixes

$f :: 'a \Rightarrow 'b$ **and**

$S\ T :: 'a\ \text{set}$ **and**

$x :: 'a$

assumes

$T \subseteq S$ **and**

$x \in T$

shows *extensional-continuation* $f\ S\ x = \text{extensional-continuation}\ f\ T\ x$

<proof>

lemma *rel-ind-by-coinciding-action-on-subset-eq-restr*:

fixes

$\varphi\ \psi :: ('a, 'b)\ \text{binary-fun}$ **and**

$S :: 'a\ \text{set}$ **and**

$T\ U :: 'b\ \text{set}$

assumes

$U \subseteq T$ **and**

$\forall x \in S. \forall y \in U. \psi\ x\ y = \varphi\ x\ y$

shows *action-induced-rel* $S\ U\ \psi = \text{restricted-rel}\ (\text{action-induced-rel}\ S\ T\ \varphi)\ U$

UNIV

<proof>

lemma *coinciding-actions-ind-equal-rel*:

fixes

$S :: 'x\ \text{set}$ **and**

$T :: 'y\ \text{set}$ **and**

$\varphi\ \psi :: ('x, 'y)\ \text{binary-fun}$

assumes $\forall x \in S. \forall y \in T. \varphi\ x\ y = \psi\ x\ y$

shows *action-induced-rel* $S\ T\ \varphi = \text{action-induced-rel}\ S\ T\ \psi$

<proof>

1.9.6 Group Actions

lemma *const-id-is-group-action:*

fixes $m :: 'x \text{ monoid}$

assumes *group m*

shows *group-action m UNIV ($\lambda x. id$)*

<proof>

theorem *group-act-induces-set-group-act:*

fixes

$m :: 'x \text{ monoid}$ **and**

$S :: 'y \text{ set}$ **and**

$\varphi :: ('x, 'y) \text{ binary-fun}$

defines $\varphi\text{-img} \equiv (\lambda x. \text{extensional-continuation } (\text{image } (\varphi x)) (Pow S))$

assumes *group-action m S φ*

shows *group-action m (Pow S) $\varphi\text{-img}$*

<proof>

1.9.7 Invariance and Equivariance

It suffices to show equivariance under the group action of a generating set of a group to show equivariance under the group action of the whole group. For example, it is enough to show invariance under transpositions to show invariance under a complete finite symmetric group.

theorem *equivar-generators-imp-equivar-group:*

fixes

$f :: 'x \Rightarrow 'y$ **and**

$m :: 'z \text{ monoid}$ **and**

$S :: 'z \text{ set}$ **and**

$T :: 'x \text{ set}$ **and**

$\varphi :: ('z, 'x) \text{ binary-fun}$ **and**

$\psi :: ('z, 'y) \text{ binary-fun}$

assumes

equivar: is-symmetry f (action-induced-equivariance S T φ ψ) **and**

action- φ : group-action m T φ **and**

action- ψ : group-action m ($f \circ T$) ψ **and**

gen: carrier m = generate m S

shows *is-symmetry f (action-induced-equivariance (carrier m) T φ ψ)*

<proof>

lemma *invar-parameterized-fun:*

fixes

$f :: 'x \Rightarrow ('x \Rightarrow 'y)$ **and**

$r :: 'x \text{ rel}$

assumes

$\forall x. \text{is-symmetry } (f x) (\text{Invariance } r)$ **and**

is-symmetry f (Invariance r)

shows *is-symmetry ($\lambda x. f x x$) (Invariance r)*

<proof>

lemma *invar-under-subset-rel*:

fixes

$f :: 'x \Rightarrow 'y$ **and**

$r\ s :: 'x\ rel$

assumes

subset: $r \subseteq s$ **and**

invar: *is-symmetry* f (*Invariance* s)

shows *is-symmetry* f (*Invariance* r)

<proof>

lemma *equivar-ind-by-act-coincide*:

fixes

$S :: 'x\ set$ **and**

$T :: 'y\ set$ **and**

$f :: 'y \Rightarrow 'z$ **and**

$\varphi\ \varphi' :: ('x, 'y)\ binary_fun$ **and**

$\psi :: ('x, 'z)\ binary_fun$

assumes $\forall\ x \in S. \forall\ y \in T. \varphi\ x\ y = \varphi'\ x\ y$

shows *is-symmetry* f (*action-induced-equivariance* $S\ T\ \varphi\ \psi$) =

is-symmetry f (*action-induced-equivariance* $S\ T\ \varphi'\ \psi$)

<proof>

lemma *equivar-under-subset*:

fixes

$f :: 'x \Rightarrow 'y$ **and**

$S\ T :: 'x\ set$ **and**

$\tau :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y))\ set$

assumes

is-symmetry f (*Equivariance* $S\ \tau$) **and**

$T \subseteq S$

shows *is-symmetry* f (*Equivariance* $T\ \tau$)

<proof>

lemma *equivar-under-subset'*:

fixes

$f :: 'x \Rightarrow 'y$ **and**

$S :: 'x\ set$ **and**

$\tau\ v :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y))\ set$

assumes

is-symmetry f (*Equivariance* $S\ \tau$) **and**

$v \subseteq \tau$

shows *is-symmetry* f (*Equivariance* $S\ v$)

<proof>

theorem *group-action-equivar-f-imp-equivar-preimg*:

fixes

$f :: 'x \Rightarrow 'y$ **and**

$\mathcal{D}_f\ S :: 'x\ set$ **and**

$m :: 'z \text{ monoid}$ **and**
 $\varphi :: ('z, 'x) \text{ binary-fun}$ **and**
 $\psi :: ('z, 'y) \text{ binary-fun}$ **and**
 $x :: 'z$
defines $\text{equivar-prop} \equiv \text{action-induced-equivariance } (\text{carrier } m) \mathcal{D}_f \varphi \psi$
assumes
 $\text{action-}\varphi$: $\text{group-action } m \ S \ \varphi$ **and**
 action-res : $\text{group-action } m \ \text{UNIV } \psi$ **and**
 dom-in-s : $\mathcal{D}_f \subseteq S$ **and**
 closed-domain :
 $\text{closed-restricted-rel } (\text{action-induced-rel } (\text{carrier } m) \ S \ \varphi) \ S \ \mathcal{D}_f$ **and**
 equivar-f : $\text{is-symmetry } f \ \text{equivar-prop}$ **and**
 group-elem-x : $x \in \text{carrier } m$
shows $\forall y. \text{preimg } f \ \mathcal{D}_f \ (\psi \ x \ y) = (\varphi \ x) \text{ ` } (\text{preimg } f \ \mathcal{D}_f \ y)$
 $\langle \text{proof} \rangle$

1.9.8 Function Composition

lemma invar-comp :
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $g :: 'y \Rightarrow 'z$ **and**
 $r :: 'x \text{ rel}$
assumes $\text{is-symmetry } f \ (\text{Invariance } r)$
shows $\text{is-symmetry } (g \circ f) \ (\text{Invariance } r)$
 $\langle \text{proof} \rangle$

lemma equivar-comp :
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $g :: 'y \Rightarrow 'z$ **and**
 $S :: 'x \text{ set}$ **and**
 $T :: 'y \text{ set}$ **and**
 $\tau :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y)) \text{ set}$ **and**
 $v :: (('y \Rightarrow 'y) \times ('z \Rightarrow 'z)) \text{ set}$
defines
 $\text{transitive-acts} \equiv$
 $\{(\varphi, \psi). \exists \chi :: 'y \Rightarrow 'y. (\varphi, \chi) \in \tau \wedge (\chi, \psi) \in v \wedge \chi \text{ ` } f \text{ ` } S \subseteq T\}$
assumes
 $f \text{ ` } S \subseteq T$ **and**
 $\text{is-symmetry } f \ (\text{Equivariance } S \ \tau)$ **and**
 $\text{is-symmetry } g \ (\text{Equivariance } T \ v)$
shows $\text{is-symmetry } (g \circ f) \ (\text{Equivariance } S \ \text{transitive-acts})$
 $\langle \text{proof} \rangle$

lemma $\text{equivar-ind-by-action-comp}$:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $g :: 'y \Rightarrow 'z$ **and**


```

    S :: 'w set and
    T :: 'x set and
    U :: 'y set and
    φ :: ('w, 'x) binary-fun and
    χ :: ('w, 'y) binary-fun and
    ψ :: ('w, 'z) binary-fun
  assumes
    f ' T ⊆ U and
    ∀ x ∈ S. χ x ' f ' T ⊆ U and
    is-symmetry f (action-induced-equivariance S T φ χ) and
    is-symmetry g (action-induced-equivariance S U χ ψ)
  shows is-symmetry (g ∘ f) (action-induced-equivariance S T φ ψ)
⟨proof⟩

lemma equivar-set-minus:
  fixes
    f g :: 'x ⇒ 'y set and
    S :: 'z set and
    T :: 'x set and
    φ :: ('z, 'x) binary-fun and
    ψ :: ('z, 'y) binary-fun
  assumes
    f-equivar: is-symmetry f (action-induced-equivariance S T φ (set-action ψ))
  and
    g-equivar: is-symmetry g (action-induced-equivariance S T φ (set-action ψ))
  and
    bij-a: ∀ a ∈ S. bij (ψ a)
  shows
    is-symmetry (λ b. f b - g b) (action-induced-equivariance S T φ (set-action ψ))
⟨proof⟩

lemma equivar-union-under-image-action:
  fixes
    f :: 'x ⇒ 'y and
    S :: 'z set and
    φ :: ('z, 'x) binary-fun
  shows is-symmetry ⋃ (action-induced-equivariance S UNIV
    (set-action (set-action φ)) (set-action φ))
⟨proof⟩

end

```

1.10 Symmetry Properties of Voting Rules

```

theory Voting-Symmetry
  imports Symmetry-Of-Functions
          Social-Choice-Result

```

*Social-Welfare-Result
Profile*

begin

1.10.1 Definitions

fun *result-action* :: ('x, 'r) *binary-fun* \Rightarrow ('x, 'r *Result*) *binary-fun* **where**
result-action ψ $x = (\lambda r. (\psi \ x \ 'elect-r \ r, \psi \ x \ 'reject-r \ r, \psi \ x \ 'defer-r \ r))$

Anonymity

Bijection group on the set of voters.

definition *bijection_{VG}* :: ('v \Rightarrow 'v) *monoid* **where**
bijection_{VG} \equiv *BijGroup* (*UNIV* :: 'v *set*)

Permutation action on the set of voters. Invariance under this action implies anonymity.

fun φ -*anon* :: ('a, 'v) *Election set* \Rightarrow ('v \Rightarrow 'v) \Rightarrow
 (('a, 'v) *Election* \Rightarrow ('a, 'v) *Election*) **where**
 φ -*anon* $\mathcal{E} \ \pi = \text{extensional-continuation } (\text{rename } \pi) \ \mathcal{E}$

fun *anonymity_R* :: ('a, 'v) *Election set* \Rightarrow ('a, 'v) *Election rel* **where**
anonymity_R $\mathcal{E} = \text{action-induced-rel } (\text{carrier } \text{bijection}_{VG}) \ \mathcal{E} \ (\varphi\text{-anon } \mathcal{E})$

Neutrality

fun *rel-rename* :: ('a \Rightarrow 'a, 'a *Preference-Relation*) *binary-fun* **where**
rel-rename $\pi \ r = \{(\pi \ a, \pi \ b) \mid a \ b. (a, b) \in r\}$

fun *alternatives-rename* :: ('a \Rightarrow 'a, ('a, 'v) *Election*) *binary-fun* **where**
alternatives-rename $\pi \ \mathcal{E} =$
 $(\pi \ '(\text{alternatives-}\mathcal{E} \ \mathcal{E}), \text{voters-}\mathcal{E} \ \mathcal{E}, (\text{rel-rename } \pi) \circ (\text{profile-}\mathcal{E} \ \mathcal{E}))$

Bijection group on the set of alternatives. Invariance under this action implies neutrality.

definition *bijection_{AG}* :: ('a \Rightarrow 'a) *monoid* **where**
bijection_{AG} \equiv *BijGroup* (*UNIV* :: 'a *set*)

Permutation action on the set of alternatives.

fun φ -*neutral* :: ('a, 'v) *Election set* \Rightarrow
 ('a \Rightarrow 'a, ('a, 'v) *Election*) *binary-fun* **where**
 φ -*neutral* $\mathcal{E} \ \pi = \text{extensional-continuation } (\text{alternatives-rename } \pi) \ \mathcal{E}$

fun *neutrality_R* :: ('a, 'v) *Election set* \Rightarrow ('a, 'v) *Election rel* **where**
neutrality_R $\mathcal{E} = \text{action-induced-rel } (\text{carrier } \text{bijection}_{AG}) \ \mathcal{E} \ (\varphi\text{-neutral } \mathcal{E})$

fun ψ -*neutral_c* :: ('a \Rightarrow 'a, 'a) *binary-fun* **where**
 ψ -*neutral_c* $\pi \ r = \pi \ r$

fun $\psi\text{-neutral}_w :: ('a \Rightarrow 'a, 'a \text{ rel}) \text{ binary-fun where}$
 $\psi\text{-neutral}_w \pi r = \text{rel-rename } \pi r$

Homogeneity

fun $\text{homogeneity}_{\mathcal{R}} :: ('a, 'v) \text{ Election set} \Rightarrow ('a, 'v) \text{ Election rel where}$
 $\text{homogeneity}_{\mathcal{R}} \mathcal{E} =$
 $\{(E, E'). E \in \mathcal{E}$
 $\wedge \text{alternatives-}\mathcal{E} E = \text{alternatives-}\mathcal{E} E'$
 $\wedge \text{finite } (\text{voters-}\mathcal{E} E) \wedge \text{finite } (\text{voters-}\mathcal{E} E')$
 $\wedge (\exists n > 0. \forall r :: 'a \text{ Preference-Relation.}$
 $\text{vote-count } r E = n * (\text{vote-count } r E'))\}$

fun $\text{copy-list} :: \text{nat} \Rightarrow 'x \text{ list} \Rightarrow 'x \text{ list where}$
 $\text{copy-list } 0 l = [] \mid$
 $\text{copy-list } (\text{Suc } n) l = \text{copy-list } n l @ l$

fun $\text{homogeneity}_{\mathcal{R}}' :: ('a, 'v :: \text{linorder}) \text{ Election set} \Rightarrow ('a, 'v) \text{ Election rel where}$
 $\text{homogeneity}_{\mathcal{R}}' \mathcal{E} =$
 $\{(E, E'). E \in \mathcal{E}$
 $\wedge \text{alternatives-}\mathcal{E} E = \text{alternatives-}\mathcal{E} E'$
 $\wedge \text{finite } (\text{voters-}\mathcal{E} E) \wedge \text{finite } (\text{voters-}\mathcal{E} E')$
 $\wedge (\exists n > 0.$
 $\text{to-list } (\text{voters-}\mathcal{E} E') (\text{profile-}\mathcal{E} E') =$
 $\text{copy-list } n (\text{to-list } (\text{voters-}\mathcal{E} E) (\text{profile-}\mathcal{E} E)))\}$

Reversal Symmetry

fun $\text{reverse-rel} :: 'a \text{ rel} \Rightarrow 'a \text{ rel where}$
 $\text{reverse-rel } r = \{(a, b). (b, a) \in r\}$

fun $\text{rel-app} :: ('a \text{ rel} \Rightarrow 'a \text{ rel}) \Rightarrow ('a, 'v) \text{ Election} \Rightarrow ('a, 'v) \text{ Election where}$
 $\text{rel-app } f (A, V, p) = (A, V, f \circ p)$

definition $\text{reversal}_{\mathcal{G}} :: ('a \text{ rel} \Rightarrow 'a \text{ rel}) \text{ monoid where}$
 $\text{reversal}_{\mathcal{G}} \equiv (\text{carrier} = \{\text{reverse-rel}, \text{id}\}, \text{monoid.mult} = \text{comp}, \text{one} = \text{id})$

fun $\varphi\text{-reverse} :: ('a, 'v) \text{ Election set}$
 $\Rightarrow ('a \text{ rel} \Rightarrow 'a \text{ rel}, ('a, 'v) \text{ Election}) \text{ binary-fun where}$
 $\varphi\text{-reverse } \mathcal{E} \varphi = \text{extensional-continuation } (\text{rel-app } \varphi) \mathcal{E}$

fun $\psi\text{-reverse} :: ('a \text{ rel} \Rightarrow 'a \text{ rel}, 'a \text{ rel}) \text{ binary-fun where}$
 $\psi\text{-reverse } \varphi r = \varphi r$

fun $\text{reversal}_{\mathcal{R}} :: ('a, 'v) \text{ Election set} \Rightarrow ('a, 'v) \text{ Election rel where}$
 $\text{reversal}_{\mathcal{R}} \mathcal{E} = \text{action-induced-rel } (\text{carrier reversal}_{\mathcal{G}}) \mathcal{E} (\varphi\text{-reverse } \mathcal{E})$

1.10.2 Auxiliary Lemmas

fun *n-app* :: *nat* \Rightarrow (*'x* \Rightarrow *'x*) \Rightarrow (*'x* \Rightarrow *'x*) **where**
n-app-id: *n-app* 0 *f* = *id* |
n-app-suc: *n-app* (*Suc* *n*) *f* = *f* \circ *n-app* *n* *f*

lemma *n-app-rewrite*:

fixes

f :: *'x* \Rightarrow *'x* **and**

n :: *nat* **and**

x :: *'x*

shows (*f* \circ *n-app* *n* *f*) *x* = (*n-app* *n* *f* \circ *f*) *x*

<proof>

lemma *n-app-leaves-set*:

fixes

A B :: *'x* *set* **and**

f :: *'x* \Rightarrow *'x* **and**

x :: *'x*

assumes

fin-A: *finite* *A* **and**

fin-B: *finite* *B* **and**

x-el: *x* \in *A* - *B* **and**

bij-f: *bij-betw* *f* *A B*

obtains *n* :: *nat* **where**

n > 0 **and**

n-app *n* *f* *x* \in *B* - *A* **and**

\forall *m* > 0. *m* < *n* \longrightarrow *n-app* *m* *f* *x* \in *A* \cap *B*

<proof>

lemma *n-app-rev*:

fixes

A B :: *'x* *set* **and**

f :: *'x* \Rightarrow *'x* **and**

m n :: *nat* **and**

x y :: *'x*

assumes

x-in-A: *x* \in *A* **and**

y-in-A: *y* \in *A* **and**

n-geq-m: *n* \geq *m* **and**

n-app-eq-m-n: *n-app* *n* *f* *x* = *n-app* *m* *f* *y* **and**

n-app-x-in-A: \forall *n'* < *n*. *n-app* *n'* *f* *x* \in *A* **and**

n-app-y-in-A: \forall *m'* < *m*. *n-app* *m'* *f* *y* \in *A* **and**

fin-A: *finite* *A* **and**

fin-B: *finite* *B* **and**

bij-f-A-B: *bij-betw* *f* *A B*

shows *n-app* (*n* - *m*) *f* *x* = *y*

<proof>

lemma *n-app-inv*:

fixes
 $A\ B :: 'x\ \text{set}$ **and**
 $f :: 'x \Rightarrow 'x$ **and**
 $n :: \text{nat}$ **and**
 $x :: 'x$
assumes
 $x \in B$ **and**
 $\forall\ m \geq 0. m < n \longrightarrow n\text{-app}\ m\ (\text{the-inv-into}\ A\ f)\ x \in B$ **and**
 $\text{bij-betw}\ f\ A\ B$
shows $n\text{-app}\ n\ f\ (n\text{-app}\ n\ (\text{the-inv-into}\ A\ f)\ x) = x$
 $\langle \text{proof} \rangle$

lemma *bij-betw-finite-ind-global-bij*:

fixes
 $A\ B :: 'x\ \text{set}$ **and**
 $f :: 'x \Rightarrow 'x$
assumes
 $\text{fin-}A$: $\text{finite}\ A$ **and**
 $\text{fin-}B$: $\text{finite}\ B$ **and**
 $\text{bij-}f$: $\text{bij-betw}\ f\ A\ B$
obtains $g :: 'x \Rightarrow 'x$ **where**
 $\text{bij}\ g$ **and**
 $\forall\ a \in A. g\ a = f\ a$ **and**
 $\forall\ b \in B - A. g\ b \in A - B \wedge (\exists\ n > 0. n\text{-app}\ n\ f\ (g\ b) = b)$ **and**
 $\forall\ x \in \text{UNIV} - A - B. g\ x = x$
 $\langle \text{proof} \rangle$

lemma *bij-betw-ext*:

fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $X :: 'x\ \text{set}$ **and**
 $Y :: 'y\ \text{set}$
assumes $\text{bij-betw}\ f\ X\ Y$
shows $\text{bij-betw}\ (\text{extensional-continuation}\ f\ X)\ X\ Y$
 $\langle \text{proof} \rangle$

1.10.3 Anonymity Lemmas

lemma *anon-rel-vote-count*:

fixes
 $\mathcal{E} :: ('a, 'v)\ \text{Election set}$ **and**
 $E\ E' :: ('a, 'v)\ \text{Election}$
assumes
 $\text{finite}\ (\text{voters-}\mathcal{E}\ E)$ **and**
 $(E, E') \in \text{anonymity}_{\mathcal{R}}\ \mathcal{E}$
shows $\text{alternatives-}\mathcal{E}\ E = \text{alternatives-}\mathcal{E}\ E' \wedge E \in \mathcal{E}$
 $\wedge (\forall\ p. \text{vote-count}\ p\ E = \text{vote-count}\ p\ E')$
 $\langle \text{proof} \rangle$

lemma *vote-count-anon-rel*:

fixes

$\mathcal{E} :: ('a, 'v)$ *Election set* **and**

$E\ E' :: ('a, 'v)$ *Election*

assumes

fin-voters-E: *finite* (*voters- \mathcal{E} E*) **and**

fin-voters-E': *finite* (*voters- \mathcal{E} E'*) **and**

default-non-v: $\forall v. v \notin \text{voters-}\mathcal{E}\ E \longrightarrow \text{profile-}\mathcal{E}\ E\ v = \{\}$ **and**

default-non-v': $\forall v. v \notin \text{voters-}\mathcal{E}\ E' \longrightarrow \text{profile-}\mathcal{E}\ E'\ v = \{\}$ **and**

eq: *alternatives- \mathcal{E} E* = *alternatives- \mathcal{E} E'* $\wedge (E, E') \in \mathcal{E} \times \mathcal{E}$

$\wedge (\forall p. \text{vote-count } p\ E = \text{vote-count } p\ E')$

shows $(E, E') \in \text{anonymity}_{\mathcal{R}}\ \mathcal{E}$

<proof>

lemma *rename-comp*:

fixes $\pi\ \pi' :: 'v \Rightarrow 'v$

assumes

bij π **and**

bij π'

shows *rename* $\pi \circ \text{rename } \pi' = \text{rename } (\pi \circ \pi')$

<proof>

interpretation *anonymous-group-action*:

*group-action bijection*_{VG} *well-formed-elections* φ -anon *well-formed-elections*

<proof>

lemma (*in result*) *anonymity-action-presv-symmetry*: *is-symmetry* $(\lambda E. \text{limit}$

*(alternatives- \mathcal{E} E) UNIV) (Invariance (*anonymity*_R *well-formed-elections*))*

<proof>

1.10.4 Neutrality Lemmas

lemma *rel-rename-helper*:

fixes

$r :: 'a\ \text{rel}$ **and**

$\pi :: 'a \Rightarrow 'a$ **and**

$a\ b :: 'a$

assumes *bij* π

shows $(\pi\ a, \pi\ b) \in \{(\pi\ x, \pi\ y) \mid x\ y. (x, y) \in r\}$

$\longleftrightarrow (a, b) \in \{(x, y) \mid x\ y. (x, y) \in r\}$

<proof>

lemma *rel-rename-comp*:

fixes $\pi\ \pi' :: 'a \Rightarrow 'a$

shows *rel-rename* $(\pi \circ \pi') = \text{rel-rename } \pi \circ \text{rel-rename } \pi'$

<proof>

lemma *rel-rename-sound*:

fixes

$\pi :: 'a \Rightarrow 'a$ **and**
 $r :: 'a \text{ rel}$ **and**
 $A :: 'a \text{ set}$
assumes *inj* π
shows
 $\text{refl-on } A \ r \longrightarrow \text{refl-on } (\pi \text{ ` } A) \ (\text{rel-rename } \pi \ r)$ **and**
 $\text{antisym } r \longrightarrow \text{antisym } (\text{rel-rename } \pi \ r)$ **and**
 $\text{total-on } A \ r \longrightarrow \text{total-on } (\pi \text{ ` } A) \ (\text{rel-rename } \pi \ r)$ **and**
 $\text{Relation.trans } r \longrightarrow \text{Relation.trans } (\text{rel-rename } \pi \ r)$
 $\langle \text{proof} \rangle$

lemma *rename-subset*:
fixes
 $r \ s :: 'a \text{ rel}$ **and**
 $a \ b :: 'a$ **and**
 $\pi :: 'a \Rightarrow 'a$
assumes
 $\text{bij-}\pi$: $\text{bij } \pi$ **and**
 $\text{rel-rename } \pi \ r = \text{rel-rename } \pi \ s$ **and**
 $(a, b) \in r$
shows $(a, b) \in s$
 $\langle \text{proof} \rangle$

lemma *rel-rename-bij*:
fixes $\pi :: 'a \Rightarrow 'a$
assumes $\text{bij-}\pi$: $\text{bij } \pi$
shows $\text{bij } (\text{rel-rename } \pi)$
 $\langle \text{proof} \rangle$

lemma *alternatives-rename-comp*:
fixes $\pi \ \pi' :: 'a \Rightarrow 'a$
shows $\text{alternatives-rename } \pi \circ \text{alternatives-rename } \pi' =$
 $\text{alternatives-rename } (\pi \circ \pi')$
 $\langle \text{proof} \rangle$

lemma *alternatives-rename-sound*:
fixes
 $A \ A' :: 'a \text{ set}$ **and**
 $V \ V' :: 'v \text{ set}$ **and**
 $p \ p' :: ('a, 'v) \text{ Profile}$ **and**
 $\pi :: 'a \Rightarrow 'a$
assumes
 $\text{bij-}\pi$: $\text{bij } \pi$ **and**
 wf-elects : $(A, V, p) \in \text{well-formed-elections}$ **and**
 renamed : $(A', V', p') = \text{alternatives-rename } \pi \ (A, V, p)$
shows $(A', V', p') \in \text{well-formed-elections}$
 $\langle \text{proof} \rangle$

lemma *alternatives-rename-bij*:

fixes $\pi :: ('a \Rightarrow 'a)$
assumes $\text{bij-}\pi$: $\text{bij } \pi$
shows $\text{bij-betw } (\text{alternatives-rename } \pi) \text{ well-formed-elections well-formed-elections}$
 $\langle \text{proof} \rangle$

interpretation φ -neutral-action: $\text{group-action bijection}_{\mathcal{AG}} \text{ well-formed-elections}$
 φ -neutral well-formed-elections
 $\langle \text{proof} \rangle$

interpretation ψ -neutral_c-action: $\text{group-action bijection}_{\mathcal{AG}} \text{ UNIV } \psi$ -neutral_c
 $\langle \text{proof} \rangle$

interpretation ψ -neutral_w-action: $\text{group-action bijection}_{\mathcal{AG}} \text{ UNIV } \psi$ -neutral_w
 $\langle \text{proof} \rangle$

lemma *neutrality-action-presv-SCF-symmetry*: $\text{is-symmetry } (\lambda \mathcal{E}. \text{limit-SCF } (\text{alternatives-}\mathcal{E} \mathcal{E}) \text{ UNIV})$
 $(\text{action-induced-equivariance } (\text{carrier bijection}_{\mathcal{AG}}) \text{ well-formed-elections}$
 $(\varphi\text{-neutral well-formed-elections}) (\text{set-action } \psi\text{-neutral}_c))$
 $\langle \text{proof} \rangle$

lemma *neutrality-action-presv-SWF-symmetry*: $\text{is-symmetry } (\lambda \mathcal{E}. \text{limit-SWF}$
 $(\text{alternatives-}\mathcal{E} \mathcal{E}) \text{ UNIV}) (\text{action-induced-equivariance } (\text{carrier bijection}_{\mathcal{AG}}) \text{ well-formed-elections}$
 $(\varphi\text{-neutral well-formed-elections}) (\text{set-action } \psi\text{-neutral}_w))$
 $\langle \text{proof} \rangle$

1.10.5 Homogeneity Lemmas

definition $\text{reflp-on}' :: 'a \text{ set} \Rightarrow 'a \text{ rel} \Rightarrow \text{bool}$ **where**
 $\text{reflp-on}' A r \equiv \text{reflp-on } A (\lambda x y. (x, y) \in r)$

lemma *refl-homogeneity_R*:
fixes $\mathcal{E} :: ('a, 'v) \text{ Election set}$
assumes $\mathcal{E} \subseteq \text{finite-elections-}\mathcal{V}$
shows $\text{reflp-on}' \mathcal{E} (\text{homogeneity}_{\mathcal{R}} \mathcal{E})$
 $\langle \text{proof} \rangle$

lemma (in result) *homogeneity-action-presv-symmetry*:
 $\text{is-symmetry } (\lambda \mathcal{E}. \text{limit } (\text{alternatives-}\mathcal{E} \mathcal{E}) \text{ UNIV})$
 $(\text{Invariance } (\text{homogeneity}_{\mathcal{R}} \text{ UNIV}))$
 $\langle \text{proof} \rangle$

lemma *refl-homogeneity'_R*:
fixes $\mathcal{E} :: ('a, 'v :: \text{linorder}) \text{ Election set}$
assumes $\mathcal{E} \subseteq \text{finite-elections-}\mathcal{V}$
shows $\text{reflp-on}' \mathcal{E} (\text{homogeneity}_{\mathcal{R}}' \mathcal{E})$
 $\langle \text{proof} \rangle$

lemma (in result) *homogeneity'-action-presv-symmetry*:

is-symmetry ($\lambda \mathcal{E}. \text{limit } (\text{alternatives-}\mathcal{E} \ \mathcal{E}) \text{ UNIV}$)
 (*Invariance* (*homogeneity* _{\mathcal{R}'} *UNIV*))
 $\langle \text{proof} \rangle$

1.10.6 Reversal Symmetry Lemmas

lemma *reverse-reverse-id*: $\text{reverse-rel} \circ \text{reverse-rel} = \text{id}$
 $\langle \text{proof} \rangle$

lemma *reverse-rel-limit*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ rel}$
shows $\text{reverse-rel } (\text{limit } A \ r) = \text{limit } A \ (\text{reverse-rel } r)$
 $\langle \text{proof} \rangle$

lemma *reverse-rel-lin-ord*:
fixes
 $A :: 'a \text{ set}$ **and**
 $r :: 'a \text{ rel}$
assumes *linear-order-on* $A \ r$
shows *linear-order-on* $A \ (\text{reverse-rel } r)$
 $\langle \text{proof} \rangle$

interpretation *reversal_G-group*: *group reversal_G*
 $\langle \text{proof} \rangle$

interpretation φ -*reverse-action*: *group-action reversal_G well-formed-elections*
 φ -*reverse well-formed-elections*
 $\langle \text{proof} \rangle$

interpretation ψ -*reverse-action*: *group-action reversal_G UNIV ψ -reverse*
 $\langle \text{proof} \rangle$

lemma *reversal-symmetry-action-presv-symmetry*: *is-symmetry* ($\lambda \mathcal{E}. \text{limit-SWF}$
 (*alternatives-}\mathcal{E} \ \mathcal{E}) \text{ UNIV})
 (*action-induced-equivariance* (*carrier reversal_G*) *well-formed-elections*
 (φ -*reverse well-formed-elections*) (*set-action* ψ -*reverse*))
 $\langle \text{proof} \rangle$*

end

1.11 Result-Dependent Voting Rule Properties

theory *Property-Interpretations*
imports *Voting-Symmetry*
Result-Interpretations

begin

1.11.1 Property Definitions

The interpretation of equivariance properties generally depends on the result type. For example, neutrality for social choice rules means that single winners are renamed when the candidates in the votes are consistently renamed. For social welfare results, the complete result rankings must be renamed. New result-type-dependent definitions for properties can be added here.

locale *result-properties* = *result* +
fixes $\psi :: ('a \Rightarrow 'a, 'b) \text{ binary-fun}$ **and**
 $\nu :: 'v \text{ itself}$
assumes
action-neutral: group-action bijection_{AG} UNIV ψ and
neutrality:
is-symmetry ($\lambda \mathcal{E} :: ('a, 'v) \text{ Election. limit (alternatives-}\mathcal{E} \mathcal{E}) \text{ UNIV}$)
(action-induced-equivariance (carrier bijection_{AG})
well-formed-elections
(φ -neutral well-formed-elections) (set-action ψ))
sublocale *result-properties* \subseteq *result*
<proof>

1.11.2 Interpretations

global-interpretation *SCF-properties: result-properties well-formed-SCF*
limit-SCF ψ -neutral_c
<proof>

global-interpretation *SWF-properties: result-properties well-formed-SWF*
limit-SWF ψ -neutral_w
<proof>

end

Chapter 2

Refined Types

2.1 Preference List

```
theory Preference-List
  imports ../Preference-Relation
           HOL-Combinatorics.Multiset-Permutations
           List-Index.List-Index
begin
```

Preference lists derive from preference relations, ordered from most to least preferred alternative.

2.1.1 Well-Formedness

```
type-synonym 'a Preference-List = 'a list
```

```
abbreviation well-formed-l :: 'a Preference-List  $\Rightarrow$  bool where
  well-formed-l l  $\equiv$  distinct l
```

2.1.2 Auxiliary Lemmas About Lists

```
lemma is-arg-min-equal:
  fixes
    f g :: 'a  $\Rightarrow$  'b :: ord and
    S :: 'a set and
    x :: 'a
  assumes  $\forall x \in S. f\ x = g\ x$ 
  shows is-arg-min f ( $\lambda s. s \in S$ ) x = is-arg-min g ( $\lambda s. s \in S$ ) x
   $\langle$ proof $\rangle$ 
```

```
lemma list-cons-presv-finiteness:
  fixes
    A :: 'a set and
    S :: 'a list set
  assumes
```

$\text{fin-}A$: $\text{finite } A$ **and**
 $\text{fin-}B$: $\text{finite } S$
shows $\text{finite } \{a\#l \mid a \text{ l. } a \in A \wedge l \in S\}$
 $\langle \text{proof} \rangle$

lemma *listset-finiteness*:
fixes $l :: 'a \text{ set list}$
assumes $\forall i :: \text{nat. } i < \text{length } l \longrightarrow \text{finite } (l[i])$
shows $\text{finite } (\text{listset } l)$
 $\langle \text{proof} \rangle$

lemma *all-ls-elems-same-len*:
fixes $l :: 'a \text{ set list}$
shows $\forall l' :: 'a \text{ list. } l' \in \text{listset } l \longrightarrow \text{length } l' = \text{length } l$
 $\langle \text{proof} \rangle$

lemma *all-ls-elems-in-ls-set*:
fixes $l :: 'a \text{ set list}$
shows $\forall l' \in \text{listset } l. \forall i :: \text{nat} < \text{length } l'. l'[i] \in l[i]$
 $\langle \text{proof} \rangle$

lemma *all-ls-in-ls-set*:
fixes $l :: 'a \text{ set list}$
shows $\forall l'. \text{length } l' = \text{length } l$
 $\wedge (\forall i < \text{length } l'. l'[i] \in l[i]) \longrightarrow l' \in \text{listset } l$
 $\langle \text{proof} \rangle$

2.1.3 Ranking

Rank 1 is the top preference, rank 2 the second, and so on. Rank 0 does not exist.

fun *rank-l* :: $'a \text{ Preference-List} \Rightarrow 'a \Rightarrow \text{nat}$ **where**
 $\text{rank-l } l \ a = (\text{if } a \in \text{set } l \text{ then } \text{index } l \ a + 1 \text{ else } 0)$

fun *rank-l-idx* :: $'a \text{ Preference-List} \Rightarrow 'a \Rightarrow \text{nat}$ **where**
 $\text{rank-l-idx } l \ a =$
 $(\text{let } i = \text{index } l \ a \text{ in}$
 $\text{if } i = \text{length } l \text{ then } 0 \text{ else } i + 1)$

lemma *rank-l-equiv*: $\text{rank-l} = \text{rank-l-idx}$
 $\langle \text{proof} \rangle$

lemma *rank-zero-imp-not-present*:
fixes
 $p :: 'a \text{ Preference-List}$ **and**
 $a :: 'a$
assumes $\text{rank-l } p \ a = 0$
shows $a \notin \text{set } p$
 $\langle \text{proof} \rangle$

definition *above-l* :: 'a Preference-List \Rightarrow 'a \Rightarrow 'a Preference-List **where**
above-l r a \equiv take (rank-l r a) r

2.1.4 Definition

fun *is-less-preferred-than-l* :: 'a \Rightarrow 'a Preference-List \Rightarrow 'a \Rightarrow bool
 (- \lesssim - [50, 1000, 51] 50) **where**
a \lesssim_l *b* = (*a* \in set *l* \wedge *b* \in set *l* \wedge index *l* *a* \geq index *l* *b*)

lemma *rank-gt-zero*:
fixes
l :: 'a Preference-List **and**
a :: 'a
assumes *a* \lesssim_l *a*
shows rank-l *l* *a* \geq 1
 <proof>

definition *pl- α* :: 'a Preference-List \Rightarrow 'a Preference-Relation **where**
pl- α *l* \equiv {(*a*, *b*). *a* \lesssim_l *b*}

lemma *rel-trans*:
fixes *l* :: 'a Preference-List
shows trans (*pl- α* *l*)
 <proof>

lemma *pl- α -lin-order*:
fixes
A :: 'a set **and**
r :: 'a rel
assumes *r* \in *pl- α* 'permutations-of-set *A*
shows linear-order-on *A* *r*
 <proof>

lemma *lin-order-pl- α* :
fixes
r :: 'a rel **and**
A :: 'a set
assumes
lin-order: linear-order-on *A* *r* **and**
fin: finite *A*
shows *r* \in *pl- α* 'permutations-of-set *A*
 <proof>

lemma *index-helper*:
fixes
l :: 'x list **and**
x :: 'x
assumes

finite (set l) and
distinct l and
x ∈ set l
shows $\text{index } l \ x = \text{card } \{y \in \text{set } l. \text{index } l \ y < \text{index } l \ x\}$
 <proof>

lemma *pl-α-eq-imp-list-eq*:
fixes $l \ l' :: 'x \text{ list}$
assumes
 fin-set-l: finite (set l) and
 set-eq: set l = set l' and
 dist-l: distinct l and
 dist-l': distinct l' and
 pl-α-eq: pl-α l = pl-α l'
shows $l = l'$
 <proof>

lemma *pl-α-bij-betw*:
fixes $X :: 'x \text{ set}$
assumes *finite X*
shows *bij-betw pl-α (permutations-of-set X) {r. linear-order-on X r}*
 <proof>

2.1.5 Limited Preference

definition *limited* :: $'a \text{ set} \Rightarrow 'a \text{ Preference-List} \Rightarrow \text{bool}$ **where**
 $\text{limited } A \ r \equiv \forall \ a. \ a \in \text{set } r \longrightarrow a \in A$

fun *limit-l* :: $'a \text{ set} \Rightarrow 'a \text{ Preference-List} \Rightarrow 'a \text{ Preference-List}$ **where**
 $\text{limit-l } A \ l = \text{List.filter } (\lambda \ a. \ a \in A) \ l$

lemma *limited-dest*:
fixes
 $A :: 'a \text{ set}$ **and**
 $l :: 'a \text{ Preference-List}$ **and**
 $a \ b :: 'a$
assumes
 $a \lesssim_l b$ **and**
 $\text{limited } A \ l$
shows $a \in A \wedge b \in A$
 <proof>

lemma *limit-equiv*:
fixes
 $A :: 'a \text{ set}$ **and**
 $l :: 'a \text{ list}$
assumes *well-formed-l l*
shows $\text{pl-}\alpha \ (\text{limit-l } A \ l) = \text{limit } A \ (\text{pl-}\alpha \ l)$
 <proof>

2.1.6 Auxiliary Definitions

definition *total-on-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
total-on-l A l $\equiv \forall a \in A. a \in \text{set } l$

definition *refl-on-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
refl-on-l A l $\equiv (\forall a. a \in \text{set } l \longrightarrow a \in A) \wedge (\forall a \in A. a \lesssim_l a)$

definition *trans* :: 'a Preference-List \Rightarrow bool **where**
trans l $\equiv \forall (a, b, c) \in \text{set } l \times \text{set } l \times \text{set } l. a \lesssim_l b \wedge b \lesssim_l c \longrightarrow a \lesssim_l c$

definition *preorder-on-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
preorder-on-l A l $\equiv \text{refl-on-l } A \ l \wedge \text{trans } l$

definition *antisym-l* :: 'a list \Rightarrow bool **where**
antisym-l l $\equiv \forall a \ b. a \lesssim_l b \wedge b \lesssim_l a \longrightarrow a = b$

definition *partial-order-on-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
partial-order-on-l A l $\equiv \text{preorder-on-l } A \ l \wedge \text{antisym-l } l$

definition *linear-order-on-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
linear-order-on-l A l $\equiv \text{partial-order-on-l } A \ l \wedge \text{total-on-l } A \ l$

definition *connex-l* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
connex-l A l $\equiv \text{limited } A \ l \wedge (\forall a \in A. \forall b \in A. a \lesssim_l b \vee b \lesssim_l a)$

abbreviation *ballot-on* :: 'a set \Rightarrow 'a Preference-List \Rightarrow bool **where**
ballot-on A l $\equiv \text{well-formed-l } l \wedge \text{linear-order-on-l } A \ l$

2.1.7 Auxiliary Lemmas

lemma *list-trans[simp]*:
fixes l :: 'a Preference-List
shows *trans* l
 <proof>

lemma *list-antisym[simp]*:
fixes l :: 'a Preference-List
shows *antisym-l* l
 <proof>

lemma *lin-order-equiv-list-of-alts*:
fixes
 A :: 'a set **and**
 l :: 'a Preference-List
shows *linear-order-on-l* A l = (A = set l)
 <proof>

lemma *connex-imp-refl*:
fixes

```

    A :: 'a set and
    l :: 'a Preference-List
  assumes connex-l A l
  shows refl-on-l A l
  <proof>

lemma lin-ord-imp-connex-l:
  fixes
    A :: 'a set and
    l :: 'a Preference-List
  assumes linear-order-on-l A l
  shows connex-l A l
  <proof>

lemma above-trans:
  fixes
    l :: 'a Preference-List and
    a b :: 'a
  assumes
    trans l and
    a  $\lesssim_l$  b
  shows set (above-l l b)  $\subseteq$  set (above-l l a)
  <proof>

lemma less-preferred-l-rel-equiv:
  fixes
    l :: 'a Preference-List and
    a b :: 'a
  shows a  $\lesssim_l$  b =
    Preference-Relation.is-less-preferred-than a (pl- $\alpha$  l) b
  <proof>

theorem above-equiv:
  fixes
    l :: 'a Preference-List and
    a :: 'a
  shows set (above-l l a) = above (pl- $\alpha$  l) a
  <proof>

theorem rank-equiv:
  fixes
    l :: 'a Preference-List and
    a :: 'a
  assumes well-formed-l l
  shows rank-l l a = rank (pl- $\alpha$  l) a
  <proof>

lemma lin-ord-equiv:
  fixes

```


$A :: 'a \text{ set}$ **and**
 $l :: 'a \text{ Preference-List}$
shows $\text{linear-order-on-}l \ A \ l = \text{linear-order-on } A \ (pl-\alpha \ l)$
 $\langle \text{proof} \rangle$

2.1.8 First Occurrence Indices

lemma *pos-in-list-yields-rank*:

fixes
 $l :: 'a \text{ Preference-List}$ **and**
 $a :: 'a$ **and**
 $n :: \text{nat}$
assumes
 $\forall (j :: \text{nat}) \leq n. \ l!j \neq a$ **and**
 $l!(n - 1) = a$
shows $\text{rank-}l \ l \ a = n$
 $\langle \text{proof} \rangle$

lemma *ranked-alt-not-at-pos-before*:

fixes
 $l :: 'a \text{ Preference-List}$ **and**
 $a :: 'a$ **and**
 $n :: \text{nat}$
assumes
 $a \in \text{set } l$ **and**
 $n < (\text{rank-}l \ l \ a) - 1$
shows $l!n \neq a$
 $\langle \text{proof} \rangle$

lemma *pos-in-list-yields-pos*:

fixes
 $l :: 'a \text{ Preference-List}$ **and**
 $a :: 'a$
assumes $a \in \text{set } l$
shows $l!(\text{rank-}l \ l \ a - 1) = a$
 $\langle \text{proof} \rangle$

lemma *rel-of-pref-pred-for-set-eq-list-to-rel*:

fixes $l :: 'a \text{ Preference-List}$
shows $\text{relation-of } (\lambda y z. y \lesssim_l z) \ (\text{set } l) = pl-\alpha \ l$
 $\langle \text{proof} \rangle$

end

2.2 Preference (List) Profile

```

theory Profile-List
  imports ../Profile
           Preference-List
begin

```

2.2.1 Definition

A profile (list) contains one ballot for each voter.

type-synonym 'a Profile-List = 'a Preference-List list

type-synonym 'a Election-List = 'a set \times 'a Profile-List

Abstraction from profile list to profile.

```

fun pl-to-pr- $\alpha$  :: 'a Profile-List  $\Rightarrow$  ('a, nat) Profile where
  pl-to-pr- $\alpha$  pl = ( $\lambda$  n. if n < length pl  $\wedge$  n  $\geq$  0
                     then (map pl- $\alpha$  pl)!n
                     else {})

```

```

lemma prof-abstr-presv-size:
  fixes p :: 'a Profile-List
  shows length p = length (to-list {0 ..< length p} (pl-to-pr- $\alpha$  p))
  <proof>

```

2.2.2 Refinement Proof

A profile on a finite set of alternatives A contains only ballots that are lists of linear orders on A.

definition profile-l :: 'a set \Rightarrow 'a Profile-List \Rightarrow bool **where**
 profile-l A p $\equiv \forall$ i < length p. ballot-on A (p!i)

```

lemma profile-list-refines-profile:
  fixes
    A :: 'a set and
    p :: 'a Profile-List
  assumes profile-l A p
  shows profile {0 ..< length p} A (pl-to-pr- $\alpha$  p)
  <proof>

end

```

2.3 Ordered Relation Type

```

theory Ordered-Relation

```

```

imports Preference-Relation
         ./Refined-Types/Preference-List
         HOL-Combinatorics.Multiset-Permutations
begin

lemma fin-ordered:
  fixes  $X :: 'x \text{ set}$ 
  assumes finite X
  obtains  $\text{ord} :: 'x \text{ rel}$  where
    linear-order-on X ord
   $\langle \text{proof} \rangle$ 

typedef  $'a \text{ Ordered-Preference} =$ 
   $\{p :: 'a :: \text{finite } \text{Preference-Relation}. \text{linear-order-on } (\text{UNIV} :: 'a \text{ set}) p\}$ 
  morphisms ord2pref pref2ord
   $\langle \text{proof} \rangle$ 

instance Ordered-Preference :: (finite) finite
   $\langle \text{proof} \rangle$ 

lemma range-ord2pref:  $\text{range ord2pref} = \{p. \text{linear-order } p\}$ 
   $\langle \text{proof} \rangle$ 

lemma card-ord-pref:  $\text{card } (\text{UNIV} :: 'a :: \text{finite } \text{Ordered-Preference set}) =$ 
   $\text{fact } (\text{card } (\text{UNIV} :: 'a \text{ set}))$ 
   $\langle \text{proof} \rangle$ 

end

```

2.4 Alternative Election Type

```

theory Quotient-Type-Election
  imports Profile
begin

lemma election-equality-equiv:
  election-equality E E and
  election-equality E E'  $\longrightarrow$  election-equality E' E and
  election-equality E E'  $\longrightarrow$  election-equality E' F
     $\longrightarrow \text{election-equality } E F$ 
   $\langle \text{proof} \rangle$ 

quotient-type  $( 'a, 'v) \text{ Election}_{\mathcal{Q}} =$ 
   $'a \text{ set} \times 'v \text{ set} \times ('a, 'v) \text{ Profile} / \text{election-equality}$ 
   $\langle \text{proof} \rangle$ 

fun  $\text{fst}_{\mathcal{Q}} :: ('a, 'v) \text{ Election}_{\mathcal{Q}} \Rightarrow 'a \text{ set}$  where

```

$fst_Q E = fst (rep-Election_Q E)$

fun $snd_Q :: ('a, 'v) Election_Q \Rightarrow 'v \text{ set} \times ('a, 'v) Profile$ **where**
 $snd_Q E = snd (rep-Election_Q E)$

abbreviation $alternatives-\mathcal{E}_Q :: ('a, 'v) Election_Q \Rightarrow 'a \text{ set}$ **where**
 $alternatives-\mathcal{E}_Q E \equiv fst_Q E$

abbreviation $voters-\mathcal{E}_Q :: ('a, 'v) Election_Q \Rightarrow 'v \text{ set}$ **where**
 $voters-\mathcal{E}_Q E \equiv snd_Q E$

abbreviation $profile-\mathcal{E}_Q :: ('a, 'v) Election_Q \Rightarrow ('a, 'v) Profile$ **where**
 $profile-\mathcal{E}_Q E \equiv snd (snd_Q E)$

end

Chapter 3

Quotient Rules

3.1 Quotients of Equivalence Relations

```
theory Relation-Quotients
imports ../Social-Choice-Types/Symmetry-Of-Functions
begin
```

3.1.1 Definitions

```
fun singleton-set :: 'x set  $\Rightarrow$  'x where
  singleton-set s = (if (card s = 1) then (the-inv ( $\lambda$  x. {x}) s) else undefined)
— This is undefined if card s  $\neq$  1. Note that "undefined = undefined" is the only
provable equality for undefined.
```

For a given function, we define a function on sets that maps each set to the unique image under *f* of its elements, if one exists. Otherwise, the result is undefined.

```
fun  $\pi_Q$  :: ('x  $\Rightarrow$  'y)  $\Rightarrow$  ('x set  $\Rightarrow$  'y) where
   $\pi_Q$  f s = singleton-set (f ` s)
```

For a given function *f* on sets and a mapping from elements to sets, we define a function on the set element type that maps each element to the image of its corresponding set under *f*. A natural mapping is from elements to their classes under a relation.

```
fun inv- $\pi_Q$  :: ('x  $\Rightarrow$  'x set)  $\Rightarrow$  ('x set  $\Rightarrow$  'y)  $\Rightarrow$  ('x  $\Rightarrow$  'y) where
  inv- $\pi_Q$  cls f x = f (cls x)
```

```
fun relation-class :: 'x rel  $\Rightarrow$  'x  $\Rightarrow$  'x set where
  relation-class r x = r `` {x}
```

3.1.2 Well-Definedness

```
lemma singleton-set-undef-if-card-neq-one:
fixes s :: 'x set
```

assumes $\text{card } s \neq 1$
shows $\text{singleton-set } s = \text{undefined}$
 $\langle \text{proof} \rangle$

lemma *singleton-set-def-if-card-one*:
fixes $s :: 'x \text{ set}$
assumes $\text{card } s = 1$
shows $\exists! x. x = \text{singleton-set } s \wedge \{x\} = s$
 $\langle \text{proof} \rangle$

If the given function is invariant under an equivalence relation, the induced function on sets is well-defined for all equivalence classes of that relation.

theorem *pass-to-quotient*:
fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $r :: 'x \text{ rel}$ **and**
 $s :: 'x \text{ set}$
assumes
 f *respects* r **and**
 $\text{equiv } s \ r$
shows $\forall t \in s // r. \forall x \in t. \pi_Q f t = f x$
 $\langle \text{proof} \rangle$

A function on sets induces a function on the element type that is invariant under a given equivalence relation.

theorem *pass-to-quotient-inv*:
fixes
 $f :: 'x \text{ set} \Rightarrow 'x$ **and**
 $r :: 'x \text{ rel}$ **and**
 $s :: 'x \text{ set}$
assumes $\text{equiv } s \ r$
defines $\text{induced-fun} \equiv (\text{inv-}\pi_Q (\text{relation-class } r) f)$
shows
 induced-fun *respects* r **and**
 $\forall A \in s // r. \pi_Q \text{ induced-fun } A = f A$
 $\langle \text{proof} \rangle$

3.1.3 Equivalence Relations

lemma *restr-equals-restricted-rel*:
fixes
 $s \ t :: 'a \text{ set}$ **and**
 $r :: 'a \text{ rel}$
assumes
 $\text{closed-restricted-rel } r \ s \ t$ **and**
 $t \subseteq s$
shows $\text{restricted-rel } r \ t \ s = \text{Restr } r \ t$
 $\langle \text{proof} \rangle$

```

lemma equiv-rel-restr:
  fixes
     $s\ t :: 'x\ set$  and
     $r :: 'x\ rel$ 
  assumes
     $equiv\ s\ r$  and
     $t \subseteq s$ 
  shows  $equiv\ t\ (Restr\ r\ t)$ 
   $\langle proof \rangle$ 

lemma rel-ind-by-group-act-equiv:
  fixes
     $m :: 'x\ monoid$  and
     $s :: 'y\ set$  and
     $\varphi :: ('x, 'y)\ binary\_fun$ 
  assumes  $group\_action\ m\ s\ \varphi$ 
  shows  $equiv\ s\ (action\_induced\_rel\ (carrier\ m)\ s\ \varphi)$ 
   $\langle proof \rangle$ 

end

```

3.2 Quotients of Election Set Equivalences

```

theory Election-Quotients
  imports Relation-Quotients
    ../Social-Choice-Types/Voting-Symmetry
    ../Social-Choice-Types/Ordered-Relation
    HOL-Analysis.Convex
    HOL-Analysis.Cartesian-Space

begin

```

3.2.1 Auxiliary Lemmas

```

lemma obtain-partition:
  fixes
     $A :: 'a\ set$  and
     $B :: 'b\ set$  and
     $f :: 'b \Rightarrow nat$ 
  assumes
     $finite\ A$  and
     $finite\ B$  and
     $(\sum x \in B. f\ x) = card\ A$ 
  shows  $\exists\ \mathcal{X}. A = \bigcup \{\mathcal{X}\ i \mid i. i \in B\} \wedge (\forall\ i \in B. card\ (\mathcal{X}\ i) = f\ i) \wedge$ 
     $(\forall\ i\ j. i \neq j \longrightarrow i \in B \wedge j \in B \longrightarrow \mathcal{X}\ i \cap \mathcal{X}\ j = \{\})$ 
   $\langle proof \rangle$ 

```

3.2.2 Anonymity Quotient: Grid

fun *anonymity*_Q :: 'a set \Rightarrow ('a, 'v) Election set set **where**
*anonymity*_Q A = quotient (elections- \mathcal{A} A) (*anonymity*_R (elections- \mathcal{A} A))

— Here, we count the occurrences of a ballot per election in a set of elections for which the occurrences of the ballot per election coincide for all elections in the set.

fun *vote-count*_Q :: 'a Preference-Relation \Rightarrow ('a, 'v) Election set \Rightarrow nat **where**
*vote-count*_Q p = π_Q (*vote-count* p)

fun *anonymity-class* :: ('a :: finite, 'v) Election set \Rightarrow
 (nat, 'a Ordered-Preference) vec **where**
anonymity-class X = (χ p. *vote-count*_Q (ord2pref p) X)

lemma *anon-rel-equiv*: equiv (elections- \mathcal{A} UNIV) (*anonymity*_R (elections- \mathcal{A} UNIV))
 <proof>

We assume that all elections consist of a fixed finite alternative set of size n and finite subsets of an infinite voter universe. Profiles are linear orders on the alternatives. Then, we can operate on the natural-number-vectors of dimension $n!$ instead of the equivalence classes of the anonymity relation: Each dimension corresponds to one possible linear order on the alternative set, i.e., the possible preferences. Each equivalence class of elections corresponds to a vector whose entries denote the amount of voters per election in that class who vote the respective corresponding preference.

theorem *anonymity*_Q-isomorphism:

assumes *infinite* (UNIV :: 'v set)

shows *bij-betw* (*anonymity-class* :: ('a :: finite, 'v) Election set
 \Rightarrow nat^{nat} ('a Ordered-Preference)) (*anonymity*_Q (UNIV :: 'a set))
 (UNIV :: (nat^{nat} ('a Ordered-Preference)) set)

<proof>

3.2.3 Homogeneity Quotient: Simplex

fun *vote-fraction* :: 'a Preference-Relation \Rightarrow ('a, 'v) Election \Rightarrow rat **where**
vote-fraction r E =

(if (finite (voters- \mathcal{E} E) \wedge voters- \mathcal{E} E \neq {})
 then (Fract (*vote-count* r E) (card (voters- \mathcal{E} E))) else 0)

fun *anonymity-homogeneity*_R :: ('a, 'v) Election set \Rightarrow ('a, 'v) Election rel **where**
*anonymity-homogeneity*_R \mathcal{E} =

{(E, E') | E E'. E \in \mathcal{E} \wedge E' \in \mathcal{E}
 \wedge (finite (voters- \mathcal{E} E) = finite (voters- \mathcal{E} E'))
 \wedge (\forall r. *vote-fraction* r E = *vote-fraction* r E')}

fun *anonymity-homogeneity*_Q :: 'a set \Rightarrow ('a, 'v) Election set set **where**
*anonymity-homogeneity*_Q A =
 quotient (elections- \mathcal{A} A) (*anonymity-homogeneity*_R (elections- \mathcal{A} A))

fun *vote-fraction*_Q :: ('a Preference-Relation \Rightarrow ('a, 'v) Election set \Rightarrow rat **where**
*vote-fraction*_Q p = π_Q (vote-fraction p)

fun *anonymity-homogeneity-class* :: ('a :: finite, 'v) Election set \Rightarrow
 (rat, 'a Ordered-Preference) vec **where**
anonymity-homogeneity-class \mathcal{E} = (χ p. *vote-fraction*_Q (ord2pref p) \mathcal{E})

Maps each rational real vector entry to the corresponding rational. If the entry is not rational, the corresponding entry will be undefined.

fun *rat-vector* :: $real^b \Rightarrow rat^b$ **where**
rat-vector v = (χ p. the-inv of-rat (v\$p))

fun *rat-vector-set* :: ($real^b$) set \Rightarrow (rat^b) set **where**
rat-vector-set V = *rat-vector* ' {v \in V. \forall i. v\$i \in Q}

definition *standard-basis* :: ($real^b$) set **where**
standard-basis \equiv {v. \exists b. v\$b = 1 \wedge (\forall c \neq b. v\$c = 0)}

The rational points in the simplex.

definition *vote-simplex* :: (rat^b) set **where**
vote-simplex \equiv
 insert 0 (*rat-vector-set* (convex hull (*standard-basis* :: ($real^b$) set)))

Auxiliary Lemmas

lemma *convex-combination-in-convex-hull*:

fixes

X :: ($real^b$) set **and**

x :: $real^b$

assumes \exists f :: ($real^b$) \Rightarrow real.

$$\left(\sum y \in X. f y \right) = 1 \wedge (\forall x \in X. f x \geq 0) \\ \wedge x = \left(\sum x \in X. (f x) *_R x \right)$$

shows x \in convex hull X

<proof>

lemma *standard-simplex-rewrite*: convex hull *standard-basis* =

$$\{v :: (real^b). (\forall i. v\$i \geq 0) \wedge (\sum y \in UNIV. v\$y) = 1\}$$

<proof>

lemma *fract-distr-helper*:

fixes a b c :: int

assumes c \neq 0

shows Fract a c + Fract b c = Fract (a + b) c

<proof>

lemma *anonymity-homogeneity-is-equivalence*:

fixes X :: ('a, 'v) Election set

assumes $\forall E \in X. \text{finite } (\text{voters-}\mathcal{E} \ E)$

shows equiv X (*anonymity-homogeneity*_R X)

$\langle proof \rangle$

lemma *fract-distr*:

fixes

$A :: 'x \text{ set}$ **and**

$f :: 'x \Rightarrow \text{int}$ **and**

$b :: \text{int}$

assumes

finite A **and**

$b \neq 0$

shows $(\sum a \in A. \text{Fract } (f a) b) = \text{Fract } (\sum x \in A. f x) b$

$\langle proof \rangle$

Simplex Bijection

We assume all our elections to consist of a fixed finite alternative set of size n and finite subsets of an infinite voter universe. Profiles are linear orders on the alternatives. Then we can work on the standard simplex of dimension $n!$ instead of the equivalence classes of the equivalence relation for anonymous + homogeneous voting rules (anon hom): Each dimension corresponds to one possible linear order on the alternative set, i.e., the possible preferences. Each equivalence class of elections corresponds to a vector whose entries denote the fraction of voters per election in that class who vote the respective corresponding preference.

theorem *anonymity-homogeneity_Q-isomorphism*:

assumes *infinite* ($UNIV :: 'v \text{ set}$)

shows

$\text{bij-betw } (\text{anonymity-homogeneity-class} :: ('a :: \text{finite}, 'v) \text{ Election set} \Rightarrow$
 $\text{rat}^{\sim}('a \text{ Ordered-Preference})) (\text{anonymity-homogeneity}_{\mathcal{Q}} (UNIV :: 'a \text{ set}))$
 $(\text{vote-simplex} :: (\text{rat}^{\sim}('a \text{ Ordered-Preference})) \text{ set})$

$\langle proof \rangle$

end

Chapter 4

Component Types

4.1 Distance

```
theory Distance
imports HOL-Library.Extended-Real
          Social-Choice-Types/Voting-Symmetry
begin
```

A general distance on a set X is a mapping $d: X \times X \mapsto R \cup \{+\infty\}$ such that for every x, y, z in X , the following four conditions are satisfied:

- $d(x, y) \geq 0$ (non-negativity);
- $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
- $d(x, y) = d(y, x)$ (symmetry);
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Moreover, a mapping that satisfies all but the second conditions is called a pseudo-distance, whereas a quasi-distance needs to satisfy the first three conditions (and not necessarily the last one).

4.1.1 Definition

```
type-synonym 'a Distance = 'a  $\Rightarrow$  'a  $\Rightarrow$  ereal
```

The un-curried version of a distance is defined on tuples.

```
fun tup :: 'a Distance  $\Rightarrow$  ('a * 'a  $\Rightarrow$  ereal) where
  tup d = ( $\lambda$  pair. d (fst pair) (snd pair))
```

```
definition distance :: 'a set  $\Rightarrow$  'a Distance  $\Rightarrow$  bool where
  distance S d  $\equiv \forall x y. x \in S \wedge y \in S \longrightarrow d x x = 0 \wedge 0 \leq d x y$ 
```

4.1.2 Conditions

definition *symmetric* :: 'a set \Rightarrow 'a Distance \Rightarrow bool **where**
symmetric $S\ d \equiv \forall\ x\ y. x \in S \wedge y \in S \longrightarrow d\ x\ y = d\ y\ x$

definition *triangle-ineq* :: 'a set \Rightarrow 'a Distance \Rightarrow bool **where**
triangle-ineq $S\ d \equiv \forall\ x\ y\ z. x \in S \wedge y \in S \wedge z \in S \longrightarrow d\ x\ z \leq d\ x\ y + d\ y\ z$

definition *eq-if-zero* :: 'a set \Rightarrow 'a Distance \Rightarrow bool **where**
eq-if-zero $S\ d \equiv \forall\ x\ y. x \in S \wedge y \in S \longrightarrow d\ x\ y = 0 \longrightarrow x = y$

definition *vote-distance* :: ('a Vote set \Rightarrow 'a Vote Distance \Rightarrow bool) \Rightarrow
'a Vote Distance \Rightarrow bool **where**
vote-distance $\pi\ d \equiv \pi\ \{(A, p). \text{linear-order-on } A\ p \wedge \text{finite } A\}\ d$

definition *election-distance* :: (('a, 'v) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool) \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
election-distance $\pi\ d \equiv \pi\ \{(A, V, p). \text{finite-profile } V\ A\ p\}\ d$

4.1.3 Standard-Distance Property

definition *standard* :: ('a, 'v) Election Distance \Rightarrow bool **where**
standard $d \equiv$
 $\forall\ A\ A'\ V\ V'\ p\ p'. A \neq A' \vee V \neq V' \longrightarrow d\ (A, V, p)\ (A', V', p') = \infty$

4.1.4 Auxiliary Lemmas

fun *arg-min-set* :: ('b \Rightarrow 'a :: ord) \Rightarrow 'b set \Rightarrow 'b set **where**
arg-min-set $f\ A = \text{Collect } (\text{is-arg-min } f\ (\lambda\ a. a \in A))$

lemma *arg-min-subset*:

fixes
 $B :: 'b\ \text{set}$ **and**
 $f :: 'b \Rightarrow 'a :: \text{ord}$
shows *arg-min-set* $f\ B \subseteq B$
 $\langle \text{proof} \rangle$

lemma *sum-monotone*:

fixes
 $A :: 'a\ \text{set}$ **and**
 $f\ g :: 'a \Rightarrow \text{int}$
assumes $\forall\ a \in A. f\ a \leq g\ a$
shows $(\sum\ a \in A. f\ a) \leq (\sum\ a \in A. g\ a)$
 $\langle \text{proof} \rangle$

lemma *distrib*:

fixes
 $A :: 'a\ \text{set}$ **and**
 $f\ g :: 'a \Rightarrow \text{int}$

shows $(\sum a \in A. f\ a) + (\sum a \in A. g\ a) = (\sum a \in A. f\ a + g\ a)$
 ⟨proof⟩

lemma *distrib-ereal*:

fixes

$A :: 'a\ set$ **and**

$f\ g :: 'a \Rightarrow int$

shows $ereal\ (real-of-int\ ((\sum a \in A. (f :: 'a \Rightarrow int)\ a) + (\sum a \in A. g\ a))) =$
 $ereal\ (real-of-int\ ((\sum a \in A. (f\ a) + (g\ a))))$

⟨proof⟩

lemma *uneq-ereal*:

fixes $x\ y :: int$

assumes $x \leq y$

shows $ereal\ (real-of-int\ x) \leq ereal\ (real-of-int\ y)$

⟨proof⟩

4.1.5 Swap Distance

fun *neq-ord* :: $'a\ Preference-Relation \Rightarrow 'a\ Preference-Relation \Rightarrow$

$'a \Rightarrow 'a \Rightarrow bool$ **where**

$neq-ord\ r\ s\ a\ b = ((a \preceq_r b \wedge b \preceq_s a) \vee (b \preceq_r a \wedge a \preceq_s b))$

fun *pairwise-disagreements* :: $'a\ set \Rightarrow 'a\ Preference-Relation \Rightarrow$

$'a\ Preference-Relation \Rightarrow ('a \times 'a)\ set$ **where**

$pairwise-disagreements\ A\ r\ s = \{(a, b) \in A \times A. a \neq b \wedge neq-ord\ r\ s\ a\ b\}$

fun *pairwise-disagreements'* :: $'a\ set \Rightarrow 'a\ Preference-Relation \Rightarrow$

$'a\ Preference-Relation \Rightarrow ('a \times 'a)\ set$ **where**

$pairwise-disagreements'\ A\ r\ s =$

$Set.filter\ (\lambda\ (a, b). a \neq b \wedge neq-ord\ r\ s\ a\ b)\ (A \times A)$

lemma *set-eq-filter*:

fixes

$X :: 'a\ set$ **and**

$P :: 'a \Rightarrow bool$

shows $\{x \in X. P\ x\} = Set.filter\ P\ X$

⟨proof⟩

lemma *pairwise-disagreements-eq*[code]: $pairwise-disagreements = pairwise-disagreements'$

⟨proof⟩

fun *swap* :: $'a\ Vote\ Distance$ **where**

$swap\ (A, r)\ (A', r') =$

$(if\ A = A'$

$then\ card\ (pairwise-disagreements\ A\ r\ r')$

$else\ \infty)$

lemma *swap-case-infinity*:

```

fixes  $x\ y :: 'a\ Vote$ 
assumes  $alts\mathcal{V}\ x \neq alts\mathcal{V}\ y$ 
shows  $swap\ x\ y = \infty$ 
 $\langle proof \rangle$ 

```

```

lemma swap-case-fin:
  fixes  $x\ y :: 'a\ Vote$ 
  assumes  $alts\mathcal{V}\ x = alts\mathcal{V}\ y$ 
  shows  $swap\ x\ y = card\ (pairwise-disagreements\ (alts\mathcal{V}\ x)\ (pref\mathcal{V}\ x)\ (pref\mathcal{V}\ y))$ 
   $\langle proof \rangle$ 

```

4.1.6 Spearman Distance

```

fun spearman ::  $'a\ Vote\ Distance$  where
  spearman  $(A, x)\ (A', y) =$ 
    (if  $A = A'$ 
     then  $\sum a \in A. abs\ (int\ (rank\ x\ a) - int\ (rank\ y\ a))$ 
     else  $\infty$ )

```

```

lemma spearman-case-inf:
  fixes  $x\ y :: 'a\ Vote$ 
  assumes  $alts\mathcal{V}\ x \neq alts\mathcal{V}\ y$ 
  shows  $spearman\ x\ y = \infty$ 
   $\langle proof \rangle$ 

```

```

lemma spearman-case-fin:
  fixes  $x\ y :: 'a\ Vote$ 
  assumes  $alts\mathcal{V}\ x = alts\mathcal{V}\ y$ 
  shows  $spearman\ x\ y =$ 
    ( $\sum a \in alts\mathcal{V}\ x. abs\ (int\ (rank\ (pref\mathcal{V}\ x)\ a) - int\ (rank\ (pref\mathcal{V}\ y)\ a))$ )
   $\langle proof \rangle$ 

```

4.1.7 Properties

Distances that are invariant under specific relations induce symmetry properties in distance rationalized voting rules.

Definitions

```

fun total-invarianceD ::  $'x\ Distance \Rightarrow 'x\ rel \Rightarrow bool$  where
  total-invarianceD  $d\ rel = is-symmetry\ (tup\ d)\ (Invariance\ (product\ rel))$ 

```

```

fun invarianceD ::  $'y\ Distance \Rightarrow 'x\ set \Rightarrow 'y\ set \Rightarrow$ 
  ( $'x, 'y$ ) binary-fun  $\Rightarrow bool$  where
  invarianceD  $d\ X\ Y\ \varphi = is-symmetry\ (tup\ d)\ (Invariance\ (equivariance\ X\ Y\ \varphi))$ 

```

```

definition distance-anonymity ::  $('a, 'v)\ Election\ Distance \Rightarrow bool$  where
  distance-anonymity  $d \equiv$ 
     $\forall A\ A'\ V\ V'\ p\ p'\ \pi :: ('v \Rightarrow 'v).$ 

```

$(bij \pi \longrightarrow$
 $(d (A, V, p) (A', V', p')) =$
 $(d (rename \pi (A, V, p)) (rename \pi (A', V', p')))$

fun *distance-anonymity'* :: ('a, 'v) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
distance-anonymity' X d = invariance _{\mathcal{D}} d (carrier bijection _{ν_G}) X (φ -anon X)

fun *distance-neutrality* :: ('a, 'v) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
distance-neutrality X d = invariance _{\mathcal{D}} d (carrier bijection _{\mathcal{AG}}) X (φ -neutral X)

fun *distance-reversal-symmetry* :: ('a, 'v) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
distance-reversal-symmetry X d =
invariance _{\mathcal{D}} d (carrier reversal _{\mathcal{G}}) X (φ -reverse X)

definition *distance-homogeneity'* :: ('a, 'v :: linorder) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
distance-homogeneity' X d \equiv total-invariance _{\mathcal{D}} d (homogeneity _{\mathcal{R}} ' X)

definition *distance-homogeneity* :: ('a, 'v) Election set \Rightarrow
('a, 'v) Election Distance \Rightarrow bool **where**
distance-homogeneity X d \equiv total-invariance _{\mathcal{D}} d (homogeneity _{\mathcal{R}} X)

Auxiliary Lemmas

lemma *rewrite-total-invariance _{\mathcal{D}}* :
fixes
d :: 'x Distance **and**
r :: 'x rel
shows total-invariance _{\mathcal{D}} d r = ($\forall (x, y) \in r. \forall (a, b) \in r. d a x = d b y$)
<proof>

lemma *rewrite-invariance _{\mathcal{D}}* :
fixes
d :: 'y Distance **and**
X :: 'x set **and**
Y :: 'y set **and**
 φ :: ('x, 'y) binary-fun
shows invariance _{\mathcal{D}} d X Y φ =
($\forall x \in X. \forall y \in Y. \forall z \in Y. d y z = d (\varphi x y) (\varphi x z)$)
<proof>

lemma *invar-dist-image*:
fixes
d :: 'y Distance **and**
G :: 'x monoid **and**
Y Y' :: 'y set **and**

```

   $\varphi :: ('x, 'y)$  binary-fun and
   $y :: 'y$  and
   $g :: 'x$ 
assumes
  invar-d: invarianceD  $d$  (carrier  $G$ )  $Y$   $\varphi$  and
  Y'-in-Y:  $Y' \subseteq Y$  and
  action- $\varphi$ : group-action  $G$   $Y$   $\varphi$  and
  g-carrier:  $g \in \text{carrier } G$  and
  y-in-Y:  $y \in Y$ 
shows  $d (\varphi g y) \text{ ' } (\varphi g) \text{ ' } Y' = d y \text{ ' } Y'$ 
<proof>

lemma swap-neutral: invarianceD swap (carrier bijectionAg)
  UNIV  $(\lambda \pi (A, q). (\pi \text{ ' } A, \text{rel-rename } \pi q))$ 
<proof>

end

```

4.2 Votewise Distance

```

theory Votewise-Distance
imports Social-Choice-Types/Norm
  Distance
begin

```

Votewise distances are a natural class of distances on elections which depend on the submitted votes in a simple and transparent manner. They are formed by using any distance d on individual orders and combining the components with a norm on \mathbb{R}^n .

4.2.1 Definition

```

fun votewise-distance :: 'a Vote Distance  $\Rightarrow$  Norm  $\Rightarrow$ 
  ('a, 'v :: linorder) Election Distance where
  votewise-distance  $d$   $n$  ( $A, V, p$ ) ( $A', V', p'$ ) =
    (if (finite  $V$ )  $\wedge V = V' \wedge (V \neq \{\}$   $\vee A = A')$ 
      then  $n$  (map2  $(\lambda q q'. d (A, q) (A', q'))$  (to-list  $V$   $p$ ) (to-list  $V'$   $p'$ ))
      else  $\infty$ )

```

4.2.2 Inference Rules

```

lemma symmetric-norm-inv-under-map-permute:
fixes
   $d :: 'a$  Vote Distance and
   $n ::$  Norm and
   $A A' :: 'a$  set and

```


$\varphi :: \text{nat} \Rightarrow \text{nat}$ **and**
 $p \ p' :: ('a \text{ Preference-Relation}) \text{ list}$
assumes
 $\text{perm}: \varphi \text{ permutes } \{0 \dots \text{length } p\}$ **and**
 $\text{len-eq}: \text{length } p = \text{length } p'$ **and**
 $\text{sym-n}: \text{symmetry } n$
shows $n \ (\text{map2 } (\lambda q \ q'. d \ (A, q) \ (A', q')) \ p \ p') =$
 $n \ (\text{map2 } (\lambda q \ q'. d \ (A, q) \ (A', q')) \ (\text{permute-list } \varphi \ p) \ (\text{permute-list } \varphi \ p'))$
 $\langle \text{proof} \rangle$

lemma *permute-invariant-under-map*:
fixes $l \ l' :: 'a \text{ list}$
assumes $l <^{\sim\sim} l'$
shows $\text{map } f \ l <^{\sim\sim} \text{map } f \ l'$
 $\langle \text{proof} \rangle$

lemma *linorder-rank-injective*:
fixes
 $V :: 'v :: \text{linorder set}$ **and**
 $v \ v' :: 'v$
assumes
 $v\text{-in-}V: v \in V$ **and**
 $v'\text{-in-}V: v' \in V$ **and**
 $v'\text{-neq-}v: v' \neq v$ **and**
 $\text{fin-}V: \text{finite } V$
shows $\text{card } \{x \in V. x < v\} \neq \text{card } \{x \in V. x < v'\}$
 $\langle \text{proof} \rangle$

lemma *permute-invariant-under-coinciding-funs*:
fixes
 $l :: 'v \text{ list}$ **and**
 $\pi_1 \ \pi_2 :: \text{nat} \Rightarrow \text{nat}$
assumes $\forall \ i < \text{length } l. \pi_1 \ i = \pi_2 \ i$
shows $\text{permute-list } \pi_1 \ l = \text{permute-list } \pi_2 \ l$
 $\langle \text{proof} \rangle$

lemma *symmetric-norm-imp-distance-anonymous*:
fixes
 $d :: 'a \text{ Vote Distance}$ **and**
 $n :: \text{Norm}$
assumes *symmetry* n
shows *distance-anonymity* $(\text{votewise-distance } d \ n)$
 $\langle \text{proof} \rangle$

lemma *neutral-dist-imp-neutral-votewise-dist*:
fixes
 $d :: 'a \text{ Vote Distance}$ **and**
 $n :: \text{Norm}$
defines $\text{vote-action} \equiv \lambda \ \pi \ (A, q). (\pi \text{ ' } A, \text{rel-rename } \pi \ q)$

```

assumes invarianceD d (carrier bijectionAG) UNIV vote-action
shows distance-neutrality well-formed-elections (votewise-distance d n)
<proof>

end

```

4.3 Consensus

```

theory Consensus
imports Social-Choice-Types/Voting-Symmetry
begin

```

An election consisting of a set of alternatives and preferential votes for each voter (a profile) is a consensus if it has an undisputed winner reflecting a certain concept of fairness in the society.

4.3.1 Definition

```

type-synonym ('a, 'v) Consensus = ('a, 'v) Election  $\Rightarrow$  bool

```

4.3.2 Consensus Conditions

Nonempty alternative set.

```

fun nonempty-setC :: ('a, 'v) Consensus where
  nonempty-setC (A, V, p) = (A  $\neq$  {})

```

Nonempty profile, i.e., nonempty voter set. Note that this is also true if $p(v) =$ holds for all voters v in V .

```

fun nonempty-profileC :: ('a, 'v) Consensus where
  nonempty-profileC (A, V, p) = (V  $\neq$  {})

```

Equal top ranked alternatives.

```

fun equal-topC' :: 'a  $\Rightarrow$  ('a, 'v) Consensus where
  equal-topC' a (A, V, p) = (a  $\in$  A  $\wedge$  ( $\forall$  v  $\in$  V. above (p v) a = {a}))

```

```

fun equal-topC :: ('a, 'v) Consensus where
  equal-topC c = ( $\exists$  a. equal-topC' a c)

```

Equal votes.

```

fun equal-voteC' :: 'a Preference-Relation  $\Rightarrow$  ('a, 'v) Consensus where
  equal-voteC' r (A, V, p) = ( $\forall$  v  $\in$  V. (p v) = r)

```

```

fun equal-voteC :: ('a, 'v) Consensus where
  equal-voteC c = ( $\exists$  r. equal-voteC' r c)

```

Unanimity condition.

fun *unanimity*_C :: ('a, 'v) Consensus **where**
*unanimity*_C c = (nonempty-set_C c ∧ nonempty-profile_C c ∧ equal-top_C c)

Strong unanimity condition.

fun *strong-unanimity*_C :: ('a, 'v) Consensus **where**
*strong-unanimity*_C c = (nonempty-set_C c ∧ nonempty-profile_C c ∧ equal-vote_C c)

4.3.3 Properties

definition *consensus-anonymity* :: ('a, 'v) Consensus ⇒ bool **where**
consensus-anonymity c ≡
 (∀ A V p π :: ('v ⇒ 'v).
 bij π ⟶
 (let (A', V', q) = (rename π (A, V, p)) in
 profile V A p ⟶ profile V' A' q
 ⟶ c (A, V, p) ⟶ c (A', V', q)))

fun *consensus-neutrality* :: ('a, 'v) Election set ⇒ ('a, 'v) Consensus ⇒ bool **where**
consensus-neutrality X c = is-symmetry c (Invariance (neutrality_R X))

4.3.4 Auxiliary Lemmas

lemma *cons-anon-conj*:
fixes c c' :: ('a, 'v) Consensus
assumes
 consensus-anonymity c **and**
 consensus-anonymity c'
shows *consensus-anonymity* (λ e. c e ∧ c' e)
 ⟨proof⟩

theorem *cons-conjunction-invariant*:
fixes
 C :: ('a, 'v) Consensus set **and**
 rel :: ('a, 'v) Election rel
defines C ≡ λ E. ∀ C' ∈ C. C' E
assumes ∀ C'. C' ∈ C ⟶ is-symmetry C' (Invariance rel)
shows is-symmetry C (Invariance rel)
 ⟨proof⟩

lemma *cons-anon-invariant*:
fixes
 c :: ('a, 'v) Consensus **and**
 A A' :: 'a set **and**
 V V' :: 'v set **and**
 p q :: ('a, 'v) Profile **and**
 π :: 'v ⇒ 'v
assumes
 anon: *consensus-anonymity* c **and**

bij- π : *bij* π **and**
prof- p : *profile* V A p **and**
renamed: *rename* π $(A, V, p) = (A', V', q)$ **and**
cond- c : c (A, V, p)
shows c (A', V', q)
 $\langle \text{proof} \rangle$

lemma *ex-anon-cons-imp-cons-anonymous:*
fixes
 $b :: ('a, 'v)$ *Consensus* **and**
 $b' :: 'b \Rightarrow ('a, 'v)$ *Consensus*
assumes
general-cond- b : $b = (\lambda E. \exists x. b' x E)$ **and**
all-cond-anon: $\forall x. \text{consensus-anonymity } (b' x)$
shows *consensus-anonymity* b
 $\langle \text{proof} \rangle$

4.3.5 Theorems

Anonymity

lemma *nonempty-set-cons-anonymous:* *consensus-anonymity nonempty-set_C*
 $\langle \text{proof} \rangle$

lemma *nonempty-profile-cons-anonymous:* *consensus-anonymity nonempty-profile_C*
 $\langle \text{proof} \rangle$

lemma *equal-top-cons'-anonymous:*
fixes $a :: 'a$
shows *consensus-anonymity* $(\text{equal-top}_C 'a)$
 $\langle \text{proof} \rangle$

lemma *eq-top-cons-anon:* *consensus-anonymity equal-top_C*
 $\langle \text{proof} \rangle$

lemma *eq-vote-cons'-anonymous:*
fixes $r :: 'a$ *Preference-Relation*
shows *consensus-anonymity* $(\text{equal-vote}_C 'r)$
 $\langle \text{proof} \rangle$

lemma *eq-vote-cons-anonymous:* *consensus-anonymity equal-vote_C*
 $\langle \text{proof} \rangle$

Neutrality

lemma *nonempty-set_C-neutral:* *consensus-neutrality well-formed-elections nonempty-set_C*
 $\langle \text{proof} \rangle$

lemma *nonempty-profile_C-neutral:* *consensus-neutrality well-formed-elections nonempty-profile_C*
 $\langle \text{proof} \rangle$

```

lemma equal-voteC-neutral: consensus-neutrality well-formed-elections equal-voteC
  <proof>

lemma strong-unanimityC-neutral: consensus-neutrality
  well-formed-elections strong-unanimityC
  <proof>

end

```

4.4 Electoral Module

```

theory Electoral-Module
  imports Social-Choice-Types/Property-Interpretations
begin

```

Electoral modules are the principal component type of the composable modules voting framework, as they are a generalization of voting rules in the sense of social choice functions. These are only the types used for electoral modules. Further restrictions are encompassed by the electoral-module predicate.

An electoral module does not need to make final decisions for all alternatives, but can instead defer the decision for some or all of them to other modules. Hence, electoral modules partition the received (possibly empty) set of alternatives into elected, rejected and deferred alternatives. In particular, any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives. Just like a voting rule, an electoral module also receives a profile which holds the voters preferences, which, unlike a voting rule, consider only the (sub-)set of alternatives that the module receives.

4.4.1 Definition

An electoral module maps an election to a result. To enable currying, the Election type is not used here because that would require tuples.

```

type-synonym ('a, 'v, 'r) Electoral-Module = 'v set  $\Rightarrow$  'a set  $\Rightarrow$ 
  ('a, 'v) Profile  $\Rightarrow$  'r

```

```

fun funE :: ('v set  $\Rightarrow$  'a set  $\Rightarrow$  ('a, 'v) Profile  $\Rightarrow$  'r)  $\Rightarrow$ 
  (('a, 'v) Election  $\Rightarrow$  'r) where
  funE m = ( $\lambda$  E. m (voters- $\mathcal{E}$  E) (alternatives- $\mathcal{E}$  E) (profile- $\mathcal{E}$  E))

```

The next three functions take an electoral module and turn it into a function only outputting the elect, reject, or defer set respectively.

abbreviation $elect :: ('a, 'v, 'r \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'r \text{ set}$ **where**
 $elect\ m\ V\ A\ p \equiv elect\text{-}r\ (m\ V\ A\ p)$

abbreviation $reject :: ('a, 'v, 'r \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'r \text{ set}$ **where**
 $reject\ m\ V\ A\ p \equiv reject\text{-}r\ (m\ V\ A\ p)$

abbreviation $defer :: ('a, 'v, 'r \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'r \text{ set}$ **where**
 $defer\ m\ V\ A\ p \equiv defer\text{-}r\ (m\ V\ A\ p)$

4.4.2 Auxiliary Definitions

Electoral modules partition a given set of alternatives A into a set of elected alternatives e , a set of rejected alternatives r , and a set of deferred alternatives d , using a profile. e , r , and d partition A . Electoral modules can be used as voting rules. They can also be composed in multiple structures to create more complex electoral modules.

fun (in *result*) $electoral\text{-}module :: ('a, 'v, ('r \text{ Result})) \text{ Electoral-Module} \Rightarrow bool$ **where**
 $electoral\text{-}module\ m = (\forall\ A\ V\ p. \text{profile}\ V\ A\ p \longrightarrow well\text{-}formed\ A\ (m\ V\ A\ p))$

fun $voters\text{-}determine\text{-}election :: ('a, 'v, ('r \text{ Result})) \text{ Electoral-Module} \Rightarrow bool$ **where**
 $voters\text{-}determine\text{-}election\ m =$
 $(\forall\ A\ V\ p\ p'. (\forall\ v \in V. p\ v = p'\ v) \longrightarrow m\ V\ A\ p = m\ V\ A\ p')$

lemma (in *result*) $electoral\text{-}modI$:
fixes $m :: ('a, 'v, ('r \text{ Result})) \text{ Electoral-Module}$
assumes $\forall\ A\ V\ p. \text{profile}\ V\ A\ p \longrightarrow well\text{-}formed\ A\ (m\ V\ A\ p)$
shows $electoral\text{-}module\ m$
 $\langle proof \rangle$

4.4.3 Properties

We only require voting rules to behave a specific way on admissible elections, i.e., elections that are valid profiles (= votes are linear orders on the alternatives). Note that we do not assume finiteness of voter or alternative sets by default.

Anonymity

An electoral module is anonymous iff the result is invariant under renamings of voters, i.e., any permutation of the voter set that does not change the preferences leads to an identical result.

definition (*in result*) *anonymity* :: ('a, 'v, ('r Result)) *Electoral-Module* \Rightarrow *bool* **where**
anonymity *m* \equiv
electoral-module *m* \wedge
 $(\forall A V p \pi :: ('v \Rightarrow 'v).$
 $\text{bij } \pi \longrightarrow (\text{let } (A', V', q) = (\text{rename } \pi (A, V, p)) \text{ in}$
 $\text{profile } V A p \wedge \text{profile } V' A' q \longrightarrow m V A p = m V' A' q))$

Anonymity can alternatively be described as invariance under the voter permutation group acting on elections via the rename function.

fun *anonymity-in* :: ('a, 'v) *Election set* \Rightarrow ('a, 'v, 'r) *Electoral-Module* \Rightarrow *bool* **where**
anonymity-in *X m* = *is-symmetry* (*fun_E* *m*) (*Invariance* (*anonymity_R* *X*))

fun *anonymity'* :: ('a, 'v, 'r) *Electoral-Module* \Rightarrow *bool* **where**
anonymity' *m* = *anonymity-in* *well-formed-elections m*

Homogeneity

A voting rule is homogeneous if copying an election does not change the result. For ordered voter types and finite elections, we use the notion of copying ballot lists to define copying an election. The more general definition of homogeneity for unordered voter types already implies anonymity.

fun *homogeneity-in* :: ('a, 'v) *Election set* \Rightarrow
('a, 'v, 'r Result) *Electoral-Module* \Rightarrow *bool* **where**
homogeneity-in *X m* = *is-symmetry* (*fun_E* *m*) (*Invariance* (*homogeneity_R* *X*))

— This does not require any specific behaviour on infinite voter sets ... It might make sense to extend the definition to that case somehow.

fun *homogeneity* :: ('a, 'v, 'b Result) *Electoral-Module* \Rightarrow *bool* **where**
homogeneity *m* = *homogeneity-in* *finite-elections- \mathcal{V} m*

fun *homogeneity'-in* :: ('a, 'v :: *linorder*) *Election set* \Rightarrow
('a, 'v, 'b Result) *Electoral-Module* \Rightarrow *bool* **where**
homogeneity'-in *X m* = *is-symmetry* (*fun_E* *m*) (*Invariance* (*homogeneity_R* ' *X*))

fun *homogeneity'* :: ('a, 'v :: *linorder*, 'b Result) *Electoral-Module* \Rightarrow *bool* **where**
homogeneity' *m* = *homogeneity'-in* *finite-elections- \mathcal{V} m*

lemma *hom-imp-anon*:

fixes

X :: ('a, 'v) *Election set* **and**

m :: ('a, 'v, ('r Result)) *Electoral-Module*

assumes

homogeneity-in *X m* **and**

$\forall E \in X. \text{finite } (\text{voters-}\mathcal{E} \ E)$

shows *anonymity-in* *X m*

<proof>

Neutrality

Neutrality is equivariance under consistent renaming of candidates in the candidate set and election results.

```
fun (in result-properties) neutrality-in :: ('a, 'v) Election set  $\Rightarrow$ 
  ('a, 'v, 'b Result) Electoral-Module  $\Rightarrow$  bool where
  neutrality-in X m =
    is-symmetry (funE m) (action-induced-equivariance (carrier bijectionAG) X
      (φ-neutral X) (result-action ψ))

fun (in result-properties) neutrality :: ('a, 'v, 'b Result) Electoral-Module  $\Rightarrow$ 
  bool where
  neutrality m = neutrality-in well-formed-elections m
```

4.4.4 Social-Welfare Properties

Reversal Symmetry

A social welfare rule is reversal symmetric if reversing all voters' preferences reverses the result rankings as well.

```
definition reversal-symmetry-in :: ('a, 'v) Election set  $\Rightarrow$ 
  ('a, 'v, 'a rel Result) Electoral-Module  $\Rightarrow$  bool where
  reversal-symmetry-in X m  $\equiv$ 
    is-symmetry (funE m) (action-induced-equivariance (carrier reversalG) X
      (φ-reverse X) (result-action ψ-reverse))
```

```
fun reversal-symmetry :: ('a, 'v, 'a rel Result) Electoral-Module  $\Rightarrow$  bool where
  reversal-symmetry m = reversal-symmetry-in well-formed-elections m
```

4.4.5 Social-Choice Modules

The following results require electoral modules to return social choice results, i.e., sets of elected, rejected and deferred alternatives. In order to export code, we use the hack provided by Locale-Code.

"defers n" is true for all electoral modules that defer exactly n alternatives, whenever there are n or more alternatives.

```
definition defers :: nat  $\Rightarrow$  ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  bool where
  defers n m  $\equiv$ 
    SCF-result.electoral-module m  $\wedge$ 
    ( $\forall$  A V p. (card A  $\geq$  n  $\wedge$  finite A  $\wedge$  profile V A p)
       $\longrightarrow$  card (defer m V A p) = n)
```

"rejects n" is true for all electoral modules that reject exactly n alternatives, whenever there are n or more alternatives.

```
definition rejects :: nat  $\Rightarrow$  ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  bool where
  rejects n m  $\equiv$ 
```


$$\begin{aligned}
& \text{SCF-result.electoral-module } m \wedge \\
& (\forall A V p. (\text{card } A \geq n \wedge \text{finite } A \wedge \text{profile } V A p) \\
& \longrightarrow \text{card } (\text{reject } m V A p) = n)
\end{aligned}$$

As opposed to "rejects", "eliminates" allows to stop rejecting if no alternatives were to remain.

definition *eliminates* :: $\text{nat} \Rightarrow ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**
eliminates $n m \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $(\forall A V p. (\text{card } A > n \wedge \text{profile } V A p) \longrightarrow \text{card } (\text{reject } m V A p) = n)$

"elects n " is true for all electoral modules that elect exactly n alternatives, whenever there are n or more alternatives.

definition *elects* :: $\text{nat} \Rightarrow ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**
elects $n m \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $(\forall A V p. (\text{card } A \geq n \wedge \text{profile } V A p) \longrightarrow \text{card } (\text{elect } m V A p) = n)$

An electoral module is independent of an alternative a iff a 's ranking does not influence the outcome.

definition *indep-of-alt* :: $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{bool})$ **where**
indep-of-alt $m V A a \equiv$
 $\text{SCF-result.electoral-module } m$
 $\wedge (\forall p q. \text{equiv-prof-except-}a V A p q a \longrightarrow m V A p = m V A q)$

definition *unique-winner-if-profile-non-empty* :: $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool})$ **where**
unique-winner-if-profile-non-empty $m \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $(\forall A V p. (A \neq \{\} \wedge V \neq \{\} \wedge \text{profile } V A p) \longrightarrow$
 $(\exists a \in A. m V A p = (\{a\}, A - \{a\}, \{\})))$

4.4.6 Equivalence Definitions

definition *prof-contains-result* :: $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'a \Rightarrow \text{bool})$ **where**
prof-contains-result $m V A p q a \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $\text{profile } V A p \wedge \text{profile } V A q \wedge a \in A \wedge$
 $(a \in \text{elect } m V A p \longrightarrow a \in \text{elect } m V A q) \wedge$
 $(a \in \text{reject } m V A p \longrightarrow a \in \text{reject } m V A q) \wedge$
 $(a \in \text{defer } m V A p \longrightarrow a \in \text{defer } m V A q)$

definition *prof-leq-result* :: $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'a \Rightarrow \text{bool})$ **where**
prof-leq-result $m V A p q a \equiv$
 $\text{SCF-result.electoral-module } m \wedge$

$profile\ V\ A\ p \wedge profile\ V\ A\ q \wedge a \in A \wedge$
 $(a \in reject\ m\ V\ A\ p \longrightarrow a \in reject\ m\ V\ A\ q) \wedge$
 $(a \in defer\ m\ V\ A\ p \longrightarrow a \notin elect\ m\ V\ A\ q)$

definition $prof\text{-}geq\text{-}result :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow 'v\ set \Rightarrow$
 $'a\ set \Rightarrow ('a, 'v)\ Profile \Rightarrow ('a, 'v)\ Profile \Rightarrow 'a \Rightarrow bool$ **where**
 $prof\text{-}geq\text{-}result\ m\ V\ A\ p\ q\ a \equiv$
 $SCF\text{-}result.electoral\text{-}module\ m \wedge$
 $profile\ V\ A\ p \wedge profile\ V\ A\ q \wedge a \in A \wedge$
 $(a \in elect\ m\ V\ A\ p \longrightarrow a \in elect\ m\ V\ A\ q) \wedge$
 $(a \in defer\ m\ V\ A\ p \longrightarrow a \notin reject\ m\ V\ A\ q)$

definition $mod\text{-}contains\text{-}result :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
 $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow 'v\ set \Rightarrow 'a\ set \Rightarrow$
 $('a, 'v)\ Profile \Rightarrow 'a \Rightarrow bool$ **where**
 $mod\text{-}contains\text{-}result\ m\ n\ V\ A\ p\ a \equiv$
 $SCF\text{-}result.electoral\text{-}module\ m \wedge$
 $SCF\text{-}result.electoral\text{-}module\ n \wedge$
 $profile\ V\ A\ p \wedge a \in A \wedge$
 $(a \in elect\ m\ V\ A\ p \longrightarrow a \in elect\ n\ V\ A\ p) \wedge$
 $(a \in reject\ m\ V\ A\ p \longrightarrow a \in reject\ n\ V\ A\ p) \wedge$
 $(a \in defer\ m\ V\ A\ p \longrightarrow a \in defer\ n\ V\ A\ p)$

definition $mod\text{-}contains\text{-}result\text{-}sym :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
 $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow 'v\ set \Rightarrow 'a\ set \Rightarrow$
 $('a, 'v)\ Profile \Rightarrow 'a \Rightarrow bool$ **where**
 $mod\text{-}contains\text{-}result\text{-}sym\ m\ n\ V\ A\ p\ a \equiv$
 $SCF\text{-}result.electoral\text{-}module\ m \wedge$
 $SCF\text{-}result.electoral\text{-}module\ n \wedge$
 $profile\ V\ A\ p \wedge a \in A \wedge$
 $(a \in elect\ m\ V\ A\ p \longleftrightarrow a \in elect\ n\ V\ A\ p) \wedge$
 $(a \in reject\ m\ V\ A\ p \longleftrightarrow a \in reject\ n\ V\ A\ p) \wedge$
 $(a \in defer\ m\ V\ A\ p \longleftrightarrow a \in defer\ n\ V\ A\ p)$

4.4.7 Auxiliary Lemmas

lemma *elect-rej-def-combination:*

fixes

$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**

$V :: 'v\ set$ **and**

$A :: 'a\ set$ **and**

$p :: ('a, 'v)\ Profile$ **and**

$e\ r\ d :: 'a\ set$

assumes

$elect\ m\ V\ A\ p = e$ **and**

$reject\ m\ V\ A\ p = r$ **and**

$defer\ m\ V\ A\ p = d$

shows $m\ V\ A\ p = (e, r, d)$

<proof>

lemma *par-comp-result-sound*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$\text{SCF-result.electoral-module } m$ **and**

$\text{profile } V \ A \ p$

shows $\text{well-formed-SCF } A \ (m \ V \ A \ p)$

$\langle \text{proof} \rangle$

lemma *result-presv-alts*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$\text{SCF-result.electoral-module } m$ **and**

$\text{profile } V \ A \ p$

shows $(\text{elect } m \ V \ A \ p) \cup (\text{reject } m \ V \ A \ p) \cup (\text{defer } m \ V \ A \ p) = A$

$\langle \text{proof} \rangle$

lemma *result-disj*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$ **and**

$V :: 'v \text{ set}$

assumes

$\text{SCF-result.electoral-module } m$ **and**

$\text{profile } V \ A \ p$

shows

$(\text{elect } m \ V \ A \ p) \cap (\text{reject } m \ V \ A \ p) = \{\} \wedge$
 $(\text{elect } m \ V \ A \ p) \cap (\text{defer } m \ V \ A \ p) = \{\} \wedge$
 $(\text{reject } m \ V \ A \ p) \cap (\text{defer } m \ V \ A \ p) = \{\}$

$\langle \text{proof} \rangle$

lemma *elect-in-alts*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$\text{SCF-result.electoral-module } m$ **and**

$\text{profile } V \ A \ p$

shows $\text{elect } m \ V \ A \ p \subseteq A$

$\langle \text{proof} \rangle$

lemma *reject-in-alts*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$SCF\text{-result.electoral-module } m$ **and**

$profile\ V\ A\ p$

shows $reject\ m\ V\ A\ p \subseteq A$

<proof>

lemma *defer-in-alts*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$SCF\text{-result.electoral-module } m$ **and**

$profile\ V\ A\ p$

shows $defer\ m\ V\ A\ p \subseteq A$

<proof>

lemma *def-presv-prof*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$SCF\text{-result.electoral-module } m$ **and**

$profile\ V\ A\ p$

shows $let\ new\text{-}A = defer\ m\ V\ A\ p\ in\ profile\ V\ new\text{-}A\ (limit\text{-}profile\ new\text{-}A\ p)$

<proof>

An electoral module can never reject, defer or elect more than $|A|$ alternatives.

lemma *upper-card-bounds-for-result*:

fixes

$m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

assumes

$SCF\text{-result.electoral-module } m$ **and**

$profile\ V\ A\ p$ **and**

$finite\ A$

shows

$upper\text{-}card\text{-}bound\text{-}for\text{-}elect: card (elect\ m\ V\ A\ p) \leq card\ A$ **and**
 $upper\text{-}card\text{-}bound\text{-}for\text{-}reject: card (reject\ m\ V\ A\ p) \leq card\ A$ **and**
 $upper\text{-}card\text{-}bound\text{-}for\text{-}defer: card (defer\ m\ V\ A\ p) \leq card\ A$
 <proof>

lemma *reject-not-elected-or-deferred:*

fixes
 $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $profile\ V\ A\ p$
shows $reject\ m\ V\ A\ p = A - (elect\ m\ V\ A\ p) - (defer\ m\ V\ A\ p)$
 <proof>

lemma *elec-and-def-not-rej:*

fixes
 $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $profile\ V\ A\ p$
shows $elect\ m\ V\ A\ p \cup defer\ m\ V\ A\ p = A - (reject\ m\ V\ A\ p)$
 <proof>

lemma *defer-not-elec-or-rej:*

fixes
 $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $profile\ V\ A\ p$
shows $defer\ m\ V\ A\ p = A - (elect\ m\ V\ A\ p) - (reject\ m\ V\ A\ p)$
 <proof>

lemma *electoral-mod-defer-elem:*

fixes
 $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$ **and**
 $a :: 'a$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**

profile $V A p$ **and**
 $a \in A$ **and**
 $a \notin \text{elect } m V A p$ **and**
 $a \notin \text{reject } m V A p$
shows $a \in \text{defer } m V A p$
 $\langle \text{proof} \rangle$

lemma *mod-contains-result-comm:*
fixes
 $m n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes *mod-contains-result* $m n V A p a$
shows *mod-contains-result* $n m V A p a$
 $\langle \text{proof} \rangle$

lemma *not-rej-imp-elec-or-defer:*
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
SCF-result.electoral-module m **and**
profile $V A p$ **and**
 $a \in A$ **and**
 $a \notin \text{reject } m V A p$
shows $a \in \text{elect } m V A p \vee a \in \text{defer } m V A p$
 $\langle \text{proof} \rangle$

lemma *single-elim-imp-red-def-set:*
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
eliminates 1 m **and**
 $\text{card } A > 1$ **and**
profile $V A p$
shows *defer* $m V A p \subset A$
 $\langle \text{proof} \rangle$

lemma *eq-alts-in-profs-imp-eq-results:*
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**

$A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p \ q :: ('a, 'v) \text{ Profile}$
assumes
 $eq: \forall a \in A. \text{prof-contains-result } m \ V \ A \ p \ q \ a$ **and**
 $mod\text{-}m: \text{SCF-result.electoral-module } m$ **and**
 $prof\text{-}p: \text{profile } V \ A \ p$ **and**
 $prof\text{-}q: \text{profile } V \ A \ q$
shows $m \ V \ A \ p = m \ V \ A \ q$
 $\langle proof \rangle$

lemma $eq\text{-}def\text{-}and\text{-}elect\text{-}imp\text{-}eq$:
fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p \ q :: ('a, 'v) \text{ Profile}$
assumes
 $mod\text{-}m: \text{SCF-result.electoral-module } m$ **and**
 $mod\text{-}n: \text{SCF-result.electoral-module } n$ **and**
 $fin\text{-}p: \text{profile } V \ A \ p$ **and**
 $fin\text{-}q: \text{profile } V \ A \ q$ **and**
 $elec\text{-}eq: \text{elect } m \ V \ A \ p = \text{elect } n \ V \ A \ q$ **and**
 $def\text{-}eq: \text{defer } m \ V \ A \ p = \text{defer } n \ V \ A \ q$
shows $m \ V \ A \ p = n \ V \ A \ q$
 $\langle proof \rangle$

4.4.8 Non-Blocking

An electoral module is non-blocking iff this module never rejects all alternatives.

definition $non\text{-}blocking :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**
 $non\text{-}blocking \ m \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $(\forall A \ V \ p. ((A \neq \{\}) \wedge \text{finite } A \wedge \text{profile } V \ A \ p) \longrightarrow \text{reject } m \ V \ A \ p \neq A)$

4.4.9 Electing

An electoral module is electing iff it always elects at least one alternative.

definition $electing :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**
 $electing \ m \equiv$
 $\text{SCF-result.electoral-module } m \wedge$
 $(\forall A \ V \ p. (A \neq \{\}) \wedge \text{finite } A \wedge \text{profile } V \ A \ p) \longrightarrow \text{elect } m \ V \ A \ p \neq \{\})$

lemma $electing\text{-}for\text{-}only\text{-}alt$:
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
 $\text{one-alt: card } A = 1$ **and**
 $\text{electing: electing } m$ **and**
 $\text{prof: profile } V \ A \ p$
shows $\text{elect } m \ V \ A \ p = A$
 $\langle \text{proof} \rangle$

theorem *electing-imp-non-blocking*:
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes *electing* m
shows *non-blocking* m
 $\langle \text{proof} \rangle$

4.4.10 Properties

An electoral module is non-electing iff it never elects an alternative.

definition *non-electing* $:: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**
 $\text{non-electing } m \equiv$
 $\text{SCF-result.electoral-module } m$
 $\wedge (\forall \ A \ V \ p. \text{profile } V \ A \ p \longrightarrow \text{elect } m \ V \ A \ p = \{\})$

lemma *single-rej-decr-def-card*:
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
 $\text{rejecting: rejects } 1 \ m$ **and**
 $\text{non-electing: non-electing } m$ **and**
 $\text{f-prof: finite-profile } V \ A \ p$
shows $\text{card } (\text{defer } m \ V \ A \ p) = \text{card } A - 1$
 $\langle \text{proof} \rangle$

lemma *single-elim-decr-def-card'*:
fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
 $\text{eliminating: eliminates } 1 \ m$ **and**
 $\text{non-electing: non-electing } m$ **and**
 $\text{not-empty: card } A > 1$ **and**
 $\text{prof-p: profile } V \ A \ p$
shows $\text{card } (\text{defer } m \ V \ A \ p) = \text{card } A - 1$
 $\langle \text{proof} \rangle$

An electoral module is defer-deciding iff this module chooses exactly 1 alternative to defer and rejects any other alternative. Note that ‘rejects n-1 m’ can be omitted due to the well-formedness property.

definition *defer-deciding* :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow bool **where**
defer-deciding m \equiv
SCF-result.electoral-module m \wedge *non-electing* m \wedge *defers* 1 m

An electoral module decrements iff this module rejects at least one alternative whenever possible ($|A| > 1$).

definition *decrementing* :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow bool **where**
decrementing m \equiv
SCF-result.electoral-module m \wedge
 $(\forall A V p. \text{profile } V A p \wedge \text{card } A > 1 \longrightarrow \text{card } (\text{reject } m V A p) \geq 1)$

definition *defer-condorcet-consistency* :: ('a, 'v, 'a Result)
Electoral-Module \Rightarrow bool **where**
defer-condorcet-consistency m \equiv
SCF-result.electoral-module m \wedge
 $(\forall A V p a. \text{condorcet-winner } V A p a \longrightarrow$
 $(m V A p = (\{\}, A - (\text{defer } m V A p), \{d \in A. \text{condorcet-winner } V A p d\})))$

definition *condorcet-compatibility* :: ('a, 'v, 'a Result)
Electoral-Module \Rightarrow bool **where**
condorcet-compatibility m \equiv
SCF-result.electoral-module m \wedge
 $(\forall A V p a. \text{condorcet-winner } V A p a \longrightarrow$
 $(a \notin \text{reject } m V A p \wedge$
 $(\forall b. \neg \text{condorcet-winner } V A p b \longrightarrow b \notin \text{elect } m V A p) \wedge$
 $(a \in \text{elect } m V A p \longrightarrow$
 $(\forall b \in A. \neg \text{condorcet-winner } V A p b \longrightarrow b \in \text{reject } m V A p))))$

An electoral module is defer-monotone iff, when a deferred alternative is lifted, this alternative remains deferred.

definition *defer-monotonicity* :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow bool **where**
defer-monotonicity m \equiv
SCF-result.electoral-module m \wedge
 $(\forall A V p q a.$
 $(a \in \text{defer } m V A p \wedge \text{lifted } V A p q a) \longrightarrow a \in \text{defer } m V A q)$

An electoral module is defer-lift-invariant iff lifting a deferred alternative does not affect the outcome.

definition *defer-lift-invariance* :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow bool **where**
defer-lift-invariance m \equiv
SCF-result.electoral-module m \wedge
 $(\forall A V p q a. (a \in (\text{defer } m V A p) \wedge \text{lifted } V A p q a)$
 $\longrightarrow m V A p = m V A q)$

fun *dli-rel* :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow ('a, 'v) Election rel **where**

$$dli\text{-}rel\ m = \{((A, V, p), (A, V, q)) \mid A \ V\ p\ q. (\exists\ a \in defer\ m\ V\ A\ p. lifted\ V\ A\ p\ q\ a)\}$$

lemma *rewrite-dli-as-invariance*:

fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$

shows

$$\begin{aligned} defer\text{-}lift\text{-}invariance\ m = \\ (SCF\text{-}result.electoral\text{-}module\ m \\ \wedge (is\text{-}symmetry\ (fun_{\mathcal{E}}\ m)\ (Invariance\ (dli\text{-}rel\ m)))) \end{aligned}$$

<proof>

Two electoral modules are disjoint-compatible if they only make decisions over disjoint sets of alternatives. Electoral modules reject alternatives for which they make no decision.

definition *disjoint-compatibility* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$

$(('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow bool\ \mathbf{where}$

disjoint-compatibility $m\ n \equiv$

$$\begin{aligned} SCF\text{-}result.electoral\text{-}module\ m \wedge SCF\text{-}result.electoral\text{-}module\ n \wedge \\ (\forall\ V. \\ (\forall\ A. \\ (\exists\ B \subseteq A. \\ (\forall\ a \in B. indep\text{-}of\text{-}alt\ m\ V\ A\ a \wedge \\ (\forall\ p. profile\ V\ A\ p \longrightarrow a \in reject\ m\ V\ A\ p)) \wedge \\ (\forall\ a \in A - B. indep\text{-}of\text{-}alt\ n\ V\ A\ a \wedge \\ (\forall\ p. profile\ V\ A\ p \longrightarrow a \in reject\ n\ V\ A\ p)))))) \end{aligned}$$

Lifting an elected alternative a from an invariant-monotone electoral module either does not change the elect set, or makes a the only elected alternative.

definition *invariant-monotonicity* :: $('a, 'v, 'a\ Result)$

$Electoral\text{-}Module \Rightarrow bool\ \mathbf{where}$

invariant-monotonicity $m \equiv$

$$\begin{aligned} SCF\text{-}result.electoral\text{-}module\ m \wedge \\ (\forall\ A\ V\ p\ q\ a. (a \in elect\ m\ V\ A\ p \wedge lifted\ V\ A\ p\ q\ a) \longrightarrow \\ (elect\ m\ V\ A\ q = elect\ m\ V\ A\ p \vee elect\ m\ V\ A\ q = \{a\})) \end{aligned}$$

Lifting a deferred alternative a from a defer-invariant-monotone electoral module either does not change the defer set, or makes a the only deferred alternative.

definition *defer-invariant-monotonicity* :: $('a, 'v, 'a\ Result)$

$Electoral\text{-}Module \Rightarrow bool\ \mathbf{where}$

defer-invariant-monotonicity $m \equiv$

$$\begin{aligned} SCF\text{-}result.electoral\text{-}module\ m \wedge non\text{-}electing\ m \wedge \\ (\forall\ A\ V\ p\ q\ a. (a \in defer\ m\ V\ A\ p \wedge lifted\ V\ A\ p\ q\ a) \longrightarrow \\ (defer\ m\ V\ A\ q = defer\ m\ V\ A\ p \vee defer\ m\ V\ A\ q = \{a\})) \end{aligned}$$

4.4.11 Inference Rules

lemma *ccomp-and-dd-imp-def-only-winner*:

fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
 $ccomp: \text{condorcet-compatibility } m$ **and**
 $dd: \text{defer-deciding } m$ **and**
 $winner: \text{condorcet-winner } V \ A \ p \ a$
shows $\text{defer } m \ V \ A \ p = \{a\}$
 $\langle \text{proof} \rangle$

theorem $ccomp\text{-and-}dd\text{-imp-}dcc[simp]:$
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 $ccomp: \text{condorcet-compatibility } m$ **and**
 $dd: \text{defer-deciding } m$
shows $\text{defer-condorcet-consistency } m$
 $\langle \text{proof} \rangle$

If m and n are disjoint compatible, so are n and m .

theorem $disj\text{-compat-comm}[simp]:$
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes $\text{disjoint-compatibility } m \ n$
shows $\text{disjoint-compatibility } n \ m$
 $\langle \text{proof} \rangle$

Every electoral module which is defer-lift-invariant is also defer-monotone.

theorem $dl\text{-inv-imp-def-mono}[simp]:$
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes $\text{defer-lift-invariance } m$
shows $\text{defer-monotonicity } m$
 $\langle \text{proof} \rangle$

4.4.12 Social-Choice Properties

Condorcet Consistency

definition $\text{condorcet-consistency} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 bool **where**
 $\text{condorcet-consistency } m \equiv$
 $SCF\text{-result.electoral-module } m \wedge$
 $(\forall \ A \ V \ p \ a. \text{condorcet-winner } V \ A \ p \ a \longrightarrow$
 $(m \ V \ A \ p = (\{e \in A. \text{condorcet-winner } V \ A \ p \ e\}, A - (\text{elect } m \ V \ A \ p), \{\})))$

lemma $\text{condorcet-consistency-equiv}:$
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
shows $\text{condorcet-consistency } m =$

```

      (SCF-result.electoral-module m  $\wedge$ 
       ( $\forall$  A V p a. condorcet-winner V A p a  $\longrightarrow$ 
        (m V A p = ({a}, A - (elect m V A p), {}))))
    <proof>

```

```

lemma condorcet-consistency-equiv':
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  shows condorcet-consistency m =
    (SCF-result.electoral-module m  $\wedge$ 
     ( $\forall$  A V p a.
      condorcet-winner V A p a  $\longrightarrow$  m V A p = ({a}, A - {a}, {})))
  <proof>

```

(Weak) Monotonicity

An electoral module is monotone iff when an elected alternative is lifted, this alternative remains elected.

```

definition monotonicity :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  bool where
  monotonicity m  $\equiv$ 
    SCF-result.electoral-module m  $\wedge$ 
    ( $\forall$  A V p q a. a  $\in$  elect m V A p  $\wedge$  lifted V A p q a  $\longrightarrow$  a  $\in$  elect m V A q)

end

```

4.5 Electoral Module on Election Quotients

```

theory Quotient-Module
  imports Quotients/Relation-Quotients
          Electoral-Module
begin

```

```

lemma invariance-is-congruence:
  fixes
    m :: ('a, 'v, 'r) Electoral-Module and
    r :: ('a, 'v) Election rel
  shows is-symmetry (funE m) (Invariance r) = funE m respects r
  <proof>

```

```

lemma invariance-is-congruence':
  fixes
    f :: 'x  $\Rightarrow$  'y and
    r :: 'x rel
  shows is-symmetry f (Invariance r) = f respects r
  <proof>

```

```

theorem pass-to-election-quotient:

```

```

fixes
   $m :: ('a, 'v, 'r)$  Electoral-Module and
   $r :: ('a, 'v)$  Election rel and
   $X :: ('a, 'v)$  Election set
assumes
   $\text{equiv } X \ r$  and
   $\text{is-symmetry } (\text{fun}_{\mathcal{E}} \ m)$  (Invariance  $r$ )
shows  $\forall A \in X \ // \ r. \forall E \in A. \pi_{\mathcal{Q}} (\text{fun}_{\mathcal{E}} \ m) \ A = \text{fun}_{\mathcal{E}} \ m \ E$ 
  <proof>
end

```

4.6 Evaluation Function

```

theory Evaluation-Function
  imports Social-Choice-Types/Profile
begin

```

This is the evaluation function. From a set of currently eligible alternatives, the evaluation function computes a numerical value that is then to be used for further (s)election, e.g., by the elimination module.

4.6.1 Definition

```

type-synonym  $('a, 'v)$  Evaluation-Function =
   $'v \text{ set} \Rightarrow 'a \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow \text{enat}$ 

```

4.6.2 Property

An Evaluation function is a Condorcet-rating iff the following holds: If a Condorcet Winner w exists, w and only w has the highest value.

```

definition condorcet-rating ::  $('a, 'v)$  Evaluation-Function  $\Rightarrow$  bool where
  condorcet-rating  $f \equiv$ 
     $\forall A \ V \ p \ w. \text{condorcet-winner } V \ A \ p \ w \longrightarrow$ 
     $(\forall l \in A. l \neq w \longrightarrow f \ V \ l \ A \ p < f \ V \ w \ A \ p)$ 

```

An Evaluation function is dependent only on the participating voters iff it is invariant under profile changes that only impact non-voters.

```

fun voters-determine-evaluation ::  $('a, 'v)$  Evaluation-Function  $\Rightarrow$  bool where
  voters-determine-evaluation  $f =$ 
     $(\forall A \ V \ p \ p'. (\forall v \in V. p \ v = p' \ v) \longrightarrow (\forall a \in A. f \ V \ a \ A \ p = f \ V \ a \ A \ p'))$ 

```

4.6.3 Theorems

If e is Condorcet-rating, the following holds: If a Condorcet winner w exists, w has the maximum evaluation value.

theorem *cond-winner-imp-max-eval-val*:
fixes
 $e :: ('a, 'v)$ *Evaluation-Function* **and**
 $A :: 'a$ *set* **and**
 $V :: 'v$ *set* **and**
 $p :: ('a, 'v)$ *Profile* **and**
 $a :: 'a$
assumes
rating: *condorcet-rating* e **and**
f-prof: *finite-profile* $V A p$ **and**
winner: *condorcet-winner* $V A p a$
shows $e V a A p = \text{Max } \{e V b A p \mid b. b \in A\}$
 $\langle \text{proof} \rangle$

If e is Condorcet-rating, the following holds: If a Condorcet Winner w exists, a non-Condorcet winner has a value lower than the maximum evaluation value.

theorem *non-cond-winner-not-max-eval*:
fixes
 $e :: ('a, 'v)$ *Evaluation-Function* **and**
 $A :: 'a$ *set* **and**
 $V :: 'v$ *set* **and**
 $p :: ('a, 'v)$ *Profile* **and**
 $a b :: 'a$
assumes
rating: *condorcet-rating* e **and**
f-prof: *finite-profile* $V A p$ **and**
winner: *condorcet-winner* $V A p a$ **and**
lin-A: $b \in A$ **and**
loser: $a \neq b$
shows $e V b A p < \text{Max } \{e V c A p \mid c. c \in A\}$
 $\langle \text{proof} \rangle$

end

4.7 Elimination Module

theory *Elimination-Module*
imports *Evaluation-Function*
Electoral-Module
begin

This is the elimination module. It rejects a set of alternatives only if these are not all alternatives. The alternatives potentially to be rejected are put in a so-called elimination set. These are all alternatives that score below a preset threshold value that depends on the specific voting rule.

4.7.1 General Definitions

type-synonym *Threshold-Value* = *enat*

type-synonym *Threshold-Relation* = *enat* \Rightarrow *enat* \Rightarrow *bool*

type-synonym (*'a*, *'v*) *Electoral-Set* = *'v set* \Rightarrow *'a set* \Rightarrow (*'a*, *'v*) *Profile* \Rightarrow *'a set*

fun *elimination-set* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow *Threshold-Value* \Rightarrow
Threshold-Relation \Rightarrow (*'a*, *'v*) *Electoral-Set* **where**
elimination-set *e t r V A p* = (if *finite A* then $\{a \in A . r (e \ V \ a \ A \ p) \ t\}$ else $\{\}$)

fun *average* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow *'v set* \Rightarrow *'a set* \Rightarrow (*'a*, *'v*) *Profile* \Rightarrow
Threshold-Value **where**
average *e V A p* = (let *sum-eval* = $(\sum x \in A. e \ V \ x \ A \ p)$ in
 (if *sum-eval* = ∞ then ∞ else *the-enat sum-eval div card A*))

4.7.2 Social-Choice Definitions

fun *elimination-module* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow *Threshold-Value* \Rightarrow
Threshold-Relation \Rightarrow (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
elimination-module *e t r V A p* =
 (if (*elimination-set* *e t r V A p*) $\neq A$
 then $(\{\}, (elimination-set \ e \ t \ r \ V \ A \ p), A - (elimination-set \ e \ t \ r \ V \ A \ p))$
 else $(\{\}, \{\}, A)$)

4.7.3 Social-Choice Eliminators

fun *less-eliminator* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow *Threshold-Value* \Rightarrow
 (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
less-eliminator *e t V A p* = *elimination-module* *e t* (*<*) *V A p*

fun *max-eliminator* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow
 (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
max-eliminator *e V A p* =
less-eliminator *e* (*Max* $\{e \ V \ x \ A \ p \mid x. x \in A\}$) *V A p*

fun *leq-eliminator* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow *Threshold-Value* \Rightarrow
 (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
leq-eliminator *e t V A p* = *elimination-module* *e t* (*\leq*) *V A p*

fun *min-eliminator* :: (*'a*, *'v*) *Evaluation-Function* \Rightarrow
 (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
min-eliminator *e V A p* =

$\text{leq-eliminator } e \text{ (Min } \{e \text{ V } x \text{ A } p \mid x. x \in A\}) \text{ V } A \text{ } p$
fun *less-average-eliminator* :: ('a, 'v) *Evaluation-Function* \Rightarrow
 ('a, 'v, 'a *Result*) *Electoral-Module* **where**
less-average-eliminator *e* *V A p* = *less-eliminator* *e* (*average* *e* *V A p*) *V A p*
fun *leq-average-eliminator* :: ('a, 'v) *Evaluation-Function* \Rightarrow
 ('a, 'v, 'a *Result*) *Electoral-Module* **where**
leq-average-eliminator *e* *V A p* = *leq-eliminator* *e* (*average* *e* *V A p*) *V A p*

4.7.4 Soundness

lemma *elim-mod-sound[simp]*:
fixes
e :: ('a, 'v) *Evaluation-Function* **and**
t :: *Threshold-Value* **and**
r :: *Threshold-Relation*
shows *SCF-result.electoral-module* (*elimination-module* *e* *t* *r*)
 <proof>

lemma *less-elim-sound[simp]*:
fixes
e :: ('a, 'v) *Evaluation-Function* **and**
t :: *Threshold-Value*
shows *SCF-result.electoral-module* (*less-eliminator* *e* *t*)
 <proof>

lemma *leq-elim-sound[simp]*:
fixes
e :: ('a, 'v) *Evaluation-Function* **and**
t :: *Threshold-Value*
shows *SCF-result.electoral-module* (*leq-eliminator* *e* *t*)
 <proof>

lemma *max-elim-sound[simp]*:
fixes *e* :: ('a, 'v) *Evaluation-Function*
shows *SCF-result.electoral-module* (*max-eliminator* *e*)
 <proof>

lemma *min-elim-sound[simp]*:
fixes *e* :: ('a, 'v) *Evaluation-Function*
shows *SCF-result.electoral-module* (*min-eliminator* *e*)
 <proof>

lemma *less-avg-elim-sound[simp]*:
fixes *e* :: ('a, 'v) *Evaluation-Function*
shows *SCF-result.electoral-module* (*less-average-eliminator* *e*)
 <proof>

lemma *leq-avg-elim-sound*[simp]:
fixes $e :: ('a, 'v)$ *Evaluation-Function*
shows *SCF-result.electoral-module* (*leq-average-eliminator* e)
 ⟨*proof*⟩

4.7.5 Independence of Non-Voters

lemma *voters-determine-elim-mod*[simp]:
fixes
 $e :: ('a, 'v)$ *Evaluation-Function* **and**
 $t ::$ *Threshold-Value* **and**
 $r ::$ *Threshold-Relation*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*elimination-module* e t r)
 ⟨*proof*⟩

lemma *voters-determine-less-elim*[simp]:
fixes
 $e :: ('a, 'v)$ *Evaluation-Function* **and**
 $t ::$ *Threshold-Value*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*less-eliminator* e t)
 ⟨*proof*⟩

lemma *voters-determine-leq-elim*[simp]:
fixes
 $e :: ('a, 'v)$ *Evaluation-Function* **and**
 $t ::$ *Threshold-Value*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*leq-eliminator* e t)
 ⟨*proof*⟩

lemma *voters-determine-max-elim*[simp]:
fixes $e :: ('a, 'v)$ *Evaluation-Function*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*max-eliminator* e)
 ⟨*proof*⟩

lemma *voters-determine-min-elim*[simp]:
fixes $e :: ('a, 'v)$ *Evaluation-Function*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*min-eliminator* e)
 ⟨*proof*⟩

lemma *voters-determine-less-avg-elim*[simp]:
fixes $e :: ('a, 'v)$ *Evaluation-Function*
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*less-average-eliminator* e)
 ⟨*proof*⟩

lemma *voters-determine-leq-avg-elim*[simp]:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
assumes *voters-determine-evaluation* e
shows *voters-determine-election* (*leq-average-eliminator* e)
 $\langle \text{proof} \rangle$

4.7.6 Non-Blocking

lemma *elim-mod-non-blocking*:
fixes
 $e :: ('a, 'v) \text{ Evaluation-Function}$ **and**
 $t :: \text{Threshold-Value}$ **and**
 $r :: \text{Threshold-Relation}$
shows *non-blocking* (*elimination-module* e t r)
 $\langle \text{proof} \rangle$

lemma *less-elim-non-blocking*:
fixes
 $e :: ('a, 'v) \text{ Evaluation-Function}$ **and**
 $t :: \text{Threshold-Value}$
shows *non-blocking* (*less-eliminator* e t)
 $\langle \text{proof} \rangle$

lemma *leq-elim-non-blocking*:
fixes
 $e :: ('a, 'v) \text{ Evaluation-Function}$ **and**
 $t :: \text{Threshold-Value}$
shows *non-blocking* (*leq-eliminator* e t)
 $\langle \text{proof} \rangle$

lemma *max-elim-non-blocking*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
shows *non-blocking* (*max-eliminator* e)
 $\langle \text{proof} \rangle$

lemma *min-elim-non-blocking*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
shows *non-blocking* (*min-eliminator* e)
 $\langle \text{proof} \rangle$

lemma *less-avg-elim-non-blocking*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
shows *non-blocking* (*less-average-eliminator* e)
 $\langle \text{proof} \rangle$

lemma *leq-avg-elim-non-blocking*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
shows *non-blocking* (*leq-average-eliminator* e)

<proof>

4.7.7 Non-Electing

lemma *elim-mod-non-electing*:
 fixes
 e :: ('a, 'v) *Evaluation-Function* **and**
 t :: *Threshold-Value* **and**
 r :: *Threshold-Relation*
 shows *non-electing* (*elimination-module e t r*)
 <proof>

lemma *less-elim-non-electing*:
 fixes
 e :: ('a, 'v) *Evaluation-Function* **and**
 t :: *Threshold-Value*
 shows *non-electing* (*less-eliminator e t*)
 <proof>

lemma *leq-elim-non-electing*:
 fixes
 e :: ('a, 'v) *Evaluation-Function* **and**
 t :: *Threshold-Value*
 shows *non-electing* (*leq-eliminator e t*)
 <proof>

lemma *max-elim-non-electing*:
 fixes *e* :: ('a, 'v) *Evaluation-Function*
 shows *non-electing* (*max-eliminator e*)
 <proof>

lemma *min-elim-non-electing*:
 fixes *e* :: ('a, 'v) *Evaluation-Function*
 shows *non-electing* (*min-eliminator e*)
 <proof>

lemma *less-avg-elim-non-electing*:
 fixes *e* :: ('a, 'v) *Evaluation-Function*
 shows *non-electing* (*less-average-eliminator e*)
 <proof>

lemma *leq-avg-elim-non-electing*:
 fixes *e* :: ('a, 'v) *Evaluation-Function*
 shows *non-electing* (*leq-average-eliminator e*)
 <proof>

4.7.8 Inference Rules

If the used evaluation function is Condorcet rating, max-eliminator is Condorcet compatible.

theorem *cr-eval-imp-ccomp-max-elim[simp]*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
assumes *condorcet-rating* e
shows *condorcet-compatibility* (*max-eliminator* e)
 $\langle \text{proof} \rangle$

If the used evaluation function is Condorcet rating, max-eliminator is defer-Condorcet-consistent.

theorem *cr-eval-imp-dcc-max-elim[simp]*:
fixes $e :: ('a, 'v) \text{ Evaluation-Function}$
assumes *condorcet-rating* e
shows *defer-condorcet-consistency* (*max-eliminator* e)
 $\langle \text{proof} \rangle$

end

4.8 Aggregator

theory *Aggregator*
imports *Social-Choice-Types/Social-Choice-Result*
begin

An aggregator gets two partitions (results of electoral modules) as input and output another partition. They are used to aggregate results of parallel composed electoral modules. They are commutative, i.e., the order of the aggregated modules does not affect the resulting aggregation. Moreover, they are conservative in the sense that the resulting decisions are subsets of the two given partitions' decisions.

4.8.1 Definition

type-synonym $'a \text{ Aggregator} = 'a \text{ set} \Rightarrow 'a \text{ Result} \Rightarrow 'a \text{ Result} \Rightarrow 'a \text{ Result}$

definition *aggregator* $agg :: 'a \text{ Aggregator} \Rightarrow \text{bool}$ **where**
 $agg \equiv$
 $\forall A e e' d d' r r'.$
 $(\text{well-formed-SCF } A (e, r, d) \wedge \text{well-formed-SCF } A (e', r', d')) \longrightarrow$
 $\text{well-formed-SCF } A (agg A (e, r, d) (e', r', d'))$

4.8.2 Properties

definition *agg-commutative* :: 'a Aggregator \Rightarrow bool **where**
agg-commutative *agg* \equiv
aggregator *agg* $\wedge (\forall A\ e\ e'\ d\ d'\ r\ r'.$
 $\text{agg } A\ (e, r, d)\ (e', r', d') = \text{agg } A\ (e', r', d')\ (e, r, d))$

definition *agg-conservative* :: 'a Aggregator \Rightarrow bool **where**
agg-conservative *agg* \equiv
aggregator *agg* \wedge
 $(\forall A\ e\ e'\ d\ d'\ r\ r'.$
 $((\text{well-formed-SCF } A\ (e, r, d) \wedge \text{well-formed-SCF } A\ (e', r', d')) \longrightarrow$
 $\text{elect-r } (\text{agg } A\ (e, r, d)\ (e', r', d')) \subseteq (e \cup e') \wedge$
 $\text{reject-r } (\text{agg } A\ (e, r, d)\ (e', r', d')) \subseteq (r \cup r') \wedge$
 $\text{defer-r } (\text{agg } A\ (e, r, d)\ (e', r', d')) \subseteq (d \cup d')))$

end

4.9 Maximum Aggregator

theory *Maximum-Aggregator*
imports *Aggregator*
begin

The max(imum) aggregator takes two partitions of an alternative set A as input. It returns a partition where every alternative receives the maximum result of the two input partitions.

4.9.1 Definition

fun *max-aggregator* :: 'a Aggregator **where**
max-aggregator *A* $(e, r, d)\ (e', r', d') =$
 $(e \cup e',$
 $A - (e \cup e' \cup d \cup d'),$
 $(d \cup d') - (e \cup e'))$

4.9.2 Auxiliary Lemma

lemma *max-agg-rej-set*:
fixes
 $A\ e\ e'\ d\ d'\ r\ r' :: 'a\ \text{set}$ **and**
 $a :: 'a$
assumes
wf-first-mod: *well-formed-SCF* *A* (e, r, d) **and**
wf-second-mod: *well-formed-SCF* *A* (e', r', d')

shows *reject-r* (*max-aggregator* *A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*)) = *r* \cap *r'*
 \langle *proof* \rangle

4.9.3 Soundness

theorem *max-agg-sound[simp]*: *aggregator max-aggregator*
 \langle *proof* \rangle

4.9.4 Properties

The max-aggregator is conservative.

theorem *max-agg-consv[simp]*: *agg-conservative max-aggregator*
 \langle *proof* \rangle

The max-aggregator is commutative.

theorem *max-agg-comm[simp]*: *agg-commutative max-aggregator*
 \langle *proof* \rangle

end

4.10 Termination Condition

theory *Termination-Condition*
imports *Social-Choice-Types/Result*
begin

The termination condition is used in loops. It decides whether or not to terminate the loop after each iteration, depending on the current state of the loop.

type-synonym *'r Termination-Condition* = *'r Result* \Rightarrow *bool*

end

4.11 Defer Equal Condition

theory *Defer-Equal-Condition*
imports *Termination-Condition*
begin

This is a family of termination conditions. For a natural number n , the according defer-equal condition is true if and only if the given result's defer-set contains exactly n elements.

```
fun defer-equal-condition :: nat  $\Rightarrow$  'a Termination-Condition where  
  defer-equal-condition n (e, r, d) = (card d = n)  
end
```

Chapter 5

Basic Modules

5.1 Defer Module

```
theory Defer-Module  
  imports Component-Types/Electoral-Module  
begin
```

The defer module is not concerned about the voter's ballots, and simply defers all alternatives. It is primarily used for defining an empty loop.

5.1.1 Definition

```
fun defer-module :: ('a, 'v, 'a Result) Electoral-Module where  
  defer-module V A p = ({}, {}, A)
```

5.1.2 Soundness

```
theorem def-mod-sound[simp]: SCF-result.electoral-module defer-module  
  <proof>
```

5.1.3 Properties

```
theorem def-mod-non-electing: non-electing defer-module  
  <proof>
```

```
theorem def-mod-def-lift-inv: defer-lift-invariance defer-module  
  <proof>
```

```
end
```

5.2 Elect-First Module

```
theory Elect-First-Module
```



```

imports Component-Types/Electoral-Module
begin

```

The elect first module elects the alternative that is most preferred on the first ballot and rejects all other alternatives.

5.2.1 Definition

```

fun least :: 'v :: wellorder set  $\Rightarrow$  'v where
  least V = (Least ( $\lambda$  v. v  $\in$  V))

```

```

fun elect-first-module :: ('a, 'v :: wellorder, 'a Result) Electoral-Module where
  elect-first-module V A p =
    ({a  $\in$  A. above (p (least V)) a = {a}},
     {a  $\in$  A. above (p (least V)) a  $\neq$  {a}},
     {})

```

5.2.2 Soundness

```

theorem elect-first-mod-sound: SCF-result.electoral-module elect-first-module
  <proof>

```

```

end

```

5.3 Consensus Class

```

theory Consensus-Class
  imports Consensus
           ../Defer-Module
           ../Elect-First-Module
begin

```

A consensus class is a pair of a set of elections and a mapping that assigns a unique alternative to each election in that set (of elections). This alternative is then called the consensus alternative (winner). Here, we model the mapping by an electoral module that defers alternatives which are not in the consensus.

5.3.1 Definition

```

type-synonym ('a, 'v, 'r) Consensus-Class =
  ('a, 'v) Consensus  $\times$  ('a, 'v, 'r) Electoral-Module

```

```

fun consensus-K :: ('a, 'v, 'r) Consensus-Class  $\Rightarrow$  ('a, 'v) Consensus where
  consensus-K K = fst K

```

fun *rule-K* :: ('a, 'v, 'r) *Consensus-Class* \Rightarrow ('a, 'v, 'r) *Electoral-Module* **where**
rule-K *K* = *snd K*

5.3.2 Consensus Choice

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

fun *K_E* :: ('a, 'v, 'r *Result*) *Consensus-Class* \Rightarrow 'r \Rightarrow ('a, 'v) *Election set* **where**
K_E *K* *w* =
 $\{(A, V, p) \mid A \ V \ p. (\text{consensus-}\mathcal{K} \ K) (A, V, p) \wedge \text{finite-profile } V \ A \ p$
 $\wedge \text{elect } (\text{rule-}\mathcal{K} \ K) \ V \ A \ p = \{w\}\}$

fun *elections-K* :: ('a, 'v, 'r *Result*) *Consensus-Class* \Rightarrow ('a, 'v) *Election set* **where**
elections-K *K* = $\bigcup ((\mathcal{K}_E \ K) \text{ 'UNIV})$

A consensus class is deemed well-formed if the result of its mapping is completely determined by its consensus, the elected set of the electoral module's result.

definition *well-formed* :: ('a, 'v) *Consensus* \Rightarrow ('a, 'v, 'r) *Electoral-Module* \Rightarrow
bool **where**
well-formed *c* *m* \equiv
 $\forall \ A \ V \ V' \ p \ p'. \text{profile } V \ A \ p \wedge \text{profile } V' \ A \ p' \wedge c \ (A, V, p) \wedge c \ (A, V', p')$
 $\longrightarrow m \ V \ A \ p = m \ V' \ A \ p'$

A sensible social choice rule for a given arbitrary consensus and social choice rule *r* is the one that chooses the result of *r* for all consensus elections and defers all candidates otherwise.

fun *consensus-choice* :: ('a, 'v) *Consensus* \Rightarrow
('a, 'v, 'a *Result*) *Electoral-Module* \Rightarrow
('a, 'v, 'a *Result*) *Consensus-Class* **where**
consensus-choice *c* *m* =
(*let*
 $w = (\lambda \ V \ A \ p. \text{if } c \ (A, V, p) \text{ then } m \ V \ A \ p \text{ else defer-module } V \ A \ p)$
in (*c*, *w*))

5.3.3 Auxiliary Lemmas

lemma *unanimity'-consensus-imp-elect-fst-mod-well-formed*:

fixes *a* :: 'a

shows *well-formed*

$(\lambda \ c. \text{nonempty-set}_C \ c \wedge \text{nonempty-profile}_C \ c$
 $\wedge \text{equal-top}_C \ 'a \ c) \text{ elect-first-module}$

<proof>

lemma *strong-unanimity'-consensus-imp-elect-fst-mod-completely-determined*:

fixes *r* :: 'a *Preference-Relation*

shows *well-formed*
 $(\lambda c. \text{nonempty-set}_C c \wedge \text{nonempty-profile}_C c \wedge \text{equal-vote}_C 'r c) \text{elect-first-module}$
 $\langle \text{proof} \rangle$

lemma *strong-unanimity'consensus-imp-elect-fst-mod-well-formed*:
fixes $r :: 'a \text{ Preference-Relation}$
shows *well-formed*
 $(\lambda c. \text{nonempty-set}_C c \wedge \text{nonempty-profile}_C c$
 $\wedge \text{equal-vote}_C 'r c) \text{elect-first-module}$
 $\langle \text{proof} \rangle$

lemma *cons-domain-valid*:
fixes $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$
shows $\text{elections-}\mathcal{K} C \subseteq \text{well-formed-elections}$
 $\langle \text{proof} \rangle$

lemma *cons-domain-finite*:
fixes $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$
shows
 $\text{finite: elections-}\mathcal{K} C \subseteq \text{finite-elections}$ **and**
 $\text{finite-voters: elections-}\mathcal{K} C \subseteq \text{finite-elections-}\mathcal{V}$
 $\langle \text{proof} \rangle$

5.3.4 Consensus Rules

definition *non-empty-set* $:: ('a, 'v, 'r) \text{ Consensus-Class} \Rightarrow \text{bool}$ **where**
 $\text{non-empty-set } c \equiv \exists K. \text{consensus-}\mathcal{K} c K$

Unanimity condition.

definition *unanimity* $:: ('a, 'v :: \text{wellorder}, 'a \text{ Result}) \text{ Consensus-Class}$ **where**
 $\text{unanimity} \equiv \text{consensus-choice unanimity}_C \text{elect-first-module}$

Strong unanimity condition.

definition *strong-unanimity* $:: ('a, 'v :: \text{wellorder}, 'a \text{ Result}) \text{ Consensus-Class}$
where
 $\text{strong-unanimity} \equiv \text{consensus-choice strong-unanimity}_C \text{elect-first-module}$

5.3.5 Properties

definition *consensus-rule-anonymity* $:: ('a, 'v, 'r) \text{ Consensus-Class} \Rightarrow \text{bool}$ **where**
 $\text{consensus-rule-anonymity } c \equiv$
 $(\forall A V p \pi :: ('v \Rightarrow 'v).$
 $\text{bij } \pi \longrightarrow$
 $(\text{let } (A', V', q) = (\text{rename } \pi (A, V, p)) \text{ in}$
 $\text{profile } V A p \longrightarrow \text{profile } V' A' q$
 $\longrightarrow \text{consensus-}\mathcal{K} c (A, V, p)$
 $\longrightarrow (\text{consensus-}\mathcal{K} c (A', V', q) \wedge (\text{rule-}\mathcal{K} c V A p = \text{rule-}\mathcal{K} c V' A' q))))$

fun *consensus-rule-anonymity'* $:: ('a, 'v) \text{ Election set} \Rightarrow$

$(\text{'a}, \text{'v}, \text{'r Result}) \text{ Consensus-Class} \Rightarrow \text{bool}$ **where**
 $\text{consensus-rule-anonymity}' X C =$
 $\text{is-symmetry } (\text{elect-r} \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} C)) (\text{Invariance } (\text{anonymity}_{\mathcal{R}} X))$

fun **(in result-properties)** $\text{consensus-rule-neutrality} :: (\text{'a}, \text{'v}) \text{ Election set} \Rightarrow$
 $(\text{'a}, \text{'v}, \text{'b Result}) \text{ Consensus-Class} \Rightarrow \text{bool}$ **where**
 $\text{consensus-rule-neutrality} X C =$
 $\text{is-symmetry } (\text{elect-r} \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} C))$
 $(\text{action-induced-equivariance } (\text{carrier bijection}_{\mathcal{AG}}) X (\varphi\text{-neutral } X) (\text{set-action } \psi))$

fun $\text{consensus-rule-reversal-symmetry} :: (\text{'a}, \text{'v}) \text{ Election set} \Rightarrow$
 $(\text{'a}, \text{'v}, \text{'a rel Result}) \text{ Consensus-Class} \Rightarrow \text{bool}$ **where**
 $\text{consensus-rule-reversal-symmetry} X C = \text{is-symmetry } (\text{elect-r} \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} C))$
 $(\text{action-induced-equivariance } (\text{carrier reversal}_G) X (\varphi\text{-reverse } X) (\text{set-action } \psi\text{-reverse}))$

5.3.6 Inference Rules

lemma *if-else-cons-equivar:*

fixes
 $m n :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **and**
 $c :: (\text{'a}, \text{'v}) \text{ Consensus}$ **and**
 $G :: \text{'b set}$ **and**
 $X :: (\text{'a}, \text{'v}) \text{ Election set}$ **and**
 $\varphi :: (\text{'b}, (\text{'a}, \text{'v}) \text{ Election}) \text{ binary-fun}$ **and**
 $\psi :: (\text{'b}, \text{'a}) \text{ binary-fun}$ **and**
 $f :: \text{'a Result} \Rightarrow \text{'a set}$

defines
 $\text{equivar} \equiv \text{action-induced-equivariance } G X \varphi (\text{set-action } \psi)$ **and**
 $\text{if-else-cons} \equiv (c, (\lambda V A p. \text{if } c (A, V, p) \text{ then } m V A p \text{ else } n V A p))$

assumes
 $\text{equivar-m: is-symmetry } (f \circ \text{fun}_{\mathcal{E}} m) \text{ equivar}$ **and**
 $\text{equivar-n: is-symmetry } (f \circ \text{fun}_{\mathcal{E}} n) \text{ equivar}$ **and**
 $\text{invar-cons: is-symmetry } c (\text{Invariance } (\text{action-induced-rel } G X \varphi))$

shows $\text{is-symmetry } (f \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} \text{ if-else-cons}))$
 $(\text{action-induced-equivariance } G X \varphi (\text{set-action } \psi))$

<proof>

lemma *consensus-choice-anonymous:*

fixes
 $\alpha \beta :: (\text{'a}, \text{'v}) \text{ Consensus}$ **and**
 $m :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **and**
 $\beta' :: \text{'b} \Rightarrow (\text{'a}, \text{'v}) \text{ Consensus}$

assumes
 $\text{beta-sat: } \beta = (\lambda E. \exists a. \beta' a E)$ **and**
 $\text{beta'-anon: } \forall x. \text{consensus-anonymity } (\beta' x)$ **and**
 $\text{anon-cons-cond: consensus-anonymity } \alpha$ **and**
 $\text{conditions-univ: } \forall x. \text{well-formed } (\lambda E. \alpha E \wedge \beta' x E) m$

shows *consensus-rule-anonymity* (*consensus-choice* ($\lambda E. \alpha E \wedge \beta E$) *m*)
 <proof>

5.3.7 Theorems

Anonymity

lemma *unanimity-anonymous: consensus-rule-anonymity unanimity*
 <proof>

lemma *strong-unanimity-anonymous: consensus-rule-anonymity strong-unanimity*
 <proof>

Neutrality

lemma *defer-winners-equivariant:*

fixes

$G :: 'b$ set **and**

$E :: ('a, 'v)$ Election set **and**

$\varphi :: ('b, ('a, 'v)$ Election) binary-fun **and**

$\psi :: ('b, 'a)$ binary-fun

shows *is-symmetry* (*elect-r* \circ *fun_E* *defer-module*)
 (*action-induced-equivariance* G E φ (*set-action* ψ))

<proof>

lemma *elect-first-winners-neutral: is-symmetry* (*elect-r* \circ *fun_E* *elect-first-module*)
 (*action-induced-equivariance* (*carrier bijection_{AG}*)
well-formed-elections (φ -neutral *well-formed-elections*)
 (*set-action* ψ -neutral_c))

<proof>

lemma *strong-unanimity-neutral:*

defines *domain* \equiv *well-formed-elections* \cap *Collect strong-unanimity_C*

— We want to show neutrality on a set as general as possible, as this implies subset neutrality.

shows *SCF-properties.consensus-rule-neutrality domain strong-unanimity*

<proof>

lemma *strong-unanimity-neutral': SCF-properties.consensus-rule-neutrality*
 (*elections-K strong-unanimity*) *strong-unanimity*

<proof>

lemma *strong-unanimity-closed-under-neutrality: closed-restricted-rel*
 (*neutrality_R* *well-formed-elections*) *well-formed-elections*
 (*elections-K strong-unanimity*)

<proof>

end

5.4 Distance Rationalization

theory *Distance-Rationalization*

imports *Social-Choice-Types/Refined-Types/Preference-List*
Consensus-Class
Distance

begin

A distance rationalization of a voting rule is its interpretation as a procedure that elects an uncontroversial winner if there is one, and otherwise elects the alternatives that are as close to becoming an uncontroversial winner as possible. Within general distance rationalization, a voting rule is characterized by a distance on profiles and a consensus class.

5.4.1 Definitions

Returns the distance of an election to the preimage of a unique winner under the given consensus elections and consensus rule.

fun *score* :: ('a, 'v) Election Distance \Rightarrow ('a, 'v, 'r Result) Consensus-Class \Rightarrow ('a, 'v) Election \Rightarrow 'r \Rightarrow ereal **where**
score *d* *K* *E* *w* = Inf (*d* *E* ' ($\mathcal{K}_{\mathcal{E}}$ *K* *w*))

fun (in result) $\mathcal{R}_{\mathcal{W}}$:: ('a, 'v) Election Distance \Rightarrow ('a, 'v, 'r Result) Consensus-Class \Rightarrow 'v set \Rightarrow 'a set \Rightarrow ('a, 'v) Profile \Rightarrow 'r set **where**
 $\mathcal{R}_{\mathcal{W}}$ *d* *K* *V* *A* *p* = arg-min-set (*score* *d* *K* (*A*, *V*, *p*)) (limit *A* UNIV)

fun (in result) *distance- \mathcal{R}* :: ('a, 'v) Election Distance \Rightarrow ('a, 'v, 'r Result) Consensus-Class \Rightarrow ('a, 'v, 'r Result) Electoral-Module **where**
distance- \mathcal{R} *d* *K* *V* *A* *p* =
($\mathcal{R}_{\mathcal{W}}$ *d* *K* *V* *A* *p*, (limit *A* UNIV) - $\mathcal{R}_{\mathcal{W}}$ *d* *K* *V* *A* *p*, {})

5.4.2 Standard Definitions

definition *standard* :: ('a, 'v) Election Distance \Rightarrow bool **where**

standard *d* \equiv
 $\forall A A' V V' p p'. (V \neq V' \vee A \neq A') \longrightarrow d(A, V, p) (A', V', p') = \infty$

definition *voters-determine-distance* :: ('a, 'v) Election Distance \Rightarrow bool **where**

voters-determine-distance *d* \equiv
 $\forall A A' V V' p q p'. (V \neq V' \vee A \neq A') \longrightarrow d(A, V, p) (A', V', p') = d(A, V, q) (A', V', p') \wedge d(A, V, p) (A', V', p') = d(A, V, q) (A', V', p')$

Creates a set of all possible profiles on a finite alternative set that are empty everywhere outside of a given finite voter set.

```

fun profiles :: 'v set  $\Rightarrow$  'a set  $\Rightarrow$  (('a, 'v) Profile) set where
  profiles V A =
    (if (infinite A  $\vee$  infinite V)
     then {} else {p. p ' V  $\subseteq$  (pl- $\alpha$  ' permutations-of-set A)})

fun  $\mathcal{K}_{\mathcal{E}}$ -std :: ('a, 'v, 'r Result) Consensus-Class  $\Rightarrow$  'r  $\Rightarrow$  'a set  $\Rightarrow$  'v set  $\Rightarrow$ 
  ('a, 'v) Election set where
   $\mathcal{K}_{\mathcal{E}}$ -std K w A V =
    ( $\lambda$  p. (A, V, p)) ' (Set.filter
      ( $\lambda$  p. (consensus- $\mathcal{K}$  K) (A, V, p)  $\wedge$  elect (rule- $\mathcal{K}$  K) V A p = {w})
      (profiles V A))

```

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

```

fun score-std :: ('a, 'v) Election Distance  $\Rightarrow$  ('a, 'v, 'r Result) Consensus-Class  $\Rightarrow$ 
  ('a, 'v) Election  $\Rightarrow$  'r  $\Rightarrow$  ereal where
  score-std d K E w =
    (if  $\mathcal{K}_{\mathcal{E}}$ -std K w (alternatives- $\mathcal{E}$  E) (voters- $\mathcal{E}$  E) = {}
     then  $\infty$  else Min (d E ' ( $\mathcal{K}_{\mathcal{E}}$ -std K w (alternatives- $\mathcal{E}$  E) (voters- $\mathcal{E}$  E))))

fun (in result)  $\mathcal{R}_{\mathcal{W}}$ -std :: ('a, 'v) Election Distance  $\Rightarrow$ 
  ('a, 'v, 'r Result) Consensus-Class  $\Rightarrow$  'v set  $\Rightarrow$  'a set  $\Rightarrow$  ('a, 'v) Profile  $\Rightarrow$ 
  'r set where
   $\mathcal{R}_{\mathcal{W}}$ -std d K V A p = arg-min-set (score-std d K (A, V, p)) (limit A UNIV)

fun (in result) distance- $\mathcal{R}$ -std :: ('a, 'v) Election Distance  $\Rightarrow$ 
  ('a, 'v, 'r Result) Consensus-Class  $\Rightarrow$ 
  ('a, 'v, 'r Result) Electoral-Module where
  distance- $\mathcal{R}$ -std d K V A p =
    ( $\mathcal{R}_{\mathcal{W}}$ -std d K V A p, (limit A UNIV) -  $\mathcal{R}_{\mathcal{W}}$ -std d K V A p, {})

```

5.4.3 Auxiliary Lemmas

lemma fin- $\mathcal{K}_{\mathcal{E}}$:
fixes C :: ('a, 'v, 'r Result) Consensus-Class
shows elections- \mathcal{K} C \subseteq finite-elections
 <proof>

lemma univ- $\mathcal{K}_{\mathcal{E}}$:
fixes C :: ('a, 'v, 'r Result) Consensus-Class
shows elections- \mathcal{K} C \subseteq UNIV
 <proof>

lemma listset-finiteness:
fixes l :: 'a set list
assumes $\forall i :: \text{nat. } i < \text{length } l \longrightarrow \text{finite } (!i)$
shows finite (listset l)
 <proof>

lemma *ls-entries-empty-imp-ls-set-empty*:

fixes $l :: 'a \text{ set list}$

assumes

$0 < \text{length } l$ **and**

$\forall i :: \text{nat. } i < \text{length } l \longrightarrow l[i] = \{\}$

shows $\text{listset } l = \{\}$

<proof>

lemma *all-ls-elems-same-len*:

fixes $l :: 'a \text{ set list}$

shows $\forall l' :: 'a \text{ list. } l' \in \text{listset } l \longrightarrow \text{length } l' = \text{length } l$

<proof>

lemma *fin-all-profs*:

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$x :: 'a \text{ Preference-Relation}$

assumes

fin-A: *finite A* **and**

fin-V: *finite V*

shows *finite (profiles V A $\cap \{p. \forall v. v \notin V \longrightarrow p \ v = x\}$)*

<proof>

lemma *profile-permutation-set*:

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$

shows *profiles V A = $\{p :: ('a, 'v) \text{ Profile. finite-profile V A } p\}$*

<proof>

5.4.4 Soundness

lemma (*in result*) *\mathcal{R} -sound*:

fixes

$K :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$ **and**

$d :: ('a, 'v) \text{ Election Distance}$

shows *electoral-module (distance- \mathcal{R} d K)*

<proof>

5.4.5 Properties

fun *distance-decisiveness* :: $('a, 'v) \text{ Election set} \Rightarrow ('a, 'v) \text{ Election Distance} \Rightarrow$

$('a, 'v, 'r \text{ Result}) \text{ Electoral-Module} \Rightarrow \text{bool}$ **where**

distance-decisiveness X d m =

$(\nexists E. E \in X$

$\wedge (\exists \delta > 0. \forall E' \in X. d \ E \ E' < \delta \longrightarrow \text{card } (\text{elect-r } (\text{fun}_{\mathcal{E}} \ m \ E')) > 1))$

5.4.6 Inference Rules

lemma (in result) *standard-distance-imp-equal-score*:

fixes

$d :: ('a, 'v)$ Election Distance **and**
 $K :: ('a, 'v, 'r)$ Result Consensus-Class **and**
 $A :: 'a$ set **and**
 $V :: 'v$ set **and**
 $p :: ('a, 'v)$ Profile **and**
 $w :: 'r$

assumes

irr-non-V: voters-determine-distance d **and**
std: standard d

shows $\text{score } d \ K \ (A, V, p) \ w = \text{score-std } d \ K \ (A, V, p) \ w$

<proof>

lemma (in result) *anonymous-distance-and-consensus-imp-rule-anonymity*:

fixes

$d :: ('a, 'v)$ Election Distance **and**
 $K :: ('a, 'v, 'r)$ Result Consensus-Class

assumes

d-anon: distance-anonymity d **and**
K-anon: consensus-rule-anonymity K

shows anonymity (distance- \mathcal{R} $d \ K$)

<proof>

end

5.5 Votewise Distance Rationalization

theory *Votewise-Distance-Rationalization*

imports *Distance-Rationalization*

Votewise-Distance

begin

A votewise distance rationalization of a voting rule is its distance rationalization with a distance function that depends on the submitted votes in a simple and a transparent manner by using a distance on individual orders and combining the components with a norm on \mathbb{R} to \mathbb{N} .

5.5.1 Common Rationalizations

fun *swap- \mathcal{R}* :: $('a, 'v :: \text{linorder}, 'a \text{ Result}) \text{ Consensus-Class} \Rightarrow$

$('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **where**

swap- \mathcal{R} $K = \text{SCF-result.distance-}\mathcal{R} \ (\text{votewise-distance swap l-one}) \ K$

5.5.2 Theorems

lemma *votewise-non-voters-irrelevant:*

fixes

$d :: 'a \text{ Vote Distance}$ **and**

$N :: \text{Norm}$

shows *voters-determine-distance* (*votewise-distance* d N)

<proof>

lemma *swap-standard: standard* (*votewise-distance* *swap* *l-one*)

<proof>

5.5.3 Equivalence Lemmas

type-synonym ($'a, 'v$) *score-type* = ($'a, 'v$) *Election Distance* \Rightarrow

($'a, 'v, 'a \text{ Result}$) *Consensus-Class* \Rightarrow ($'a, 'v$) *Election* $\Rightarrow 'a \Rightarrow \text{ereal}$

type-synonym ($'a, 'v$) *dist-rat-type* = ($'a, 'v$) *Election Distance* \Rightarrow

($'a, 'v, 'a \text{ Result}$) *Consensus-Class* $\Rightarrow 'v \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a, 'v) \text{ Profile} \Rightarrow 'a \text{ set}$

type-synonym ($'a, 'v$) *dist-rat-std-type* = ($'a, 'v$) *Election Distance* \Rightarrow

($'a, 'v, 'a \text{ Result}$) *Consensus-Class* $\Rightarrow ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$

type-synonym ($'a, 'v$) *dist-type* = ($'a, 'v$) *Election Distance* \Rightarrow

($'a, 'v, 'a \text{ Result}$) *Consensus-Class* $\Rightarrow ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$

lemma *equal-score-swap:* ($\text{score} :: ('a, 'v :: \text{linorder}) \text{ score-type}$)

(*votewise-distance* *swap* *l-one*) = *score-std* (*votewise-distance* *swap* *l-one*)

<proof>

lemma *swap- \mathcal{R} -code*[*code*]: *swap- \mathcal{R}* =

(*SCF-result.distance- \mathcal{R} -std* :: ($'a, 'v :: \text{linorder}$) *dist-rat-std-type*)

(*votewise-distance* *swap* *l-one*)

<proof>

end

5.6 Symmetry in Distance-Rationalizable Rules

theory *Distance-Rationalization-Symmetry*

imports *Distance-Rationalization*

begin

5.6.1 Minimizer Function

fun *distance-infimum* :: $'a \text{ Distance} \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{ereal}$ **where**

distance-infimum d A $a = \text{Inf } (d \text{ a } 'A)$

fun *closest-preimg-distance* :: ('a \Rightarrow 'b) \Rightarrow 'a set \Rightarrow 'a Distance \Rightarrow
 'a \Rightarrow 'b \Rightarrow ereal **where**
 closest-preimg-distance f domain_f d a b = distance-infimum d (preimg f domain_f
 b) a

fun *minimizer* :: ('a \Rightarrow 'b) \Rightarrow 'a set \Rightarrow 'a Distance \Rightarrow 'b set \Rightarrow 'a \Rightarrow 'b set **where**
 minimizer f domain_f d A a = arg-min-set (*closest-preimg-distance* f domain_f d
 a) A

Auxiliary Lemmas

lemma *rewrite-arg-min-set*:

fixes
 f :: 'a \Rightarrow 'b :: linorder **and**
 A :: 'a set
shows arg-min-set f A = \bigcup (preimg f A ' {y \in f ' A. \forall z \in f ' A. y \leq z})
 <proof>

Equivariance

abbreviation *Restrp* :: 'a rel \Rightarrow 'a set \Rightarrow 'a rel **where**

Restrp r A \equiv r Int (A \times UNIV)

lemma *restr-induced-rel*:

fixes
 A :: 'a set **and**
 B B' :: 'b set **and**
 φ :: ('a, 'b) binary-fun
assumes B' \subseteq B
shows *Restrp* (action-induced-rel A B φ) B' = action-induced-rel A B' φ
 <proof>

theorem *group-action-invar-dist-and-equivar-f-imp-equivar-minimizer*:

fixes
 f :: 'a \Rightarrow 'b **and**
 domain_f X :: 'a set **and**
 d :: 'a Distance **and**
 well-formed-img :: 'a \Rightarrow 'b set **and**
 G :: 'c monoid **and**
 φ :: ('c, 'a) binary-fun **and**
 ψ :: ('c, 'b) binary-fun
defines *equivar-prop-set-valued* \equiv
 action-induced-equivariance (carrier G) X φ (set-action ψ)
assumes
 action- φ : group-action G X φ **and**
 group-action-res: group-action G UNIV ψ **and**
 dom-in-X: domain_f \subseteq X **and**
 closed-domain:
 closed-restricted-rel (action-induced-rel (carrier G) X φ) X domain_f **and**
 equivar-img: is-symmetry well-formed-img *equivar-prop-set-valued* **and**

invar-d: invariance_D d (carrier G) X φ and
equivar-f:
is-symmetry f (action-induced-equivariance (carrier G) domain_f φ ψ)
shows *is-symmetry (λx . minimizer f domain_f d (well-formed-img x) x) equivar-prop-set-valued*
<proof>

Invariance

lemma *closest-dist-invar-under-refl-rel-and-tot-invar-dist:*

fixes
f :: 'a \Rightarrow 'b and
domain_f :: 'a set and
d :: 'a Distance and
rel :: 'a rel
assumes
reflp-on' domain_f (Restr_p rel domain_f) and
total-invariance_D d rel
shows *is-symmetry (closest-preimg-distance f domain_f d) (Invariance rel)*
<proof>

lemma *refl-rel-and-tot-invar-dist-imp-invar-minimizer:*

fixes
f :: 'a \Rightarrow 'b and
domain_f :: 'a set and
d :: 'a Distance and
rel :: 'a rel and
img :: 'b set
assumes
reflp-on' domain_f (Restr_p rel domain_f) and
total-invariance_D d rel
shows *is-symmetry (minimizer f domain_f d img) (Invariance rel)*
<proof>

theorem *group-act-invar-dist-and-invar-f-imp-invar-minimizer:*

fixes
f :: 'a \Rightarrow 'b and
domain_f A :: 'a set and
d :: 'a Distance and
img :: 'b set and
G :: 'c monoid and
 φ :: ('c, 'a) binary-fun
defines
rel \equiv action-induced-rel (carrier G) A φ and
rel' \equiv action-induced-rel (carrier G) domain_f φ
assumes
action- φ : group-action G A φ and
dom-in-A: domain_f \subseteq A and
closed-domain: closed-restricted-rel rel A domain_f and
invar-d: invariance_D d (carrier G) A φ and

$\text{invar-f: is-symmetry } f \text{ (Invariance rel')}$
shows $\text{is-symmetry (minimizer } f \text{ domain}_f \text{ d img) (Invariance rel)}$
 $\langle \text{proof} \rangle$

5.6.2 Minimizer Translation

lemma $\mathcal{K}_{\mathcal{E}}$ -is-preimg:

fixes
 $d :: ('a, 'v) \text{ Election Distance and}$
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class and}$
 $E :: ('a, 'v) \text{ Election and}$
 $w :: 'r$
shows $\text{preimg (elect-r } \circ \text{ fun}_{\mathcal{E}} \text{ (rule-}\mathcal{K} \text{ } C)) \text{ (elections-}\mathcal{K} \text{ } C) \{w\} = \mathcal{K}_{\mathcal{E}} \text{ } C \text{ } w$
 $\langle \text{proof} \rangle$

lemma score-is-closest-preimg-dist:

fixes
 $d :: ('a, 'v) \text{ Election Distance and}$
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class and}$
 $E :: ('a, 'v) \text{ Election and}$
 $w :: 'r$
shows $\text{score } d \text{ } C \text{ } E \text{ } w =$
 $\text{closest-preimg-distance (elect-r } \circ \text{ fun}_{\mathcal{E}} \text{ (rule-}\mathcal{K} \text{ } C)) \text{ (elections-}\mathcal{K} \text{ } C) \text{ } d \text{ } E \text{ } \{w\}$
 $\langle \text{proof} \rangle$

lemma (in result) $\mathcal{R}_{\mathcal{W}}$ -is-minimizer:

fixes
 $d :: ('a, 'v) \text{ Election Distance and}$
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$
shows $\text{fun}_{\mathcal{E}} (\mathcal{R}_{\mathcal{W}} \text{ } d \text{ } C) =$
 $(\lambda E. \bigcup (\text{minimizer (elect-r } \circ \text{ fun}_{\mathcal{E}} \text{ (rule-}\mathcal{K} \text{ } C)) \text{ (elections-}\mathcal{K} \text{ } C) \text{ } d$
 $\text{(singleton-set-system (limit (alternatives-}\mathcal{E} \text{ } E) \text{ UNIV)) } E))$
 $\langle \text{proof} \rangle$

Invariance

theorem (in result) tot-invar-dist-imp-invar-dr-rule:

fixes
 $d :: ('a, 'v) \text{ Election Distance and}$
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class and}$
 $\text{rel} :: ('a, 'v) \text{ Election rel}$
assumes
 $r\text{-refl: reflp-on' (elections-}\mathcal{K} \text{ } C) \text{ (Restrp rel (elections-}\mathcal{K} \text{ } C)) \text{ and}$
 $\text{tot-invar-d: total-invariance}_{\mathcal{D}} \text{ } d \text{ rel and}$
 invar-res:
 $\text{is-symmetry } (\lambda E. \text{limit (alternatives-}\mathcal{E} \text{ } E) \text{ UNIV}) \text{ (Invariance rel)}$
shows $\text{is-symmetry (fun}_{\mathcal{E}} \text{ (distance-}\mathcal{R} \text{ } d \text{ } C)) \text{ (Invariance rel)}$
 $\langle \text{proof} \rangle$

theorem (in result) invar-dist-cons-imp-invar-dr-rule:

fixes
 $d :: ('a, 'v)$ Election Distance **and**
 $C :: ('a, 'v, 'r)$ Result Consensus-Class **and**
 $G :: 'b$ monoid **and**
 $\varphi :: ('b, ('a, 'v)$ Election) binary-fun **and**
 $B :: ('a, 'v)$ Election set

defines
 $rel \equiv action-induced-rel (carrier\ G)\ B\ \varphi$ **and**
 $rel' \equiv action-induced-rel (carrier\ G)\ (elections\text{-}\mathcal{K}\ C)\ \varphi$

assumes
 $action\text{-}\varphi$: group-action $G\ B\ \varphi$ **and**
 $consensus\text{-}C\text{-in-}B$: elections- $\mathcal{K}\ C \subseteq B$ **and**
 $closed\text{-}domain$:
 $closed\text{-}restricted\text{-}rel\ rel\ B\ (elections\text{-}\mathcal{K}\ C)$ **and**
 $invar\text{-}res$:
 $is\text{-}symmetry\ (\lambda\ E.\ limit\ (alternatives\text{-}\mathcal{E}\ E)\ UNIV)\ (Invariance\ rel)$ **and**
 $invar\text{-}d$: invariance $\mathcal{D}\ d\ (carrier\ G)\ B\ \varphi$ **and**
 $invar\text{-}C\text{-winners}$: $is\text{-}symmetry\ (elect\text{-}r \circ fun_{\mathcal{E}}\ (rule\text{-}\mathcal{K}\ C))\ (Invariance\ rel')$

shows $is\text{-}symmetry\ (fun_{\mathcal{E}}\ (distance\text{-}\mathcal{R}\ d\ C))\ (Invariance\ rel)$
 $\langle proof \rangle$

Equivariance

theorem (in result) $invar\text{-}dist\text{-}equivar\text{-}cons\text{-}imp\text{-}equivar\text{-}dr\text{-}rule$:

fixes
 $d :: ('a, 'v)$ Election Distance **and**
 $C :: ('a, 'v, 'r)$ Result Consensus-Class **and**
 $G :: 'b$ monoid **and**
 $\varphi :: ('b, ('a, 'v)$ Election) binary-fun **and**
 $\psi :: ('b, 'r)$ binary-fun **and**
 $B :: ('a, 'v)$ Election set

defines
 $rel \equiv action-induced-rel (carrier\ G)\ B\ \varphi$ **and**
 $rel' \equiv action-induced-rel (carrier\ G)\ (elections\text{-}\mathcal{K}\ C)\ \varphi$ **and**
 $equivar\text{-}prop \equiv$
 $action-induced-equivariance\ (carrier\ G)\ (elections\text{-}\mathcal{K}\ C)$
 $\varphi\ (set\text{-}action\ \psi)$ **and**
 $equivar\text{-}prop\text{-}global\text{-}set\text{-}valued \equiv$
 $action-induced-equivariance\ (carrier\ G)\ B\ \varphi\ (set\text{-}action\ \psi)$ **and**
 $equivar\text{-}prop\text{-}global\text{-}result\text{-}valued \equiv$
 $action-induced-equivariance\ (carrier\ G)\ B\ \varphi\ (result\text{-}action\ \psi)$

assumes
 $action\text{-}\varphi$: group-action $G\ B\ \varphi$ **and**
 $group\text{-}act\text{-}res$: group-action $G\ UNIV\ \psi$ **and**
 $cons\text{-}elect\text{-}set$: elections- $\mathcal{K}\ C \subseteq B$ **and**
 $closed\text{-}domain$: $closed\text{-}restricted\text{-}rel\ rel\ B\ (elections\text{-}\mathcal{K}\ C)$ **and**
 $equivar\text{-}res$:
 $is\text{-}symmetry\ (\lambda\ E.\ limit\ (alternatives\text{-}\mathcal{E}\ E)\ UNIV)$
 $equivar\text{-}prop\text{-}global\text{-}set\text{-}valued$ **and**

invar-d: invariance_D d (carrier G) B φ and
equivar-C-winners: is-symmetry (elect-r \circ fun_E (rule-K C)) equivar-prop
shows *is-symmetry (fun_E (distance- \mathcal{R} d C)) equivar-prop-global-result-valued*
 <proof>

5.6.3 Inference Rules

theorem (in result) *anon-dist-and-cons-imp-anon-dr:*

fixes

d :: ('a, 'v) Election Distance and
C :: ('a, 'v, 'r Result) Consensus-Class

assumes

anon-d: distance-anonymity' well-formed-elections d and
anon-C: consensus-rule-anonymity' (elections-K C) C and
closed-C: closed-restricted-rel (anonymity_R well-formed-elections)
well-formed-elections (elections-K C)

shows *anonymity' (distance- \mathcal{R} d C)*

<proof>

theorem (in result-properties) *neutr-dist-and-cons-imp-neutr-dr:*

fixes

d :: ('a, 'v) Election Distance and
C :: ('a, 'v, 'b Result) Consensus-Class

assumes

neutral-d: distance-neutrality well-formed-elections d and
neutral-C: consensus-rule-neutrality (elections-K C) C and
closed-C: closed-restricted-rel (neutrality_R well-formed-elections)
well-formed-elections (elections-K C)

shows *neutrality (distance- \mathcal{R} d C)*

<proof>

theorem *reversal-sym-dist-and-cons-imp-reversal-sym-dr:*

fixes

d :: ('a, 'c) Election Distance and
C :: ('a, 'c, 'a rel Result) Consensus-Class

assumes

reverse-sym-d: distance-reversal-symmetry well-formed-elections d and
reverse-sym-C: consensus-rule-reversal-symmetry (elections-K C) C and
closed-C: closed-restricted-rel (reversal_R well-formed-elections)
well-formed-elections (elections-K C)

shows *reversal-symmetry (SWF-result.distance- \mathcal{R} d C)*

<proof>

theorem (in result) *tot-hom-dist-imp-hom-dr:*

fixes

d :: ('a, nat) Election Distance and
C :: ('a, nat, 'r Result) Consensus-Class

assumes *distance-homogeneity finite-elections- \mathcal{V} d*

shows *homogeneity (distance- \mathcal{R} d C)*

<proof>

theorem (in result) tot-hom-dist-imp-hom-dr':

fixes

$d :: ('a, 'v :: \text{linorder}) \text{ Election Distance}$ and

$C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$

assumes distance-homogeneity' finite-elections- \mathcal{V} d

shows homogeneity' (distance- \mathcal{R} d C)

<proof>

end

5.7 Distance Rationalization on Election Quotients

theory Quotient-Distance-Rationalization

imports Quotient-Module

Distance-Rationalization-Symmetry

begin

5.7.1 Distances

fun distance $_{\mathcal{Q}}$:: 'x Distance \Rightarrow 'x set Distance **where**

distance $_{\mathcal{Q}}$ d A $B = (\text{if } (A = \{\} \wedge B = \{\}) \text{ then } 0 \text{ else}$
 $(\text{if } (A = \{\} \vee B = \{\}) \text{ then } \infty \text{ else}$
 $\pi_{\mathcal{Q}} (\text{tup } d) (A \times B)))$

fun relation-paths :: 'x rel \Rightarrow 'x list set **where**

relation-paths $r =$
 $\{p. \exists k. (\text{length } p = 2 * k \wedge (\forall i < k. (p!(2 * i), p!(2 * i + 1)) \in r))\}$

fun admissible-paths :: 'x rel \Rightarrow 'x set \Rightarrow 'x set \Rightarrow 'x list set **where**

admissible-paths r X $Y =$
 $\{x \# p @ [y] \mid x \ y \ p. x \in X \wedge y \in Y \wedge p \in \text{relation-paths } r\}$

fun path-length :: 'x list \Rightarrow 'x Distance \Rightarrow ereal **where**

path-length $[]$ $d = 0 \mid$
path-length $[x]$ $d = 0 \mid$
path-length $(x \# y \# xs)$ $d = d \ x \ y + \text{path-length } xs \ d$

fun quotient-dist :: 'x rel \Rightarrow 'x Distance \Rightarrow 'x set Distance **where**

quotient-dist r d A $B =$
 $\text{Inf } (\bigcup \{\{\text{path-length } p \ d \mid p. p \in \text{admissible-paths } r \ A \ B\}\})$

fun distance-infimum $_{\mathcal{Q}}$:: 'x Distance \Rightarrow 'x set Distance **where**

distance-infimum $_{\mathcal{Q}}$ d A $B = \text{Inf } \{d \ a \ b \mid a \ b. a \in A \wedge b \in B\}$

fun simple :: 'x rel \Rightarrow 'x set \Rightarrow 'x Distance \Rightarrow bool **where**

$\text{simple } r \ X \ d =$
 $(\forall \ A \in X \ // \ r.$
 $\quad (\exists \ a \in A. \forall \ B \in X \ // \ r.$
 $\quad \quad \text{distance-infimum}_{\mathcal{Q}} \ d \ A \ B = \text{Inf } \{d \ a \ b \mid b. b \in B\}))$

— We call a distance simple with respect to a relation if for all relation classes, there is an a in A that minimizes the infimum distance between A and all B such that the infimum distance between these sets coincides with the infimum distance over all b in B for a fixed a .

fun $\text{product}' :: 'x \text{ rel} \Rightarrow ('x * 'x) \text{ rel}$ **where**
 $\text{product}' \ r = \{(p_1, p_2). ((fst \ p_1, fst \ p_2) \in r \wedge snd \ p_1 = snd \ p_2)$
 $\quad \vee ((snd \ p_1, snd \ p_2) \in r \wedge fst \ p_1 = fst \ p_2)\}$

Auxiliary Lemmas

lemma *tot-dist-invariance-is-congruence:*

fixes
 $d :: 'x \text{ Distance}$ **and**
 $r :: 'x \text{ rel}$
shows $(\text{total-invariance}_{\mathcal{D}} \ d \ r) = (\text{tup } d \text{ respects } (\text{product } r))$
 $\langle \text{proof} \rangle$

lemma *product-helper:*

fixes
 $r :: 'x \text{ rel}$ **and**
 $X :: 'x \text{ set}$
shows
 $\text{trans-imp: } \text{Relation.trans } r \longrightarrow \text{Relation.trans } (\text{product } r)$ **and**
 $\text{refl-imp: } \text{refl-on } X \ r \longrightarrow \text{refl-on } (X \times X) \ (\text{product } r)$ **and**
 $\text{sym: } \text{sym-on } X \ r \longrightarrow \text{sym-on } (X \times X) \ (\text{product } r)$
 $\langle \text{proof} \rangle$

theorem *dist-pass-to-quotient:*

fixes
 $d :: 'x \text{ Distance}$ **and**
 $r :: 'x \text{ rel}$ **and**
 $X :: 'x \text{ set}$
assumes
 $\text{equiv-X-r: } \text{equiv } X \ r$ **and**
 $\text{tot-inv-dist-d-r: } \text{total-invariance}_{\mathcal{D}} \ d \ r$
shows $\forall \ A \ B. A \in X \ // \ r \wedge B \in X \ // \ r$
 $\longrightarrow (\forall \ a \ b. a \in A \wedge b \in B \longrightarrow \text{distance}_{\mathcal{Q}} \ d \ A \ B = d \ a \ b)$
 $\langle \text{proof} \rangle$

lemma *relation-paths-subset:*

fixes
 $n :: \text{nat}$ **and**
 $p :: 'x \text{ list}$ **and**
 $r :: 'x \text{ rel}$ **and**

$X :: 'x \text{ set}$
assumes $r \subseteq X \times X$
shows $\forall p. p \in \text{relation-paths } r \longrightarrow (\forall i < \text{length } p. p[i] \in X)$
 $\langle \text{proof} \rangle$

lemma *admissible-path-len*:

fixes
 $d :: 'x \text{ Distance}$ **and**
 $r :: 'x \text{ rel}$ **and**
 $X :: 'x \text{ set}$ **and**
 $a \ b :: 'x$ **and**
 $p :: 'x \text{ list}$
assumes *refl-on* $X \ r$
shows $\text{triangle-ineq } X \ d \wedge p \in \text{relation-paths } r \wedge \text{total-invariance}_{\mathcal{D}} \ d \ r$
 $\wedge a \in X \wedge b \in X \longrightarrow \text{path-length } (a \# p @ [b]) \ d \geq d \ a \ b$
 $\langle \text{proof} \rangle$

lemma *quotient-dist-coincides-with-dist_Q*:

fixes
 $d :: 'x \text{ Distance}$ **and**
 $r :: 'x \text{ rel}$ **and**
 $X :: 'x \text{ set}$
assumes
 $\text{equiv: equiv } X \ r$ **and**
 $\text{tri: triangle-ineq } X \ d$ **and**
 $\text{invar: total-invariance}_{\mathcal{D}} \ d \ r$
shows $\forall A \in X // r. \forall B \in X // r. \text{quotient-dist } r \ d \ A \ B = \text{distance}_{\mathcal{Q}} \ d \ A \ B$
 $\langle \text{proof} \rangle$

lemma *inf-dist-coincides-with-dist_Q*:

fixes
 $d :: 'x \text{ Distance}$ **and**
 $r :: 'x \text{ rel}$ **and**
 $X :: 'x \text{ set}$
assumes
 $\text{equiv-X-r: equiv } X \ r$ **and**
 $\text{tot-inv-d-r: total-invariance}_{\mathcal{D}} \ d \ r$
shows $\forall A \in X // r. \forall B \in X // r.$
 $\text{distance-infimum}_{\mathcal{Q}} \ d \ A \ B = \text{distance}_{\mathcal{Q}} \ d \ A \ B$
 $\langle \text{proof} \rangle$

lemma *inf-helper*:

fixes
 $A \ B :: 'x \text{ set}$ **and**
 $d :: 'x \text{ Distance}$
shows $\text{Inf } \{d \ a \ b \mid a \ b. a \in A \wedge b \in B\} =$
 $\text{Inf } \{\text{Inf } \{d \ a \ b \mid b. b \in B\} \mid a. a \in A\}$
 $\langle \text{proof} \rangle$

lemma *invar-dist-simple*:
fixes
 $d :: 'y$ Distance **and**
 $G :: 'x$ monoid **and**
 $Y :: 'y$ set **and**
 $\varphi :: ('x, 'y)$ binary-fun
assumes
action- φ : group-action G Y φ **and**
invar: invariance _{\mathcal{D}} d (carrier G) Y φ
shows simple (action-induced-rel (carrier G) Y φ) Y d
 \langle proof \rangle

lemma *tot-invar-dist-simple*:
fixes
 $d :: 'x$ Distance **and**
 $r :: 'x$ rel **and**
 $X :: 'x$ set
assumes
equiv-on- X : equiv X r **and**
invar: total-invariance _{\mathcal{D}} d r
shows simple r X d
 \langle proof \rangle

5.7.2 Consensus and Results

fun elections- $\mathcal{K}_{\mathcal{Q}} :: ('a, 'v)$ Election rel $\Rightarrow ('a, 'v, 'r$ Result) Consensus-Class \Rightarrow
 $('a, 'v)$ Election set set **where**
elections- $\mathcal{K}_{\mathcal{Q}}$ r $C = (\text{elections-}\mathcal{K} \ C) \ // \ r$

fun (in result) limit _{\mathcal{Q}} :: ('a, 'v) Election set $\Rightarrow 'r$ set $\Rightarrow 'r$ set **where**
limit _{\mathcal{Q}} X res = $\bigcap \{ \text{limit} \ (\text{alternatives-}\mathcal{E} \ E) \ \text{res} \mid E. E \in X \}$

Auxiliary Lemmas

lemma *closed-under-equiv-rel-subset*:
fixes
 $X \ Y \ Z :: 'x$ set **and**
 $r :: 'x$ rel
assumes
equiv X r **and**
 $Y \subseteq X$ **and**
 $Z \subseteq X$ **and**
 $Z \in Y \ // \ r$ **and**
closed-restricted-rel r X Y
shows $Z \subseteq Y$
 \langle proof \rangle

lemma (in result) *limit-invar*:
fixes
 $d :: ('a, 'v)$ Election Distance **and**

$r :: ('a, 'v)$ Election rel **and**
 $C :: ('a, 'v, 'r)$ Result Consensus-Class **and**
 $X A :: ('a, 'v)$ Election set
assumes
 quot-class: $A \in X // r$ **and**
 equiv-rel: equiv $X r$ **and**
 cons-subset: elections- \mathcal{K} $C \subseteq X$ **and**
 invar-res: is-symmetry $(\lambda E. \text{limit } (\text{alternatives-}\mathcal{E} \ E) \ \text{UNIV})$ (Invariance r)
shows $\forall a \in A. \text{limit } (\text{alternatives-}\mathcal{E} \ a) \ \text{UNIV} = \text{limit}_{\mathcal{Q}} A \ \text{UNIV}$
 <proof>

lemma (in result) preimg-invar:

fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $\text{domain}_f X :: 'x$ set **and**
 $d :: 'x$ Distance **and**
 $r :: 'x$ rel
assumes
 equiv-rel: equiv $X r$ **and**
 cons-subset: $\text{domain}_f \subseteq X$ **and**
 closed-domain: closed-restricted-rel $r X \text{domain}_f$ **and**
 invar-f: is-symmetry f (Invariance (Restr $r \text{domain}_f$))
shows $\forall y. (\text{preimg } f \text{domain}_f y) // r = \text{preimg } (\pi_{\mathcal{Q}} f) (\text{domain}_f // r) y$
 <proof>

lemma minimizer-helper:

fixes
 $f :: 'x \Rightarrow 'y$ **and**
 $\text{domain}_f :: 'x$ set **and**
 $d :: 'x$ Distance **and**
 $Y :: 'y$ set **and**
 $x :: 'x$ **and**
 $y :: 'y$
shows $y \in \text{minimizer } f \text{domain}_f d Y \ x =$
 $(y \in Y \wedge (\forall y' \in Y. \text{Inf } (d \ x \ ' (\text{preimg } f \text{domain}_f y)) \leq \text{Inf } (d \ x \ ' (\text{preimg } f \text{domain}_f y'))))$
 <proof>

lemma rewr-singleton-set-system-union:

fixes
 $Y :: 'x$ set set **and**
 $X :: 'x$ set
assumes $Y \subseteq \text{singleton-set-system } X$
shows
 singleton-set-union: $x \in \bigcup Y \longleftrightarrow \{x\} \in Y$ **and**
 obtain-singleton: $A \in \text{singleton-set-system } X \longleftrightarrow (\exists x \in X. A = \{x\})$
 <proof>

lemma union-inf:

fixes $X :: \text{ereal set set}$
shows $\text{Inf } \{\text{Inf } A \mid A. A \in X\} = \text{Inf } (\bigcup X)$
 $\langle \text{proof} \rangle$

5.7.3 Distance Rationalization

fun (**in result**) $\mathcal{R}_{\mathcal{Q}} :: ('a, 'v) \text{ Election rel} \Rightarrow ('a, 'v) \text{ Election Distance} \Rightarrow$
 $('a, 'v, 'r \text{ Result}) \text{ Consensus-Class} \Rightarrow ('a, 'v) \text{ Election set} \Rightarrow 'r \text{ set}$ **where**
 $\mathcal{R}_{\mathcal{Q}} r d C A =$
 $\bigcup (\text{minimizer } (\pi_{\mathcal{Q}} (\text{elect-}r \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} C))) (\text{elections-}\mathcal{K}_{\mathcal{Q}} r C)$
 $(\text{distance-infimum}_{\mathcal{Q}} d) (\text{singleton-set-system } (\text{limit}_{\mathcal{Q}} A \text{ UNIV})) A)$

fun (**in result**) $\text{distance-}\mathcal{R}_{\mathcal{Q}} :: ('a, 'v) \text{ Election rel} \Rightarrow ('a, 'v) \text{ Election Distance} \Rightarrow$
 $('a, 'v, 'r \text{ Result}) \text{ Consensus-Class} \Rightarrow ('a, 'v) \text{ Election set} \Rightarrow 'r \text{ Result}$ **where**
 $\text{distance-}\mathcal{R}_{\mathcal{Q}} r d C A =$
 $(\mathcal{R}_{\mathcal{Q}} r d C A,$
 $\pi_{\mathcal{Q}} (\lambda E. \text{limit } (\text{alternatives-}\mathcal{E} E) \text{ UNIV}) A - \mathcal{R}_{\mathcal{Q}} r d C A,$
 $\{\})$

Proposition 4.17 by Hadjibeyli and Wilson [3].

theorem (**in result**) *invar-dr-simple-dist-imp-quotient-dr-winners:*

fixes

$d :: ('a, 'v) \text{ Election Distance}$ **and**
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$ **and**
 $r :: ('a, 'v) \text{ Election rel}$ **and**
 $X A :: ('a, 'v) \text{ Election set}$

assumes

simple: $\text{simple } r X d$ **and**
closed-domain: $\text{closed-restricted-rel } r X (\text{elections-}\mathcal{K} C)$ **and**
invar-res:
 $\text{is-symmetry } (\lambda E. \text{limit } (\text{alternatives-}\mathcal{E} E) \text{ UNIV}) (\text{Invariance } r)$ **and**
 $\text{invar-}C: \text{is-symmetry } (\text{elect-}r \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} C))$
 $(\text{Invariance } (\text{Restr } r (\text{elections-}\mathcal{K} C)))$ **and**
invar-dr: $\text{is-symmetry } (\text{fun}_{\mathcal{E}} (\mathcal{R}_{\mathcal{W}} d C)) (\text{Invariance } r)$ **and**
quot-class: $A \in X // r$ **and**
equiv-rel: $\text{equiv } X r$ **and**
cons-subset: $\text{elections-}\mathcal{K} C \subseteq X$

shows $\pi_{\mathcal{Q}} (\text{fun}_{\mathcal{E}} (\mathcal{R}_{\mathcal{W}} d C)) A = \mathcal{R}_{\mathcal{Q}} r d C A$

$\langle \text{proof} \rangle$

theorem (**in result**) *invar-dr-simple-dist-imp-quotient-dr:*

fixes

$d :: ('a, 'v) \text{ Election Distance}$ **and**
 $C :: ('a, 'v, 'r \text{ Result}) \text{ Consensus-Class}$ **and**
 $r :: ('a, 'v) \text{ Election rel}$ **and**
 $X A :: ('a, 'v) \text{ Election set}$

assumes

simple: $\text{simple } r X d$ **and**
closed-domain: $\text{closed-restricted-rel } r X (\text{elections-}\mathcal{K} C)$ **and**

```

invar-res:
  is-symmetry ( $\lambda E. \text{limit } (\text{alternatives-}\mathcal{E} \ E) \ \text{UNIV}$ )
    (Invariance  $r$ ) and
invar-C: is-symmetry ( $\text{elect-}r \circ \text{fun}_{\mathcal{E}} (\text{rule-}\mathcal{K} \ C)$ )
    (Invariance ( $\text{Restr } r \ (\text{elections-}\mathcal{K} \ C)$ )) and
invar-dr: is-symmetry ( $\text{fun}_{\mathcal{E}} (\mathcal{R}_{\mathcal{W}} \ d \ C)$ ) (Invariance  $r$ ) and
quot-class:  $A \in X // r$  and
equiv-rel: equiv  $X \ r$  and
cons-subset:  $\text{elections-}\mathcal{K} \ C \subseteq X$ 
shows  $\pi_{\mathcal{Q}} (\text{fun}_{\mathcal{E}} (\text{distance-}\mathcal{R} \ d \ C)) \ A = \text{distance-}\mathcal{R}_{\mathcal{Q}} \ r \ d \ C \ A$ 
<proof>

end

```

5.8 Code Generation Interpretations for Results and Properties

```

theory Interpretation-Code
  imports Electoral-Module
    Distance-Rationalization
begin
<ML>

```

5.8.1 Code Lemmas

Lemmas stating the explicit instantiations of interpreted abstract functions from locales.

```

lemma electoral-module-SCF-code-lemma:
  fixes  $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ 
  shows  $\text{SCF-result.electoral-module } m =$ 
    ( $\forall A \ V \ p. \text{profile } V \ A \ p \longrightarrow \text{well-formed-SCF } A \ (m \ V \ A \ p)$ )
  <proof>

```

```

lemma  $\mathcal{R}_{\mathcal{W}}$ -SCF-code-lemma:
  fixes
     $d :: ('a, 'v) \text{ Election Distance}$  and
     $K :: ('a, 'v, 'a \text{ Result}) \text{ Consensus-Class}$  and
     $V :: 'v \text{ set}$  and
     $A :: 'a \text{ set}$  and
     $p :: ('a, 'v) \text{ Profile}$ 
  shows  $\text{SCF-result.}\mathcal{R}_{\mathcal{W}} \ d \ K \ V \ A \ p =$ 
     $\text{arg-min-set } (\text{score } d \ K \ (A, V, p)) \ (\text{limit-SCF } A \ \text{UNIV})$ 
  <proof>

```

```

lemma distance- $\mathcal{R}$ -SCF-code-lemma:
  fixes

```

$d :: ('a, 'v)$ Election Distance **and**
 $K :: ('a, 'v, 'a \text{ Result})$ Consensus-Class **and**
 $V :: 'v \text{ set}$ **and**
 $A :: 'a \text{ set}$ **and**
 $p :: ('a, 'v)$ Profile
shows $SCF\text{-result.distance-}\mathcal{R} \ d \ K \ V \ A \ p =$
 $(SCF\text{-result.}\mathcal{R}_{\mathcal{W}} \ d \ K \ V \ A \ p,$
 $(\text{limit-}SCF \ A \ UNIV) - SCF\text{-result.}\mathcal{R}_{\mathcal{W}} \ d \ K \ V \ A \ p,$
 $\{\})$
 $\langle \text{proof} \rangle$

lemma $\mathcal{R}_{\mathcal{W}}\text{-std-}SCF\text{-code-lemma}$:

fixes
 $d :: ('a, 'v)$ Election Distance **and**
 $K :: ('a, 'v, 'a \text{ Result})$ Consensus-Class **and**
 $V :: 'v \text{ set}$ **and**
 $A :: 'a \text{ set}$ **and**
 $p :: ('a, 'v)$ Profile
shows $SCF\text{-result.}\mathcal{R}_{\mathcal{W}}\text{-std} \ d \ K \ V \ A \ p =$
 $\text{arg-min-set} \ (\text{score-std} \ d \ K \ (A, V, p)) \ (\text{limit-}SCF \ A \ UNIV)$
 $\langle \text{proof} \rangle$

lemma $\text{distance-}\mathcal{R}\text{-std-}SCF\text{-code-lemma}$:

fixes
 $d :: ('a, 'v)$ Election Distance **and**
 $K :: ('a, 'v, 'a \text{ Result})$ Consensus-Class **and**
 $V :: 'v \text{ set}$ **and**
 $A :: 'a \text{ set}$ **and**
 $p :: ('a, 'v)$ Profile
shows $SCF\text{-result.distance-}\mathcal{R}\text{-std} \ d \ K \ V \ A \ p =$
 $(SCF\text{-result.}\mathcal{R}_{\mathcal{W}}\text{-std} \ d \ K \ V \ A \ p,$
 $(\text{limit-}SCF \ A \ UNIV) - SCF\text{-result.}\mathcal{R}_{\mathcal{W}}\text{-std} \ d \ K \ V \ A \ p,$
 $\{\})$
 $\langle \text{proof} \rangle$

lemma $\text{anonymity-}SCF\text{-code-lemma}$: $SCF\text{-result.anonymity} =$

$(\lambda \ m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module.}$
 $SCF\text{-result.electoral-module} \ m \wedge$
 $(\forall \ A \ V \ p \ \pi :: ('v \Rightarrow 'v).$
 $\text{bij } \pi \longrightarrow (\text{let } (A', V', q) = (\text{rename } \pi \ (A, V, p)) \text{ in}$
 $\text{profile } V \ A \ p \wedge \text{profile } V' \ A' \ q \longrightarrow m \ V \ A \ p = m \ V' \ A' \ q)))$
 $\langle \text{proof} \rangle$

5.8.2 Interpretation Declarations and Constants

Declarations for replacing interpreted abstract functions from locales by their explicit instantiations.

declare $[[lc\text{-add } SCF\text{-result.electoral-module} \ \text{electoral-module-}SCF\text{-code-lemma}]]$

declare $[[lc\text{-add } SCF\text{-result.}\mathcal{R}_{\mathcal{W}} \ \mathcal{R}_{\mathcal{W}}\text{-}SCF\text{-code-lemma}]]$

```

declare [[lc-add SCF-result.RW-std RW-std-SCF-code-lemma]]
declare [[lc-add SCF-result.distance-R distance-R-SCF-code-lemma]]
declare [[lc-add SCF-result.distance-R-std distance-R-std-SCF-code-lemma]]
declare [[lc-add SCF-result.anonymity anonymity-SCF-code-lemma]]

```

Constant aliases to use instead of the interpreted functions.

```

definition RW-SCF-code  $\equiv$  SCF-result.RW
definition RW-std-SCF-code  $\equiv$  SCF-result.RW-std
definition distance-R-SCF-code  $\equiv$  SCF-result.distance-R
definition distance-R-std-SCF-code  $\equiv$  SCF-result.distance-R-std
definition electoral-module-SCF-code  $\equiv$  SCF-result.electoral-module
definition anonymity-SCF-code  $\equiv$  SCF-result.anonymity

```

$\langle ML \rangle$

end

5.9 Drop Module

```

theory Drop-Module
  imports Component-Types/Electoral-Module
           Component-Types/Social-Choice-Types/Result
begin

```

This is a family of electoral modules. For a natural number n and a lexicon (linear order) r of all alternatives, the according drop module rejects the lexicographically first n alternatives (from A) and defers the rest. It is primarily used as counterpart to the pass module in a parallel composition, in order to segment the alternatives into two groups.

5.9.1 Definition

```

fun drop-module :: nat  $\Rightarrow$  'a Preference-Relation  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module where
  drop-module n r V A p =
    ({},
     {a  $\in$  A. rank (limit A r) a  $\leq$  n},
     {a  $\in$  A. rank (limit A r) a  $>$  n})

```

5.9.2 Soundness

```

theorem drop-mod-sound[simp]:
  fixes
    r :: 'a Preference-Relation and

```



```

    n :: nat
  shows SCF-result.electoral-module (drop-module n r)
  <proof>

```

```

lemma voters-determine-drop-mod:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  shows voters-determine-election (drop-module n r)
  <proof>

```

5.9.3 Non-Electing

The drop module is non-electing.

```

theorem drop-mod-non-electing[simp]:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  shows non-electing (drop-module n r)
  <proof>

```

5.9.4 Properties

The drop module is strictly defer-monotone.

```

theorem drop-mod-def-lift-inv[simp]:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  shows defer-lift-invariance (drop-module n r)
  <proof>

```

end

5.10 Pass Module

```

theory Pass-Module
  imports Component-Types/Electoral-Module
begin

```

This is a family of electoral modules. For a natural number n and a lexicon (linear order) r of all alternatives, the according pass module defers the lexicographically first n alternatives (from A) and rejects the rest. It is primarily used as counterpart to the drop module in a parallel composition in order to segment the alternatives into two groups.

5.10.1 Definition

```

fun pass-module :: nat  $\Rightarrow$  'a Preference-Relation  $\Rightarrow$ 
  ('a, 'v, 'a Result) Electoral-Module where
  pass-module n r V A p =
    ({},
     {a  $\in$  A. rank (limit A r) a > n},
     {a  $\in$  A. rank (limit A r) a  $\leq$  n})

```

5.10.2 Soundness

```

theorem pass-mod-sound[simp]:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  shows SCF-result.electoral-module (pass-module n r)
  <proof>

```

```

lemma voters-determine-pass-mod:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  shows voters-determine-election (pass-module n r)
  <proof>

```

5.10.3 Non-Blocking

The pass module is non-blocking.

```

theorem pass-mod-non-blocking[simp]:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  assumes
    order: linear-order r and
    greater-zero: n > 0
  shows non-blocking (pass-module n r)
  <proof>

```

5.10.4 Non-Electing

The pass module is non-electing.

```

theorem pass-mod-non-electing[simp]:
  fixes
    r :: 'a Preference-Relation and
    n :: nat
  assumes linear-order r
  shows non-electing (pass-module n r)
  <proof>

```

5.10.5 Properties

The pass module is strictly defer-monotone.

```
theorem pass-mod-dl-inv[simp]:  
  fixes  
    r :: 'a Preference-Relation and  
    n :: nat  
  assumes linear-order r  
  shows defer-lift-invariance (pass-module n r)  
  ⟨proof⟩
```

```
theorem pass-zero-mod-def-zero[simp]:  
  fixes r :: 'a Preference-Relation  
  assumes linear-order r  
  shows defers 0 (pass-module 0 r)  
  ⟨proof⟩
```

For any natural number n and any linear order, the according pass module defers n alternatives (if there are n alternatives). NOTE: The induction proof is still missing. The following are the proofs for $n=1$ and $n=2$.

```
theorem pass-one-mod-def-one[simp]:  
  fixes r :: 'a Preference-Relation  
  assumes linear-order r  
  shows defers 1 (pass-module 1 r)  
  ⟨proof⟩
```

```
theorem pass-two-mod-def-two:  
  fixes r :: 'a Preference-Relation  
  assumes linear-order r  
  shows defers 2 (pass-module 2 r)  
  ⟨proof⟩
```

end

5.11 Elect Module

```
theory Elect-Module  
  imports Component-Types/Electoral-Module  
begin
```

The elect module is not concerned about the voter's ballots, and just elects all alternatives. It is primarily used in sequence after an electoral module that only defers alternatives to finalize the decision, thereby inducing a proper voting rule in the social choice sense.

5.11.1 Definition

fun *elect-module* :: ('a, 'v, 'a Result) Electoral-Module **where**
elect-module V A p = (A, {}, {})

5.11.2 Soundness

theorem *elect-mod-sound[simp]*: SCF-result.electoral-module *elect-module*
 ⟨proof⟩

lemma *elect-mod-only-voters: voters-determine-election elect-module*
 ⟨proof⟩

5.11.3 Electing

theorem *elect-mod-electing[simp]*: *electing elect-module*
 ⟨proof⟩

end

5.12 Plurality Module

theory *Plurality-Module*
imports *Component-Types/Elimination-Module*
begin

The plurality module implements the plurality voting rule. The plurality rule elects all modules with the maximum amount of top preferences among all alternatives, and rejects all the other alternatives. It is electing and induces the classical plurality (voting) rule from social-choice theory.

5.12.1 Definition

fun *plurality-score* :: ('a, 'v) Evaluation-Function **where**
plurality-score V x A p = win-count V p x

fun *plurality* :: ('a, 'v, 'a Result) Electoral-Module **where**
plurality V A p = max-eliminator *plurality-score* V A p

fun *plurality'* :: ('a, 'v, 'a Result) Electoral-Module **where**
plurality' V A p =
 (if finite A
 then {},
 {a ∈ A. ∃ x ∈ A. win-count V p x > win-count V p a},
 {a ∈ A. ∀ x ∈ A. win-count V p x ≤ win-count V p a})

else ($\{\}, \{\}, A$)

lemma *enat-leq-enat-set-max*:

fixes

$x :: \text{enat}$ **and**

$X :: \text{enat set}$

assumes

$x \in X$ **and**

finite X

shows $x \leq \text{Max } X$

$\langle \text{proof} \rangle$

lemma *plurality-mod-equiv*:

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p :: ('a, 'v) \text{ Profile}$

shows $\text{plurality } V A p = \text{plurality}' V A p$

$\langle \text{proof} \rangle$

5.12.2 Soundness

theorem *plurality-sound[simp]*: *SCF-result.electoral-module plurality*

$\langle \text{proof} \rangle$

theorem *plurality'-sound[simp]*: *SCF-result.electoral-module plurality'*

$\langle \text{proof} \rangle$

lemma *voters-determine-plurality-score*: *voters-determine-evaluation plurality-score*

$\langle \text{proof} \rangle$

lemma *voters-determine-plurality*: *voters-determine-election plurality*

$\langle \text{proof} \rangle$

lemma *voters-determine-plurality'*: *voters-determine-election plurality'*

$\langle \text{proof} \rangle$

5.12.3 Non-Blocking

The plurality module is non-blocking.

theorem *plurality-mod-non-blocking[simp]*: *non-blocking plurality*

$\langle \text{proof} \rangle$

theorem *plurality'-mod-non-blocking[simp]*: *non-blocking plurality'*

$\langle \text{proof} \rangle$

5.12.4 Non-Electing

The plurality module is non-electing.

theorem *plurality-non-electing[simp]: non-electing plurality*
 ⟨proof⟩

theorem *plurality'-non-electing[simp]: non-electing plurality'*
 ⟨proof⟩

5.12.5 Property

lemma *plurality-def-inv-mono-alts:*

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p \ q :: ('a, 'v) \text{ Profile}$ **and**

$a :: 'a$

assumes

defer-a: $a \in \text{defer plurality } V \ A \ p$ **and**

lift-a: $\text{lifted } V \ A \ p \ q \ a$

shows $\text{defer plurality } V \ A \ q = \text{defer plurality } V \ A \ p$
 $\vee \text{defer plurality } V \ A \ q = \{a\}$

⟨proof⟩

lemma *plurality'-def-inv-mono-alts:*

fixes

$A :: 'a \text{ set}$ **and**

$V :: 'v \text{ set}$ **and**

$p \ q :: ('a, 'v) \text{ Profile}$ **and**

$a :: 'a$

assumes

$a \in \text{defer plurality}' \ V \ A \ p$ **and**

lifted $V \ A \ p \ q \ a$

shows $\text{defer plurality}' \ V \ A \ q = \text{defer plurality}' \ V \ A \ p$
 $\vee \text{defer plurality}' \ V \ A \ q = \{a\}$

⟨proof⟩

The plurality rule is invariant-monotone.

theorem *plurality-mod-def-inv-mono[simp]: defer-invariant-monotonicity plurality*
 ⟨proof⟩

theorem *plurality'-mod-def-inv-mono[simp]: defer-invariant-monotonicity plurality'*
 ⟨proof⟩

end

5.13 Borda Module

```
theory Borda-Module
imports Component-Types/Elimination-Module
begin
```

This is the Borda module used by the Borda rule. The Borda rule is a voting rule, where on each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.13.1 Definition

```
fun borda-score :: ('a, 'v) Evaluation-Function where
  borda-score V x A p = ( $\sum$  y  $\in$  A. (prefer-count V p x y))

fun borda :: ('a, 'v, 'a Result) Electoral-Module where
  borda V A p = max-eliminator borda-score V A p
```

5.13.2 Soundness

```
theorem borda-sound: SCF-result.electoral-module borda
  <proof>
```

5.13.3 Non-Blocking

The Borda module is non-blocking.

```
theorem borda-mod-non-blocking[simp]: non-blocking borda
  <proof>
```

5.13.4 Non-Electing

The Borda module is non-electing.

```
theorem borda-mod-non-electing[simp]: non-electing borda
  <proof>
```

```
end
```

5.14 Condorcet Module

```
theory Condorcet-Module
```

```

imports Component-Types/Elimination-Module
begin

```

This is the Condorcet module used by the Condorcet (voting) rule. The Condorcet rule is a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.14.1 Definition

```

fun condorcet-score :: ('a, 'v) Evaluation-Function where
  condorcet-score V x A p =
    (if (condorcet-winner V A p x) then 1 else 0)

```

```

fun condorcet :: ('a, 'v, 'a Result) Electoral-Module where
  condorcet V A p = (max-eliminator condorcet-score) V A p

```

5.14.2 Soundness

```

theorem condorcet-sound: SCF-result.electoral-module condorcet
  <proof>

```

5.14.3 Property

```

theorem condorcet-score-is-condorcet-rating: condorcet-rating condorcet-score
  <proof>

```

```

theorem condorcet-is-dcc: defer-condorcet-consistency condorcet
  <proof>

```

```

end

```

5.15 Copeland Module

```

theory Copeland-Module
  imports Component-Types/Elimination-Module
begin

```

This is the Copeland module used by the Copeland voting rule. The Copeland rule elects the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses. The module implemented herein only rejects the alternatives not elected by the voting

rule, and defers the alternatives that would be elected by the full voting rule.

5.15.1 Definition

fun *copeland-score* :: ('a, 'v) *Evaluation-Function* **where**
copeland-score V x A p =
 card {y ∈ A . wins V x p y} − card {y ∈ A . wins V y p x}

fun *copeland* :: ('a, 'v, 'a *Result*) *Electoral-Module* **where**
copeland V A p = *max-eliminator copeland-score* V A p

5.15.2 Soundness

theorem *copeland-sound*: *SCF-result.electoral-module copeland*
 ⟨*proof*⟩

5.15.3 Lemmas

lemma *voters-determine-copeland-score*: *voters-determine-evaluation copeland-score*
 ⟨*proof*⟩

theorem *voters-determine-copeland*: *voters-determine-election copeland*
 ⟨*proof*⟩

For a Condorcet winner w, we have: " $|\{y \in A . \text{wins } V w p y\}| = |A| - 1$ ".

lemma *cond-winner-imp-win-count*:

fixes
 A :: 'a *set* **and**
 V :: 'v *set* **and**
 p :: ('a, 'v) *Profile* **and**
 w :: 'a
assumes *condorcet-winner* V A p w
shows card {a ∈ A. wins V w p a} = card A − 1
 ⟨*proof*⟩

For a Condorcet winner w, we have: " $|\{y \in A . \text{wins } V y p w\}| = 0$ ".

lemma *cond-winner-imp-loss-count*:

fixes
 A :: 'a *set* **and**
 V :: 'v *set* **and**
 p :: ('a, 'v) *Profile* **and**
 w :: 'a
assumes *condorcet-winner* V A p w
shows card {a ∈ A. wins V a p w} = 0
 ⟨*proof*⟩

Copeland score of a Condorcet winner.

lemma *cond-winner-imp-copeland-score*:

```

fixes
   $A :: 'a \text{ set}$  and
   $V :: 'v \text{ set}$  and
   $p :: ('a, 'v) \text{ Profile}$  and
   $w :: 'a$ 
assumes condorcet-winner  $V A p w$ 
shows copeland-score  $V w A p = \text{card } A - 1$ 
 $\langle \text{proof} \rangle$ 

```

For a non-Condorcet winner l , we have: $|\{y \in A . \text{wins } V l p y\}| = |A| - 2$.

```

lemma non-cond-winner-imp-win-count:
fixes
   $A :: 'a \text{ set}$  and
   $V :: 'v \text{ set}$  and
   $p :: ('a, 'v) \text{ Profile}$  and
   $w l :: 'a$ 
assumes
  winner: condorcet-winner  $V A p w$  and
  loser:  $l \neq w$  and
  l-in-A:  $l \in A$ 
shows  $\text{card } \{a \in A . \text{wins } V l p a\} \leq \text{card } A - 2$ 
 $\langle \text{proof} \rangle$ 

```

5.15.4 Property

The Copeland score is Condorcet rating.

```

theorem copeland-score-is-cr: condorcet-rating copeland-score
 $\langle \text{proof} \rangle$ 

```

```

theorem copeland-is-dcc: defer-condorcet-consistency copeland
 $\langle \text{proof} \rangle$ 

```

end

5.16 Minimax Module

```

theory Minimax-Module
imports Component-Types/Elimination-Module
begin

```

This is the Minimax module used by the Minimax voting rule. The Minimax rule elects the alternatives with the highest Minimax score. The module implemented herein only rejects the alternatives not elected by the voting

rule, and defers the alternatives that would be elected by the full voting rule.

5.16.1 Definition

fun *minimax-score* :: ('a, 'v) *Evaluation-Function* **where**
minimax-score $V\ x\ A\ p =$
 $Min\ \{prefer-count\ V\ p\ x\ y\ |\ y.\ y \in A - \{x\}\}$

fun *minimax* :: ('a, 'v, 'a *Result*) *Electoral-Module* **where**
minimax $A\ p = max-eliminator\ minimax-score\ A\ p$

5.16.2 Soundness

theorem *minimax-sound*: *SCF-result.electoral-module minimax*
 $\langle proof \rangle$

5.16.3 Lemma

lemma *non-cond-winner-minimax-score*:
fixes
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$ **and**
 $w\ l :: 'a$
assumes
 $prof: profile\ V\ A\ p$ **and**
 $winner: condorcet-winner\ V\ A\ p\ w$ **and**
 $l-in-A: l \in A$ **and**
 $l-neq-w: l \neq w$
shows $minimax-score\ V\ l\ A\ p \leq prefer-count\ V\ p\ l\ w$
 $\langle proof \rangle$

5.16.4 Property

theorem *minimax-score-cond-rating*: *condorcet-rating minimax-score*
 $\langle proof \rangle$

theorem *minimax-is-dcc*: *defer-condorcet-consistency minimax*
 $\langle proof \rangle$

end

Chapter 6

Compositional Structures

6.1 Drop- and Pass-Compatibility

```
theory Drop-And-Pass-Compatibility
  imports Basic-Modules/Drop-Module
           Basic-Modules/Pass-Module
begin
```

This is a collection of properties about the interplay and compatibility of both the drop module and the pass module.

```
theorem drop-zero-mod-rej-zero[simp]:
  fixes  $r :: 'a \text{ Preference-Relation}$ 
  assumes linear-order  $r$ 
  shows rejects 0 (drop-module 0  $r$ )
  <proof>
```

The drop module rejects n alternatives (if there are at least n alternatives).

```
theorem drop-two-mod-rej-n[simp]:
  fixes  $r :: 'a \text{ Preference-Relation}$ 
  assumes linear-order  $r$ 
  shows rejects  $n$  (drop-module  $n$   $r$ )
  <proof>
```

The pass and drop module are (disjoint-)compatible.

```
theorem drop-pass-disj-compat[simp]:
  fixes
     $r :: 'a \text{ Preference-Relation}$  and
     $n :: \text{nat}$ 
  assumes linear-order  $r$ 
  shows disjoint-compatibility (drop-module  $n$   $r$ ) (pass-module  $n$   $r$ )
  <proof>
```

```
end
```

6.2 Revision Composition

```
theory Revision-Composition
  imports Basic-Modules/Component-Types/Electoral-Module
begin
```

A revised electoral module rejects all originally rejected or deferred alternatives, and defers the originally elected alternatives. It does not elect any alternatives.

6.2.1 Definition

```
fun revision-composition :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
  ('a, 'v, 'a Result) Electoral-Module where
  revision-composition m V A p = ({}, A - elect m V A p, elect m V A p)
```

```
abbreviation rev :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
  ('a, 'v, 'a Result) Electoral-Module (- $\downarrow$  50) where
   $m\downarrow \equiv \text{revision-composition } m$ 
```

6.2.2 Soundness

```
theorem rev-comp-sound[simp]:
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  assumes SCF-result.electoral-module m
  shows SCF-result.electoral-module (revision-composition m)
   $\langle \text{proof} \rangle$ 
```

```
lemma voters-determine-rev-comp:
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  assumes voters-determine-election m
  shows voters-determine-election (revision-composition m)
   $\langle \text{proof} \rangle$ 
```

6.2.3 Composition Rules

An electoral module received by revision is never electing.

```
theorem rev-comp-non-electing[simp]:
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  assumes SCF-result.electoral-module m
  shows non-electing ( $m\downarrow$ )
   $\langle \text{proof} \rangle$ 
```

Revising an electing electoral module results in a non-blocking electoral module.

theorem *rev-comp-non-blocking[simp]*:
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes *electing* m
shows *non-blocking* $(m \downarrow)$
 $\langle \text{proof} \rangle$

Revising an invariant monotone electoral module results in a defer-invariant-monotone electoral module.

theorem *rev-comp-def-inv-mono[simp]*:
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes *invariant-monotonicity* m
shows *defer-invariant-monotonicity* $(m \downarrow)$
 $\langle \text{proof} \rangle$

end

6.3 Sequential Composition

theory *Sequential-Composition*
imports *Basic-Modules/Component-Types/Electoral-Module*
begin

The sequential composition creates a new electoral module from two electoral modules. In a sequential composition, the second electoral module makes decisions over alternatives deferred by the first electoral module.

6.3.1 Definition

fun *sequential-composition* :: $('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \text{ where}$
 $\text{sequential-composition } m \ n \ V \ A \ p =$
 $(\text{let } \text{new-}A = \text{defer } m \ V \ A \ p;$
 $\text{new-}p = \text{limit-profile } \text{new-}A \ p \text{ in } ($
 $\text{elect } m \ V \ A \ p) \cup (\text{elect } n \ V \ \text{new-}A \ \text{new-}p),$
 $(\text{reject } m \ V \ A \ p) \cup (\text{reject } n \ V \ \text{new-}A \ \text{new-}p),$
 $\text{defer } n \ V \ \text{new-}A \ \text{new-}p))$

abbreviation *sequence* :: $('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 $(('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
 $(\text{infix } \triangleright \ 50) \text{ where}$
 $m \triangleright n \equiv \text{sequential-composition } m \ n$

fun *sequential-composition'* :: $('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$

$(\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module} \Rightarrow$
 $(\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module} \text{ where}$
 $\text{sequential-composition}' m n V A p =$
 $(\text{let } (m\text{-e}, m\text{-r}, m\text{-d}) = m \text{ V } A p; \text{new-A} = m\text{-d};$
 $\text{new-p} = \text{limit-profile new-A } p;$
 $(n\text{-e}, n\text{-r}, n\text{-d}) = n \text{ V } \text{new-A new-p in}$
 $(m\text{-e} \cup n\text{-e}, m\text{-r} \cup n\text{-r}, n\text{-d}))$

lemma *voters-determine-seq-comp*:

fixes $m n :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$
assumes *voters-determine-election* $m \wedge \text{voters-determine-election } n$
shows *voters-determine-election* $(m \triangleright n)$
 $\langle \text{proof} \rangle$

lemma *seq-comp-presv-disj*:

fixes
 $m n :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **and**
 $A :: \text{'a set}$ **and**
 $V :: \text{'v set}$ **and**
 $p :: (\text{'a}, \text{'v}) \text{ Profile}$
assumes
 $\text{module-m: SCF-result.electoral-module } m$ **and**
 $\text{module-n: SCF-result.electoral-module } n$ **and**
 $\text{prof: profile } V A p$
shows *disjoint3* $((m \triangleright n) \text{ V } A p)$
 $\langle \text{proof} \rangle$

lemma *seq-comp-presv-alt*:

fixes
 $m n :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **and**
 $A :: \text{'a set}$ **and**
 $V :: \text{'v set}$ **and**
 $p :: (\text{'a}, \text{'v}) \text{ Profile}$
assumes
 $\text{module-m: SCF-result.electoral-module } m$ **and**
 $\text{module-n: SCF-result.electoral-module } n$ **and**
 $\text{prof: profile } V A p$
shows *set-equals-partition* $A ((m \triangleright n) \text{ V } A p)$
 $\langle \text{proof} \rangle$

lemma *seq-comp-alt-eq*[*fundef-cong*, *code*]: *sequential-composition* = *sequential-composition'*
 $\langle \text{proof} \rangle$

6.3.2 Soundness

theorem *seq-comp-sound*[*simp*]:

fixes $m n :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$
assumes
 $\text{SCF-result.electoral-module } m$ **and**

$SCF\text{-}result.electoral\text{-}module\ n$
shows $SCF\text{-}result.electoral\text{-}module\ (m \triangleright n)$
 $\langle proof \rangle$

6.3.3 Lemmas

lemma *seq-comp-decrease-only-defer*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $module\text{-}m: SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $module\text{-}n: SCF\text{-}result.electoral\text{-}module\ n$ **and**
 $prof: profile\ V\ A\ p$ **and**
 $empty\text{-}defer: defer\ m\ V\ A\ p = \{\}$
shows $(m \triangleright n)\ V\ A\ p = m\ V\ A\ p$
 $\langle proof \rangle$

lemma *seq-comp-def-then-elect*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $n\text{-}electing\text{-}m: non\text{-}electing\ m$ **and**
 $def\text{-}one\text{-}m: defers\ 1\ m$ **and**
 $electing\text{-}n: electing\ n$ **and**
 $f\text{-}prof: finite\text{-}profile\ V\ A\ p$
shows $elect\ (m \triangleright n)\ V\ A\ p = defer\ m\ V\ A\ p$
 $\langle proof \rangle$

lemma *seq-comp-def-card-bounded*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $SCF\text{-}result.electoral\text{-}module\ n$ **and**
 $finite\text{-}profile\ V\ A\ p$
shows $card\ (defer\ (m \triangleright n)\ V\ A\ p) \leq card\ (defer\ m\ V\ A\ p)$
 $\langle proof \rangle$

lemma *seq-comp-def-set-bounded*:
fixes

$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $SCF\text{-}result.electoral\text{-}module\ n$ **and**
 $profile\ V\ A\ p$
shows $defer\ (m \triangleright n)\ V\ A\ p \subseteq defer\ m\ V\ A\ p$
 $\langle proof \rangle$

lemma *seq-comp-defers-def-set*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
shows $defer\ (m \triangleright n)\ V\ A\ p =$
 $defer\ n\ V\ (defer\ m\ V\ A\ p)\ (limit\text{-}profile\ (defer\ m\ V\ A\ p)\ p)$
 $\langle proof \rangle$

lemma *seq-comp-def-then-elect-elec-set*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
shows $elect\ (m \triangleright n)\ V\ A\ p =$
 $elect\ n\ V\ (defer\ m\ V\ A\ p)\ (limit\text{-}profile\ (defer\ m\ V\ A\ p)\ p) \cup (elect\ m\ V\ A\ p)$
 $\langle proof \rangle$

lemma *seq-comp-elim-one-red-def-set*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$
assumes
 $SCF\text{-}result.electoral\text{-}module\ m$ **and**
 $eliminates\ 1\ n$ **and**
 $profile\ V\ A\ p$ **and**
 $card\ (defer\ m\ V\ A\ p) > 1$
shows $defer\ (m \triangleright n)\ V\ A\ p \subset defer\ m\ V\ A\ p$
 $\langle proof \rangle$

lemma *seq-comp-def-set-trans*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**

$A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
 $a \in (\text{defer } (m \triangleright n) \ V \ A \ p)$ **and**
 $\text{SCF-result.electoral-module } m \wedge \text{SCF-result.electoral-module } n$ **and**
 $\text{profile } V \ A \ p$
shows $a \in \text{defer } n \ V \ (\text{defer } m \ V \ A \ p) \ (\text{limit-profile } (\text{defer } m \ V \ A \ p) \ p) \wedge$
 $a \in \text{defer } m \ V \ A \ p$
 $\langle \text{proof} \rangle$

6.3.4 Composition Rules

The sequential composition preserves the non-blocking property.

theorem *seq-comp-presv-non-blocking[simp]*:
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 $\text{non-blocking-}m$: $\text{non-blocking } m$ **and**
 $\text{non-blocking-}n$: $\text{non-blocking } n$
shows $\text{non-blocking } (m \triangleright n)$
 $\langle \text{proof} \rangle$

Sequential composition preserves the non-electing property.

theorem *seq-comp-presv-non-electing[simp]*:
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 $\text{non-electing } m$ **and**
 $\text{non-electing } n$
shows $\text{non-electing } (m \triangleright n)$
 $\langle \text{proof} \rangle$

Composing an electoral module that defers exactly 1 alternative in sequence after an electoral module that is electing results (still) in an electing electoral module.

theorem *seq-comp-electing[simp]*:
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 $\text{def-one-}m$: $\text{defers } 1 \ m$ **and**
 $\text{electing-}n$: $\text{electing } n$
shows $\text{electing } (m \triangleright n)$
 $\langle \text{proof} \rangle$

lemma *def-lift-inv-seq-comp-help*:
fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**

$p \ q :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
monotone-m: defer-lift-invariance m **and**
monotone-n: defer-lift-invariance n **and**
voters-determine-n: voters-determine-election n **and**
def-and-lifted: $a \in (\text{defer } (m \triangleright n) \ V \ A \ p) \wedge \text{lifted } V \ A \ p \ q \ a$
shows $(m \triangleright n) \ V \ A \ p = (m \triangleright n) \ V \ A \ q$
 <proof>

Sequential composition preserves the property defer-lift-invariance.

theorem *seq-comp-presv-def-lift-inv[simp]:*
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
defer-lift-invariance m **and**
defer-lift-invariance n **and**
voters-determine-election n
shows *defer-lift-invariance $(m \triangleright n)$*
 <proof>

Composing a non-blocking, non-electing electoral module in sequence with an electoral module that defers exactly one alternative results in an electoral module that defers exactly one alternative.

theorem *seq-comp-def-one[simp]:*
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
non-blocking-m: non-blocking m **and**
non-electing-m: non-electing m **and**
def-one-n: defers 1 n
shows *defers 1 $(m \triangleright n)$*
 <proof>

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

theorem *disj-compat-seq[simp]:*
fixes $m \ m' \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
compatible: disjoint-compatibility m n **and**
module-m': SCF-result.electoral-module m' **and**
voters-determine-m': voters-determine-election m'
shows *disjoint-compatibility $(m \triangleright m') \ n$*
 <proof>

theorem *seq-comp-cond-compat[simp]:*
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
dcc-m: defer-condorcet-consistency m **and**

nb-n: non-blocking n **and**
ne-n: non-electing n
shows *condorcet-compatibility* ($m \triangleright n$)
 ⟨*proof*⟩

Composing a defer-condorcet-consistent electoral module in sequence with a non-blocking and non-electing electoral module results in a defer-condorcet-consistent module.

theorem *seq-comp-dcc[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
dcc-m: defer-condorcet-consistency m **and**
nb-n: non-blocking n **and**
ne-n: non-electing n
shows *defer-condorcet-consistency* ($m \triangleright n$)
 ⟨*proof*⟩

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

theorem *seq-comp-mono[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
def-monotone-m: defer-lift-invariance m **and**
non-ele-m: non-electing m **and**
def-one-m: defers 1 m **and**
electing-n: electing n
shows *monotonicity* ($m \triangleright n$)
 ⟨*proof*⟩

Composing a defer-invariant-monotone electoral module in sequence before a non-electing, defer-monotone electoral module that defers exactly 1 alternative results in a defer-lift-invariant electoral module.

theorem *def-inv-mono-imp-def-lift-inv[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
strong-def-mon-m: defer-invariant-monotonicity m **and**
non-electing-n: non-electing n **and**
defers-one: defers 1 n **and**
defer-monotone-n: defer-monotonicity n **and**
voters-determine-n: voters-determine-election n
shows *defer-lift-invariance* ($m \triangleright n$)
 ⟨*proof*⟩

end

6.4 Parallel Composition

```

theory Parallel-Composition
  imports Basic-Modules/Component-Types/Aggregator
           Basic-Modules/Component-Types/Electoral-Module
begin

```

The parallel composition composes a new electoral module from two electoral modules combined with an aggregator. Therein, the two modules each make a decision and the aggregator combines them to a single (aggregated) result.

6.4.1 Definition

```

fun parallel-composition :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  'a Aggregator  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module where
    parallel-composition m n agg V A p = agg A (m V A p) (n V A p)

abbreviation parallel :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  'a Aggregator  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  ('a, 'v, 'a Result) Electoral-Module
    (- ||- - [50, 1000, 51] 50) where
    m ||a n  $\equiv$  parallel-composition m n a

```

6.4.2 Soundness

```

theorem par-comp-sound[simp]:
  fixes
    m n :: ('a, 'v, 'a Result) Electoral-Module and
    a :: 'a Aggregator
  assumes
    SCF-result.electoral-module m and
    SCF-result.electoral-module n and
    aggregator a
  shows SCF-result.electoral-module (m ||a n)
  <proof>

```

6.4.3 Composition Rule

Using a conservative aggregator, the parallel composition preserves the property non-electing.

```

theorem conserv-agg-presv-non-electing[simp]:
  fixes
    m n :: ('a, 'v, 'a Result) Electoral-Module and
    a :: 'a Aggregator
  assumes
    non-electing-m: non-electing m and

```

```

    non-electing-n: non-electing n and
    conservative: agg-conservative a
shows non-electing (m  $\parallel_a$  n)
<proof>

end

```

6.5 Loop Composition

```

theory Loop-Composition
imports Basic-Modules/Component-Types/Termination-Condition
        Basic-Modules/Defer-Module
        Sequential-Composition
begin

```

The loop composition uses the same module in sequence, combined with a termination condition, until either

- the termination condition is met or
- no new decisions are made (i.e., a fixed point is reached).

6.5.1 Definition

```

lemma loop-termination-helper:
fixes
  m acc :: ('a, 'v, 'a Result) Electoral-Module and
  t :: 'a Termination-Condition and
  A :: 'a set and
  V :: 'v set and
  p :: ('a, 'v) Profile
assumes
   $\neg t (acc \ V \ A \ p)$  and
  defer (acc  $\triangleright$  m) V A p  $\subset$  defer acc V A p and
  finite (defer acc V A p)
shows ((acc  $\triangleright$  m, m, t, V, A, p), (acc, m, t, V, A, p))  $\in$ 
  measure ( $\lambda (acc, m, t, V, A, p). \text{card } (\text{defer } acc \ V \ A \ p)$ )
<proof>

```

This function handles the accumulator for the following loop composition function.

```

function loop-comp-helper :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
  ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$  'a Termination-Condition  $\Rightarrow$ 

```

('a, 'v, 'a Result) Electoral-Module **where**
 loop-comp-helper-finite:
 finite (defer acc V A p) \wedge (defer (acc \triangleright m) V A p) \subset (defer acc V A p)
 \longrightarrow t (acc V A p) \implies
 loop-comp-helper acc m t V A p = acc V A p |
 loop-comp-helper-infinite:
 \neg (finite (defer acc V A p) \wedge (defer (acc \triangleright m) V A p) \subset (defer acc V A p))
 \longrightarrow t (acc V A p) \implies
 loop-comp-helper acc m t V A p = loop-comp-helper (acc \triangleright m) m t V A p
 <proof>
termination
 <proof>

lemma loop-comp-code-helper[code]:
fixes
 m acc :: ('a, 'v, 'a Result) Electoral-Module **and**
 t :: 'a Termination-Condition **and**
 A :: 'a set **and**
 V :: 'v set **and**
 p :: ('a, 'v) Profile
shows
 loop-comp-helper acc m t V A p =
 (if (t (acc V A p) \vee \neg ((defer (acc \triangleright m) V A p) \subset (defer acc V A p))
 \vee infinite (defer acc V A p))
 then (acc V A p) else (loop-comp-helper (acc \triangleright m) m t V A p))
 <proof>

function loop-composition :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow
 'a Termination-Condition \Rightarrow ('a, 'v, 'a Result) Electoral-Module **where**
 t ({}, {}, A)
 \implies loop-composition m t V A p = defer-module V A p |
 \neg (t ({}, {}, A))
 \implies loop-composition m t V A p = (loop-comp-helper m m t) V A p
 <proof>
termination
 <proof>

abbreviation loop :: ('a, 'v, 'a Result) Electoral-Module \Rightarrow
 'a Termination-Condition \Rightarrow ('a, 'v, 'a Result) Electoral-Module
 (- \odot 50) **where**
 m $\odot_t \equiv$ loop-composition m t

lemma loop-comp-code[code]:
fixes
 m :: ('a, 'v, 'a Result) Electoral-Module **and**
 t :: 'a Termination-Condition **and**
 A :: 'a set **and**
 V :: 'v set **and**
 p :: ('a, 'v) Profile

shows *loop-composition* $m\ t\ V\ A\ p =$
 (*if* ($t\ (\{\}, \{\}, A)$)
 then (*defer-module* $V\ A\ p$) *else* (*loop-comp-helper* $m\ m\ t$) $V\ A\ p$)
<proof>

lemma *loop-comp-helper-imp-partit*:

fixes
 $m\ acc :: ('a, 'v, 'a\ Result)\ Electoral\ Module$ **and**
 $t :: 'a\ Termination\ Condition$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$ **and**
 $n :: nat$
assumes
 module-m: *SCF-result.electoral-module* m **and**
 profile: *profile* $V\ A\ p$ **and**
 module-acc: *SCF-result.electoral-module* acc **and**
 defer-card-n: $n = card\ (defer\ acc\ V\ A\ p)$
shows *well-formed-SCF* $A\ (loop-comp-helper\ acc\ m\ t\ V\ A\ p)$
<proof>

6.5.2 Soundness

theorem *loop-comp-sound*:

fixes
 $m :: ('a, 'v, 'a\ Result)\ Electoral\ Module$ **and**
 $t :: 'a\ Termination\ Condition$
assumes *SCF-result.electoral-module* m
shows *SCF-result.electoral-module* $(m\ \odot_t)$
<proof>

lemma *loop-comp-helper-imp-no-def-incr*:

fixes
 $m\ acc :: ('a, 'v, 'a\ Result)\ Electoral\ Module$ **and**
 $t :: 'a\ Termination\ Condition$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$ **and**
 $n :: nat$
assumes
 module-m: *SCF-result.electoral-module* m **and**
 profile: *profile* $V\ A\ p$ **and**
 mod-acc: *SCF-result.electoral-module* acc **and**
 card-n-defer-acc: $n = card\ (defer\ acc\ V\ A\ p)$
shows *defer* (*loop-comp-helper* $acc\ m\ t$) $V\ A\ p \subseteq defer\ acc\ V\ A\ p$
<proof>

6.5.3 Lemmas

lemma *loop-comp-helper-def-lift-inv-helper*:

fixes
m acc :: ('a, 'v, 'a Result) Electoral-Module **and**
t :: 'a Termination-Condition **and**
A :: 'a set **and**
V :: 'v set **and**
p :: ('a, 'v) Profile **and**
n :: nat
assumes
monotone-m: defer-lift-invariance *m* **and**
prof: profile *V A p* **and**
dli-acc: defer-lift-invariance *acc* **and**
card-n-defer: *n* = card (defer *acc V A p*) **and**
defer-finite: finite (defer *acc V A p*) **and**
voters-determine-m: voters-determine-election *m*
shows
 $\forall q a. a \in (\text{defer } (\text{loop-comp-helper } \text{acc } m \ t) \ V \ A \ p) \wedge \text{lifted } V \ A \ p \ q \ a \longrightarrow$
 $(\text{loop-comp-helper } \text{acc } m \ t) \ V \ A \ p = (\text{loop-comp-helper } \text{acc } m \ t) \ V \ A \ q$
 <proof>

lemma *loop-comp-helper-def-lift-inv*:

fixes
m acc :: ('a, 'v, 'a Result) Electoral-Module **and**
t :: 'a Termination-Condition **and**
A :: 'a set **and**
V :: 'v set **and**
p q :: ('a, 'v) Profile **and**
a :: 'a
assumes
defer-lift-invariance m **and**
voters-determine-election m **and**
defer-lift-invariance acc **and**
profile V A p **and**
lifted V A p q a **and**
a ∈ defer (loop-comp-helper *acc m t*) *V A p*
shows (loop-comp-helper *acc m t*) *V A p* = (loop-comp-helper *acc m t*) *V A q*
 <proof>

lemma *lifted-imp-fin-prof*:

fixes
A :: 'a set **and**
V :: 'v set **and**
p q :: ('a, 'v) Profile **and**
a :: 'a
assumes *lifted V A p q a*
shows *finite-profile V A p*
 <proof>

lemma *loop-comp-helper-presv-def-lift-inv*:

fixes

$m \text{ acc} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$
assumes
 $\text{defer-lift-invariance } m$ **and**
 $\text{voters-determine-election } m$ **and**
 $\text{defer-lift-invariance } acc$
shows $\text{defer-lift-invariance } (\text{loop-comp-helper } acc \ m \ t)$
 $\langle \text{proof} \rangle$

lemma $\text{loop-comp-presv-non-electing-helper}$:
fixes
 $m \text{ acc} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $n :: \text{nat}$
assumes
 $\text{non-electing-}m$: $\text{non-electing } m$ **and**
 $\text{non-electing-}acc$: $\text{non-electing } acc$ **and**
 prof : $\text{profile } V \ A \ p$ **and**
 acc-defer-card : $n = \text{card } (\text{defer } acc \ V \ A \ p)$
shows $\text{elect } (\text{loop-comp-helper } acc \ m \ t) \ V \ A \ p = \{\}$
 $\langle \text{proof} \rangle$

lemma $\text{loop-comp-helper-iter-elim-def-n-helper}$:
fixes
 $m \text{ acc} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $n \ x :: \text{nat}$
assumes
 $\text{non-electing-}m$: $\text{non-electing } m$ **and**
 $\text{single-elimination}$: $\text{eliminates } 1 \ m$ **and**
 $\text{terminate-if-n-left}$: $\forall \ r. \ t \ r = (\text{card } (\text{defer-r } r) = x)$ **and**
 x-greater-zero : $x > 0$ **and**
 prof : $\text{profile } V \ A \ p$ **and**
 n-acc-defer-card : $n = \text{card } (\text{defer } acc \ V \ A \ p)$ **and**
 n-ge-x : $n \geq x$ **and**
 def-card-gt-one : $\text{card } (\text{defer } acc \ V \ A \ p) > 1$ **and**
 acc-nonelect : $\text{non-electing } acc$
shows $\text{card } (\text{defer } (\text{loop-comp-helper } acc \ m \ t) \ V \ A \ p) = x$
 $\langle \text{proof} \rangle$

lemma $\text{loop-comp-helper-iter-elim-def-n}$:
fixes

$m \text{ acc} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $x :: \text{nat}$
assumes
 $\text{non-electing } m$ **and**
 $\text{eliminates } 1 \ m$ **and**
 $\forall r. (t \ r) = (\text{card } (\text{defer-r } r) = x)$ **and**
 $x > 0$ **and**
 $\text{profile } V \ A \ p$ **and**
 $\text{card } (\text{defer acc } V \ A \ p) \geq x$ **and**
 non-electing acc
shows $\text{card } (\text{defer } (\text{loop-comp-helper acc } m \ t) \ V \ A \ p) = x$
 $\langle \text{proof} \rangle$

lemma *iter-elim-def-n-helper*:

fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $x :: \text{nat}$
assumes
 $\text{non-electing-m: non-electing } m$ **and**
 $\text{single-elimination: eliminates } 1 \ m$ **and**
 $\text{terminate-if-n-left: } \forall r. (t \ r) = (\text{card } (\text{defer-r } r) = x)$ **and**
 $\text{x-greater-zero: } x > 0$ **and**
 $\text{prof: profile } V \ A \ p$ **and**
 $\text{enough-alternatives: card } A \geq x$
shows $\text{card } (\text{defer } (m \ \odot_t) \ V \ A \ p) = x$
 $\langle \text{proof} \rangle$

6.5.4 Composition Rules

The loop composition preserves defer-lift-invariance.

theorem *loop-comp-presv-def-lift-inv[simp]*:

fixes
 $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $t :: 'a \text{ Termination-Condition}$
assumes
 $\text{defer-lift-invariance } m$ **and**
 $\text{voters-determine-election } m$
shows $\text{defer-lift-invariance } (m \ \odot_t)$
 $\langle \text{proof} \rangle$

The loop composition preserves the property non-electing.

```

theorem loop-comp-presv-non-electing[simp]:
  fixes
    m :: ('a, 'v, 'a Result) Electoral-Module and
    t :: 'a Termination-Condition
  assumes non-electing m
  shows non-electing (m ∘t)
  ⟨proof⟩

theorem iter-elim-def-n[simp]:
  fixes
    m :: ('a, 'v, 'a Result) Electoral-Module and
    t :: 'a Termination-Condition and
    n :: nat
  assumes
    non-electing-m: non-electing m and
    single-elimination: eliminates 1 m and
    terminate-if-n-left:  $\forall r. t\ r = (\text{card } (\text{defer-}r\ r) = n)$  and
    x-greater-zero:  $n > 0$ 
  shows defers n (m ∘t)
  ⟨proof⟩

end

```

6.6 Maximum Parallel Composition

```

theory Maximum-Parallel-Composition
  imports Basic-Modules/Component-Types/Maximum-Aggregator
           Parallel-Composition
begin

```

This is a family of parallel compositions. It composes a new electoral module from two electoral modules combined with the maximum aggregator. Therein, the two modules each make a decision and then a partition is returned where every alternative receives the maximum result of the two input partitions. This means that, if any alternative is elected by at least one of the modules, then it gets elected, if any non-elected alternative is deferred by at least one of the modules, then it gets deferred, only alternatives rejected by both modules get rejected.

6.6.1 Definition

```

fun maximum-parallel-composition :: ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module  $\Rightarrow$ 
    ('a, 'v, 'a Result) Electoral-Module where

```

maximum-parallel-composition $m \ n =$
 (let $a = \text{max-aggregator}$ in $(m \parallel_a n)$)

abbreviation *max-parallel* :: ($'a, 'v, 'a \text{ Result}$) *Electoral-Module* \Rightarrow
 ($'a, 'v, 'a \text{ Result}$) *Electoral-Module* \Rightarrow
 ($'a, 'v, 'a \text{ Result}$) *Electoral-Module* (**infix** $\parallel_{\uparrow} 50$) **where**
 $m \parallel_{\uparrow} n \equiv \text{maximum-parallel-composition } m \ n$

6.6.2 Soundness

theorem *max-par-comp-sound*:
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 SCF-result.electoral-module m **and**
 SCF-result.electoral-module n
shows *SCF-result.electoral-module* $(m \parallel_{\uparrow} n)$
 $\langle \text{proof} \rangle$

lemma *voters-determine-max-par-comp*:
fixes $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 voters-determine-election m **and**
 voters-determine-election n
shows *voters-determine-election* $(m \parallel_{\uparrow} n)$
 $\langle \text{proof} \rangle$

6.6.3 Lemmas

lemma *max-agg-eq-result*:
fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
 module-m: *SCF-result.electoral-module* m **and**
 module-n: *SCF-result.electoral-module* n **and**
 prof-p: *profile* $V \ A \ p$ **and**
 a-in-A: $a \in A$
shows *mod-contains-result* $(m \parallel_{\uparrow} n) \ m \ V \ A \ p \ a \vee$
 mod-contains-result $(m \parallel_{\uparrow} n) \ n \ V \ A \ p \ a$
 $\langle \text{proof} \rangle$

lemma *max-agg-rej-iff-both-reject*:
fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**

$a :: 'a$
assumes
finite-profile $V A p$ **and**
SCF-result.electoral-module m **and**
SCF-result.electoral-module n
shows $(a \in \text{reject } (m \parallel_{\uparrow} n) V A p) =$
 $(a \in \text{reject } m V A p \wedge a \in \text{reject } n V A p)$
 $\langle \text{proof} \rangle$

lemma *max-agg-rej-fst-imp-seq-contained:*
fixes
 $m n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
f-prof: *finite-profile* $V A p$ **and**
module-m: *SCF-result.electoral-module* m **and**
module-n: *SCF-result.electoral-module* n **and**
rejected: $a \in \text{reject } n V A p$
shows *mod-contains-result* $m (m \parallel_{\uparrow} n) V A p a$
 $\langle \text{proof} \rangle$

lemma *max-agg-rej-fst-equiv-seq-contained:*
fixes
 $m n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
finite-profile $V A p$ **and**
SCF-result.electoral-module m **and**
SCF-result.electoral-module n **and**
 $a \in \text{reject } n V A p$
shows *mod-contains-result-sym* $(m \parallel_{\uparrow} n) m V A p a$
 $\langle \text{proof} \rangle$

lemma *max-agg-rej-snd-imp-seq-contained:*
fixes
 $m n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **and**
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$ **and**
 $a :: 'a$
assumes
f-prof: *finite-profile* $V A p$ **and**
module-m: *SCF-result.electoral-module* m **and**

module-n: SCF-result.electoral-module n and
rejected: $a \in \text{reject } m \ V \ A \ p$
shows *mod-contains-result $n \ (m \parallel_{\uparrow} n) \ V \ A \ p \ a$*
<proof>

lemma *max-agg-rej-snd-equiv-seq-contained:*

fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ and
 $A :: 'a \text{ set}$ and
 $V :: 'v \text{ set}$ and
 $p :: ('a, 'v) \text{ Profile}$ and
 $a :: 'a$
assumes
finite-profile $V \ A \ p$ and
SCF-result.electoral-module m and
SCF-result.electoral-module n and
 $a \in \text{reject } m \ V \ A \ p$
shows *mod-contains-result-sym $(m \parallel_{\uparrow} n) \ n \ V \ A \ p \ a$*
<proof>

lemma *max-agg-rej-intersect:*

fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ and
 $A :: 'a \text{ set}$ and
 $V :: 'v \text{ set}$ and
 $p :: ('a, 'v) \text{ Profile}$
assumes
SCF-result.electoral-module m and
SCF-result.electoral-module n and
profile $V \ A \ p$ and
finite A
shows *$\text{reject } (m \parallel_{\uparrow} n) \ V \ A \ p = (\text{reject } m \ V \ A \ p) \cap (\text{reject } n \ V \ A \ p)$*
<proof>

lemma *dcompat-dec-by-one-mod:*

fixes
 $m \ n :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ and
 $A :: 'a \text{ set}$ and
 $V :: 'v \text{ set}$ and
 $a :: 'a$
assumes
disjoint-compatibility $m \ n$ and
 $a \in A$
shows
 $(\forall p. \text{finite-profile } V \ A \ p \longrightarrow \text{mod-contains-result } m \ (m \parallel_{\uparrow} n) \ V \ A \ p \ a)$
 $\vee (\forall p. \text{finite-profile } V \ A \ p \longrightarrow \text{mod-contains-result } n \ (m \parallel_{\uparrow} n) \ V \ A \ p \ a)$
<proof>

6.6.4 Composition Rules

Using a conservative aggregator, the parallel composition preserves the property non-electing.

theorem *conserv-max-agg-presv-non-electing[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
 non-electing m **and**
 non-electing n
shows *non-electing* $(m \parallel_{\uparrow} n)$
<proof>

Using the max aggregator, composing two compatible electoral modules in parallel preserves defer-lift-invariance.

theorem *par-comp-def-lift-inv[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
 compatible: disjoint-compatibility $m\ n$ **and**
 monotone-m: defer-lift-invariance m **and**
 monotone-n: defer-lift-invariance n
shows *defer-lift-invariance* $(m \parallel_{\uparrow} n)$
<proof>

lemma *par-comp-rej-card*:
fixes
 $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$ **and**
 $A :: 'a\ set$ **and**
 $V :: 'v\ set$ **and**
 $p :: ('a, 'v)\ Profile$ **and**
 $c :: nat$
assumes
 compatible: disjoint-compatibility $m\ n$ **and**
 prof: profile $V\ A\ p$ **and**
 fin-A: finite A **and**
 reject-sum: card (reject $m\ V\ A\ p) + card (reject\ n\ V\ A\ p) = card\ A + c$
shows *card (reject* $(m \parallel_{\uparrow} n)\ V\ A\ p) = c$
<proof>

Using the max-aggregator for composing two compatible modules in parallel, whereof the first one is non-electing and defers exactly one alternative, and the second one rejects exactly two alternatives, the composition results in an electoral module that eliminates exactly one alternative.

theorem *par-comp-elim-one[simp]*:
fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\ Module$
assumes
 defers-m-one: defers $1\ m$ **and**
 non-elec-m: non-electing m **and**
 rejec-n-two: rejects $2\ n$ **and**


```

    disj-comp: disjoint-compatibility m n
    shows eliminates 1 (m ||↑ n)
    ⟨proof⟩

end

```

6.7 Elect Composition

```

theory Elect-Composition
  imports Basic-Modules/Elect-Module
           Sequential-Composition
begin

```

The elect composition sequences an electoral module and the elect module. It finalizes the module's decision as it simply elects all their non-rejected alternatives. Thereby, any such elect-composed module induces a proper voting rule in the social choice sense, as all alternatives are either rejected or elected.

6.7.1 Definition

```

fun elector :: ('a, 'v, 'a Result) Electoral-Module ⇒
    ('a, 'v, 'a Result) Electoral-Module where
    elector m = (m ▷ elect-module)

```

6.7.2 Auxiliary Lemmas

```

lemma elector-seqcomp-assoc:
  fixes a b :: ('a, 'v, 'a Result) Electoral-Module
  shows (a ▷ (elector b)) = (elector (a ▷ b))
  ⟨proof⟩

```

6.7.3 Soundness

```

theorem elector-sound[simp]:
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  assumes SCF-result.electoral-module m
  shows SCF-result.electoral-module (elector m)
  ⟨proof⟩

```

```

lemma voters-determine-elect:
  fixes m :: ('a, 'v, 'a Result) Electoral-Module
  assumes voters-determine-election m
  shows voters-determine-election (elector m)
  ⟨proof⟩

```

6.7.4 Electing

theorem *elector-electing[simp]*:
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes
 $\text{module-}m: \text{SCF-result.electoral-module } m$ **and**
 $\text{non-block-}m: \text{non-blocking } m$
shows $\text{electing } (\text{elector } m)$
 $\langle \text{proof} \rangle$

6.7.5 Composition Rule

If m is defer-Condorcet-consistent, then $\text{elector}(m)$ is Condorcet consistent.

lemma *dcc-imp-cc-elector*:
fixes $m :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$
assumes $\text{defer-condorcet-consistency } m$
shows $\text{condorcet-consistency } (\text{elector } m)$
 $\langle \text{proof} \rangle$

end

6.8 Defer-One Loop Composition

theory *Defer-One-Loop-Composition*
imports *Basic-Modules/Component-Types/Defer-Equal-Condition*
 Loop-Composition
 Elect-Composition
begin

This is a family of loop compositions. It uses the same module in sequence until either no new decisions are made or only one alternative is remaining in the defer-set. The second family herein uses the above family and subsequently elects the remaining alternative.

fun $\text{iter} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 $('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ **where**
 $\text{iter } m =$
 $(\text{let } t = \text{defer-equal-condition } 1 \text{ in}$
 $(m \circlearrowleft_t))$

abbreviation $\text{defer-one-loop} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$
 $('a, 'v, 'a \text{ Result}) \text{ Electoral-Module}$ $(-\circlearrowleft_{\exists!_d} 50)$ **where**
 $m \circlearrowleft_{\exists!_d} \equiv \text{iter } m$

fun $\text{iter-elect} :: ('a, 'v, 'a \text{ Result}) \text{ Electoral-Module} \Rightarrow$

$(\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **where**
iter-elect $m = \text{elector } (m \circ_{\exists} !d)$

6.8.1 Soundness

theorem *defer-one-loop-comp-sound*:

fixes

$m :: (\text{'a}, \text{'v}, \text{'a Result}) \text{ Electoral-Module}$ **and**

$t :: \text{'a Termination-Condition}$

assumes *SCF-result.electoral-module* m

shows *SCF-result.electoral-module* $(m \circ_{\exists} !d)$

$\langle \text{proof} \rangle$

end

Chapter 7

Voting Rules

7.1 Plurality Rule

```
theory Plurality-Rule
  imports Compositional-Structures/Basic-Modules/Plurality-Module
           Compositional-Structures/Revision-Composition
           Compositional-Structures/Elect-Composition
begin
```

This is a definition of the plurality voting rule as elimination module as well as directly. In the former one, the max operator of the set of the scores of all alternatives is evaluated and is used as the threshold value.

7.1.1 Definition

```
fun plurality-rule :: ('a, 'v, 'a Result) Electoral-Module where
  plurality-rule V A p = elector plurality V A p

fun plurality-rule' :: ('a, 'v, 'a Result) Electoral-Module where
  plurality-rule' V A p =
    (if finite A
     then ({a ∈ A. ∀ x ∈ A. win-count V p x ≤ win-count V p a},
           {a ∈ A. ∃ x ∈ A. win-count V p x > win-count V p a},
           {}))
     else (A, {}, {}))

lemma plurality-revision-equiv:
  fixes
    A :: 'a set and
    V :: 'v set and
    p :: ('a, 'v) Profile
  shows plurality V A p = (plurality-rule↓) V A p
  ⟨proof⟩

lemma plurality'-revision-equiv:
```

fixes
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
shows $\text{plurality}' V A p = (\text{plurality-rule}'\downarrow) V A p$
 $\langle \text{proof} \rangle$

lemma *plurality-rule-equiv*:
fixes
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
shows $\text{plurality-rule } V A p = \text{plurality-rule}' V A p$
 $\langle \text{proof} \rangle$

7.1.2 Soundness

theorem *plurality-rule-sound[simp]*: *SCF-result.electoral-module plurality-rule*
 $\langle \text{proof} \rangle$

theorem *plurality-rule'-sound[simp]*: *SCF-result.electoral-module plurality-rule'*
 $\langle \text{proof} \rangle$

lemma *voters-determine-plurality-rule*: *voters-determine-election plurality-rule*
 $\langle \text{proof} \rangle$

lemma *voters-determine-plurality-rule'*: *voters-determine-election plurality-rule'*
 $\langle \text{proof} \rangle$

7.1.3 Electing

lemma *plurality-rule-elect-non-empty*:
fixes
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
 $A \neq \{\}$ **and**
 $\text{finite } A$
shows $\text{elect plurality-rule } V A p \neq \{\}$
 $\langle \text{proof} \rangle$

lemma *plurality-rule'-elect-non-empty*:
fixes
 $A :: 'a \text{ set}$ **and**
 $V :: 'v \text{ set}$ **and**
 $p :: ('a, 'v) \text{ Profile}$
assumes
 $A \neq \{\}$ **and**
 $\text{profile } V A p$ **and**

finite A
shows *elect plurality-rule' V A p ≠ {}*
 ⟨*proof*⟩

The plurality module is electing.

theorem *plurality-rule-electing[simp]: electing plurality-rule*
 ⟨*proof*⟩

theorem *plurality-rule'-electing[simp]: electing plurality-rule'*
 ⟨*proof*⟩

7.1.4 Properties

lemma *plurality-rule-inv-mono-eq:*

fixes

A :: 'a set **and**
V :: 'v set **and**
p q :: ('a, 'v) Profile **and**
a :: 'a

assumes

elect-a: *a ∈ elect plurality-rule V A p* **and**
lifted-a: *lifted V A p q a*

shows *elect plurality-rule V A q = elect plurality-rule V A p*
 ∨ elect plurality-rule V A q = {a}
 ⟨*proof*⟩

lemma *plurality-rule'-inv-mono-eq:*

fixes

A :: 'a set **and**
V :: 'v set **and**
p q :: ('a, 'v) Profile **and**
a :: 'a

assumes

a ∈ elect plurality-rule' V A p **and**
lifted V A p q a

shows *elect plurality-rule' V A q = elect plurality-rule' V A p*
 ∨ elect plurality-rule' V A q = {a}
 ⟨*proof*⟩

The plurality rule is invariant-monotone.

theorem *plurality-rule-inv-mono[simp]: invariant-monotonicity plurality-rule*
 ⟨*proof*⟩

theorem *plurality-rule'-inv-mono[simp]: invariant-monotonicity plurality-rule'*
 ⟨*proof*⟩

(Weak) Monotonicity

theorem *plurality-rule-monotone: monotonicity plurality-rule*

<proof>

end

7.2 Borda Rule

theory *Borda-Rule*

imports *Compositional-Structures/Basic-Modules/Borda-Module*

Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization

Compositional-Structures/Elect-Composition

begin

This is the Borda rule. On each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected.

7.2.1 Definition

fun *borda-rule* :: ('a, 'v, 'a Result) Electoral-Module **where**
 borda-rule V A p = *elector borda* V A p

fun *borda-rule_R* :: ('a, 'v :: wellorder, 'a Result) Electoral-Module **where**
 borda-rule_R V A p = *swap-R unanimity* V A p

7.2.2 Soundness

theorem *borda-rule-sound*: *SCF-result.electoral-module borda-rule*
<proof>

theorem *borda-rule_R-sound*: *SCF-result.electoral-module borda-rule_R*
<proof>

7.2.3 Anonymity

theorem *borda-rule_R-anonymous*: *SCF-result.anonymity borda-rule_R*
<proof>

end

7.3 Pairwise Majority Rule

```
theory Pairwise-Majority-Rule
  imports Compositional-Structures/Basic-Modules/Condorcet-Module
           Compositional-Structures/Defer-One-Loop-Composition
begin
```

This is the pairwise majority rule, a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives.

7.3.1 Definition

```
fun pairwise-majority-rule :: ('a, 'v, 'a Result) Electoral-Module where
  pairwise-majority-rule V A p = elector condorcet V A p

fun condorcet' :: ('a, 'v, 'a Result) Electoral-Module where
  condorcet' V A p = ((min-eliminator condorcet-score)  $\circ_{\exists!d}$ ) V A p

fun pairwise-majority-rule' :: ('a, 'v, 'a Result) Electoral-Module where
  pairwise-majority-rule' V A p = iter-elect condorcet' V A p
```

7.3.2 Soundness

```
theorem pairwise-majority-rule-sound: SCF-result.electoral-module pairwise-majority-rule
  <proof>

theorem condorcet'-sound: SCF-result.electoral-module condorcet'
  <proof>

theorem pairwise-majority-rule'-sound: SCF-result.electoral-module pairwise-majority-rule'
  <proof>
```

7.3.3 Condorcet Consistency

```
theorem condorcet-condorcet: condorcet-consistency pairwise-majority-rule
  <proof>

end
```

7.4 Copeland Rule

```
theory Copeland-Rule
  imports Compositional-Structures/Basic-Modules/Copeland-Module
           Compositional-Structures/Elect-Composition
begin
```


This is the Copeland voting rule. The idea is to elect the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses.

7.4.1 Definition

```
fun copeland-rule :: ('a, 'v, 'a Result) Electoral-Module where
  copeland-rule V A p = elector copeland V A p
```

7.4.2 Soundness

```
theorem copeland-rule-sound: SCF-result.electoral-module copeland-rule
  ⟨proof⟩
```

7.4.3 Condorcet Consistency

```
theorem copeland-condorcet: condorcet-consistency copeland-rule
  ⟨proof⟩
```

end

7.5 Minimax Rule

```
theory Minimax-Rule
  imports Compositional-Structures/Basic-Modules/Minimax-Module
           Compositional-Structures/Elect-Composition
begin
```

This is the Minimax voting rule. It elects the alternatives with the highest Minimax score.

7.5.1 Definition

```
fun minimax-rule :: ('a, 'v, 'a Result) Electoral-Module where
  minimax-rule V A p = elector minimax V A p
```

7.5.2 Soundness

```
theorem minimax-rule-sound: SCF-result.electoral-module minimax-rule
  ⟨proof⟩
```

7.5.3 Condorcet Consistency

```
theorem minimax-condorcet: condorcet-consistency minimax-rule
  ⟨proof⟩
```

end

7.6 Black's Rule

```
theory Blacks-Rule
  imports Pairwise-Majority-Rule
           Borda-Rule
begin
```

This is Black's voting rule. It is composed of a function that determines the Condorcet winner, i.e., the Pairwise Majority rule, and the Borda rule. Whenever there exists no Condorcet winner, it elects the choice made by the Borda rule, otherwise the Condorcet winner is elected.

7.6.1 Definition

```
fun black :: ('a, 'v, 'a Result) Electoral-Module where
  black A p = (condorcet  $\triangleright$  borda) A p

fun blacks-rule :: ('a, 'v, 'a Result) Electoral-Module where
  blacks-rule A p = elector black A p
```

7.6.2 Soundness

```
theorem blacks-sound: SCF-result.electoral-module black
   $\langle$ proof $\rangle$ 
```

```
theorem blacks-rule-sound: SCF-result.electoral-module blacks-rule
   $\langle$ proof $\rangle$ 
```

7.6.3 Condorcet Consistency

```
theorem black-is-dcc: defer-condorcet-consistency black
   $\langle$ proof $\rangle$ 
```

```
theorem black-condorcet: condorcet-consistency blacks-rule
   $\langle$ proof $\rangle$ 
```

end

7.7 Nanson-Baldwin Rule

```
theory Nanson-Baldwin-Rule
  imports Compositional-Structures/Basic-Modules/Borda-Module
           Compositional-Structures/Defer-One-Loop-Composition
begin
```

This is the Nanson-Baldwin voting rule. It excludes alternatives with the lowest Borda score from the set of possible winners and then adjusts the Borda score to the new (remaining) set of still eligible alternatives.

7.7.1 Definition

```
fun nanson-baldwin-rule :: ('a, 'v, 'a Result) Electoral-Module where
  nanson-baldwin-rule A p =
    ((min-eliminator borda-score)  $\circ_{\exists 1d}$ ) A p
```

7.7.2 Soundness

```
theorem nanson-baldwin-rule-sound: SCF-result.electoral-module nanson-baldwin-rule
  <proof>

end
```

7.8 Classic Nanson Rule

```
theory Classic-Nanson-Rule
  imports Compositional-Structures/Basic-Modules/Borda-Module
           Compositional-Structures/Defer-One-Loop-Composition
begin
```

This is the classic Nanson's voting rule, i.e., the rule that was originally invented by Nanson, but not the Nanson-Baldwin rule. The idea is similar, however, as alternatives with a Borda score less or equal than the average Borda score are excluded. The Borda scores of the remaining alternatives are hence adjusted to the new set of (still) eligible alternatives.

7.8.1 Definition

```
fun classic-nanson-rule :: ('a, 'v, 'a Result) Electoral-Module where
  classic-nanson-rule V A p =
    ((leq-average-eliminator borda-score)  $\circ_{\exists 1d}$ ) V A p
```

7.8.2 Soundness

theorem *classic-nanson-rule-sound: SCF-result.electoral-module classic-nanson-rule*
 <proof>
end

7.9 Schwartz Rule

theory *Schwartz-Rule*
 imports *Compositional-Structures/Basic-Modules/Borda-Module*
 Compositional-Structures/Defer-One-Loop-Composition
begin

This is the Schwartz voting rule. Confusingly, it is sometimes also referred as Nanson's rule. The Schwartz rule proceeds as in the classic Nanson's rule, but excludes alternatives with a Borda score that is strictly less than the average Borda score.

7.9.1 Definition

fun *schwartz-rule* :: ('a, 'v, 'a Result) Electoral-Module **where**
 schwartz-rule V A p =
 ((*less-average-eliminator* borda-score) $\circ_{\exists!d}$) V A p

7.9.2 Soundness

theorem *schwartz-rule-sound: SCF-result.electoral-module schwartz-rule*
 <proof>
end

7.10 Sequential Majority Comparison

theory *Sequential-Majority-Comparison*
 imports *Plurality-Rule*
 Compositional-Structures/Drop-And-Pass-Compatibility
 Compositional-Structures/Revision-Composition
 Compositional-Structures/Maximum-Parallel-Composition
 Compositional-Structures/Defer-One-Loop-Composition
begin

Sequential majority comparison compares two alternatives by plurality voting. The loser gets rejected, and the winner is compared to the next alternative. This process is repeated until only a single alternative is left, which is then elected.

7.10.1 Definition

fun *smc* :: 'a Preference-Relation \Rightarrow ('a, 'v, 'a Result) Electoral-Module **where**
smc *x* *V* *A* *p* =
 ((*elector* (((*pass-module* 2 *x*) \triangleright ((*plurality-rule* \downarrow) \triangleright (*pass-module* 1 *x*))) \parallel_{\uparrow}
 (*drop-module* 2 *x*) $\cup_{\exists !d}$)) *V* *A* *p*)

7.10.2 Soundness

As all base components are electoral modules (, aggregators, or termination conditions), and all used compositional structures create electoral modules, sequential majority comparison unsurprisingly is an electoral module.

theorem *smc-sound*:
fixes *x* :: 'a Preference-Relation
shows *SCF-result.electoral-module* (*smc* *x*)
<proof>

7.10.3 Electing

The sequential majority comparison electoral module is electing. This property is needed to convert electoral modules to a social choice function. Apart from the very last proof step, it is a part of the monotonicity proof below.

theorem *smc-electing*:
fixes *x* :: 'a Preference-Relation
assumes *linear-order* *x*
shows *electing* (*smc* *x*)
<proof>

7.10.4 (Weak) Monotonicity

The following proof is a fully modular proof for weak monotonicity of sequential majority comparison. It is composed of many small steps.

theorem *smc-monotone*:
fixes *x* :: 'a Preference-Relation
assumes *linear-order* *x*
shows *monotonicity* (*smc* *x*)
<proof>

end

7.11 Kemeny Rule

theory *Kemeny-Rule*

imports

Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization
Compositional-Structures/Basic-Modules/Component-Types/Distance-Rationalization-Symmetry

begin

This is the Kemeny rule. It creates a complete ordering of alternatives and evaluates each ordering of the alternatives in terms of the sum of preference reversals on each ballot that would have to be performed in order to produce that transitive ordering. The complete ordering which requires the fewest preference reversals is the final result of the method.

7.11.1 Definition

fun *kemeny-rule* :: ('a, 'v :: wellorder, 'a Result) Electoral-Module **where**
kemeny-rule V A p = *swap- \mathcal{R} strong-unanimity* V A p

7.11.2 Soundness

theorem *kemeny-rule-sound*: *SCF-result.electoral-module kemeny-rule*
<proof>

7.11.3 Anonymity

theorem *kemeny-rule-anonymous*: *SCF-result.anonymity kemeny-rule*
<proof>

7.11.4 Neutrality

lemma *swap-dist-neutral*: *distance-neutrality well-formed-elections*
(votewise-distance swap l-one)
<proof>

theorem *kemeny-rule-neutral*: *SCF-properties.neutrality kemeny-rule*
<proof>

end

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