Verified Construction of Fair Voting Rules

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Abstract

Voting rules aggregate multiple individual preferences in order to make a collective decision. Commonly, these mechanisms are expected to respect a multitude of different notions of fairness and reliability, which must be carefully balanced to avoid inconsistencies.

This article contains a formalisation of a framework for the construction of such fair voting rules using composable modules [1, 2]. The framework is a formal and systematic approach for the flexible and verified construction of voting rules from individual composable modules to respect such social-choice properties by construction. Formal composition rules guarantee resulting social-choice properties from properties of the individual components which are of generic nature to be reused for various voting rules. We provide proofs for a selected set of structures and composition rules. The approach can be readily extended in order to support more voting rules, e.g., from the literature by extending the sets of modules and composition rules.

Contents

Chapter 1

Social-Choice Types

1.1 Auxiliary Lemmas

theory *Auxiliary-Lemmas*
  imports *Main*
begin

Summation function is invariant under application of a (bijective) permutation on the elements.

lemma *sum-comp*:
  fixes
    $f :: {'}x \Rightarrow ({'}z :: comm\text{-}monoid\text{-}add)$ **and**
    $g :: {'}y \Rightarrow {'}x$ **and**
    $X :: {'}x\ set$ **and**
    $Y :: {'}y\ set$
  assumes *bij-betw g Y X*
  shows $(\sum\ x \in X.\ f\ x) = (\sum\ x \in Y.\ (f \circ g)\ x)$
  $\langle proof \rangle$

The inversion of a composition of injective functions is equivalent to the composition of the two individual inverted functions.

lemma *the-inv-comp*:
  fixes
    $X :: {'}x\ set$ **and**
    $Y :: {'}y\ set$ **and**
    $Z :: {'}z\ set$ **and**
    $f :: {'}y \Rightarrow {'}x$ **and**
    $g :: {'}z \Rightarrow {'}y$ **and**
    $x :: {'}x$
  assumes
    bij-betw f Y X **and**
    bij-betw g Z Y **and**
    $x \in X$
  shows *the-inv-into Z* $(f \circ g)$ $x = (($*the-inv-into Z g*$) \circ ($*the-inv-into Y f*$))$ x
  $\langle proof \rangle$

end

1.2 Preference Relation

theory *Preference-Relation*
  imports *Main*
begin

The very core of the composable modules voting framework: types and functions, derivations, lemmas, operations on preference relations, etc.

1.2.1 Definition

Each voter expresses pairwise relations between all alternatives, thereby inducing a linear order.

type-synonym $'a$ *Preference-Relation* $= 'a$ *rel*

type-synonym $'a$ *Vote* $= 'a$ *set* $\times$ $'a$ *Preference-Relation*

fun *is-less-preferred-than* :: $'a \Rightarrow 'a$ *Preference-Relation* $\Rightarrow 'a \Rightarrow bool$
     $(\text{-} \preceq_{\text{-}} \text{-} [50, 1000, 51]\ 50)$ **where**
  $a \preceq_r b = ((a,\ b) \in r)$

fun *alts-$\mathcal{V}$* :: $'a$ *Vote* $\Rightarrow 'a$ *set* **where**
  alts-$\mathcal{V}$ $V = fst\ V$

fun *pref-$\mathcal{V}$* :: $'a$ *Vote* $\Rightarrow 'a$ *Preference-Relation* **where**
  pref-$\mathcal{V}$ $V = snd\ V$

lemma *lin-imp-antisym*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation*
  assumes *linear-order-on* A r
  shows *antisym* r
  $\langle proof \rangle$

lemma *lin-imp-trans*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation*
  assumes *linear-order-on* A r
  shows *trans* r

$\langle proof \rangle$

1.2.2 Ranking

fun *rank* :: *'a Preference-Relation* $\Rightarrow$ *'a* $\Rightarrow$ *nat* **where**
  rank r a = card (above r a)

lemma *rank-gt-zero*:
  fixes
    r :: *'a Preference-Relation* **and**
    a :: *'a*
  assumes
    refl: *a* $\preceq_r$ *a* **and**
    fin: *finite r*
  shows *rank r a* $\geq$ *1*
$\langle proof \rangle$

1.2.3 Limited Preference

definition *limited* :: *'a set* $\Rightarrow$ *'a Preference-Relation* $\Rightarrow$ *bool* **where**
  limited A r $\equiv$ *r* $\subseteq$ *A* $\times$ *A*

lemma *limited-dest*:
  fixes
    A :: *'a set* **and**
    r :: *'a Preference-Relation* **and**
    a b :: *'a*
  assumes
    a $\preceq_r$ *b* **and**
    limited A r
  shows *a* $\in$ *A* $\wedge$ *b* $\in$ *A*
  $\langle proof \rangle$

fun *limit* :: *'a set* $\Rightarrow$ *'a Preference-Relation* $\Rightarrow$ *'a Preference-Relation* **where**
  limit A r = {(a, b) $\in$ *r. a* $\in$ *A* $\wedge$ *b* $\in$ *A}*

definition *connex* :: *'a set* $\Rightarrow$ *'a Preference-Relation* $\Rightarrow$ *bool* **where**
  connex A r $\equiv$ *limited A r* $\wedge$ *(*$\forall$ *a* $\in$ *A.* $\forall$ *b* $\in$ *A. a* $\preceq_r$ *b* $\vee$ *b* $\preceq_r$ *a)*

lemma *connex-imp-refl*:
  fixes
    A :: *'a set* **and**
    r :: *'a Preference-Relation*
  assumes *connex A r*
  shows *refl-on A r*
  $\langle proof \rangle$

lemma *lin-ord-imp-connex*:
  fixes
    A :: *'a set* **and**

$r :: \,'a$ *Preference-Relation*
  assumes *linear-order-on A r*
  shows *connex A r*
⟨*proof*⟩

lemma *connex-antsym-and-trans-imp-lin-ord*:
  fixes
    $A :: \,'a$ *set* **and**
    $r :: \,'a$ *Preference-Relation*
  assumes
    connex-r: *connex A r* **and**
    antisym-r: *antisym r* **and**
    trans-r: *trans r*
  shows *linear-order-on A r*
⟨*proof*⟩

lemma *limit-to-limits*:
  fixes
    $A :: \,'a$ *set* **and**
    $r :: \,'a$ *Preference-Relation*
  shows *limited A* (*limit A r*)
  ⟨*proof*⟩

lemma *limit-presv-connex*:
  fixes
    $A \; B :: \,'a$ *set* **and**
    $r :: \,'a$ *Preference-Relation*
  assumes
    connex: *connex B r* **and**
    subset: $A \subseteq B$
  shows *connex A* (*limit A r*)
⟨*proof*⟩

lemma *limit-presv-antisym*:
  fixes
    $A :: \,'a$ *set* **and**
    $r :: \,'a$ *Preference-Relation*
  assumes *antisym r*
  shows *antisym* (*limit A r*)
  ⟨*proof*⟩

lemma *limit-presv-trans*:
  fixes
    $A :: \,'a$ *set* **and**
    $r :: \,'a$ *Preference-Relation*
  assumes *trans r*
  shows *trans* (*limit A r*)
  ⟨*proof*⟩

lemma *limit-presv-lin-ord*:
  fixes
    $A\ B :: {}'a\ set$ **and**
    $r :: {}'a\ Preference\text{-}Relation$
  assumes
    linear-order-on B r **and**
    $A \subseteq B$
  shows *linear-order-on A* (*limit A r*)
  $\langle proof \rangle$

lemma *limit-presv-prefs*:
  fixes
    $A :: {}'a\ set$ **and**
    $r :: {}'a\ Preference\text{-}Relation$ **and**
    $a\ b :: {}'a$
  assumes
    $a \preceq_r b$ **and**
    $a \in A$ **and**
    $b \in A$
  shows *let s = limit A r in* $a \preceq_s b$
  $\langle proof \rangle$

lemma *limit-rel-presv-prefs*:
  fixes
    $A :: {}'a\ set$ **and**
    $r :: {}'a\ Preference\text{-}Relation$ **and**
    $a\ b :: {}'a$
  assumes $(a,\ b) \in limit\ A\ r$
  shows $a \preceq_r b$
  $\langle proof \rangle$

lemma *limit-trans*:
  fixes
    $A\ B :: {}'a\ set$ **and**
    $r :: {}'a\ Preference\text{-}Relation$
  assumes $A \subseteq B$
  shows *limit A r = limit A* (*limit B r*)
  $\langle proof \rangle$

lemma *lin-ord-not-empty*:
  fixes $r :: {}'a\ Preference\text{-}Relation$
  assumes $r \neq \{\}$
  shows $\neg$ *linear-order-on* $\{\}$ *r*
  $\langle proof \rangle$

lemma *lin-ord-singleton*:
  fixes $a :: {}'a$
  shows $\forall\ r.$ *linear-order-on* $\{a\}\ r \longrightarrow r = \{(a,\ a)\}$
$\langle proof \rangle$

1.2.4 Auxiliary Lemmas

lemma *above-trans*:
  fixes
    r :: $'a$ *Preference-Relation* **and**
    $a\ b$ :: $'a$
  assumes
    trans r **and**
    $(a,\ b) \in r$
  shows *above r b* $\subseteq$ *above r a*
  $\langle proof \rangle$

lemma *above-refl*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes
    refl-on A r **and**
    $a \in A$
  shows $a \in$ *above r a*
  $\langle proof \rangle$

lemma *above-subset-geq-one*:
  fixes
    A :: $'a$ *set* **and**
    $r\ r'$ :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes
    linear-order-on A r **and**
    linear-order-on A r' **and**
    above r a $\subseteq$ *above r' a* **and**
    above r' a $= \{a\}$
  shows *above r a* $= \{a\}$
  $\langle proof \rangle$

lemma *above-connex*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes
    connex A r **and**
    $a \in A$
  shows $a \in$ *above r a*
  $\langle proof \rangle$

lemma *pref-imp-in-above*:
  fixes
    r :: $'a$ *Preference-Relation* **and**

$a\ b :: {'}a$
  shows $(a \preceq_r b) = (b \in above\ r\ a)$
  $\langle proof \rangle$

lemma *limit-presv-above*:
  fixes
    $A :: {'}a\ set$ **and**
    $r :: {'}a\ Preference\text{-}Relation$ **and**
    $a\ b :: {'}a$
  assumes
    $b \in above\ r\ a$ **and**
    $a \in A$ **and**
    $b \in A$
  shows $b \in above\ (limit\ A\ r)\ a$
  $\langle proof \rangle$

lemma *limit-rel-presv-above*:
  fixes
    $A\ B :: {'}a\ set$ **and**
    $r :: {'}a\ Preference\text{-}Relation$ **and**
    $a\ b :: {'}a$
  assumes $b \in above\ (limit\ B\ r)\ a$
  shows $b \in above\ r\ a$
  $\langle proof \rangle$

lemma *above-one*:
  fixes
    $A :: {'}a\ set$ **and**
    $r :: {'}a\ Preference\text{-}Relation$
  assumes
    lin-ord-r: $linear\text{-}order\text{-}on\ A\ r$ **and**
    fin-A: $finite\ A$ **and**
    non-empty-A: $A \neq \{\}$
  shows $\exists\ a \in A.\ above\ r\ a = \{a\} \wedge (\forall\ a' \in A.\ above\ r\ a' = \{a'\} \longrightarrow a' = a)$
$\langle proof \rangle$

lemma *above-one-eq*:
  fixes
    $A :: {'}a\ set$ **and**
    $r :: {'}a\ Preference\text{-}Relation$ **and**
    $a\ b :: {'}a$
  assumes
    lin-ord: $linear\text{-}order\text{-}on\ A\ r$ **and**
    fin-A: $finite\ A$ **and**
    not-empty-A: $A \neq \{\}$ **and**
    above-a: $above\ r\ a = \{a\}$ **and**
    above-b: $above\ r\ b = \{b\}$
  shows $a = b$
$\langle proof \rangle$

lemma *above-one-imp-rank-one*:
  fixes
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes *above r a* = $\{a\}$
  shows *rank r a* = *1*
  $\langle proof \rangle$

lemma *rank-one-imp-above-one*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes
    lin-ord: *linear-order-on A r* **and**
    rank-one: *rank r a* = *1*
  shows *above r a* = $\{a\}$
$\langle proof \rangle$

theorem *above-rank*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  assumes *linear-order-on A r*
  shows (*above r a* = $\{a\}$) = (*rank r a* = *1*)
  $\langle proof \rangle$

lemma *rank-unique*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    $a\ b$ :: $'a$
  assumes
    lin-ord: *linear-order-on A r* **and**
    fin-A: *finite A* **and**
    a-in-A: $a \in A$ **and**
    b-in-A: $b \in A$ **and**
    a-neq-b: $a \neq b$
  shows *rank r a* $\neq$ *rank r b*
$\langle proof \rangle$

lemma *above-presv-limit*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *Preference-Relation* **and**
    a :: $'a$
  shows *above* (*limit A r*) $a \subseteq A$

⟨*proof*⟩

1.2.5 Lifting Property

definition *equiv-rel-except-a* :: *′a set* ⇒ *′a Preference-Relation* ⇒
      ′a Preference-Relation ⇒ *′a* ⇒ *bool* **where**
  equiv-rel-except-a A r r′ a ≡
    linear-order-on A r ∧ *linear-order-on A r′* ∧ *a* ∈ *A* ∧
    (∀ *a′* ∈ *A* − {*a*}. ∀ *b′* ∈ *A* − {*a*}. (*a′* ⪯$_r$ *b′*) = (*a′* ⪯$_r$′ *b′*))

definition *lifted* :: *′a set* ⇒ *′a Preference-Relation* ⇒ *′a Preference-Relation* ⇒
      ′a ⇒ *bool* **where**
  lifted A r r′ a ≡
    equiv-rel-except-a A r r′ a ∧ (∃ *a′* ∈ *A* − {*a*}. *a* ⪯$_r$ *a′* ∧ *a′* ⪯$_r$′ *a*)

lemma *trivial-equiv-rel*:
  fixes
    A :: *′a set* **and**
    r :: *′a Preference-Relation*
  assumes *linear-order-on A r*
  shows ∀ *a* ∈ *A*. *equiv-rel-except-a A r r a*
  ⟨*proof*⟩

lemma *lifted-imp-equiv-rel-except-a*:
  fixes
    A :: *′a set* **and**
    r r′ :: *′a Preference-Relation* **and**
    a :: *′a*
  assumes *lifted A r r′ a*
  shows *equiv-rel-except-a A r r′ a*
  ⟨*proof*⟩

lemma *lifted-imp-switched*:
  fixes
    A :: *′a set* **and**
    r r′ :: *′a Preference-Relation* **and**
    a :: *′a*
  assumes *lifted A r r′ a*
  shows ∀ *a′* ∈ *A* − {*a*}. ¬ (*a′* ⪯$_r$ *a* ∧ *a* ⪯$_r$′ *a′*)
⟨*proof*⟩

lemma *lifted-mono*:
  fixes
    A :: *′a set* **and**
    r r′ :: *′a Preference-Relation* **and**
    a a′ :: *′a*
  assumes
    lifted: *lifted A r r′ a* **and**
    a′-pref-a: *a′* ⪯$_r$ *a*

shows $a' \preceq_r{}' a$
⟨*proof*⟩

lemma *lifted-above-subset*:
  fixes
    $A :: {}'a$ *set* **and**
    $r \; r' :: {}'a$ *Preference-Relation* **and**
    $a :: {}'a$
  assumes *lifted* $A \; r \; r' \; a$
  shows *above* $r' \; a \subseteq$ *above* $r \; a$
⟨*proof*⟩

lemma *lifted-above-mono*:
  fixes
    $A :: {}'a$ *set* **and**
    $r \; r' :: {}'a$ *Preference-Relation* **and**
    $a \; a' :: {}'a$
  assumes
    lifted-a: *lifted* $A \; r \; r' \; a$ **and**
    a'-in-A-sub-a: $a' \in A - \{a\}$
  shows *above* $r \; a' \subseteq$ *above* $r' \; a' \cup \{a\}$
⟨*proof*⟩

lemma *limit-lifted-imp-eq-or-lifted*:
  fixes
    $A \; A' :: {}'a$ *set* **and**
    $r \; r' :: {}'a$ *Preference-Relation* **and**
    $a :: {}'a$
  assumes
    lifted: *lifted* $A' \; r \; r' \; a$ **and**
    subset: $A \subseteq A'$
  shows *limit* $A \; r =$ *limit* $A \; r' \lor$ *lifted* A (*limit* $A \; r$) (*limit* $A \; r'$) a
⟨*proof*⟩

lemma *negl-diff-imp-eq-limit*:
  fixes
    $A \; A' :: {}'a$ *set* **and**
    $r \; r' :: {}'a$ *Preference-Relation* **and**
    $a :: {}'a$
  assumes
    change: *equiv-rel-except-a* $A' \; r \; r' \; a$ **and**
    subset: $A \subseteq A'$ **and**
    not-in-A: $a \notin A$
  shows *limit* $A \; r =$ *limit* $A \; r'$
⟨*proof*⟩

theorem *lifted-above-winner-alts*:
  fixes
    $A :: {}'a$ *set* **and**

   r r′ :: ′a Preference-Relation **and**
   a a′ :: ′a
 assumes
  lifted-a: *lifted A r r′ a* **and**
  a′-above-a′: *above r a′ = {a′}* **and**
  fin-A: *finite A*
 shows *above r′ a′ = {a′} ∨ above r′ a = {a}*
⟨*proof*⟩

theorem *lifted-above-winner-single*:
 fixes
   A :: ′a set **and**
   r r′ :: ′a Preference-Relation **and**
   a :: ′a
 assumes
  lifted A r r′ a **and**
  above r a = {a} **and**
  finite A
 shows *above r′ a = {a}*
 ⟨*proof*⟩

theorem *lifted-above-winner-other*:
 fixes
   A :: ′a set **and**
   r r′ :: ′a Preference-Relation **and**
   a a′ :: ′a
 assumes
  lifted-a: *lifted A r r′ a* **and**
  a′-above-a′: *above r′ a′ = {a′}* **and**
  fin-A: *finite A* **and**
  a-not-a′: *a ≠ a′*
 shows *above r a′ = {a′}*
⟨*proof*⟩

  end

1.3   Norm

theory *Norm*
 imports *HOL−Library.Extended-Real*
     HOL−Combinatorics.List-Permutation
     Auxiliary-Lemmas
begin

A norm on R to n is a mapping N: $R \mapsto n$ on R that has the following
properties for all mappings u (and v) in R to n:

- positive scalability: $N(a * u) = |a| * N(u)$ for all a in R.

- positive semidefiniteness: $N(u) \geq 0$ with $N(u) = 0$ if and only if $u = (0, 0, \ldots, 0)$.

- triangle inequality: $N(u + v) \leq N(u) + N(v)$.

1.3.1 Definition

type-synonym *Norm = ereal list ⇒ ereal*

definition *norm :: Norm ⇒ bool* **where**
  norm n ≡ ∀ x :: ereal list. n x ≥ 0 ∧ (∀ i < length x. x!i = 0 ⟶ n x = 0)

1.3.2 Auxiliary Lemmas

lemma *sum-over-image-of-bijection*:
  fixes
    A :: 'a set **and**
    A' :: 'b set **and**
    f :: 'a ⇒ 'b **and**
    g :: 'a ⇒ ereal
  assumes *bij-betw f A A'*
  shows $(\sum a \in A.\ g\ a) = (\sum a' \in A'.\ g\ (\text{the-inv-into}\ A\ f\ a'))$
  ⟨*proof*⟩

1.3.3 Common Norms

fun *l-one :: Norm* **where**
  l-one x = $(\sum i < length\ x.\ |x!i|)$

1.3.4 Properties

definition *symmetry :: Norm ⇒ bool* **where**
  symmetry n ≡ ∀ x y. x <~~> y ⟶ n x = n y

1.3.5 Theorems

theorem *l-one-is-sym: symmetry l-one*
⟨*proof*⟩

end

1.4 Electoral Result

theory *Result*
  imports *Main*
begin

An electoral result is the principal result type of the composable modules voting framework, as it is a generalization of the set of winning alternatives from social choice functions. Electoral results are selections of the received (possibly empty) set of alternatives into the three disjoint groups of elected, rejected and deferred alternatives. Any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives.

1.4.1 Auxiliary Functions

type-synonym $'r\ Result = 'r\ set * 'r\ set * 'r\ set$

A partition of a set A are pairwise disjoint sets that "set equals partition" A. For this specific predicate, we have three disjoint sets in a three-tuple.

fun *disjoint3* :: $'r\ Result \Rightarrow bool$ **where**
  disjoint3 $(e,\ r,\ d) =$
    $((e \cap r = \{\}) \land$
      $(e \cap d = \{\}) \land$
      $(r \cap d = \{\}))$

fun *set-equals-partition* :: $'r\ set \Rightarrow 'r\ Result \Rightarrow bool$ **where**
  set-equals-partition $X\ (e,\ r,\ d) = (e \cup r \cup d = X)$

1.4.2 Definition

A result generally is related to the alternative set A (of type 'a). A result should be well-formed on the alternatives. Also it should be possible to limit a well-formed result to a subset of the alternatives.

Specific result types like social choice results (sets of alternatives) can be realized via sublocales of the result locale.

locale *result* $=$
  fixes
    well-formed :: $'a\ set \Rightarrow ('r\ Result) \Rightarrow bool$ **and**
    limit :: $'a\ set \Rightarrow 'r\ set \Rightarrow 'r\ set$
  assumes $\forall\ (A :: 'a\ set)\ (r :: 'r\ Result).$
    $(set\text{-}equals\text{-}partition\ (limit\ A\ UNIV)\ r \land disjoint3\ r) \longrightarrow well\text{-}formed\ A\ r$

These three functions return the elect, reject, or defer set of a result.

fun (**in** *result*) $limit_{\mathcal{R}}$ :: $'a\ set \Rightarrow 'r\ Result \Rightarrow 'r\ Result$ **where**
  $limit_{\mathcal{R}}\ A\ (e,\ r,\ d) = (limit\ A\ e,\ limit\ A\ r,\ limit\ A\ d)$

abbreviation *elect-r* :: *$'r$ Result $\Rightarrow$ $'r$ set* **where**
  elect-r r $\equiv$ fst r

abbreviation *reject-r* :: *$'r$ Result $\Rightarrow$ $'r$ set* **where**
  reject-r r $\equiv$ fst (snd r)

abbreviation *defer-r* :: *$'r$ Result $\Rightarrow$ $'r$ set* **where**
  defer-r r $\equiv$ snd (snd r)

end

1.5 Preference Profile

theory *Profile*
  imports *Preference-Relation*
        Auxiliary-Lemmas
        HOL$-$Library.Extended-Nat
        HOL$-$Combinatorics.Permutations
begin

Preference profiles denote the decisions made by the individual voters on the eligible alternatives. They are represented in the form of one preference relation (e.g., selected on a ballot) per voter, collectively captured in a mapping of voters onto their respective preference relations. If there are finitely many voters, they can be enumerated and the mapping can be interpreted as a list of preference relations. Unlike the common preference profiles in the social-choice sense, the profiles described here consider only the (sub-)set of alternatives that are received.

1.5.1 Definition

A profile contains one ballot for each voter. An election consists of a set of participating voters, a set of eligible alternatives, and a corresponding profile.

type-synonym *($'a$, $'v$) Profile = $'v$ $\Rightarrow$ ($'a$ Preference-Relation)*

type-synonym *($'a$, $'v$) Election = $'a$ set $\times$ $'v$ set $\times$ ($'a$, $'v$) Profile*

fun *alternatives-$\mathcal{E}$* :: *($'a$, $'v$) Election $\Rightarrow$ $'a$ set* **where**
  alternatives-$\mathcal{E}$ E = fst E

fun *voters-$\mathcal{E}$* :: *($'a$, $'v$) Election $\Rightarrow$ $'v$ set* **where**

voters-$\mathcal{E}$ E = fst (snd E)

fun *profile-$\mathcal{E}$ ::* ($'a$, $'v$) *Election* $\Rightarrow$ ($'a$, $'v$) *Profile* **where**
  profile-$\mathcal{E}$ E = snd (snd E)

fun *election-equality ::* ($'a$, $'v$) *Election* $\Rightarrow$ ($'a$, $'v$) *Election* $\Rightarrow$ *bool* **where**
  election-equality (A, V, p) (A$'$, V$'$, p$'$) =
    (A = A$'$ $\wedge$ V = V$'$ $\wedge$ ($\forall$ v $\in$ V. p v = p$'$ v))

A profile on a set of alternatives A and a voter set V consists of ballots that
are linear orders on A for all voters in V. A finite profile is one with finitely
many alternatives and voters.

definition *profile ::* $'v$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ *bool* **where**
  profile V A p $\equiv$ $\forall$ v $\in$ V. *linear-order-on* A (p v)

abbreviation *finite-profile ::* $'v$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ *bool* **where**
  finite-profile V A p $\equiv$ *finite* A $\wedge$ *finite* V $\wedge$ *profile V A p*

abbreviation *finite-election ::* ($'a$, $'v$) *Election* $\Rightarrow$ *bool* **where**
  finite-election E $\equiv$ *finite-profile* (*voters-$\mathcal{E}$ E*) (*alternatives-$\mathcal{E}$ E*) (*profile-$\mathcal{E}$ E*)

definition *finite-elections-$\mathcal{V}$ ::* ($'a$, $'v$) *Election set* **where**
  finite-elections-$\mathcal{V}$ $\equiv$ {E :: ($'a$, $'v$) *Election. finite* (*voters-$\mathcal{E}$ E*)}

definition *finite-elections ::* ($'a$, $'v$) *Election set* **where**
  finite-elections $\equiv$ {E :: ($'a$, $'v$) *Election. finite-election E*}

definition *well-formed-elections ::* ($'a$, $'v$) *Election set* **where**
  well-formed-elections $\equiv$ {E. *profile* (*voters-$\mathcal{E}$ E*) (*alternatives-$\mathcal{E}$ E*) (*profile-$\mathcal{E}$ E*)}

— This function subsumes elections with fixed alternatives, finite voters, and a
default value for the profile value on non-voters.
fun *elections-$\mathcal{A}$ ::* $'a$ *set* $\Rightarrow$ ($'a$, $'v$) *Election set* **where**
  elections-$\mathcal{A}$ A =
      well-formed-elections
    $\cap$ {E. *alternatives-$\mathcal{E}$ E = A* $\wedge$ *finite* (*voters-$\mathcal{E}$ E*)
        $\wedge$ ($\forall$ v. v $\notin$ *voters-$\mathcal{E}$ E* $\longrightarrow$ *profile-$\mathcal{E}$ E v = {}*)}

— Here, we count the occurrences of a ballot in an election, i.e., how many voters
specifically chose that exact ballot.
fun *vote-count ::* $'a$ *Preference-Relation* $\Rightarrow$ ($'a$, $'v$) *Election* $\Rightarrow$ *nat* **where**
  vote-count p E = card {v $\in$ (*voters-$\mathcal{E}$ E*). (*profile-$\mathcal{E}$ E*) v = p}

1.5.2   Vote Count

lemma *vote-count-sum*:
  fixes E :: ($'a$, $'v$) *Election*
  assumes
    finite (*voters-$\mathcal{E}$ E*) **and**

finite (UNIV :: ('a × 'a) set)
  shows $(\sum p \in UNIV.\ vote\text{-}count\ p\ E) = card\ (voters\text{-}\mathcal{E}\ E)$
⟨*proof*⟩

1.5.3   Voter Permutations

A common action of interest on elections is renaming the voters, e.g., when talking about anonymity.

fun *rename* :: $('v \Rightarrow 'v) \Rightarrow ('a, 'v)\ Election \Rightarrow ('a, 'v)\ Election$ **where**
  rename π $(A, V, p) = (A, \pi\ `\ V, p \circ (the\text{-}inv\ \pi))$

lemma *rename-sound*:
  fixes
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)\ Profile$ **and**
    π :: $'v \Rightarrow 'v$
  assumes
    prof: *profile V A p* **and**
    renamed: $(A, V', q) = rename\ \pi\ (A, V, p)$ **and**
    bij-perm: *bij* π
  shows *profile V ′ A q*
⟨*proof*⟩

lemma *rename-prof*:
  fixes
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)\ Profile$ **and**
    π :: $'v \Rightarrow 'v$
  assumes
    profile V A p **and**
    $(A, V', q) = rename\ \pi\ (A, V, p)$ **and**
    bij π
  shows *profile V ′ A q*
  ⟨*proof*⟩

lemma *rename-finite*:
  fixes
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)\ Profile$ **and**
    π :: $'v \Rightarrow 'v$
  assumes
    finite V **and**
    $(A, V', q) = rename\ \pi\ (A, V, p)$ **and**
    bij π
  shows *finite V ′*
  ⟨*proof*⟩

lemma *rename-inv*:
  fixes
    $\pi :: \ 'v \Rightarrow {}'v$ **and**
    $A :: \ 'a \ set$ **and**
    $V :: \ 'v \ set$ **and**
    $p :: \ ('a, \ 'v) \ Profile$
  assumes *bij* π
  shows *rename* π (*rename* (*the-inv* π) (A, V, p)) = (A, V, p)
⟨*proof*⟩

lemma *rename-inj*:
  fixes $\pi :: \ 'v \Rightarrow {}'v$
  assumes *bij* π
  shows *inj* (*rename* π)
⟨*proof*⟩

lemma *rename-surj*:
  fixes $\pi :: \ 'v \Rightarrow {}'v$
  assumes *bij* π
  shows
    rename π ' *well-formed-elections* = *well-formed-elections* **and**
    rename π ' *finite-elections* = *finite-elections*
⟨*proof*⟩

1.5.4 List Representation

A profile on a voter set that has a natural order can be viewed as a list of ballots.

fun *to-list* :: $'v$ :: *linorder set* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$
    ($'a$ *Preference-Relation*) *list* **where**
  to-list V p = (*if finite* V
              then map p (*sorted-list-of-set* V)
              else [])

lemma *map-helper*:
  fixes
    $f :: \ 'x \Rightarrow {}'y \Rightarrow {}'z$ **and**
    $g :: \ 'x \Rightarrow {}'x$ **and**
    $h :: \ 'y \Rightarrow {}'y$ **and**
    $l :: \ 'x \ list$ **and**
    $l' :: \ 'y \ list$
  shows *map2* f (*map* g l) (*map* h l') = *map2* (λ x y. f (g x) (h y)) l l'
⟨*proof*⟩

lemma *to-list-simp*:
  fixes
    $i :: \ nat$ **and**
    $V :: \ 'v$ :: *linorder set* **and**

$p :: ('a, 'v)$ *Profile*
  assumes $i < card\ V$
  shows $(to\text{-}list\ V\ p)!i = p\ ((sorted\text{-}list\text{-}of\text{-}set\ V)!i)$
  $\langle proof \rangle$

lemma *to-list-comp*:
  fixes
    $V :: {'v} :: linorder\ set$ **and**
    $p :: ('a, 'v)$ *Profile* **and**
    $f :: {'a}\ rel \Rightarrow {'a}\ rel$
  shows $to\text{-}list\ V\ (f \circ p) = map\ f\ (to\text{-}list\ V\ p)$
  $\langle proof \rangle$

lemma *set-card-upper-bound*:
  fixes
    $i :: nat$ **and**
    $V :: nat\ set$
  assumes
    fin-V: $finite\ V$ **and**
    bound-v: $\forall\ v \in V.\ v < i$
  shows $card\ V \le i$
$\langle proof \rangle$

lemma *sorted-list-of-set-nth-equals-card*:
  fixes
    $V :: {'v} :: linorder\ set$ **and**
    $x :: {'v}$
  assumes
    fin-V: $finite\ V$ **and**
    x-V: $x \in V$
  shows $sorted\text{-}list\text{-}of\text{-}set\ V!(card\ \{v \in V.\ v < x\}) = x$
$\langle proof \rangle$

lemma *to-list-permutes-under-bij*:
  fixes
    $\pi :: {'v} :: linorder \Rightarrow {'v}$ **and**
    $V :: {'v}\ set$ **and**
    $p :: ('a, 'v)$ *Profile*
  assumes $bij\ \pi$
  shows
    $let\ \varphi = \lambda\ i.\ card\ \{v \in \pi\ \text{‘}\ V.\ v < \pi\ ((sorted\text{-}list\text{-}of\text{-}set\ V)!i)\}$
    $in\ (to\text{-}list\ V\ p) = permute\text{-}list\ \varphi\ (to\text{-}list\ (\pi\ \text{‘}\ V)\ (\lambda\ x.\ p\ (the\text{-}inv\ \pi\ x)))$
$\langle proof \rangle$

1.5.5   Preference Counts

The win count for an alternative a with respect to a finite voter set V in a
profile p is the amount of ballots from V in p that rank alternative a in first
position. If the voter set is infinite, counting is not generally possible.

fun *win-count* :: $'v$ *set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ *enat* **where**
  win-count V p a = (*if finite V*
    then card {$v \in V$. *above* $(p\ v)\ a = \{a\}$} *else* ∞)

fun *prefer-count* :: $'v$ *set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *enat* **where**
  prefer-count V p x y = (*if finite V*
    then card {$v \in V$. *let* $r = (p\ v)$ *in* $(y \preceq_r x)$} *else* ∞)

lemma *pref-count-voter-set-card*:
  fixes
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile* **and**
    $a\ b$ :: $'a$
  assumes *finite V*
  shows *prefer-count V p a b* $\leq$ *card V*
  $\langle proof \rangle$

lemma *set-compr*:
  fixes
    A :: $'a$ *set* **and**
    f :: $'a \Rightarrow 'a$ *set*
  shows {$f\ x \mid x.\ x \in A$} = f ' A
  $\langle proof \rangle$

lemma *pref-count-set-compr*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile* **and**
    a :: $'a$
  shows {*prefer-count V p a a'* $\mid$ *a'.* $a' \in A - \{a\}$} =
      (*prefer-count V p a*) ' $(A - \{a\})$
  $\langle proof \rangle$

lemma *pref-count*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile* **and**
    $a\ b$ :: $'a$
  assumes
    prof: *profile V A p* **and**
    fin: *finite V* **and**
    a-in-A: $a \in A$ **and**
    b-in-A: $b \in A$ **and**
    neq: $a \neq b$
  shows *prefer-count V p a b* = *card V* $-$ (*prefer-count V p b a*)
$\langle proof \rangle$

lemma *pref-count-sym*:
  fixes
    p :: $('a, 'v)$ *Profile* **and**
    V :: $'v$ *set* **and**
    a b c :: $'a$
  assumes
    pref-count-ineq: *prefer-count* V p a c $\geq$ *prefer-count* V p c b **and**
    prof: *profile* V A p **and**
    a-in-A: $a \in A$ **and**
    b-in-A: $b \in A$ **and**
    c-in-A: $c \in A$ **and**
    a-neq-c: $a \neq c$ **and**
    c-neq-b: $c \neq b$
  shows *prefer-count* V p b c $\geq$ *prefer-count* V p c a
$\langle$*proof*$\rangle$

lemma *empty-prof-imp-zero-pref-count*:
  fixes
    p :: $('a, 'v)$ *Profile* **and**
    V :: $'v$ *set* **and**
    a b :: $'a$
  assumes $V = \{\}$
  shows *prefer-count* V p a b = 0
$\langle$*proof*$\rangle$

fun *wins* :: $'v$ *set* $\Rightarrow$ $'a$ $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
  wins V a p b =
    (*prefer-count* V p a b > *prefer-count* V p b a)

lemma *wins-inf-voters*:
  fixes
    p :: $('a, 'v)$ *Profile* **and**
    a b :: $'a$ **and**
    V :: $'v$ *set*
  assumes *infinite* V
  shows $\neg$ *wins* V b p a
$\langle$*proof*$\rangle$

Having alternative a win against b implies that b does not win against a.

lemma *wins-antisym*:
  fixes
    p :: $('a, 'v)$ *Profile* **and**
    a b :: $'a$ **and**
    V :: $'v$ *set*
  assumes *wins* V a p b — This already implies that V is finite.
  shows $\neg$ *wins* V b p a
$\langle$*proof*$\rangle$

lemma *wins-irreflex*:

fixes
  $p :: ('a, 'v)$ *Profile* **and**
  $a :: 'a$ **and**
  $V :: 'v$ *set*
shows $\neg$ *wins V a p a*
$\langle proof \rangle$

1.5.6 Condorcet Winner

fun *condorcet-winner* :: $'v$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
  condorcet-winner V A p a =
    (finite-profile V A p $\wedge$ $a \in A$ $\wedge$ $(\forall\ x \in A - \{a\}.\ \textit{wins V a p x}))$

lemma *cond-winner-unique-eq*:
  fixes
    $V :: 'v$ *set* **and**
    $A :: 'a$ *set* **and**
    $p :: ('a, 'v)$ *Profile* **and**
    $a\ b :: 'a$
  assumes
    condorcet-winner V A p a **and**
    condorcet-winner V A p b
  shows $b = a$
$\langle proof \rangle$

lemma *cond-winner-unique*:
  fixes
    $A :: 'a$ *set* **and**
    $p :: ('a, 'v)$ *Profile* **and**
    $a :: 'a$
  assumes *condorcet-winner V A p a*
  shows $\{a' \in A.\ \textit{condorcet-winner V A p a}'\} = \{a\}$
$\langle proof \rangle$

lemma *cond-winner-unique$'$*:
  fixes
    $V :: 'v$ *set* **and**
    $A :: 'a$ *set* **and**
    $p :: ('a, 'v)$ *Profile* **and**
    $a\ b :: 'a$
  assumes
    condorcet-winner V A p a **and**
    $b \neq a$
  shows $\neg$ *condorcet-winner V A p b*
  $\langle proof \rangle$

1.5.7 Limited Profile

This function restricts a profile p to a set A of alternatives and a set V of voters s.t. voters outside of V do not have any preferences or do not cast a vote. This keeps all of A's preferences.

fun *limit-profile* :: *$'a$ set $\Rightarrow$ ($'a$, $'v$) Profile $\Rightarrow$ ($'a$, $'v$) Profile* **where**
  limit-profile A p = (λ v. limit A (p v))

lemma *limit-prof-trans*:
  fixes
    A B C :: *$'a$ set* **and**
    p :: *($'a$, $'v$) Profile*
  assumes
    B $\subseteq$ A **and**
    C $\subseteq$ B
  shows *limit-profile C p = limit-profile C (limit-profile B p)*
  $\langle proof \rangle$

lemma *limit-profile-sound*:
  fixes
    A B :: *$'a$ set* **and**
    V :: *$'v$ set* **and**
    p :: *($'a$, $'v$) Profile*
  assumes
    profile V B p **and**
    A $\subseteq$ B
  shows *profile V A (limit-profile A p)*
$\langle proof \rangle$

1.5.8 Lifting Property

definition *equiv-prof-except-a* :: *$'v$ set $\Rightarrow$ $'a$ set $\Rightarrow$ ($'a$, $'v$) Profile $\Rightarrow$*
      ($'a$, $'v$) Profile $\Rightarrow$ $'a$ $\Rightarrow$ bool **where**
  equiv-prof-except-a V A p p' a $\equiv$
    profile V A p $\wedge$ profile V A p' $\wedge$ a $\in$ A $\wedge$
      ($\forall$ v $\in$ V. equiv-rel-except-a A (p v) (p' v) a)

An alternative gets lifted from one profile to another iff its ranking increases in at least one ballot, and nothing else changes.

definition *lifted* :: *$'v$ set $\Rightarrow$ $'a$ set $\Rightarrow$ ($'a$, $'v$) Profile $\Rightarrow$*
      ($'a$, $'v$) Profile $\Rightarrow$ $'a$ $\Rightarrow$ bool **where**
  lifted V A p p' a $\equiv$
    finite-profile V A p $\wedge$ finite-profile V A p' $\wedge$ a $\in$ A
      $\wedge$ ($\forall$ v $\in$ V. $\neg$ Preference-Relation.lifted A (p v) (p' v) a $\longrightarrow$ (p v) = (p' v))
      $\wedge$ ($\exists$ v $\in$ V. Preference-Relation.lifted A (p v) (p' v) a)

lemma *lifted-imp-equiv-prof-except-a*:
  fixes
    A :: *$'a$ set* **and**

   $V :: 'v\ set$ **and**
   $p\ p' :: ('a,\ 'v)\ Profile$ **and**
   $a :: 'a$
 assumes *lifted V A p p′ a*
 shows *equiv-prof-except-a V A p p′ a*
⟨*proof*⟩

lemma *negl-diff-imp-eq-limit-prof*:
 fixes
   $A\ A' :: 'a\ set$ **and**
   $V :: 'v\ set$ **and**
   $p\ p' :: ('a,\ 'v)\ Profile$ **and**
   $a :: 'a$
 assumes
   change: *equiv-prof-except-a V A′ p q a* **and**
   subset: $A \subseteq A'$ **and**
   not-in-A: $a \notin A$
 shows $\forall\ v \in V.$ (*limit-profile A p*) $v =$ (*limit-profile A q*) v
 — With the current definitions of *equiv-prof-except-a* and *limit-prof*, we can only conclude that the limited profiles coincide on the given voter set, since *limit-prof* may change the profiles everywhere, while *equiv-prof-except-a* only makes statements about the voter set.
⟨*proof*⟩

lemma *limit-prof-eq-or-lifted*:
 fixes
   $A\ A' :: 'a\ set$ **and**
   $V :: 'v\ set$ **and**
   $p\ p' :: ('a,\ 'v)\ Profile$ **and**
   $a :: 'a$
 assumes
   lifted-a: *lifted V A′ p p′ a* **and**
   subset: $A \subseteq A'$
 shows ($\forall\ v \in V.$ *limit-profile A p v = limit-profile A p′ v*)
    $\lor$ *lifted V A* (*limit-profile A p*) (*limit-profile A p′*) *a*
⟨*proof*⟩

end

1.6 Social Choice Result

theory *Social-Choice-Result*
 imports *Result*
begin

1.6.1 Definition

A social choice result contains three sets of alternatives: elected, rejected, and deferred alternatives.

fun *well-formed-SCF* :: *′a set ⇒ ′a Result ⇒ bool* **where**
  well-formed-SCF A res = (disjoint3 res ∧ set-equals-partition A res)

fun *limit-SCF* :: *′a set ⇒ ′a set ⇒ ′a set* **where**
  limit-SCF A r = A ∩ r

1.6.2 Auxiliary Lemmas

lemma *result-imp-rej*:
  fixes *A e r d* :: *′a set*
  assumes *well-formed-SCF A (e, r, d)*
  shows *A − (e ∪ d) = r*
⟨*proof*⟩

lemma *result-count*:
  fixes *A e r d* :: *′a set*
  assumes
    wf-result: *well-formed-SCF A (e, r, d)* **and**
    fin-A: *finite A*
  shows *card A = card e + card r + card d*
⟨*proof*⟩

lemma *defer-subset*:
  fixes
    A :: *′a set* **and**
    r :: *′a Result*
  assumes *well-formed-SCF A r*
  shows *defer-r r ⊆ A*
⟨*proof*⟩

lemma *elect-subset*:
  fixes
    A :: *′a set* **and**
    r :: *′a Result*
  assumes *well-formed-SCF A r*
  shows *elect-r r ⊆ A*
⟨*proof*⟩

lemma *reject-subset*:
  fixes
    A :: *′a set* **and**
    r :: *′a Result*
  assumes *well-formed-SCF A r*
  shows *reject-r r ⊆ A*
⟨*proof*⟩

end

1.7 Social Welfare Result

theory *Social-Welfare-Result*
  imports *Result*
       Preference-Relation
begin

A social welfare result contains three sets of relations: elected, rejected, and deferred. A well-formed social welfare result consists only of linear orders on the alternatives.

fun *well-formed-$\mathcal{SWF}$ :: 'a set $\Rightarrow$ ('a Preference-Relation) Result $\Rightarrow$ bool* **where**
  well-formed-$\mathcal{SWF}$ A res = (disjoint3 res $\wedge$
                    set-equals-partition {r. linear-order-on A r} res)

fun *limit-$\mathcal{SWF}$ :: 'a set $\Rightarrow$ ('a Preference-Relation) set $\Rightarrow$*
     ('a Preference-Relation) set **where**
  limit-$\mathcal{SWF}$ A res = {limit A r | r. r $\in$ res $\wedge$ linear-order-on A (limit A r)}

end

1.8 Electoral Result Types

theory *Result-Interpretations*
  imports *Social-Choice-Result*
      Social-Welfare-Result
      Collections.Locale-Code
begin

Interpretations of the result locale are placed inside a Locale-Code block in order to enable code generation of later definitions in the locale. Those definitions need to be added via a Locale-Code block as well.

⟨*ML*⟩

Results from social choice functions ($\mathcal{SCF}s$), for the purpose of composability and modularity given as three sets of (potentially tied) alternatives. See `Social_Choice_Result.thy` for details.

global-interpretation *$\mathcal{SCF}$-result*: *result well-formed-$\mathcal{SCF}$ limit-$\mathcal{SCF}$*
⟨*proof*⟩

Results from committee functions, for the purpose of composability and

modularity given as three sets of (potentially tied) sets of alternatives or committees. [[*Not actually used yet.*]]

global-interpretation *committee-result*: *result*
    λ *A r. set-equals-partition* (*Pow A*) *r* ∧ *disjoint3 r*
    λ *A rs.* {*r* ∩ *A* | *r. r* ∈ *rs*}
⟨*proof*⟩

Results from social welfare functions ($\mathcal{SWF}s$), for the purpose of composability and modularity given as three sets of (potentially tied) linear orders over the alternatives. See `Social_Welfare_Result.thy` for details.

global-interpretation $\mathcal{SWF}$-*result*: *result well-formed-$\mathcal{SWF}$ limit-$\mathcal{SWF}$*
⟨*proof*⟩

⟨*ML*⟩

end

1.9 Symmetry Properties of Functions

theory *Symmetry-Of-Functions*
  imports *HOL−Algebra.Group-Action*
        HOL−Algebra.Generated-Groups
begin

1.9.1 Functions

type-synonym ($'x$, $'y$) *binary-fun* = $'x \Rightarrow 'y \Rightarrow 'y$

fun *extensional-continuation* :: ($'x \Rightarrow 'y$) $\Rightarrow$ $'x$ *set* $\Rightarrow$ ($'x \Rightarrow 'y$) **where**
  extensional-continuation f S = (λ *x. if x* ∈ *S then f x else undefined*)

fun *preimg* :: ($'x \Rightarrow 'y$) $\Rightarrow$ $'x$ *set* $\Rightarrow$ $'y$ $\Rightarrow$ $'x$ *set* **where**
  preimg f S y = {*x* ∈ *S. f x* = *y*}

1.9.2 Relations for Symmetry Constructions

fun *restricted-rel* :: $'x$ *rel* $\Rightarrow$ $'x$ *set* $\Rightarrow$ $'x$ *set* $\Rightarrow$ $'x$ *rel* **where**
  restricted-rel r S S′ = *r* ∩ (*S* × *S′*)

fun *closed-restricted-rel* :: $'x$ *rel* $\Rightarrow$ $'x$ *set* $\Rightarrow$ $'x$ *set* $\Rightarrow$ *bool* **where**
  closed-restricted-rel r S T = ((*restricted-rel r T S*) '' *T* ⊆ *T*)

fun *action-induced-rel* :: $'x$ *set* $\Rightarrow$ $'y$ *set* $\Rightarrow$ ($'x$, $'y$) *binary-fun* $\Rightarrow$ $'y$ *rel* **where**
  action-induced-rel S T φ = {(*y, y′*). *y* ∈ *T* ∧ (∃ *x* ∈ *S. φ x y* = *y′*)}

fun *product* :: $'x$ *rel* $\Rightarrow$ ($'x$ * $'x$) *rel* **where**

product r = {(p, p'). (fst p, fst p') ∈ r ∧ (snd p, snd p') ∈ r}

fun *equivariance :: 'x set ⇒ 'y set ⇒ ('x, 'y) binary-fun ⇒ ('y * 'y) rel* **where**
  equivariance S T φ =
    {((u, v), (x, y)). (u, v) ∈ T × T ∧ (∃ z ∈ S. x = φ z u ∧ y = φ z v)}

fun *singleton-set-system :: 'x set ⇒ 'x set set* **where**
  singleton-set-system S = {{x} | x. x ∈ S}

fun *set-action :: ('x, 'r) binary-fun ⇒ ('x, 'r set) binary-fun* **where**
  set-action ψ x = image (ψ x)

1.9.3 Invariance and Equivariance

Invariance and equivariance are symmetry properties of functions: Invariance means that related preimages have identical images and equivariance denotes consistent changes.

datatype *('x, 'y) symmetry =*
  Invariance 'x rel |
  Equivariance 'x set (('x ⇒ 'x) × ('y ⇒ 'y)) set

fun *is-symmetry :: ('x ⇒ 'y) ⇒ ('x, 'y) symmetry ⇒ bool* **where**
  is-symmetry f (Invariance r) = (∀ x. ∀ y. (x, y) ∈ r ⟶ f x = f y) |
  is-symmetry f (Equivariance s τ) = (∀ (φ, ψ) ∈ τ. ∀ x ∈ s. f (φ x) = ψ (f x))

definition *action-induced-equivariance :: 'z set ⇒ 'x set ⇒ ('z, 'x) binary-fun ⇒*
    ('z, 'y) binary-fun ⇒ ('x, 'y) symmetry **where**
  action-induced-equivariance T S φ ψ ≡ Equivariance S {(φ z, ψ z) | z. z ∈ T}

1.9.4 Auxiliary Lemmas

lemma *un-left-inv-singleton-set-system*: ⋃ ∘ *singleton-set-system = id*
⟨*proof*⟩

lemma *preimg-comp*:
  fixes
    f :: 'x ⇒ 'y **and**
    g :: 'x ⇒ 'x **and**
    S :: 'x set **and**
    x :: 'y
  shows *preimg f (g ' S) x = g ' preimg (f ∘ g) S x*
⟨*proof*⟩

1.9.5 Rewrite Rules

theorem *rewrite-invar-as-equivar*:
  fixes
    f :: 'x ⇒ 'y **and**
    S :: 'x set **and**

$T :: \prime z \ set$ **and**

$\varphi :: (\prime z, \ \prime x) \ binary\text{-}fun$

shows *is-symmetry f* (*Invariance* (*action-induced-rel T S φ*)) =

is-symmetry f (*action-induced-equivariance T S φ* ($\lambda \ g. \ id$))

$\langle proof \rangle$

lemma *rewrite-invar-ind-by-act*:

fixes

$f :: \prime x \Rightarrow \prime y$ **and**

$S :: \prime z \ set$ **and**

$T :: \prime x \ set$ **and**

$\varphi :: (\prime z, \ \prime x) \ binary\text{-}fun$

shows *is-symmetry f* (*Invariance* (*action-induced-rel S T φ*)) =

($\forall \ x \in S. \ \forall \ y \in T. \ f \ y = f \ (\varphi \ x \ y)$)

$\langle proof \rangle$

lemma *rewrite-equivariance*:

fixes

$f :: \prime x \Rightarrow \prime y$ **and**

$S :: \prime z \ set$ **and**

$T :: \prime x \ set$ **and**

$\varphi :: (\prime z, \ \prime x) \ binary\text{-}fun$ **and**

$\psi :: (\prime z, \ \prime y) \ binary\text{-}fun$

shows *is-symmetry f* (*action-induced-equivariance S T $\varphi \ \psi$*) =

($\forall \ x \in S. \ \forall \ y \in T. \ f \ (\varphi \ x \ y) = \psi \ x \ (f \ y)$)

$\langle proof \rangle$

lemma *rewrite-group-action-img*:

fixes

$m :: \prime x \ monoid$ **and**

$S \ T :: \prime y \ set$ **and**

$\varphi :: (\prime x, \ \prime y) \ binary\text{-}fun$ **and**

$x \ y :: \prime x$

assumes

$T \subseteq S$ **and**

$x \in carrier \ m$ **and**

$y \in carrier \ m$ **and**

group-action m S φ

shows $\varphi \ (x \otimes_m y) \ \lq \ T = \varphi \ x \ \lq \ \varphi \ y \ \lq \ T$

$\langle proof \rangle$

lemma *rewrite-carrier*: *carrier* (*BijGroup UNIV*) = $\{f. \ bij \ f\}$

$\langle proof \rangle$

lemma *universal-set-carrier-imp-bij-group*:

fixes $f :: \prime a \Rightarrow \prime a$

assumes $f \in carrier$ (*BijGroup UNIV*)

shows *bij f*

$\langle proof \rangle$

lemma *rewrite-sym-group*:
  fixes
    $f\ g :: \ 'a \Rightarrow \ 'a$ **and**
    $S :: \ 'a\ set$
  assumes
    $f \in carrier\ (BijGroup\ S)$ **and**
    $g \in carrier\ (BijGroup\ S)$
  shows
    rewrite-mult: $f \otimes_{BijGroup\ S} g = extensional\text{-}continuation\ (f \circ g)\ S$ **and**
    rewrite-mult-univ: $S = UNIV \longrightarrow f \otimes_{BijGroup\ S} g = f \circ g$
  ⟨*proof*⟩

lemma *simp-extensional-univ*:
  fixes $f :: \ 'a \Rightarrow \ 'b$
  shows $extensional\text{-}continuation\ f\ UNIV = f$
  ⟨*proof*⟩

lemma *extensional-continuation-subset*:
  fixes
    $f :: \ 'a \Rightarrow \ 'b$ **and**
    $S\ T :: \ 'a\ set$ **and**
    $x :: \ 'a$
  assumes
    $T \subseteq S$ **and**
    $x \in T$
  shows $extensional\text{-}continuation\ f\ S\ x = extensional\text{-}continuation\ f\ T\ x$
  ⟨*proof*⟩

lemma *rel-ind-by-coinciding-action-on-subset-eq-restr*:
  fixes
    $\varphi\ \psi :: (\ 'a,\ 'b)\ binary\text{-}fun$ **and**
    $S :: \ 'a\ set$ **and**
    $T\ U :: \ 'b\ set$
  assumes
    $U \subseteq T$ **and**
    $\forall\ x \in S.\ \forall\ y \in U.\ \psi\ x\ y = \varphi\ x\ y$
  shows $action\text{-}induced\text{-}rel\ S\ U\ \psi = restricted\text{-}rel\ (action\text{-}induced\text{-}rel\ S\ T\ \varphi)\ U$
$UNIV$
⟨*proof*⟩

lemma *coinciding-actions-ind-equal-rel*:
  fixes
    $S :: \ 'x\ set$ **and**
    $T :: \ 'y\ set$ **and**
    $\varphi\ \psi :: (\ 'x,\ 'y)\ binary\text{-}fun$
  assumes $\forall\ x \in S.\ \forall\ y \in T.\ \varphi\ x\ y = \psi\ x\ y$
  shows $action\text{-}induced\text{-}rel\ S\ T\ \varphi = action\text{-}induced\text{-}rel\ S\ T\ \psi$
  ⟨*proof*⟩

1.9.6 Group Actions

lemma *const-id-is-group-action*:
  fixes $m :: {'x}$ *monoid*
  assumes *group m*
  shows *group-action m UNIV* (λ *x. id*)
  $\langle proof \rangle$

theorem *group-act-induces-set-group-act*:
  fixes
    $m :: {'x}$ *monoid* **and**
    $S :: {'y}$ *set* **and**
    $\varphi :: ({'x}, {'y})$ *binary-fun*
  defines $\varphi\text{-}img \equiv (\lambda$ *x. extensional-continuation* (*image* (φ *x*)) (*Pow S*))
  assumes *group-action m S φ*
  shows *group-action m* (*Pow S*) *φ-img*
$\langle proof \rangle$

1.9.7 Invariance and Equivariance

It suffices to show equivariance under the group action of a generating set
of a group to show equivariance under the group action of the whole group.
For example, it is enough to show invariance under transpositions to show
invariance under a complete finite symmetric group.

theorem *equivar-generators-imp-equivar-group*:
  fixes
    $f :: {'x} \Rightarrow {'y}$ **and**
    $m :: {'z}$ *monoid* **and**
    $S :: {'z}$ *set* **and**
    $T :: {'x}$ *set* **and**
    $\varphi :: ({'z}, {'x})$ *binary-fun* **and**
    $\psi :: ({'z}, {'y})$ *binary-fun*
  assumes
    equivar: *is-symmetry f* (*action-induced-equivariance S T φ ψ*) **and**
    action-φ: *group-action m T φ* **and**
    action-ψ: *group-action m* (*f ' T*) *ψ* **and**
    gen: *carrier m = generate m S*
  shows *is-symmetry f* (*action-induced-equivariance* (*carrier m*) *T φ ψ*)
$\langle proof \rangle$

lemma *invar-parameterized-fun*:
  fixes
    $f :: {'x} \Rightarrow ({'x} \Rightarrow {'y})$ **and**
    $r :: {'x}$ *rel*
  assumes
    $\forall$ *x. is-symmetry* (*f x*) (*Invariance r*) **and**
    is-symmetry f (*Invariance r*)
  shows *is-symmetry* (λ *x. f x x*) (*Invariance r*)
  $\langle proof \rangle$

lemma *invar-under-subset-rel*:
  fixes
    $f :: {}'x \Rightarrow {}'y$ **and**
    $r\ s :: {}'x\ rel$
  assumes
    subset: $r \subseteq s$ **and**
    invar: *is-symmetry f* (*Invariance s*)
  shows *is-symmetry f* (*Invariance r*)
  $\langle proof \rangle$

lemma *equivar-ind-by-act-coincide*:
  fixes
    $S :: {}'x\ set$ **and**
    $T :: {}'y\ set$ **and**
    $f :: {}'y \Rightarrow {}'z$ **and**
    $\varphi\ \varphi' :: ({}'x,\ {}'y)\ binary\text{-}fun$ **and**
    $\psi :: ({}'x,\ {}'z)\ binary\text{-}fun$
  assumes $\forall\ x \in S.\ \forall\ y \in T.\ \varphi\ x\ y = \varphi'\ x\ y$
  shows *is-symmetry f* (*action-induced-equivariance S T φ ψ*) =
          is-symmetry f (*action-induced-equivariance S T φ' ψ*)
  $\langle proof \rangle$

lemma *equivar-under-subset*:
  fixes
    $f :: {}'x \Rightarrow {}'y$ **and**
    $S\ T :: {}'x\ set$ **and**
    $\tau :: (({}'x \Rightarrow {}'x) \times ({}'y \Rightarrow {}'y))\ set$
  assumes
    is-symmetry f (*Equivariance S τ*) **and**
    $T \subseteq S$
  shows *is-symmetry f* (*Equivariance T τ*)
  $\langle proof \rangle$

lemma *equivar-under-subset$'$*:
  fixes
    $f :: {}'x \Rightarrow {}'y$ **and**
    $S :: {}'x\ set$ **and**
    $\tau\ \upsilon :: (({}'x \Rightarrow {}'x) \times ({}'y \Rightarrow {}'y))\ set$
  assumes
    is-symmetry f (*Equivariance S τ*) **and**
    $\upsilon \subseteq \tau$
  shows *is-symmetry f* (*Equivariance S υ*)
  $\langle proof \rangle$

theorem *group-action-equivar-f-imp-equivar-preimg*:
  fixes
    $f :: {}'x \Rightarrow {}'y$ **and**
    $\mathcal{D}_f\ S :: {}'x\ set$ **and**

m :: $'z$ *monoid* **and**
φ :: $('z, 'x)$ *binary-fun* **and**
ψ :: $('z, 'y)$ *binary-fun* **and**
x :: $'z$
defines *equivar-prop* $\equiv$ *action-induced-equivariance* (*carrier m*) $\mathcal{D}_f$ φ ψ
assumes
  action-φ: *group-action m S φ* **and**
  action-res: *group-action m UNIV ψ* **and**
  dom-in-s: $\mathcal{D}_f \subseteq S$ **and**
  closed-domain:
    closed-restricted-rel (*action-induced-rel* (*carrier m*) S φ) S $\mathcal{D}_f$ **and**
  equivar-f: *is-symmetry f equivar-prop* **and**
  group-elem-x: $x \in$ *carrier m*
shows $\forall$ $y.$ *preimg* f $\mathcal{D}_f$ (ψ x y) $= (\varphi$ $x)$ ' (*preimg* f $\mathcal{D}_f$ y)
$\langle proof \rangle$

1.9.8 Function Composition

lemma *invar-comp*:
  fixes
    f :: $'x \Rightarrow 'y$ **and**
    g :: $'y \Rightarrow 'z$ **and**
    r :: $'x$ *rel*
  assumes *is-symmetry f* (*Invariance r*)
  shows *is-symmetry* ($g \circ f$) (*Invariance r*)
  $\langle proof \rangle$

lemma *equivar-comp*:
  fixes
    f :: $'x \Rightarrow 'y$ **and**
    g :: $'y \Rightarrow 'z$ **and**
    S :: $'x$ *set* **and**
    T :: $'y$ *set* **and**
    τ :: $(('x \Rightarrow 'x) \times ('y \Rightarrow 'y))$ *set* **and**
    υ :: $(('y \Rightarrow 'y) \times ('z \Rightarrow 'z))$ *set*
  defines
    transitive-acts $\equiv$
      $\{(\varphi, \psi). \exists$ χ :: $'y \Rightarrow 'y. (\varphi, \chi) \in \tau \wedge (\chi, \psi) \in \upsilon \wedge \chi$ ' f ' $S \subseteq T\}$
  assumes
    f ' $S \subseteq T$ **and**
    is-symmetry f (*Equivariance S τ*) **and**
    is-symmetry g (*Equivariance T υ*)
  shows *is-symmetry* ($g \circ f$) (*Equivariance S transitive-acts*)
  $\langle proof \rangle$

lemma *equivar-ind-by-action-comp*:
  fixes
    f :: $'x \Rightarrow 'y$ **and**
    g :: $'y \Rightarrow 'z$ **and**

$S ::\ 'w\ set$ **and**

$T ::\ 'x\ set$ **and**

$U ::\ 'y\ set$ **and**

$\varphi ::\ ('w,\ 'x)\ binary\text{-}fun$ **and**

$\chi ::\ ('w,\ 'y)\ binary\text{-}fun$ **and**

$\psi ::\ ('w,\ 'z)\ binary\text{-}fun$

assumes

$f\ `\ T \subseteq U$ **and**

$\forall\ x \in S.\ \chi\ x\ `\ f\ `\ T \subseteq U$ **and**

is-symmetry f (*action-induced-equivariance S T* φ χ) **and**

is-symmetry g (*action-induced-equivariance S U* χ ψ)

shows *is-symmetry* $(g \circ f)$ (*action-induced-equivariance S T* φ ψ)

⟨*proof*⟩

lemma *equivar-set-minus*:

fixes

$f\ g ::\ 'x \Rightarrow 'y\ set$ **and**

$S ::\ 'z\ set$ **and**

$T ::\ 'x\ set$ **and**

$\varphi ::\ ('z,\ 'x)\ binary\text{-}fun$ **and**

$\psi ::\ ('z,\ 'y)\ binary\text{-}fun$

assumes

f-equivar: *is-symmetry f* (*action-induced-equivariance S T* φ (*set-action* ψ))

and

g-equivar: *is-symmetry g* (*action-induced-equivariance S T* φ (*set-action* ψ))

and

bij-a: $\forall\ a \in S.\ bij\ (\psi\ a)$

shows

is-symmetry $(\lambda\ b.\ f\ b - g\ b)$ (*action-induced-equivariance S T* φ (*set-action* ψ))

⟨*proof*⟩

lemma *equivar-union-under-image-action*:

fixes

$f ::\ 'x \Rightarrow 'y$ **and**

$S ::\ 'z\ set$ **and**

$\varphi ::\ ('z,\ 'x)\ binary\text{-}fun$

shows *is-symmetry* $\bigcup$ (*action-induced-equivariance S UNIV*

(*set-action* (*set-action* φ)) (*set-action* φ))

⟨*proof*⟩

end

1.10   Symmetry Properties of Voting Rules

theory *Voting-Symmetry*

imports *Symmetry-Of-Functions*

   Social-Choice-Result

Social-Welfare-Result
Profile

begin

1.10.1   Definitions

fun *result-action* :: (′x, ′r) *binary-fun* ⇒ (′x, ′r Result) *binary-fun* **where**
  result-action ψ x = (λ *r*. (ψ *x* ' *elect-r r*, ψ *x* ' *reject-r r*, ψ *x* ' *defer-r r*))

Anonymity

Bijection group on the set of voters.

definition *bijection*$_{\mathcal{VG}}$ :: (′v ⇒ ′v) *monoid* **where**
  bijection$_{\mathcal{VG}}$ ≡ *BijGroup* (*UNIV* :: ′v *set*)

Permutation action on the set of voters. Invariance under this action implies anonymity.

fun *φ-anon* :: (′a, ′v) *Election set* ⇒ (′v ⇒ ′v) ⇒
    ((′a, ′v) *Election* ⇒ (′a, ′v) *Election*) **where**
  φ-anon $\mathcal{E}$ *π* = *extensional-continuation* (*rename π*) $\mathcal{E}$

fun *anonymity*$_{\mathcal{R}}$ :: (′a, ′v) *Election set* ⇒ (′a, ′v) *Election rel* **where**
  anonymity$_{\mathcal{R}}$ $\mathcal{E}$ = *action-induced-rel* (*carrier bijection*$_{\mathcal{VG}}$) $\mathcal{E}$ (*φ-anon* $\mathcal{E}$)

Neutrality

fun *rel-rename* :: (′a ⇒ ′a, ′a *Preference-Relation*) *binary-fun* **where**
  rel-rename π r = {(π *a*, π *b*) | *a b*. (*a*, *b*) ∈ *r*}

fun *alternatives-rename* :: (′a ⇒ ′a, (′a, ′v) *Election*) *binary-fun* **where**
  alternatives-rename π $\mathcal{E}$ =
    (π ' (*alternatives-*$\mathcal{E}$ $\mathcal{E}$), *voters-*$\mathcal{E}$ $\mathcal{E}$, (*rel-rename π*) ∘ (*profile-*$\mathcal{E}$ $\mathcal{E}$))

Bijection group on the set of alternatives. Invariance under this action implies neutrality.

definition *bijection*$_{\mathcal{AG}}$ :: (′a ⇒ ′a) *monoid* **where**
  bijection$_{\mathcal{AG}}$ ≡ *BijGroup* (*UNIV* :: ′a *set*)

Permutation action on the set of alternatives.

fun *φ-neutral* :: (′a, ′v) *Election set* ⇒
    (′a ⇒ ′a, (′a, ′v) *Election*) *binary-fun* **where**
  φ-neutral $\mathcal{E}$ *π* = *extensional-continuation* (*alternatives-rename π*) $\mathcal{E}$

fun *neutrality*$_{\mathcal{R}}$ :: (′a, ′v) *Election set* ⇒ (′a, ′v) *Election rel* **where**
  neutrality$_{\mathcal{R}}$ $\mathcal{E}$ = *action-induced-rel* (*carrier bijection*$_{\mathcal{AG}}$) $\mathcal{E}$ (*φ-neutral* $\mathcal{E}$)

fun *ψ-neutral*$_c$ :: (′a ⇒ ′a, ′a) *binary-fun* **where**
  ψ-neutral$_c$ *π r* = π *r*

fun ψ-neutral$_\text{w}$ :: $('a \Rightarrow 'a, 'a\ rel)\ binary\text{-}fun$ **where**
  ψ-neutral$_\text{w}$ $\pi\ r$ = rel-rename $\pi\ r$

Homogeneity

fun homogeneity$_\mathcal{R}$ :: $('a, 'v)\ Election\ set \Rightarrow ('a, 'v)\ Election\ rel$ **where**
  homogeneity$_\mathcal{R}$ $\mathcal{E}$ =
    $\{(E, E').\ E \in \mathcal{E}$
      $\wedge$ alternatives-$\mathcal{E}$ E = alternatives-$\mathcal{E}$ E'
      $\wedge$ finite (voters-$\mathcal{E}$ E) $\wedge$ finite (voters-$\mathcal{E}$ E')
      $\wedge\ (\exists\ n > 0.\ \forall\ r :: 'a\ Preference\text{-}Relation.$
          vote-count $r\ E$ = $n * (vote\text{-}count\ r\ E'))\}$

fun copy-list :: $nat \Rightarrow 'x\ list \Rightarrow 'x\ list$ **where**
  copy-list $0\ l$ = $[]$ $\mid$
  copy-list $(Suc\ n)\ l$ = copy-list $n\ l$ @ l

fun homogeneity$_\mathcal{R}$$'$ :: $('a, 'v :: linorder)\ Election\ set \Rightarrow ('a, 'v)\ Election\ rel$ **where**
  homogeneity$_\mathcal{R}$$'$ $\mathcal{E}$ =
    $\{(E, E').\ E \in \mathcal{E}$
      $\wedge$ alternatives-$\mathcal{E}$ E = alternatives-$\mathcal{E}$ E'
      $\wedge$ finite (voters-$\mathcal{E}$ E) $\wedge$ finite (voters-$\mathcal{E}$ E')
      $\wedge\ (\exists\ n > 0.$
        to-list (voters-$\mathcal{E}$ E') (profile-$\mathcal{E}$ E') =
        copy-list n (to-list (voters-$\mathcal{E}$ E) (profile-$\mathcal{E}$ $E)))\}$

Reversal Symmetry

fun reverse-rel :: $'a\ rel \Rightarrow 'a\ rel$ **where**
  reverse-rel r = $\{(a, b).\ (b, a) \in r\}$

fun rel-app :: $('a\ rel \Rightarrow 'a\ rel) \Rightarrow ('a, 'v)\ Election \Rightarrow ('a, 'v)\ Election$ **where**
  rel-app $f\ (A,\ V,\ p)$ = $(A,\ V,\ f \circ p)$

definition reversal$_\mathcal{G}$ :: $('a\ rel \Rightarrow 'a\ rel)\ monoid$ **where**
  reversal$_\mathcal{G}$ $\equiv$ $(\!|carrier = \{reverse\text{-}rel,\ id\},\ monoid.mult = comp,\ one = id|\!)$

fun φ-reverse :: $('a, 'v)\ Election\ set$
        $\Rightarrow ('a\ rel \Rightarrow 'a\ rel, ('a, 'v)\ Election)\ binary\text{-}fun$ **where**
  φ-reverse $\mathcal{E}$ φ = extensional-continuation (rel-app φ) $\mathcal{E}$

fun ψ-reverse :: $('a\ rel \Rightarrow 'a\ rel, 'a\ rel)\ binary\text{-}fun$ **where**
  ψ-reverse φ r = $\varphi\ r$

fun reversal$_\mathcal{R}$ :: $('a, 'v)\ Election\ set \Rightarrow ('a, 'v)\ Election\ rel$ **where**
  reversal$_\mathcal{R}$ $\mathcal{E}$ = action-induced-rel (carrier reversal$_\mathcal{G}$) $\mathcal{E}$ (φ-reverse $\mathcal{E}$)

1.10.2 Auxiliary Lemmas

fun *n-app* :: *nat* $\Rightarrow$ (*'x* $\Rightarrow$ *'x*) $\Rightarrow$ (*'x* $\Rightarrow$ *'x*) **where**
  n-app-id: *n-app 0 f = id* |
  n-app-suc: *n-app* (*Suc n*) *f = f* $\circ$ *n-app n f*

lemma *n-app-rewrite*:
  fixes
    f :: *'x* $\Rightarrow$ *'x* **and**
    n :: *nat* **and**
    x :: *'x*
  shows (*f* $\circ$ *n-app n f*) *x* = (*n-app n f* $\circ$ *f*) *x*
$\langle proof \rangle$

lemma *n-app-leaves-set*:
  fixes
    A B :: *'x set* **and**
    f :: *'x* $\Rightarrow$ *'x* **and**
    x :: *'x*
  assumes
    fin-A: *finite A* **and**
    fin-B: *finite B* **and**
    x-el: *x* $\in$ *A* $-$ *B* **and**
    bij-f: *bij-betw f A B*
  obtains *n* :: *nat* **where**
    n > *0* **and**
    n-app n f x $\in$ *B* $-$ *A* **and**
    $\forall$ *m* > *0*. *m* < *n* $\longrightarrow$ *n-app m f x* $\in$ *A* $\cap$ *B*
$\langle proof \rangle$

lemma *n-app-rev*:
  fixes
    A B :: *'x set* **and**
    f :: *'x* $\Rightarrow$ *'x* **and**
    m n :: *nat* **and**
    x y :: *'x*
  assumes
    x-in-A: *x* $\in$ *A* **and**
    y-in-A: *y* $\in$ *A* **and**
    n-geq-m: *n* $\geq$ *m* **and**
    n-app-eq-m-n: *n-app n f x = n-app m f y* **and**
    n-app-x-in-A: $\forall$ *n'* < *n*. *n-app n' f x* $\in$ *A* **and**
    n-app-y-in-A: $\forall$ *m'* < *m*. *n-app m' f y* $\in$ *A* **and**
    fin-A: *finite A* **and**
    fin-B: *finite B* **and**
    bij-f-A-B: *bij-betw f A B*
  shows *n-app* (*n* $-$ *m*) *f x = y*
  $\langle proof \rangle$

lemma *n-app-inv*:

fixes
  $A\ B :: \prime x\ set$ **and**
  $f :: \prime x \Rightarrow \prime x$ **and**
  $n :: nat$ **and**
  $x :: \prime x$
assumes
  $x \in B$ **and**
  $\forall\ m \geq 0.\ m < n \longrightarrow n\text{-}app\ m\ (the\text{-}inv\text{-}into\ A\ f)\ x \in B$ **and**
  $bij\text{-}betw\ f\ A\ B$
shows $n\text{-}app\ n\ f\ (n\text{-}app\ n\ (the\text{-}inv\text{-}into\ A\ f)\ x) = x$
$\langle proof \rangle$

lemma *bij-betw-finite-ind-global-bij*:
  fixes
    $A\ B :: \prime x\ set$ **and**
    $f :: \prime x \Rightarrow \prime x$
  assumes
    fin-A: *finite A* **and**
    fin-B: *finite B* **and**
    bij-f: *bij-betw f A B*
  obtains $g :: \prime x \Rightarrow \prime x$ **where**
    $bij\ g$ **and**
    $\forall\ a \in A.\ g\ a = f\ a$ **and**
    $\forall\ b \in B - A.\ g\ b \in A - B \land (\exists\ n > 0.\ n\text{-}app\ n\ f\ (g\ b) = b)$ **and**
    $\forall\ x \in UNIV - A - B.\ g\ x = x$
$\langle proof \rangle$

lemma *bij-betw-ext*:
  fixes
    $f :: \prime x \Rightarrow \prime y$ **and**
    $X :: \prime x\ set$ **and**
    $Y :: \prime y\ set$
  assumes *bij-betw f X Y*
  shows *bij-betw (extensional-continuation f X) X Y*
$\langle proof \rangle$

1.10.3   Anonymity Lemmas

lemma *anon-rel-vote-count*:
  fixes
    $\mathcal{E} :: (\prime a,\ \prime v)\ Election\ set$ **and**
    $E\ E\prime :: (\prime a,\ \prime v)\ Election$
  assumes
    $finite\ (voters\text{-}\mathcal{E}\ E)$ **and**
    $(E,\ E\prime) \in anonymity_{\mathcal{R}}\ \mathcal{E}$
  shows $alternatives\text{-}\mathcal{E}\ E = alternatives\text{-}\mathcal{E}\ E\prime \land E \in \mathcal{E}$
        $\land\ (\forall\ p.\ vote\text{-}count\ p\ E = vote\text{-}count\ p\ E\prime)$
$\langle proof \rangle$

lemma *vote-count-anon-rel*:
  fixes
    $\mathcal{E}$ :: $('a,\ 'v)$ *Election set* **and**
    $E\ E'$ :: $('a,\ 'v)$ *Election*
  assumes
    fin-voters-E: *finite* (*voters-$\mathcal{E}$ E*) **and**
    fin-voters-E': *finite* (*voters-$\mathcal{E}$ E'*) **and**
    default-non-v: $\forall\ v.\ v \notin$ *voters-$\mathcal{E}$ E* $\longrightarrow$ *profile-$\mathcal{E}$ E v* = {} **and**
    default-non-v': $\forall\ v.\ v \notin$ *voters-$\mathcal{E}$ E'* $\longrightarrow$ *profile-$\mathcal{E}$ E' v* = {} **and**
    eq: *alternatives-$\mathcal{E}$ E* = *alternatives-$\mathcal{E}$ E'* $\wedge$ $(E,\ E') \in \mathcal{E} \times \mathcal{E}$
        $\wedge$ ($\forall\ p.$ *vote-count p E* = *vote-count p E'*)
  shows $(E,\ E') \in$ *anonymity$_\mathcal{R}$ $\mathcal{E}$*
$\langle proof \rangle$

lemma *rename-comp*:
  fixes $\pi\ \pi'$ :: $'v \Rightarrow 'v$
  assumes
    bij π **and**
    bij π'
  shows *rename* $\pi \circ$ *rename* π' = *rename* $(\pi \circ \pi')$
$\langle proof \rangle$

interpretation *anonymous-group-action*:
  group-action bijection$_{\mathcal{VG}}$ well-formed-elections φ-anon well-formed-elections
$\langle proof \rangle$

lemma (**in** *result*) *anonymity-action-presv-symmetry*: *is-symmetry* ($\lambda\ E.$ *limit*
  (*alternatives-$\mathcal{E}$ E*) *UNIV*) (*Invariance* (*anonymity$_\mathcal{R}$ well-formed-elections*))
  $\langle proof \rangle$

1.10.4 Neutrality Lemmas

lemma *rel-rename-helper*:
  fixes
    r :: $'a\ rel$ **and**
    π :: $'a \Rightarrow 'a$ **and**
    $a\ b$ :: $'a$
  assumes *bij* π
  shows $(\pi\ a,\ \pi\ b) \in \{(\pi\ x,\ \pi\ y) \mid x\ y.\ (x,\ y) \in r\}$
      $\longleftrightarrow (a,\ b) \in \{(x,\ y) \mid x\ y.\ (x,\ y) \in r\}$
$\langle proof \rangle$

lemma *rel-rename-comp*:
  fixes $\pi\ \pi'$ :: $'a \Rightarrow 'a$
  shows *rel-rename* $(\pi \circ \pi')$ = *rel-rename* $\pi \circ$ *rel-rename* π'
$\langle proof \rangle$

lemma *rel-rename-sound*:
  fixes

$\pi :: {}'a \Rightarrow {}'a$ **and**
$r :: {}'a\ rel$ **and**
$A :: {}'a\ set$
assumes *inj* π
shows
refl-on $A\ r \longrightarrow$ *refl-on* (π ' A) (*rel-rename* $\pi\ r$) **and**
antisym $r \longrightarrow$ *antisym* (*rel-rename* $\pi\ r$) **and**
total-on $A\ r \longrightarrow$ *total-on* (π ' A) (*rel-rename* $\pi\ r$) **and**
Relation.trans $r \longrightarrow$ *Relation.trans* (*rel-rename* $\pi\ r$)
⟨*proof*⟩

lemma *rename-subset*:
  fixes
    $r\ s :: {}'a\ rel$ **and**
    $a\ b :: {}'a$ **and**
    $\pi :: {}'a \Rightarrow {}'a$
  assumes
    bij-π: *bij* π **and**
    rel-rename $\pi\ r =$ *rel-rename* $\pi\ s$ **and**
    $(a,\ b) \in r$
  shows $(a,\ b) \in s$
⟨*proof*⟩

lemma *rel-rename-bij*:
  fixes $\pi :: {}'a \Rightarrow {}'a$
  assumes *bij-π*: *bij* π
  shows *bij* (*rel-rename* π)
⟨*proof*⟩

lemma *alternatives-rename-comp*:
  fixes $\pi\ \pi' :: {}'a \Rightarrow {}'a$
  shows *alternatives-rename* $\pi \circ$ *alternatives-rename* $\pi' =$
          alternatives-rename ($\pi \circ \pi'$)
⟨*proof*⟩

lemma *alternatives-rename-sound*:
  fixes
    $A\ A' :: {}'a\ set$ **and**
    $V\ V' :: {}'v\ set$ **and**
    $p\ p' :: ({}'a,\ {}'v)\ Profile$ **and**
    $\pi :: {}'a \Rightarrow {}'a$
  assumes
    bij-π: *bij* π **and**
    wf-elects: $(A,\ V,\ p) \in$ *well-formed-elections* **and**
    renamed: $(A',\ V',\ p') =$ *alternatives-rename* π $(A,\ V,\ p)$
  shows $(A',\ V',\ p') \in$ *well-formed-elections*
⟨*proof*⟩

lemma *alternatives-rename-bij*:

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fixes $\pi :: ('a \Rightarrow 'a)$
assumes *bij-π*: *bij* π
shows *bij-betw* (*alternatives-rename* π) *well-formed-elections well-formed-elections*
⟨*proof*⟩

interpretation φ-*neutral-action*: *group-action bijection*$_{\mathcal{AG}}$ *well-formed-elections*
  φ-*neutral well-formed-elections*
⟨*proof*⟩

interpretation ψ-*neutral*$_c$-*action*: *group-action bijection*$_{\mathcal{AG}}$ *UNIV* ψ-*neutral*$_c$
⟨*proof*⟩

interpretation ψ-*neutral*$_w$-*action*: *group-action bijection*$_{\mathcal{AG}}$ *UNIV* ψ-*neutral*$_w$
⟨*proof*⟩

lemma *neutrality-action-presv-$\mathcal{SCF}$-symmetry*: *is-symmetry* (λ $\mathcal{E}$. *limit-$\mathcal{SCF}$* (*alternatives-$\mathcal{E}$*
$\mathcal{E}$) *UNIV*)
        (*action-induced-equivariance* (*carrier bijection*$_{\mathcal{AG}}$) *well-formed-elections*
                    (φ-*neutral well-formed-elections*) (*set-action* ψ-*neutral*$_c$))
⟨*proof*⟩

lemma *neutrality-action-presv-$\mathcal{SWF}$-symmetry*: *is-symmetry* (λ $\mathcal{E}$. *limit-$\mathcal{SWF}$*
  (*alternatives-$\mathcal{E}$* $\mathcal{E}$) *UNIV*) (*action-induced-equivariance* (*carrier bijection*$_{\mathcal{AG}}$) *well-formed-elections*
                    (φ-*neutral well-formed-elections*) (*set-action* ψ-*neutral*$_w$))
⟨*proof*⟩

1.10.5   Homogeneity Lemmas

definition *reflp-on'* :: $'a$ *set* $\Rightarrow$ $'a$ *rel* $\Rightarrow$ *bool* **where**
  reflp-on' A r $\equiv$ *reflp-on* A (λ x y. (x, y) $\in$ r)

lemma *refl-homogeneity*$_{\mathcal{R}}$:
  fixes $\mathcal{E} :: ('a, 'v)$ *Election set*
  assumes $\mathcal{E} \subseteq$ *finite-elections-$\mathcal{V}$*
  shows *reflp-on'* $\mathcal{E}$ (*homogeneity*$_{\mathcal{R}}$ $\mathcal{E}$)
  ⟨*proof*⟩

lemma (**in** *result*) *homogeneity-action-presv-symmetry*:
  is-symmetry (λ $\mathcal{E}$. *limit* (*alternatives-$\mathcal{E}$* $\mathcal{E}$) *UNIV*)
      (*Invariance* (*homogeneity*$_{\mathcal{R}}$ *UNIV*))
  ⟨*proof*⟩

lemma *refl-homogeneity*$_{\mathcal{R}}$$'$:
  fixes $\mathcal{E} :: ('a, 'v :: linorder)$ *Election set*
  assumes $\mathcal{E} \subseteq$ *finite-elections-$\mathcal{V}$*
  shows *reflp-on'* $\mathcal{E}$ (*homogeneity*$_{\mathcal{R}}$$'$ $\mathcal{E}$)
  ⟨*proof*⟩

lemma (**in** *result*) *homogeneity'-action-presv-symmetry*:

is-symmetry (λ $\mathcal{E}$. *limit* (*alternatives-$\mathcal{E}$ $\mathcal{E}$) UNIV*)
   (*Invariance* (*homogeneity$_\mathcal{R}$' UNIV*))
$\langle proof \rangle$

1.10.6 Reversal Symmetry Lemmas

lemma *reverse-reverse-id*: *reverse-rel* $\circ$ *reverse-rel* $=$ *id*
 $\langle proof \rangle$

lemma *reverse-rel-limit*:
 fixes
   A :: $'a$ *set* **and**
   r :: $'a$ *rel*
 shows *reverse-rel* (*limit A r*) $=$ *limit A* (*reverse-rel r*)
 $\langle proof \rangle$

lemma *reverse-rel-lin-ord*:
 fixes
   A :: $'a$ *set* **and**
   r :: $'a$ *rel*
 assumes *linear-order-on A r*
 shows *linear-order-on A* (*reverse-rel r*)
 $\langle proof \rangle$

interpretation *reversal$_\mathcal{G}$-group*: *group reversal$_\mathcal{G}$*
$\langle proof \rangle$

interpretation *φ-reverse-action*: *group-action reversal$_\mathcal{G}$ well-formed-elections*
   φ-reverse well-formed-elections
$\langle proof \rangle$

interpretation *ψ-reverse-action*: *group-action reversal$_\mathcal{G}$ UNIV ψ-reverse*
$\langle proof \rangle$

lemma *reversal-symmetry-action-presv-symmetry*: *is-symmetry* (λ $\mathcal{E}$. *limit-$\mathcal{SWF}$*
(*alternatives-$\mathcal{E}$ $\mathcal{E}$) UNIV*)
   (*action-induced-equivariance* (*carrier reversal$_\mathcal{G}$*) *well-formed-elections*
      (*φ-reverse well-formed-elections*) (*set-action ψ-reverse*))
$\langle proof \rangle$

end

1.11 Result-Dependent Voting Rule Properties

theory *Property-Interpretations*
 imports *Voting-Symmetry*
      Result-Interpretations

begin

1.11.1 Property Definitions

The interpretation of equivariance properties generally depends on the result type. For example, neutrality for social choice rules means that single winners are renamed when the candidates in the votes are consistently renamed. For social welfare results, the complete result rankings must be renamed.

New result-type-dependent definitions for properties can be added here.

locale *result-properties = result +*
  fixes $\psi :: ('a \Rightarrow 'a, 'b)$ *binary-fun* **and**
      $\nu :: 'v$ *itself*
  assumes
   action-neutral: *group-action bijection$_{\mathcal{AG}}$ UNIV ψ* **and**
   neutrality:
    is-symmetry $(\lambda \; \mathcal{E} :: ('a, 'v) \; Election. \; limit \; (alternatives\text{-}\mathcal{E} \; \mathcal{E}) \; UNIV)$
         $(action\text{-}induced\text{-}equivariance \; (carrier \; bijection_{\mathcal{AG}})$
            well-formed-elections
            $(\varphi\text{-}neutral \; well\text{-}formed\text{-}elections) \; (set\text{-}action \; \psi))$

sublocale *result-properties* $\subseteq$ *result*
  $\langle proof \rangle$

1.11.2 Interpretations

global-interpretation $\mathcal{SCF}$*-properties*: *result-properties well-formed-$\mathcal{SCF}$*
     limit-$\mathcal{SCF}$ ψ-neutral$_\mathrm{c}$
  $\langle proof \rangle$

global-interpretation $\mathcal{SWF}$*-properties*: *result-properties well-formed-$\mathcal{SWF}$*
     limit-$\mathcal{SWF}$ ψ-neutral$_\mathrm{w}$
  $\langle proof \rangle$

end

Chapter 2

Refined Types

2.1 Preference List

theory *Preference-List*
  imports *../Preference-Relation*
         HOL−Combinatorics.Multiset-Permutations
         List−Index.List-Index
begin

Preference lists derive from preference relations, ordered from most to least preferred alternative.

2.1.1 Well-Formedness

type-synonym *$'a$ Preference-List = $'a$ list*

abbreviation *well-formed-l :: $'a$ Preference-List $\Rightarrow$ bool* **where**
  well-formed-l l $\equiv$ distinct l

2.1.2 Auxiliary Lemmas About Lists

lemma *is-arg-min-equal*:
  fixes
    f g :: $'a \Rightarrow 'b$ :: ord **and**
    S :: $'a$ set **and**
    x :: $'a$
  assumes $\forall\ x \in S.\ f\ x = g\ x$
  shows *is-arg-min f (λ s. s $\in$ S) x = is-arg-min g (λ s. s $\in$ S) x*
⟨*proof*⟩

lemma *list-cons-presv-finiteness*:
  fixes
    A :: $'a$ set **and**
    S :: $'a$ list set
  assumes

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fin-A: *finite A* **and**
    fin-B: *finite S*
  shows *finite {a#l | a l. a ∈ A ∧ l ∈ S}*
⟨*proof*⟩

lemma *listset-finiteness*:
  fixes *l* :: *'a set list*
  assumes ∀ *i* :: *nat. i < length l* ⟶ *finite (l!i)*
  shows *finite (listset l)*
  ⟨*proof*⟩

lemma *all-ls-elems-same-len*:
  fixes *l* :: *'a set list*
  shows ∀ *l'* :: *'a list. l'* ∈ *listset l* ⟶ *length l' = length l*
⟨*proof*⟩

lemma *all-ls-elems-in-ls-set*:
  fixes *l* :: *'a set list*
  shows ∀ *l'* ∈ *listset l.* ∀ *i* :: *nat < length l'. l'!i* ∈ *l!i*
⟨*proof*⟩

lemma *all-ls-in-ls-set*:
  fixes *l* :: *'a set list*
  shows ∀ *l'. length l' = length l*
        ∧ (∀ *i < length l'. l'!i* ∈ *l!i*) ⟶ *l'* ∈ *listset l*
⟨*proof*⟩

2.1.3 Ranking

Rank 1 is the top preference, rank 2 the second, and so on. Rank 0 does not exist.

fun *rank-l* :: *'a Preference-List* ⇒ *'a* ⇒ *nat* **where**
  rank-l l a = (if a ∈ *set l then index l a + 1 else 0)*

fun *rank-l-idx* :: *'a Preference-List* ⇒ *'a* ⇒ *nat* **where**
  rank-l-idx l a =
    (*let i = index l a in*
      if i = length l then 0 else i + 1)

lemma *rank-l-equiv*: *rank-l = rank-l-idx*
  ⟨*proof*⟩

lemma *rank-zero-imp-not-present*:
  fixes
    p :: *'a Preference-List* **and**
    a :: *'a*
  assumes *rank-l p a = 0*
  shows *a* ∉ *set p*
  ⟨*proof*⟩

definition *above-l* :: $'a$ *Preference-List* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *Preference-List* **where**
  above-l r a $\equiv$ *take* (*rank-l r a*) *r*

2.1.4 Definition

fun *is-less-preferred-than-l* :: $'a$ $\Rightarrow$ $'a$ *Preference-List* $\Rightarrow$ $'a$ $\Rightarrow$ *bool*
      (- $\precsim$- - [*50*, *1000*, *51*] *50*) **where**
  $a \precsim_l b = (a \in set\ l \wedge b \in set\ l \wedge index\ l\ a \geq index\ l\ b)$

lemma *rank-gt-zero*:
  fixes
    l :: $'a$ *Preference-List* **and**
    a :: $'a$
  assumes $a \precsim_l a$
  shows *rank-l l a* $\geq$ *1*
  $\langle proof \rangle$

definition *pl-α* :: $'a$ *Preference-List* $\Rightarrow$ $'a$ *Preference-Relation* **where**
  pl-α l $\equiv$ $\{(a,\ b).\ a \precsim_l b\}$

lemma *rel-trans*:
  fixes *l* :: $'a$ *Preference-List*
  shows *trans* (*pl-α l*)
  $\langle proof \rangle$

lemma *pl-α-lin-order*:
  fixes
    A :: $'a$ *set* **and**
    r :: $'a$ *rel*
  assumes $r \in$ *pl-α ' permutations-of-set A*
  shows *linear-order-on A r*
$\langle proof \rangle$

lemma *lin-order-pl-α*:
  fixes
    r :: $'a$ *rel* **and**
    A :: $'a$ *set*
  assumes
    lin-order: *linear-order-on A r* **and**
    fin: *finite A*
  shows $r \in$ *pl-α ' permutations-of-set A*
$\langle proof \rangle$

lemma *index-helper*:
  fixes
    l :: $'x$ *list* **and**
    x :: $'x$
  assumes

$finite\ (set\ l)$ **and**
    $distinct\ l$ **and**
    $x \in set\ l$
  shows $index\ l\ x = card\ \{y \in set\ l.\ index\ l\ y < index\ l\ x\}$
$\langle proof \rangle$

lemma *pl-α-eq-imp-list-eq*:
  fixes $l\ l' :: \ 'x\ list$
  assumes
    fin-set-l: $finite\ (set\ l)$ **and**
    set-eq: $set\ l = set\ l'$ **and**
    dist-l: $distinct\ l$ **and**
    dist-l': $distinct\ l'$ **and**
    pl-α-eq: $pl\text{-}α\ l = pl\text{-}α\ l'$
  shows $l = l'$
$\langle proof \rangle$

lemma *pl-α-bij-betw*:
  fixes $X :: \ 'x\ set$
  assumes $finite\ X$
  shows $bij\text{-}betw\ pl\text{-}α\ (permutations\text{-}of\text{-}set\ X)\ \{r.\ linear\text{-}order\text{-}on\ X\ r\}$
$\langle proof \rangle$

2.1.5 Limited Preference

definition $limited :: \ 'a\ set \Rightarrow \ 'a\ Preference\text{-}List \Rightarrow bool$ **where**
  $limited\ A\ r \equiv \forall\ a.\ a \in set\ r \longrightarrow\ a \in A$

fun $limit\text{-}l :: \ 'a\ set \Rightarrow \ 'a\ Preference\text{-}List \Rightarrow \ 'a\ Preference\text{-}List$ **where**
  $limit\text{-}l\ A\ l = List.filter\ (\lambda\ a.\ a \in A)\ l$

lemma *limited-dest*:
  fixes
    $A :: \ 'a\ set$ **and**
    $l :: \ 'a\ Preference\text{-}List$ **and**
    $a\ b :: \ 'a$
  assumes
    $a \lesssim_l b$ **and**
    $limited\ A\ l$
  shows $a \in A \land b \in A$
  $\langle proof \rangle$

lemma *limit-equiv*:
  fixes
    $A :: \ 'a\ set$ **and**
    $l :: \ 'a\ list$
  assumes $well\text{-}formed\text{-}l\ l$
  shows $pl\text{-}α\ (limit\text{-}l\ A\ l) = limit\ A\ (pl\text{-}α\ l)$
  $\langle proof \rangle$

2.1.6 Auxiliary Definitions

definition *total-on-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  total-on-l A l $\equiv$ $\forall$ a $\in$ A. a $\in$ set l

definition *refl-on-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  refl-on-l A l $\equiv$ ($\forall$ a. a $\in$ set l $\longrightarrow$ a $\in$ A) $\wedge$ ($\forall$ a $\in$ A. a $\precsim_l$ a)

definition *trans* :: *'a Preference-List $\Rightarrow$ bool* **where**
  trans l $\equiv$ $\forall$ (a, b, c) $\in$ set l $\times$ set l $\times$ set l. a $\precsim_l$ b $\wedge$ b $\precsim_l$ c $\longrightarrow$ a $\precsim_l$ c

definition *preorder-on-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  preorder-on-l A l $\equiv$ refl-on-l A l $\wedge$ trans l

definition *antisym-l* :: *'a list $\Rightarrow$ bool* **where**
  antisym-l l $\equiv$ $\forall$ a b. a $\precsim_l$ b $\wedge$ b $\precsim_l$ a $\longrightarrow$ a = b

definition *partial-order-on-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  partial-order-on-l A l $\equiv$ preorder-on-l A l $\wedge$ antisym-l l

definition *linear-order-on-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  linear-order-on-l A l $\equiv$ partial-order-on-l A l $\wedge$ total-on-l A l

definition *connex-l* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  connex-l A l $\equiv$ limited A l $\wedge$ ($\forall$ a $\in$ A. $\forall$ b $\in$ A. a $\precsim_l$ b $\vee$ b $\precsim_l$ a)

abbreviation *ballot-on* :: *'a set $\Rightarrow$ 'a Preference-List $\Rightarrow$ bool* **where**
  ballot-on A l $\equiv$ well-formed-l l $\wedge$ linear-order-on-l A l

2.1.7 Auxiliary Lemmas

lemma *list-trans*[*simp*]:
  fixes *l* :: *'a Preference-List*
  shows *trans l*
  $\langle proof \rangle$

lemma *list-antisym*[*simp*]:
  fixes *l* :: *'a Preference-List*
  shows *antisym-l l*
  $\langle proof \rangle$

lemma *lin-order-equiv-list-of-alts*:
  fixes
    A :: *'a set* **and**
    l :: *'a Preference-List*
  shows *linear-order-on-l A l = (A = set l)*
  $\langle proof \rangle$

lemma *connex-imp-refl*:
  fixes

    A :: $'a$ *set* **and**
    l :: $'a$ *Preference-List*
  assumes *connex-l A l*
  shows *refl-on-l A l*
  ⟨*proof*⟩

lemma *lin-ord-imp-connex-l*:
  fixes
    A :: $'a$ *set* **and**
    l :: $'a$ *Preference-List*
  assumes *linear-order-on-l A l*
  shows *connex-l A l*
  ⟨*proof*⟩

lemma *above-trans*:
  fixes
    l :: $'a$ *Preference-List* **and**
    a b :: $'a$
  assumes
    trans l **and**
    $a \lesssim_l b$
  shows *set* (*above-l l b*) $\subseteq$ *set* (*above-l l a*)
  ⟨*proof*⟩

lemma *less-preferred-l-rel-equiv*:
  fixes
    l :: $'a$ *Preference-List* **and**
    a b :: $'a$
  shows $a \lesssim_l b =$
    Preference-Relation.is-less-preferred-than a (*pl-α l*) *b*
  ⟨*proof*⟩

theorem *above-equiv*:
  fixes
    l :: $'a$ *Preference-List* **and**
    a :: $'a$
  shows *set* (*above-l l a*) $=$ *above* (*pl-α l*) *a*
⟨*proof*⟩

theorem *rank-equiv*:
  fixes
    l :: $'a$ *Preference-List* **and**
    a :: $'a$
  assumes *well-formed-l l*
  shows *rank-l l a = rank* (*pl-α l*) *a*
⟨*proof*⟩

lemma *lin-ord-equiv*:
  fixes

$A :: {}'a$ *set* **and**
  $l :: {}'a$ *Preference-List*
shows *linear-order-on-l A l = linear-order-on A (pl-α l)*
⟨*proof*⟩

2.1.8 First Occurrence Indices

lemma *pos-in-list-yields-rank*:
  fixes
    $l :: {}'a$ *Preference-List* **and**
    $a :: {}'a$ **and**
    $n ::$ *nat*
  assumes
    $\forall (j :: nat) \leq n.\ l!j \neq a$ **and**
    $l!(n - 1) = a$
  shows *rank-l l a = n*
  ⟨*proof*⟩

lemma *ranked-alt-not-at-pos-before*:
  fixes
    $l :: {}'a$ *Preference-List* **and**
    $a :: {}'a$ **and**
    $n ::$ *nat*
  assumes
    $a \in$ *set l* **and**
    $n < (rank\text{-}l\ l\ a) - 1$
  shows $l!n \neq a$
  ⟨*proof*⟩

lemma *pos-in-list-yields-pos*:
  fixes
    $l :: {}'a$ *Preference-List* **and**
    $a :: {}'a$
  assumes $a \in$ *set l*
  shows $l!(rank\text{-}l\ l\ a - 1) = a$
  ⟨*proof*⟩

lemma *rel-of-pref-pred-for-set-eq-list-to-rel*:
  fixes $l :: {}'a$ *Preference-List*
  shows *relation-of* $(\lambda\ y\ z.\ y \precsim_l z)$ *(set l) = pl-α l*
⟨*proof*⟩

end

2.2 Preference (List) Profile

theory *Profile-List*
  imports *../Profile*
      Preference-List
begin

2.2.1 Definition

A profile (list) contains one ballot for each voter.

type-synonym $'a$ *Profile-List* $= \,'a$ *Preference-List list*

type-synonym $'a$ *Election-List* $= \,'a$ *set* $\times \,'a$ *Profile-List*

Abstraction from profile list to profile.

fun *pl-to-pr-α* $:: \,'a$ *Profile-List* $\Rightarrow (\,'a,\ nat)$ *Profile* **where**
  pl-to-pr-α pl $= (\lambda\ n.\ if\ n < length\ pl \wedge n \geq 0$
                  then (*map pl-α pl*)!*n*
                  else {})

lemma *prof-abstr-presv-size*:
  fixes $p :: \,'a$ *Profile-List*
  shows *length* $p = length$ (*to-list* {0 ..$< length\ p$} (*pl-to-pr-α p*))
  ⟨*proof*⟩

2.2.2 Refinement Proof

A profile on a finite set of alternatives A contains only ballots that are lists
of linear orders on A.

definition *profile-l* $:: \,'a$ *set* $\Rightarrow \,'a$ *Profile-List* $\Rightarrow$ *bool* **where**
  profile-l A p $\equiv \forall\ i < length\ p.\ ballot\text{-}on\ A\ (p!i)$

lemma *profile-list-refines-profile*:
  fixes
    $A :: \,'a$ *set* **and**
    $p :: \,'a$ *Profile-List*
  assumes *profile-l A p*
  shows *profile* {0 ..$< length\ p$} A (*pl-to-pr-α p*)
⟨*proof*⟩

end

2.3 Ordered Relation Type

theory *Ordered-Relation*

imports *Preference-Relation*
       ./Refined-Types/Preference-List
       HOL−Combinatorics.Multiset-Permutations
begin

lemma *fin-ordered*:
  fixes X :: $'x$ *set*
  assumes *finite X*
  obtains *ord* :: $'x$ *rel* **where**
    linear-order-on X ord
⟨*proof*⟩

typedef $'a$ *Ordered-Preference* =
  $\{p ::$ $'a ::$ *finite Preference-Relation. linear-order-on* $(UNIV ::$ $'a$ *set*$)$ $p\}$
  morphisms *ord2pref pref2ord*
⟨*proof*⟩

instance *Ordered-Preference* :: (*finite*) *finite*
⟨*proof*⟩

lemma *range-ord2pref*: *range ord2pref* = $\{p.$ *linear-order* $p\}$
  ⟨*proof*⟩

lemma *card-ord-pref*: *card* $(UNIV ::$ $'a ::$ *finite Ordered-Preference set*$)$ =
                fact $(card$ $(UNIV ::$ $'a$ *set*$))$
⟨*proof*⟩

end

2.4 Alternative Election Type

theory *Quotient-Type-Election*
  imports *Profile*
begin

lemma *election-equality-equiv*:
  election-equality E E **and**
  election-equality E E′ ⟶ *election-equality E′ E* **and**
  election-equality E E′ ⟶ *election-equality E′ F*
    ⟶ *election-equality E F*
⟨*proof*⟩

quotient-type $('a,$ $'v)$ *Election*$_\mathcal{Q}$ =
  $'a$ *set* × $'v$ *set* × $('a,$ $'v)$ *Profile* / *election-equality*
  ⟨*proof*⟩

fun *fst*$_\mathcal{Q}$ :: $('a,$ $'v)$ *Election*$_\mathcal{Q}$ ⇒ $'a$ *set* **where**

$fst_{\mathcal{Q}} \ E = fst \ (rep\text{-}Election_{\mathcal{Q}} \ E)$

fun $snd_{\mathcal{Q}} :: ({}'a, {}'v) \ Election_{\mathcal{Q}} \Rightarrow {}'v \ set \times ({}'a, {}'v) \ Profile$ **where**
$\quad snd_{\mathcal{Q}} \ E = snd \ (rep\text{-}Election_{\mathcal{Q}} \ E)$

abbreviation $alternatives\text{-}\mathcal{E}_{\mathcal{Q}} :: ({}'a, {}'v) \ Election_{\mathcal{Q}} \Rightarrow {}'a \ set$ **where**
$\quad alternatives\text{-}\mathcal{E}_{\mathcal{Q}} \ E \equiv fst_{\mathcal{Q}} \ E$

abbreviation $voters\text{-}\mathcal{E}_{\mathcal{Q}} :: ({}'a, {}'v) \ Election_{\mathcal{Q}} \Rightarrow {}'v \ set$ **where**
$\quad voters\text{-}\mathcal{E}_{\mathcal{Q}} \ E \equiv fst \ (snd_{\mathcal{Q}} \ E)$

abbreviation $profile\text{-}\mathcal{E}_{\mathcal{Q}} :: ({}'a, {}'v) \ Election_{\mathcal{Q}} \Rightarrow ({}'a, {}'v) \ Profile$ **where**
$\quad profile\text{-}\mathcal{E}_{\mathcal{Q}} \ E \equiv snd \ (snd_{\mathcal{Q}} \ E)$

end

Chapter 3

Quotient Rules

3.1 Quotients of Equivalence Relations

theory *Relation-Quotients*
  imports *../Social-Choice-Types/Symmetry-Of-Functions*
begin

3.1.1 Definitions

fun *singleton-set* :: $'x$ *set* $\Rightarrow$ $'x$ **where**
  singleton-set s = (*if card s* = *1 then the-inv* (λ *x.* $\{x\}$) *s else undefined*)
— This is undefined if *card s* $\neq$ *1*. Note that "*undefined = undefined*" is the only provable equality for *undefined*.

For a given function, we define a function on sets that maps each set to the unique image under f of its elements, if one exists. Otherwise, the result is undefined.

fun $\pi_{\mathcal{Q}}$ :: $('x \Rightarrow 'y) \Rightarrow ('x \ set \Rightarrow 'y)$ **where**
  $\pi_{\mathcal{Q}}$ *f s* = *singleton-set* (*f* ' *s*)

For a given function f on sets and a mapping from elements to sets, we define a function on the set element type that maps each element to the image of its corresponding set under f. A natural mapping is from elements to their classes under a relation.

fun *inv-$\pi_{\mathcal{Q}}$* :: $('x \Rightarrow 'x \ set) \Rightarrow ('x \ set \Rightarrow 'y) \Rightarrow ('x \Rightarrow 'y)$ **where**
  inv-$\pi_{\mathcal{Q}}$ cls f x = *f* (*cls x*)

fun *relation-class* :: $'x \ rel \Rightarrow 'x \Rightarrow 'x \ set$ **where**
  relation-class r x = *r* '' $\{x\}$

3.1.2 Well-Definedness

lemma *singleton-set-undef-if-card-neq-one*:
  fixes *s* :: $'x \ set$

assumes *card s ≠ 1*
shows *singleton-set s = undefined*
⟨*proof*⟩

lemma *singleton-set-def-if-card-one*:
  fixes $s :: {'x}\ set$
  assumes *card s = 1*
  shows $∃!\ x.\ x = singleton\text{-}set\ s ∧ \{x\} = s$
⟨*proof*⟩

If the given function is invariant under an equivalence relation, the induced function on sets is well-defined for all equivalence classes of that relation.

theorem *pass-to-quotient*:
  fixes
    $f :: {'x} ⇒ {'y}$ **and**
    $r :: {'x}\ rel$ **and**
    $s :: {'x}\ set$
  assumes
    f respects r **and**
    equiv s r
  shows $∀\ t ∈ s\ //\ r.\ ∀\ x ∈ t.\ π_{\mathcal{Q}}\ f\ t = f\ x$
⟨*proof*⟩

A function on sets induces a function on the element type that is invariant under a given equivalence relation.

theorem *pass-to-quotient-inv*:
  fixes
    $f :: {'x}\ set ⇒ {'x}$ **and**
    $r :: {'x}\ rel$ **and**
    $s :: {'x}\ set$
  assumes *equiv s r*
  defines $induced\text{-}fun ≡ (inv\text{-}π_{\mathcal{Q}}\ (relation\text{-}class\ r)\ f)$
  shows
    induced-fun respects r **and**
    $∀\ A ∈ s\ //\ r.\ π_{\mathcal{Q}}\ induced\text{-}fun\ A = f\ A$
⟨*proof*⟩

3.1.3 Equivalence Relations

lemma *restr-equals-restricted-rel*:
  fixes
    $s\ t :: {'a}\ set$ **and**
    $r :: {'a}\ rel$
  assumes
    closed-restricted-rel r s t **and**
    $t ⊆ s$
  shows *restricted-rel r t s = Restr r t*
⟨*proof*⟩

lemma *equiv-rel-restr*:
  fixes
    s t :: *'x set* **and**
    r :: *'x rel*
  assumes
    equiv s r **and**
    t ⊆ s
  shows *equiv t* (*Restr r t*)
⟨*proof*⟩

lemma *rel-ind-by-group-act-equiv*:
  fixes
    m :: *'x monoid* **and**
    s :: *'y set* **and**
    φ :: (*'x, 'y*) *binary-fun*
  assumes *group-action m s φ*
  shows *equiv s* (*action-induced-rel* (*carrier m*) *s φ*)
⟨*proof*⟩

end

3.2   Quotients of Election Set Equivalences

theory *Election-Quotients*
  imports *Relation-Quotients*
        ../Social-Choice-Types/Voting-Symmetry
        ../Social-Choice-Types/Ordered-Relation
        HOL−Analysis.Convex
        HOL−Analysis.Cartesian-Space
begin

3.2.1   Auxiliary Lemmas

lemma *obtain-partition*:
  fixes
    A :: *'a set* **and**
    B :: *'b set* **and**
    f :: *'b ⇒ nat*
  assumes
    finite A **and**
    finite B **and**
    ($\sum$ *x ∈ B. f x*) = *card A*
  shows $\exists$ *𝒳. A* = $\bigcup$ {*𝒳 i* | *i. i ∈ B*} ∧ ($\forall$ *i ∈ B. card* (*𝒳 i*) = *f i*) ∧
            ($\forall$ *i j. i ≠ j ⟶ i ∈ B ∧ j ∈ B ⟶ 𝒳 i ∩ 𝒳 j* = {})
  ⟨*proof*⟩

3.2.2 Anonymity Quotient: Grid

fun $anonymity_{\mathcal{Q}}$ :: $'a\ set \Rightarrow ('a,\ 'v)\ Election\ set\ set$ **where**
  $anonymity_{\mathcal{Q}}\ A = quotient\ (elections\text{-}\mathcal{A}\ A)\ (anonymity_{\mathcal{R}}\ (elections\text{-}\mathcal{A}\ A))$

— Here, we count the occurrences of a ballot per election in a set of elections for which the occurrences of the ballot per election coincide for all elections in the set.
fun $vote\text{-}count_{\mathcal{Q}}$ :: $'a\ Preference\text{-}Relation \Rightarrow ('a,\ 'v)\ Election\ set \Rightarrow nat$ **where**
  $vote\text{-}count_{\mathcal{Q}}\ p = \pi_{\mathcal{Q}}\ (vote\text{-}count\ p)$

fun $anonymity\text{-}class$ :: $('a :: finite,\ 'v)\ Election\ set \Rightarrow$
      $(nat,\ 'a\ Ordered\text{-}Preference)\ vec$ **where**
  $anonymity\text{-}class\ X = (\chi\ p.\ vote\text{-}count_{\mathcal{Q}}\ (ord2pref\ p)\ X)$

lemma $anon\text{-}rel\text{-}equiv$: $equiv\ (elections\text{-}\mathcal{A}\ UNIV)\ (anonymity_{\mathcal{R}}\ (elections\text{-}\mathcal{A}\ UNIV))$
$\langle proof \rangle$

We assume that all elections consist of a fixed finite alternative set of size n and finite subsets of an infinite voter universe. Profiles are linear orders on the alternatives. Then, we can operate on the natural-number-vectors of dimension $n!$ instead of the equivalence classes of the anonymity relation: Each dimension corresponds to one possible linear order on the alternative set, i.e., the possible preferences. Each equivalence class of elections corresponds to a vector whose entries denote the amount of voters per election in that class who vote the respective corresponding preference.

theorem $anonymity_{\mathcal{Q}}\text{-}isomorphism$:
  assumes $infinite\ (UNIV :: 'v\ set)$
  shows $bij\text{-}betw\ (anonymity\text{-}class :: ('a :: finite,\ 'v)\ Election\ set$
      $\Rightarrow nat\ \frown('a\ Ordered\text{-}Preference))\ (anonymity_{\mathcal{Q}}\ (UNIV :: 'a\ set))$
        $(UNIV :: (nat\ \frown('a\ Ordered\text{-}Preference))\ set)$
$\langle proof \rangle$

3.2.3 Homogeneity Quotient: Simplex

fun $vote\text{-}fraction$ :: $'a\ Preference\text{-}Relation \Rightarrow ('a,\ 'v)\ Election \Rightarrow rat$ **where**
  $vote\text{-}fraction\ r\ E =$
    $(if\ finite\ (voters\text{-}\mathcal{E}\ E) \wedge voters\text{-}\mathcal{E}\ E \neq \{\}$
      $then\ Fract\ (vote\text{-}count\ r\ E)\ (card\ (voters\text{-}\mathcal{E}\ E))\ else\ 0)$

fun $anonymity\text{-}homogeneity_{\mathcal{R}}$ :: $('a,\ 'v)\ Election\ set \Rightarrow ('a,\ 'v)\ Election\ rel$ **where**
  $anonymity\text{-}homogeneity_{\mathcal{R}}\ \mathcal{E} =$
    $\{(E,\ E') \mid E\ E'.\ E \in \mathcal{E} \wedge E' \in \mathcal{E}$
             $\wedge\ finite\ (voters\text{-}\mathcal{E}\ E) = finite\ (voters\text{-}\mathcal{E}\ E')$
             $\wedge\ (\forall\ r.\ vote\text{-}fraction\ r\ E = vote\text{-}fraction\ r\ E')\}$

fun $anonymity\text{-}homogeneity_{\mathcal{Q}}$ :: $'a\ set \Rightarrow ('a,\ 'v)\ Election\ set\ set$ **where**
  $anonymity\text{-}homogeneity_{\mathcal{Q}}\ A =$
    $quotient\ (elections\text{-}\mathcal{A}\ A)\ (anonymity\text{-}homogeneity_{\mathcal{R}}\ (elections\text{-}\mathcal{A}\ A))$

fun *vote-fraction$_\mathbb{Q}$* :: *'a Preference-Relation $\Rightarrow$ ('a, 'v) Election set $\Rightarrow$ rat* **where**
  vote-fraction$_\mathbb{Q}$ p = $\pi_\mathbb{Q}$ (vote-fraction p)

fun *anonymity-homogeneity-class* :: *('a :: finite, 'v) Election set $\Rightarrow$*
    (rat, 'a Ordered-Preference) vec **where**
  anonymity-homogeneity-class $\mathcal{E}$ = (χ p. vote-fraction$_\mathbb{Q}$ (ord2pref p) $\mathcal{E}$)

Maps each rational real vector entry to the corresponding rational. If the entry is not rational, the corresponding entry will be undefined.

fun *rat-vector* :: *real$^\frown$'b $\Rightarrow$ rat$^\frown$'b* **where**
  rat-vector v = (χ p. the-inv of-rat (v\$p))

fun *rat-vector-set* :: *(real$^\frown$'b) set $\Rightarrow$ (rat$^\frown$'b) set* **where**
  rat-vector-set V = rat-vector ' {v $\in$ V. $\forall$ i. v\$i $\in$ $\mathbb{Q}$}

definition *standard-basis* :: *(real$^\frown$'b) set* **where**
  standard-basis $\equiv$ {v. $\exists$ b. v\$b = 1 $\wedge$ ($\forall$ c $\neq$ b. v\$c = 0)}

The rational points in the simplex.

definition *vote-simplex* :: *(rat$^\frown$'b) set* **where**
  vote-simplex $\equiv$
    insert 0 (rat-vector-set (convex hull standard-basis :: (real$^\frown$'b) set))

Auxiliary Lemmas

lemma *convex-combination-in-convex-hull*:
  fixes
    X :: *(real$^\frown$'b) set* **and**
    x :: *real$^\frown$'b*
  assumes $\exists$ *f* :: *real$^\frown$'b $\Rightarrow$ real.*
        ($\sum$ y $\in$ X. f y) = 1 $\wedge$ ($\forall$ x $\in$ X. f x $\geq$ 0)
          $\wedge$ x = ($\sum$ x $\in$ X. f x $*_R$ x)
  shows *x $\in$ convex hull X*
  $\langle$*proof*$\rangle$

lemma *standard-simplex-rewrite*: *convex hull standard-basis =*
    {v :: real$^\frown$'b. ($\forall$ i. v\$i $\geq$ 0) $\wedge$ ($\sum$ y $\in$ UNIV. v\$y) = 1}
$\langle$*proof*$\rangle$

lemma *fract-distr-helper*:
  fixes *a b c* :: *int*
  assumes *c $\neq$ 0*
  shows *Fract a c + Fract b c = Fract (a + b) c*
  $\langle$*proof*$\rangle$

lemma *anonymity-homogeneity-is-equivalence*:
  fixes *X* :: *('a, 'v) Election set*
  assumes $\forall$ *E $\in$ X. finite (voters-$\mathcal{E}$ E)*
  shows *equiv X (anonymity-homogeneity$_\mathcal{R}$ X)*

⟨*proof*⟩

lemma *fract-distr*:
  fixes
    A :: $'x$ *set* **and**
    f :: $'x \Rightarrow int$ **and**
    b :: *int*
  assumes
    finite A **and**
    $b \neq 0$
  shows $(\sum\ a \in A.\ Fract\ (f\ a)\ b) = Fract\ (\sum\ x \in A.\ f\ x)\ b$
⟨*proof*⟩

Simplex Bijection

We assume all our elections to consist of a fixed finite alternative set of size
n and finite subsets of an infinite voter universe. Profiles are linear orders on
the alternatives. Then we can work on the standard simplex of dimension n!
instead of the equivalence classes of the equivalence relation for anonymous
+ homogeneous voting rules (anon hom): Each dimension corresponds to
one possible linear order on the alternative set, i.e., the possible preferences.
Each equivalence class of elections corresponds to a vector whose entries
denote the fraction of voters per election in that class who vote the respective
corresponding preference.

theorem *anonymity-homogeneity$_\mathcal{Q}$-isomorphism*:
  assumes *infinite* (*UNIV* :: $'v$ *set*)
  shows
    bij-betw (*anonymity-homogeneity-class* :: ($'a$ :: *finite*, $'v$) *Election set* $\Rightarrow$
      $rat^{\frown\prime}a$ *Ordered-Preference*) (*anonymity-homogeneity$_\mathcal{Q}$* (*UNIV* :: $'a$ *set*))
        (*vote-simplex* :: ($rat^{\frown\prime}a$ *Ordered-Preference*) *set*)
⟨*proof*⟩

end

Chapter 4

Component Types

4.1 Distance

theory *Distance*
  imports *HOL−Library.Extended-Real*
        Social-Choice-Types/Voting-Symmetry
begin

A general distance on a set X is a mapping d: $X \times X \mapsto R \cup \{+\infty\}$ such that for every x, y, z in X, the following four conditions are satisfied:

- $d(x, y) \geq 0$ (non-negativity);

- $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);

- $d(x, y) = d(y, x)$ (symmetry);

- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

  Moreover, a mapping that satisfies all but the second conditions is called a pseudo-distance, whereas a quasi-distance needs to satisfy the first three conditions (and not necessarily the last one).

4.1.1 Definition

type-synonym *'a Distance = 'a $\Rightarrow$ 'a $\Rightarrow$ ereal*

The un-curried version of a distance is defined on tuples.

fun *tup* :: *'a Distance $\Rightarrow$ ('a $*$ 'a $\Rightarrow$ ereal)* **where**
  tup d = (λ pair. d (fst pair) (snd pair))

definition *distance* :: *'a set $\Rightarrow$ 'a Distance $\Rightarrow$ bool* **where**
  distance S d $\equiv$ $\forall$ x y. x $\in$ S $\wedge$ y $\in$ S $\longrightarrow$ d x x = 0 $\wedge$ 0 $\leq$ d x y

4.1.2 Conditions

definition *symmetric* :: $'a$ *set* $\Rightarrow$ $'a$ *Distance* $\Rightarrow$ *bool* **where**
  symmetric S d $\equiv$ $\forall$ *x y. x* $\in$ *S* $\wedge$ *y* $\in$ *S* $\longrightarrow$ *d x y = d y x*

definition *triangle-ineq* :: $'a$ *set* $\Rightarrow$ $'a$ *Distance* $\Rightarrow$ *bool* **where**
  triangle-ineq S d $\equiv$ $\forall$ *x y z. x* $\in$ *S* $\wedge$ *y* $\in$ *S* $\wedge$ *z* $\in$ *S* $\longrightarrow$ *d x z* $\leq$ *d x y + d y z*

definition *eq-if-zero* :: $'a$ *set* $\Rightarrow$ $'a$ *Distance* $\Rightarrow$ *bool* **where**
  eq-if-zero S d $\equiv$ $\forall$ *x y. x* $\in$ *S* $\wedge$ *y* $\in$ *S* $\longrightarrow$ *d x y = 0* $\longrightarrow$ *x = y*

definition *vote-distance* :: ($'a$ *Vote set* $\Rightarrow$ $'a$ *Vote Distance* $\Rightarrow$ *bool*) $\Rightarrow$
      $'a$ *Vote Distance* $\Rightarrow$ *bool* **where**
  vote-distance π *d* $\equiv$ π {(*A, p*). *linear-order-on A p* $\wedge$ *finite A*} *d*

definition *election-distance* :: (($'a$, $'v$) *Election set* $\Rightarrow$
      ($'a$, $'v$) *Election Distance* $\Rightarrow$ *bool*) $\Rightarrow$
        ($'a$, $'v$) *Election Distance* $\Rightarrow$ *bool* **where**
  election-distance π *d* $\equiv$ π {(*A, V, p*). *finite-profile V A p*} *d*

4.1.3 Standard-Distance Property

definition *standard* :: ($'a$, $'v$) *Election Distance* $\Rightarrow$ *bool* **where**
 standard d $\equiv$
    $\forall$ *A A$'$ V V$'$ p p$'$. A* $\neq$ *A$'$* $\vee$ *V* $\neq$ *V$'$* $\longrightarrow$ *d* (*A, V, p*) (*A$'$, V$'$, p$'$*) $= \infty$

4.1.4 Auxiliary Lemmas

fun *arg-min-set* :: ($'b$ $\Rightarrow$ $'a$ :: *ord*) $\Rightarrow$ $'b$ *set* $\Rightarrow$ $'b$ *set* **where**
  arg-min-set f A = Collect (*is-arg-min f* (λ *a. a* $\in$ *A*))

lemma *arg-min-subset*:
  fixes
    B :: $'b$ *set* **and**
    f :: $'b$ $\Rightarrow$ $'a$ :: *ord*
  shows *arg-min-set f B* $\subseteq$ *B*
  $\langle proof \rangle$

lemma *sum-monotone*:
  fixes
    A :: $'a$ *set* **and**
    f g :: $'a$ $\Rightarrow$ *int*
  assumes $\forall$ *a* $\in$ *A. f a* $\leq$ *g a*
  shows ($\sum$ *a* $\in$ *A. f a*) $\leq$ ($\sum$ *a* $\in$ *A. g a*)
  $\langle proof \rangle$

lemma *distrib*:
  fixes
    A :: $'a$ *set* **and**
    f g :: $'a$ $\Rightarrow$ *int*

shows $(\sum\ a \in A.\ f\ a) + (\sum\ a \in A.\ g\ a) = (\sum\ a \in A.\ f\ a + g\ a)$
⟨*proof*⟩

lemma *distrib-ereal*:
  fixes
    $A :: {}'a\ set$ **and**
    $f\ g :: {}'a \Rightarrow int$
  shows *ereal* (*real-of-int* $((\sum\ a \in A.\ (f :: {}'a \Rightarrow int)\ a) + (\sum\ a \in A.\ g\ a)))$ =
    ereal (*real-of-int* $(\sum\ a \in A.\ f\ a + g\ a))$
  ⟨*proof*⟩

lemma *uneq-ereal*:
  fixes $x\ y :: int$
  assumes $x \le y$
  shows *ereal* (*real-of-int* x) $\le$ *ereal* (*real-of-int* y)
  ⟨*proof*⟩

4.1.5 Swap Distance

fun *neq-ord* $:: {}'a\ Preference\text{-}Relation \Rightarrow {}'a\ Preference\text{-}Relation \Rightarrow$
    ${}'a \Rightarrow {}'a \Rightarrow bool$ **where**
  neq-ord $r\ s\ a\ b = ((a \preceq_r b \land b \preceq_s a) \lor (b \preceq_r a \land a \preceq_s b))$

fun *pairwise-disagreements* $:: {}'a\ set \Rightarrow {}'a\ Preference\text{-}Relation \Rightarrow$
    ${}'a\ Preference\text{-}Relation \Rightarrow ({}'a \times {}'a)\ set$ **where**
  pairwise-disagreements $A\ r\ s = \{(a,\ b) \in A \times A.\ a \ne b \land neq\text{-}ord\ r\ s\ a\ b\}$

fun *pairwise-disagreements'* $:: {}'a\ set \Rightarrow {}'a\ Preference\text{-}Relation \Rightarrow$
    ${}'a\ Preference\text{-}Relation \Rightarrow ({}'a \times {}'a)\ set$ **where**
  pairwise-disagreements' $A\ r\ s =$
    Set.filter $(\lambda\ (a,\ b).\ a \ne b \land neq\text{-}ord\ r\ s\ a\ b)\ (A \times A)$

lemma *set-eq-filter*:
  fixes
    $X :: {}'a\ set$ **and**
    $P :: {}'a \Rightarrow bool$
  shows $\{x \in X.\ P\ x\} = Set.filter\ P\ X$
  ⟨*proof*⟩

lemma *pairwise-disagreements-eq*[*code*]: *pairwise-disagreements* = *pairwise-disagreements'*
  ⟨*proof*⟩

fun *swap* $:: {}'a\ Vote\ Distance$ **where**
  swap $(A,\ r)\ (A',\ r') =$
    (*if* $A = A'$
    then *card* (*pairwise-disagreements* $A\ r\ r'$)
    else ∞)

lemma *swap-case-infinity*:

fixes $x\ y :: {'}a\ Vote$
assumes *alts-$\mathcal{V}$ x $\neq$ alts-$\mathcal{V}$ y*
shows *swap x y = ∞*
$\langle proof \rangle$

lemma *swap-case-fin*:
  fixes $x\ y :: {'}a\ Vote$
  assumes *alts-$\mathcal{V}$ x = alts-$\mathcal{V}$ y*
  shows *swap x y = card (pairwise-disagreements (alts-$\mathcal{V}$ x) (pref-$\mathcal{V}$ x) (pref-$\mathcal{V}$ y))*
  $\langle proof \rangle$

4.1.6   Spearman Distance

fun *spearman* :: ${'}a\ Vote\ Distance$ **where**
  spearman $(A,\ x)\ (A',\ y) =$
    (if $A = A'$
    then $\sum\ a \in A.$ abs (int (rank x a) $-$ int (rank y a))
    else ∞)

lemma *spearman-case-inf*:
  fixes $x\ y :: {'}a\ Vote$
  assumes *alts-$\mathcal{V}$ x $\neq$ alts-$\mathcal{V}$ y*
  shows *spearman x y = ∞*
  $\langle proof \rangle$

lemma *spearman-case-fin*:
  fixes $x\ y :: {'}a\ Vote$
  assumes *alts-$\mathcal{V}$ x = alts-$\mathcal{V}$ y*
  shows *spearman x y =*
    ($\sum\ a \in$ alts-$\mathcal{V}$ x. abs (int (rank (pref-$\mathcal{V}$ x) a) $-$ int (rank (pref-$\mathcal{V}$ y) a)))
  $\langle proof \rangle$

4.1.7   Properties

Distances that are invariant under specific relations induce symmetry properties in distance rationalized voting rules.

Definitions

fun *total-invariance$_\mathcal{D}$* :: ${'}x\ Distance \Rightarrow {'}x\ rel \Rightarrow bool$ **where**
  total-invariance$_\mathcal{D}$ d rel = is-symmetry (tup d) (Invariance (product rel))

fun *invariance$_\mathcal{D}$* :: ${'}y\ Distance \Rightarrow {'}x\ set \Rightarrow {'}y\ set \Rightarrow$
    $({'}x,\ {'}y)\ binary\text{-}fun \Rightarrow bool$ **where**
  invariance$_\mathcal{D}$ d X Y φ = is-symmetry (tup d) (Invariance (equivariance X Y φ))

definition *distance-anonymity* :: $({'}a,\ {'}v)\ Election\ Distance \Rightarrow bool$ **where**
  distance-anonymity d $\equiv$
    $\forall\ A\ A'\ V\ V'\ p\ p'\ \pi :: ({'}v \Rightarrow {'}v).$

$(bij\ \pi \longrightarrow$
$(d\ (A,\ V,\ p)\ (A',\ V',\ p')) =$
$(d\ (rename\ \pi\ (A,\ V,\ p)))\ (rename\ \pi\ (A',\ V',\ p')))$

fun *distance-anonymity′* :: $('a,\ 'v)$ *Election set* $\Rightarrow$
$\quad('a,\ 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
distance-anonymity′ $X\ d = invariance_\mathcal{D}\ d\ (carrier\ bijection_{\mathcal{VG}})\ X\ (\varphi\text{-}anon\ X)$

fun *distance-neutrality* :: $('a,\ 'v)$ *Election set* $\Rightarrow$
$\quad('a,\ 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
distance-neutrality $X\ d = invariance_\mathcal{D}\ d\ (carrier\ bijection_{\mathcal{AG}})\ X\ (\varphi\text{-}neutral\ X)$

fun *distance-reversal-symmetry* :: $('a,\ 'v)$ *Election set* $\Rightarrow$
$\quad('a,\ 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
distance-reversal-symmetry $X\ d =$
$\quad invariance_\mathcal{D}\ d\ (carrier\ reversal_\mathcal{G})\ X\ (\varphi\text{-}reverse\ X)$

definition *distance-homogeneity′* :: $('a,\ 'v :: linorder)$ *Election set* $\Rightarrow$
$\quad('a,\ 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
distance-homogeneity′ $X\ d \equiv total\text{-}invariance_\mathcal{D}\ d\ (homogeneity_\mathcal{R}'\ X)$

definition *distance-homogeneity* :: $('a,\ 'v)$ *Election set* $\Rightarrow$
$\quad('a,\ 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
distance-homogeneity $X\ d \equiv total\text{-}invariance_\mathcal{D}\ d\ (homogeneity_\mathcal{R}\ X)$

Auxiliary Lemmas

lemma *rewrite-total-invariance$_\mathcal{D}$*:
  fixes
    $d :: 'x\ Distance$ **and**
    $r :: 'x\ rel$
    shows $total\text{-}invariance_\mathcal{D}\ d\ r = (\forall\ (x,\ y) \in r.\ \forall\ (a,\ b) \in r.\ d\ a\ x = d\ b\ y)$
$\langle proof \rangle$

lemma *rewrite-invariance$_\mathcal{D}$*:
  fixes
    $d :: 'y\ Distance$ **and**
    $X :: 'x\ set$ **and**
    $Y :: 'y\ set$ **and**
    $\varphi :: ('x,\ 'y)\ binary\text{-}fun$
    shows $invariance_\mathcal{D}\ d\ X\ Y\ \varphi =$
        $(\forall\ x \in X.\ \forall\ y \in Y.\ \forall\ z \in Y.\ d\ y\ z = d\ (\varphi\ x\ y)\ (\varphi\ x\ z))$
$\langle proof \rangle$

lemma *invar-dist-image*:
  fixes
    $d :: 'y\ Distance$ **and**
    $G :: 'x\ monoid$ **and**
    $Y\ Y' :: 'y\ set$ **and**

$\varphi :: ('x, 'y)$ *binary-fun* **and**
$y :: 'y$ **and**
$g :: 'x$
assumes
  invar-d: *invariance*$_\mathcal{D}$ *d* (*carrier G*) *Y* φ **and**
  Y′-in-Y: $Y' \subseteq Y$ **and**
  action-φ: *group-action G Y* φ **and**
  g-carrier: $g \in$ *carrier G* **and**
  y-in-Y: $y \in Y$
shows *d* (φ *g y*) ' (φ *g*) ' $Y' = d$ *y* ' Y'
⟨*proof*⟩

lemma *swap-neutral*: *invariance*$_\mathcal{D}$ *swap* (*carrier bijection*$_{\mathcal{AG}}$)
                    UNIV (λ π (*A*, *q*). (π ' *A*, *rel-rename* π *q*))

⟨*proof*⟩

end

4.2   Votewise Distance

theory *Votewise-Distance*
  imports *Social-Choice-Types/Norm*
        Distance
begin

Votewise distances are a natural class of distances on elections which depend
on the submitted votes in a simple and transparent manner. They are formed
by using any distance d on individual orders and combining the components
with a norm on $\mathbb{R}^n$.

4.2.1   Definition

fun *votewise-distance* :: $'a$ *Vote Distance* $\Rightarrow$ *Norm* $\Rightarrow$
      ($'a$, $'v ::$ *linorder*) *Election Distance* **where**
  votewise-distance d n (*A*, *V*, *p*) (*A′*, *V′*, *p′*) =
    (*if finite V* $\wedge$ *V* = *V′* $\wedge$ (*V* $\neq$ {} $\vee$ *A* = *A′*)
    then n (*map2* (λ *q q′*. *d* (*A*, *q*) (*A′*, *q′*)) (*to-list V p*) (*to-list V′ p′*))
    else ∞)

4.2.2   Inference Rules

lemma *symmetric-norm-inv-under-map-permute*:
  fixes
    d :: $'a$ *Vote Distance* **and**
    n :: *Norm* **and**
    A A′ :: $'a$ *set* **and**

$\varphi :: nat \Rightarrow nat$ **and**
$p\ p' :: 'a\ Preference\text{-}Relation\ list$
assumes
  perm: φ *permutes* $\{0\ ..< length\ p\}$ **and**
  len-eq: *length* p = *length* p' **and**
  sym-n: *symmetry* n
shows n (*map2* ($\lambda\ q\ q'.\ d\ (A,\ q)\ (A',\ q')$) $p\ p'$) =
    n (*map2* ($\lambda\ q\ q'.\ d\ (A,\ q)\ (A',\ q')$) (*permute-list* $\varphi\ p$) (*permute-list* $\varphi\ p'$))
⟨*proof*⟩

lemma *permute-invariant-under-map*:
  fixes $l\ l' :: 'a\ list$
  assumes $l <\tilde{}\tilde{}> l'$
  shows *map* $f\ l <\tilde{}\tilde{}> map\ f\ l'$
  ⟨*proof*⟩

lemma *linorder-rank-injective*:
  fixes
    $V :: 'v :: linorder\ set$ **and**
    $v\ v' :: 'v$
  assumes
    v-in-V: $v \in V$ **and**
    v'-in-V: $v' \in V$ **and**
    v'-neq-v: $v' \neq v$ **and**
    fin-V: *finite* V
  shows *card* $\{x \in V.\ x < v\} \neq card\ \{x \in V.\ x < v'\}$
⟨*proof*⟩

lemma *permute-invariant-under-coinciding-funs*:
  fixes
    $l :: 'v\ list$ **and**
    $\pi_1\ \pi_2 :: nat \Rightarrow nat$
  assumes $\forall\ i < length\ l.\ \pi_1\ i = \pi_2\ i$
  shows *permute-list* $\pi_1\ l$ = *permute-list* $\pi_2\ l$
  ⟨*proof*⟩

lemma *symmetric-norm-imp-distance-anonymous*:
  fixes
    $d :: 'a\ Vote\ Distance$ **and**
    $n :: Norm$
  assumes *symmetry* n
  shows *distance-anonymity* (*votewise-distance* $d\ n$)
⟨*proof*⟩

lemma *neutral-dist-imp-neutral-votewise-dist*:
  fixes
    $d :: 'a\ Vote\ Distance$ **and**
    $n :: Norm$
  defines *vote-action* $\equiv \lambda\ \pi\ (A,\ q).\ (\pi\ `\ A,\ rel\text{-}rename\ \pi\ q)$

assumes *invariance$_\mathcal{D}$ d (carrier bijection$_{\mathcal{AG}}$) UNIV vote-action*
shows *distance-neutrality well-formed-elections (votewise-distance d n)*
⟨*proof*⟩

end

4.3   Consensus

theory *Consensus*
  imports *Social-Choice-Types/Voting-Symmetry*
begin

An election consisting of a set of alternatives and preferential votes for each
voter (a profile) is a consensus if it has an undisputed winner reflecting a
certain concept of fairness in the society.

4.3.1   Definition

type-synonym *($'a$, $'v$) Consensus = ($'a$, $'v$) Election ⇒ bool*

4.3.2   Consensus Conditions

Nonempty alternative set.

fun *nonempty-set$_\mathcal{C}$ :: ($'a$, $'v$) Consensus* **where**
  nonempty-set$_\mathcal{C}$ (A, V, p) = (A ≠ {})

Nonempty profile, i.e., nonempty voter set. Note that this is also true if
p(v) = holds for all voters v in V.

fun *nonempty-profile$_\mathcal{C}$ :: ($'a$, $'v$) Consensus* **where**
  nonempty-profile$_\mathcal{C}$ (A, V, p) = (V ≠ {})

Equal top ranked alternatives.

fun *equal-top$_\mathcal{C}$′ :: $'a$ ⇒ ($'a$, $'v$) Consensus* **where**
  equal-top$_\mathcal{C}$′ a (A, V, p) = (a ∈ A ∧ (∀ v ∈ V. above (p v) a = {a}))

fun *equal-top$_\mathcal{C}$ :: ($'a$, $'v$) Consensus* **where**
  equal-top$_\mathcal{C}$ c = (∃ a. equal-top$_\mathcal{C}$′ a c)

Equal votes.

fun *equal-vote$_\mathcal{C}$′ :: $'a$ Preference-Relation ⇒ ($'a$, $'v$) Consensus* **where**
  equal-vote$_\mathcal{C}$′ r (A, V, p) = (∀ v ∈ V. (p v) = r)

fun *equal-vote$_\mathcal{C}$ :: ($'a$, $'v$) Consensus* **where**
  equal-vote$_\mathcal{C}$ c = (∃ r. equal-vote$_\mathcal{C}$′ r c)

Unanimity condition.

fun *unanimity$_\mathcal{C}$* :: $('a, 'v)$ *Consensus* **where**
  unanimity$_\mathcal{C}$ c = (nonempty-set$_\mathcal{C}$ c $\wedge$ nonempty-profile$_\mathcal{C}$ c $\wedge$ equal-top$_\mathcal{C}$ c)

Strong unanimity condition.

fun *strong-unanimity$_\mathcal{C}$* :: $('a, 'v)$ *Consensus* **where**
  strong-unanimity$_\mathcal{C}$ c = (nonempty-set$_\mathcal{C}$ c $\wedge$ nonempty-profile$_\mathcal{C}$ c $\wedge$ equal-vote$_\mathcal{C}$ c)

4.3.3 Properties

definition *consensus-anonymity* :: $('a, 'v)$ *Consensus* $\Rightarrow$ *bool* **where**
  consensus-anonymity c $\equiv$
    $(\forall$ *A V p π* :: $('v \Rightarrow 'v)$.
      bij π $\longrightarrow$
        (let $(A', V', q) = (rename \; \pi \; (A, V, p))$ in
          profile V A p $\longrightarrow$ profile V$'$ A$'$ q
          $\longrightarrow$ *c (A, V, p) $\longrightarrow$ c (A$'$, V$'$, q)))*

fun *consensus-neutrality* :: $('a, 'v)$ *Election set* $\Rightarrow$ $('a, 'v)$ *Consensus* $\Rightarrow$ *bool* **where**
  consensus-neutrality X c = is-symmetry c (Invariance (neutrality$_\mathcal{R}$ X))

4.3.4 Auxiliary Lemmas

lemma *cons-anon-conj*:
  fixes *c c$'$* :: $('a, 'v)$ *Consensus*
  assumes
    consensus-anonymity c **and**
    consensus-anonymity c$'$
  shows *consensus-anonymity (λ e. c e $\wedge$ c$'$ e)*
$\langle proof \rangle$

theorem *cons-conjunction-invariant*:
  fixes
    $\mathfrak{C}$ :: $('a, 'v)$ *Consensus set* **and**
    rel :: $('a, 'v)$ *Election rel*
  defines $C \equiv \lambda \; E. \; \forall \; C' \in \mathfrak{C}. \; C' \; E$
  assumes $\forall$ *C$'$. C$'$ $\in \mathfrak{C}$ $\longrightarrow$ is-symmetry C$'$ (Invariance rel)*
  shows *is-symmetry C (Invariance rel)*
$\langle proof \rangle$

lemma *cons-anon-invariant*:
  fixes
    c :: $('a, 'v)$ *Consensus* **and**
    A A$'$ :: $'a$ *set* **and**
    V V$'$ :: $'v$ *set* **and**
    p q :: $('a, 'v)$ *Profile* **and**
    π :: $'v \Rightarrow 'v$
  assumes
    anon: *consensus-anonymity c* **and**

bij-π: *bij π* **and**
 prof-p: *profile V A p* **and**
 renamed: *rename π (A, V, p) = (A′, V′, q)* **and**
 cond-c: *c (A, V, p)*
 shows *c (A′, V′, q)*
⟨*proof*⟩

lemma *ex-anon-cons-imp-cons-anonymous*:
 fixes
  b :: *('a, 'v) Consensus* **and**
  b′:: *'b ⇒ ('a, 'v) Consensus*
 assumes
  general-cond-b: *b = (λ E. ∃ x. b′ x E)* **and**
  all-cond-anon: *∀ x. consensus-anonymity (b′ x)*
 shows *consensus-anonymity b*
⟨*proof*⟩

4.3.5 Theorems

Anonymity

lemma *nonempty-set-cons-anonymous*: *consensus-anonymity nonempty-set$_C$*
 ⟨*proof*⟩

lemma *nonempty-profile-cons-anonymous*: *consensus-anonymity nonempty-profile$_C$*
⟨*proof*⟩

lemma *equal-top-cons′-anonymous*:
 fixes *a* :: *'a*
 shows *consensus-anonymity (equal-top$_C$′ a)*
⟨*proof*⟩

lemma *eq-top-cons-anon*: *consensus-anonymity equal-top$_C$*
 ⟨*proof*⟩

lemma *eq-vote-cons′-anonymous*:
 fixes *r* :: *'a Preference-Relation*
 shows *consensus-anonymity (equal-vote$_C$′ r)*
⟨*proof*⟩

lemma *eq-vote-cons-anonymous*: *consensus-anonymity equal-vote$_C$*
 ⟨*proof*⟩

Neutrality

lemma *nonempty-set$_C$-neutral*: *consensus-neutrality well-formed-elections nonempty-set$_C$*
 ⟨*proof*⟩

lemma *nonempty-profile$_C$-neutral*: *consensus-neutrality well-formed-elections nonempty-profile$_C$*
 ⟨*proof*⟩

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lemma *equal-vote$_{\mathcal{C}}$ -neutral*: *consensus-neutrality well-formed-elections equal-vote$_{\mathcal{C}}$*
⟨*proof*⟩

lemma *strong-unanimity$_{\mathcal{C}}$ -neutral*: *consensus-neutrality*
    well-formed-elections strong-unanimity$_{\mathcal{C}}$
  ⟨*proof*⟩

end

4.4 Electoral Module

theory *Electoral-Module*
  imports *Social-Choice-Types/Property-Interpretations*
begin

Electoral modules are the principal component type of the composable modules voting framework, as they are a generalization of voting rules in the sense of social choice functions. These are only the types used for electoral modules. Further restrictions are encompassed by the electoral-module predicate.

An electoral module does not need to make final decisions for all alternatives, but can instead defer the decision for some or all of them to other modules. Hence, electoral modules partition the received (possibly empty) set of alternatives into elected, rejected and deferred alternatives. In particular, any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives. Just like a voting rule, an electoral module also receives a profile which holds the voters preferences, which, unlike a voting rule, consider only the (sub-)set of alternatives that the module receives.

4.4.1 Definition

An electoral module maps an election to a result. To enable currying, the Election type is not used here because that would require tuples.

type-synonym (*'a*, *'v*, *'r*) *Electoral-Module* = *'v set* ⇒ *'a set* ⇒
    (*'a*, *'v*) *Profile* ⇒ *'r*

fun *fun$_{\mathcal{E}}$* :: (*'v set* ⇒ *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒ *'r*) ⇒
    ((*'a*, *'v*) *Election* ⇒ *'r*) **where**
  fun$_{\mathcal{E}}$ *m* = (λ *E*. *m* (*voters-$\mathcal{E}$ E*) (*alternatives-$\mathcal{E}$ E*) (*profile-$\mathcal{E}$ E*))

The next three functions take an electoral module and turn it into a function only outputting the elect, reject, or defer set respectively.

abbreviation *elect* :: $('a, 'v, 'r\ Result)$ *Electoral-Module* $\Rightarrow$ $'v\ set \Rightarrow 'a\ set \Rightarrow$
$\qquad ('a, 'v)\ Profile \Rightarrow 'r\ set$ **where**
$\quad elect\ m\ V\ A\ p \equiv elect\text{-}r\ (m\ V\ A\ p)$

abbreviation *reject* :: $('a, 'v, 'r\ Result)$ *Electoral-Module* $\Rightarrow$ $'v\ set \Rightarrow 'a\ set \Rightarrow$
$\qquad ('a, 'v)\ Profile \Rightarrow 'r\ set$ **where**
$\quad reject\ m\ V\ A\ p \equiv reject\text{-}r\ (m\ V\ A\ p)$

abbreviation *defer* :: $('a, 'v, 'r\ Result)$ *Electoral-Module* $\Rightarrow$ $'v\ set \Rightarrow 'a\ set \Rightarrow$
$\qquad ('a, 'v)\ Profile \Rightarrow 'r\ set$ **where**
$\quad defer\ m\ V\ A\ p \equiv defer\text{-}r\ (m\ V\ A\ p)$

4.4.2 Auxiliary Definitions

Electoral modules partition a given set of alternatives A into a set of elected alternatives e, a set of rejected alternatives r, and a set of deferred alternatives d, using a profile. e, r, and d partition A. Electoral modules can be used as voting rules. They can also be composed in multiple structures to create more complex electoral modules.

fun (**in** *result*) *electoral-module* :: $('a, 'v, ('r\ Result))$ *Electoral-Module* $\Rightarrow$
$\qquad bool$ **where**
$\quad electoral\text{-}module\ m = (\forall\ A\ V\ p.\ profile\ V\ A\ p \longrightarrow well\text{-}formed\ A\ (m\ V\ A\ p))$

fun *voters-determine-election* :: $('a, 'v, ('r\ Result))$ *Electoral-Module* $\Rightarrow bool$ **where**
$\quad voters\text{-}determine\text{-}election\ m =$
$\quad (\forall\ A\ V\ p\ p'.\ (\forall\ v \in V.\ p\ v = p'\ v) \longrightarrow m\ V\ A\ p = m\ V\ A\ p')$

lemma (**in** *result*) *electoral-modI*:
$\quad$ **fixes** m :: $('a, 'v, ('r\ Result))$ *Electoral-Module*
$\quad$ **assumes** $\forall\ A\ V\ p.\ profile\ V\ A\ p \longrightarrow well\text{-}formed\ A\ (m\ V\ A\ p)$
$\quad$ **shows** *electoral-module* m
$\quad \langle proof \rangle$

4.4.3 Properties

We only require voting rules to behave a specific way on admissible elections, i.e., elections that are valid profiles (= votes are linear orders on the alternatives). Note that we do not assume finiteness of voter or alternative sets by default.

Anonymity

An electoral module is anonymous iff the result is invariant under renamings of voters, i.e., any permutation of the voter set that does not change the preferences leads to an identical result.

definition (**in** *result*) *anonymity* :: $('a, 'v, ('r\ Result))$ *Electoral-Module* $\Rightarrow$
    bool **where**
anonymity m $\equiv$
  electoral-module m $\wedge$
    $(\forall\ A\ V\ p\ \pi :: ('v \Rightarrow 'v).$
      bij $\pi \longrightarrow$ (*let* $(A',\ V',\ q) = (rename\ \pi\ (A,\ V,\ p))$ *in*
        profile V A p $\wedge$ *profile* $V'\ A'\ q \longrightarrow m\ V\ A\ p = m\ V'\ A'\ q))$

Anonymity can alternatively be described as invariance under the voter permutation group acting on elections via the rename function.

fun *anonymity-in* :: $('a, 'v)$ *Election set* $\Rightarrow ('a, 'v, 'r)$ *Electoral-Module* $\Rightarrow$
    bool **where**
anonymity-in X m = is-symmetry $(fun_{\mathcal{E}}\ m)$ $(Invariance\ (anonymity_{\mathcal{R}}\ X))$

fun *anonymity′* :: $('a, 'v, 'r)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
anonymity′ m = anonymity-in well-formed-elections m

Homogeneity

A voting rule is homogeneous if copying an election does not change the result. For ordered voter types and finite elections, we use the notion of copying ballot lists to define copying an election. The more general definition of homogeneity for unordered voter types already implies anonymity.

fun *homogeneity-in* :: $('a, 'v)$ *Election set* $\Rightarrow$
    $('a, 'v, 'r\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
homogeneity-in X m = is-symmetry $(fun_{\mathcal{E}}\ m)$ $(Invariance\ (homogeneity_{\mathcal{R}}\ X))$
— This does not require any specific behaviour on infinite voter sets . . . It might make sense to extend the definition to that case somehow.

fun *homogeneity* :: $('a, 'v, 'b\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
homogeneity m = homogeneity-in finite-elections-$\mathcal{V}$ *m*

fun *homogeneity′-in* :: $('a, 'v :: linorder)$ *Election set* $\Rightarrow$
    $('a, 'v, 'b\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
homogeneity′-in X m = is-symmetry $(fun_{\mathcal{E}}\ m)$ $(Invariance\ (homogeneity_{\mathcal{R}}'\ X))$

fun *homogeneity′* :: $('a, 'v :: linorder, 'b\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
homogeneity′ m = homogeneity′-in finite-elections-$\mathcal{V}$ *m*

lemma *hom-imp-anon*:
  fixes
    $X :: ('a, 'v)$ *Election set* **and**
    $m :: ('a, 'v, ('r\ Result))$ *Electoral-Module*
  assumes
    homogeneity-in X m **and**
    $\forall\ E \in X.$ *finite* $(voters\text{-}\mathcal{E}\ E)$
  shows *anonymity-in X m*
$\langle proof \rangle$

Neutrality

Neutrality is equivariance under consistent renaming of candidates in the candidate set and election results.

fun (**in** *result-properties*) *neutrality-in* :: $('a, 'v)$ *Election set* $\Rightarrow$
    $('a, 'v, 'b\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
  neutrality-in X m =
    is-symmetry $(fun_{\mathcal{E}}\ m)$ $(action\text{-}induced\text{-}equivariance$ $(carrier\ bijection_{\mathcal{AG}})\ X$
      $(\varphi\text{-}neutral\ X)\ (result\text{-}action\ \psi))$

fun (**in** *result-properties*) *neutrality* :: $('a, 'v, 'b\ Result)$ *Electoral-Module* $\Rightarrow$
    bool **where**
  neutrality m = neutrality-in well-formed-elections m

4.4.4 Social-Welfare Properties

Reversal Symmetry

A social welfare rule is reversal symmetric if reversing all voters' preferences reverses the result rankings as well.

definition *reversal-symmetry-in* :: $('a, 'v)$ *Election set* $\Rightarrow$
    $('a, 'v, 'a\ rel\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
  reversal-symmetry-in X m $\equiv$
    is-symmetry $(fun_{\mathcal{E}}\ m)$ $(action\text{-}induced\text{-}equivariance$ $(carrier\ reversal_{\mathcal{G}})\ X$
      $(\varphi\text{-}reverse\ X)\ (result\text{-}action\ \psi\text{-}reverse))$

fun *reversal-symmetry* :: $('a, 'v, 'a\ rel\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
  reversal-symmetry m = reversal-symmetry-in well-formed-elections m

4.4.5 Social-Choice Modules

The following results require electoral modules to return social choice results, i.e., sets of elected, rejected and deferred alternatives. In order to export code, we use the hack provided by Locale-Code.

"defers n" is true for all electoral modules that defer exactly n alternatives, whenever there are n or more alternatives.

definition *defers* :: *nat* $\Rightarrow$ $('a, 'v, 'a\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
  defers n m $\equiv$
    $\mathcal{SCF}$*-result.electoral-module m* $\wedge$
      $(\forall\ A\ V\ p.\ (card\ A \geq n \wedge finite\ A \wedge profile\ V\ A\ p)$
        $\longrightarrow card\ (defer\ m\ V\ A\ p) = n)$

"rejects n" is true for all electoral modules that reject exactly n alternatives, whenever there are n or more alternatives.

definition *rejects* :: *nat* $\Rightarrow$ $('a, 'v, 'a\ Result)$ *Electoral-Module* $\Rightarrow$ *bool* **where**
  rejects n m $\equiv$

$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$
$(\forall\ A\ V\ p.\ (card\ A \geq n \land finite\ A \land profile\ V\ A\ p)$
$\longrightarrow\ card\ (reject\ m\ V\ A\ p) = n)$

As opposed to "rejects", "eliminates" allows to stop rejecting if no alternatives were to remain.

definition *eliminates* :: *nat* $\Rightarrow$ ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* $\Rightarrow$ *bool* **where**
  eliminates n m $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$
    $(\forall\ A\ V\ p.\ (card\ A > n \land profile\ V\ A\ p) \longrightarrow card\ (reject\ m\ V\ A\ p) = n)$

"elects n" is true for all electoral modules that elect exactly n alternatives, whenever there are n or more alternatives.

definition *elects* :: *nat* $\Rightarrow$ ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* $\Rightarrow$ *bool* **where**
  elects n m $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$
    $(\forall\ A\ V\ p.\ (card\ A \geq n \land profile\ V\ A\ p) \longrightarrow card\ (elect\ m\ V\ A\ p) = n)$

An electoral module is independent of an alternative a iff a's ranking does not influence the outcome.

definition *indep-of-alt* :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* $\Rightarrow$ $'v$ *set* $\Rightarrow$
    $'a$ *set* $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
  indep-of-alt m V A a $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$
    $\land\ (\forall\ p\ q.\ equiv\text{-}prof\text{-}except\text{-}a\ V\ A\ p\ q\ a \longrightarrow m\ V\ A\ p = m\ V\ A\ q)$

definition *unique-winner-if-profile-non-empty* :: ($'a$, $'v$, $'a$ *Result*)
    Electoral-Module $\Rightarrow$ *bool* **where**
  unique-winner-if-profile-non-empty m $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$
    $(\forall\ A\ V\ p.\ (A \neq \{\} \land V \neq \{\} \land profile\ V\ A\ p) \longrightarrow$
        $(\exists\ a \in A.\ m\ V\ A\ p = (\{a\},\ A - \{a\},\ \{\})))$

4.4.6 Equivalence Definitions

definition *prof-contains-result* :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* $\Rightarrow$ $'v$ *set* $\Rightarrow$
    $'a$ *set* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
  prof-contains-result m V A p q a $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$
    profile V A p $\land$ *profile V A q* $\land$ $a \in A\ \land$
    $(a \in elect\ m\ V\ A\ p \longrightarrow a \in elect\ m\ V\ A\ q)\ \land$
    $(a \in reject\ m\ V\ A\ p \longrightarrow a \in reject\ m\ V\ A\ q)\ \land$
    $(a \in defer\ m\ V\ A\ p \longrightarrow a \in defer\ m\ V\ A\ q)$

definition *prof-leq-result* :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* $\Rightarrow$ $'v$ *set* $\Rightarrow$
    $'a$ *set* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ ($'a$, $'v$) *Profile* $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
  prof-leq-result m V A p q a $\equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \land$

profile V A p ∧ profile V A q ∧ a ∈ A ∧
(a ∈ reject m V A p ⟶ a ∈ reject m V A q) ∧
(a ∈ defer m V A p ⟶ a ∉ elect m V A q)

definition *prof-geq-result* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒ ′*v set* ⇒
  ′*a set* ⇒ (′*a*, ′*v*) *Profile* ⇒ (′*a*, ′*v*) *Profile* ⇒ ′*a* ⇒ *bool* **where**
 prof-geq-result m V A p q a ≡
  SCF-result.electoral-module m ∧
  profile V A p ∧ profile V A q ∧ a ∈ A ∧
  (a ∈ elect m V A p ⟶ a ∈ elect m V A q) ∧
  (a ∈ defer m V A p ⟶ a ∉ reject m V A q)

definition *mod-contains-result* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒
  (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒ ′*v set* ⇒ ′*a set* ⇒
  (′*a*, ′*v*) *Profile* ⇒ ′*a* ⇒ *bool* **where**
 mod-contains-result m n V A p a ≡
  SCF-result.electoral-module m ∧
  SCF-result.electoral-module n ∧
  profile V A p ∧ a ∈ A ∧
  (a ∈ elect m V A p ⟶ a ∈ elect n V A p) ∧
  (a ∈ reject m V A p ⟶ a ∈ reject n V A p) ∧
  (a ∈ defer m V A p ⟶ a ∈ defer n V A p)

definition *mod-contains-result-sym* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒
  (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒ ′*v set* ⇒ ′*a set* ⇒
  (′*a*, ′*v*) *Profile* ⇒ ′*a* ⇒ *bool* **where**
 mod-contains-result-sym m n V A p a ≡
  SCF-result.electoral-module m ∧
  SCF-result.electoral-module n ∧
  profile V A p ∧ a ∈ A ∧
  (a ∈ elect m V A p ⟷ a ∈ elect n V A p) ∧
  (a ∈ reject m V A p ⟷ a ∈ reject n V A p) ∧
  (a ∈ defer m V A p ⟷ a ∈ defer n V A p)

4.4.7 Auxiliary Lemmas

lemma *elect-rej-def-combination*:
 fixes
  m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
  V :: ′*v set* **and**
  A :: ′*a set* **and**
  p :: (′*a*, ′*v*) *Profile* **and**
  e r d :: ′*a set*
 assumes
  elect m V A p = e **and**
  reject m V A p = r **and**
  defer m V A p = d
 shows *m V A p = (e, r, d)*
 ⟨*proof*⟩

lemma *par-comp-result-sound*:
  fixes
    m :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
    A :: *'a set* **and**
    p :: (*'a*, *'v*) *Profile*
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile V A p
  shows *well-formed-*$\mathcal{SCF}$ *A* (*m V A p*)
  ⟨*proof*⟩

lemma *result-presv-alts*:
  fixes
    m :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
    A :: *'a set* **and**
    V :: *'v set* **and**
    p :: (*'a*, *'v*) *Profile*
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile V A p
  shows (*elect m V A p*) ∪ (*reject m V A p*) ∪ (*defer m V A p*) = *A*
⟨*proof*⟩

lemma *result-disj*:
  fixes
    m :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
    A :: *'a set* **and**
    p :: (*'a*, *'v*) *Profile* **and**
    V :: *'v set*
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile V A p
  shows
    (*elect m V A p*) ∩ (*reject m V A p*) = {} ∧
      (*elect m V A p*) ∩ (*defer m V A p*) = {} ∧
      (*reject m V A p*) ∩ (*defer m V A p*) = {}
⟨*proof*⟩

lemma *elect-in-alts*:
  fixes
    m :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
    A :: *'a set* **and**
    p :: (*'a*, *'v*) *Profile*
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile V A p
  shows *elect m V A p* ⊆ *A*
  ⟨*proof*⟩

lemma *reject-in-alts*:
  fixes
    m :: ($'a$, $'v$, $'a$ $Result$) $Electoral\text{-}Module$ **and**
    A :: $'a$ set **and**
    V :: $'v$ set **and**
    p :: ($'a$, $'v$) $Profile$
  assumes
    $\mathcal{SCF}$-$result.electoral\text{-}module\ m$ **and**
    $profile\ V\ A\ p$
  shows $reject\ m\ V\ A\ p \subseteq A$
  $\langle proof \rangle$

lemma *defer-in-alts*:
  fixes
    m :: ($'a$, $'v$, $'a$ $Result$) $Electoral\text{-}Module$ **and**
    A :: $'a$ set **and**
    V :: $'v$ set **and**
    p :: ($'a$, $'v$) $Profile$
  assumes
    $\mathcal{SCF}$-$result.electoral\text{-}module\ m$ **and**
    $profile\ V\ A\ p$
  shows $defer\ m\ V\ A\ p \subseteq A$
  $\langle proof \rangle$

lemma *def-presv-prof*:
  fixes
    m :: ($'a$, $'v$, $'a$ $Result$) $Electoral\text{-}Module$ **and**
    A :: $'a$ set **and**
    p :: ($'a$, $'v$) $Profile$
  assumes
    $\mathcal{SCF}$-$result.electoral\text{-}module\ m$ **and**
    $profile\ V\ A\ p$
  shows $let\ new\text{-}A = defer\ m\ V\ A\ p\ in\ profile\ V\ new\text{-}A\ (limit\text{-}profile\ new\text{-}A\ p)$
  $\langle proof \rangle$

An electoral module can never reject, defer or elect more than |A| alternatives.

lemma *upper-card-bounds-for-result*:
  fixes
    m :: ($'a$, $'v$, $'a$ $Result$) $Electoral\text{-}Module$ **and**
    A :: $'a$ set **and**
    V :: $'v$ set **and**
    p :: ($'a$, $'v$) $Profile$
  assumes
    $\mathcal{SCF}$-$result.electoral\text{-}module\ m$ **and**
    $profile\ V\ A\ p$ **and**
    $finite\ A$
  shows

upper-card-bound-for-elect: *card* (*elect m V A p*) ≤ *card A* **and**
　upper-card-bound-for-reject: *card* (*reject m V A p*) ≤ *card A* **and**
　upper-card-bound-for-defer: *card* (*defer m V A p*) ≤ *card A*
⟨*proof*⟩

lemma *reject-not-elected-or-deferred*:
　fixes
　　m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
　　A :: ′*a set* **and**
　　V :: ′*v set* **and**
　　p :: (′*a*, ′*v*) *Profile*
　assumes
　　$\mathcal{SCF}$-*result.electoral-module m* **and**
　　profile V A p
　shows *reject m V A p* = *A* − (*elect m V A p*) − (*defer m V A p*)
⟨*proof*⟩

lemma *elec-and-def-not-rej*:
　fixes
　　m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
　　A :: ′*a set* **and**
　　V :: ′*v set* **and**
　　p :: (′*a*, ′*v*) *Profile*
　assumes
　　$\mathcal{SCF}$-*result.electoral-module m* **and**
　　profile V A p
　shows *elect m V A p* ∪ *defer m V A p* = *A* − (*reject m V A p*)
⟨*proof*⟩

lemma *defer-not-elec-or-rej*:
　fixes
　　m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
　　A :: ′*a set* **and**
　　p :: (′*a*, ′*v*) *Profile*
　assumes
　　$\mathcal{SCF}$-*result.electoral-module m* **and**
　　profile V A p
　shows *defer m V A p* = *A* − (*elect m V A p*) − (*reject m V A p*)
⟨*proof*⟩

lemma *electoral-mod-defer-elem*:
　fixes
　　m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
　　A :: ′*a set* **and**
　　V :: ′*v set* **and**
　　p :: (′*a*, ′*v*) *Profile* **and**
　　a :: ′*a*
　assumes
　　$\mathcal{SCF}$-*result.electoral-module m* **and**

   profile V A p **and**
   a ∈ A **and**
   a ∉ elect m V A p **and**
   a ∉ reject m V A p
  shows *a ∈ defer m V A p*
  ⟨*proof*⟩

lemma *mod-contains-result-comm*:
  fixes
   m n :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
   A :: ′*a set* **and**
   V :: ′*v set* **and**
   p :: (′*a*, ′*v*) *Profile* **and**
   a :: ′*a*
  assumes *mod-contains-result m n V A p a*
  shows *mod-contains-result n m V A p a*
⟨*proof*⟩

lemma *not-rej-imp-elec-or-defer*:
  fixes
   m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
   A :: ′*a set* **and**
   V :: ′*v set* **and**
   p :: (′*a*, ′*v*) *Profile* **and**
   a :: ′*a*
  assumes
   $\mathcal{SCF}$-*result.electoral-module m* **and**
   profile V A p **and**
   a ∈ A **and**
   a ∉ reject m V A p
  shows *a ∈ elect m V A p ∨ a ∈ defer m V A p*
  ⟨*proof*⟩

lemma *single-elim-imp-red-def-set*:
  fixes
   m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**
   A :: ′*a set* **and**
   V :: ′*v set* **and**
   p :: (′*a*, ′*v*) *Profile*
  assumes
   eliminates 1 m **and**
   card A > 1 **and**
   profile V A p
  shows *defer m V A p ⊂ A*
  ⟨*proof*⟩

lemma *eq-alts-in-profs-imp-eq-results*:
  fixes
   m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **and**

   A :: *′a set* **and**
   V :: *′v set* **and**
   p q :: (*′a, ′v*) *Profile*
 assumes
   eq: ∀ *a* ∈ *A. prof-contains-result m V A p q a* **and**
   mod-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
   prof-p: *profile V A p* **and**
   prof-q: *profile V A q*
 shows *m V A p = m V A q*
⟨*proof*⟩

lemma *eq-def-and-elect-imp-eq*:
 fixes
   m n :: (*′a, ′v, ′a Result*) *Electoral-Module* **and**
   A :: *′a set* **and**
   V :: *′v set* **and**
   p q :: (*′a, ′v*) *Profile*
 assumes
   mod-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
   mod-n: $\mathcal{SCF}$-*result.electoral-module n* **and**
   fin-p: *profile V A p* **and**
   fin-q: *profile V A q* **and**
   elec-eq: *elect m V A p = elect n V A q* **and**
   def-eq: *defer m V A p = defer n V A q*
 shows *m V A p = n V A q*
⟨*proof*⟩

4.4.8 Non-Blocking

An electoral module is non-blocking iff this module never rejects all alternatives.

definition *non-blocking* :: (*′a, ′v, ′a Result*) *Electoral-Module* ⇒ *bool* **where**
 non-blocking m ≡
  $\mathcal{SCF}$-*result.electoral-module m* ∧
   (∀ *A V p*. ((*A* ≠ {} ∧ *finite A* ∧ *profile V A p*) ⟶ *reject m V A p* ≠ *A*))

4.4.9 Electing

An electoral module is electing iff it always elects at least one alternative.

definition *electing* :: (*′a, ′v, ′a Result*) *Electoral-Module* ⇒ *bool* **where**
 electing m ≡
  $\mathcal{SCF}$-*result.electoral-module m* ∧
   (∀ *A V p*. (*A* ≠ {} ∧ *finite A* ∧ *profile V A p*) ⟶ *elect m V A p* ≠ {})

lemma *electing-for-only-alt*:
 fixes
   m :: (*′a, ′v, ′a Result*) *Electoral-Module* **and**
   A :: *′a set* **and**

   V :: *'v set* **and**
   p :: (*'a, 'v*) *Profile*
 assumes
  one-alt: *card A = 1* **and**
  electing: *electing m* **and**
  prof: *profile V A p*
 shows *elect m V A p = A*
⟨*proof*⟩

theorem *electing-imp-non-blocking*:
 fixes *m* :: (*'a, 'v, 'a Result*) *Electoral-Module*
 assumes *electing m*
 shows *non-blocking m*
⟨*proof*⟩

4.4.10 Properties

An electoral module is non-electing iff it never elects an alternative.

definition *non-electing* :: (*'a, 'v, 'a Result*) *Electoral-Module* ⇒ *bool* **where**
 non-electing m ≡
  $\mathcal{SCF}$-*result.electoral-module m*
   ∧ (∀ *A V p*. *profile V A p* ⟶ *elect m V A p* = {})

lemma *single-rej-decr-def-card*:
 fixes
  m :: (*'a, 'v, 'a Result*) *Electoral-Module* **and**
  A :: *'a set* **and**
  V :: *'v set* **and**
  p :: (*'a, 'v*) *Profile*
 assumes
  rejecting: *rejects 1 m* **and**
  non-electing: *non-electing m* **and**
  f-prof: *finite-profile V A p*
 shows *card* (*defer m V A p*) = *card A − 1*
⟨*proof*⟩

lemma *single-elim-decr-def-card'*:
 fixes
  m :: (*'a, 'v, 'a Result*) *Electoral-Module* **and**
  A :: *'a set* **and**
  V :: *'v set* **and**
  p :: (*'a, 'v*) *Profile*
 assumes
  eliminating: *eliminates 1 m* **and**
  non-electing: *non-electing m* **and**
  not-empty: *card A > 1* **and**
  prof-p: *profile V A p*
 shows *card* (*defer m V A p*) = *card A − 1*
⟨*proof*⟩

An electoral module is defer-deciding iff this module chooses exactly 1 alternative to defer and rejects any other alternative. Note that 'rejects n-1 m' can be omitted due to the well-formedness property.

definition *defer-deciding* :: (′a, ′v, ′a Result) Electoral-Module ⇒ bool **where**
  defer-deciding m ≡
    SCF-result.electoral-module m ∧ non-electing m ∧ defers 1 m

An electoral module decrements iff this module rejects at least one alternative whenever possible ($|A| > 1$).

definition *decrementing* :: (′a, ′v, ′a Result) Electoral-Module ⇒ bool **where**
  decrementing m ≡
    SCF-result.electoral-module m ∧
      (∀ A V p. profile V A p ∧ card A > 1 ⟶ card (reject m V A p) ≥ 1)

definition *defer-condorcet-consistency* :: (′a, ′v, ′a Result)
      Electoral-Module ⇒ bool **where**
  defer-condorcet-consistency m ≡
    SCF-result.electoral-module m ∧
    (∀ A V p a. condorcet-winner V A p a ⟶
    (m V A p = ({}, A − (defer m V A p), {d ∈ A. condorcet-winner V A p d})))

definition *condorcet-compatibility* :: (′a, ′v, ′a Result)
      Electoral-Module ⇒ bool **where**
  condorcet-compatibility m ≡
    SCF-result.electoral-module m ∧
    (∀ A V p a. condorcet-winner V A p a ⟶
      (a ∉ reject m V A p ∧
        (∀ b. ¬ condorcet-winner V A p b ⟶ b ∉ elect m V A p) ∧
          (a ∈ elect m V A p ⟶
            (∀ b ∈ A. ¬ condorcet-winner V A p b ⟶ b ∈ reject m V A p))))

An electoral module is defer-monotone iff, when a deferred alternative is lifted, this alternative remains deferred.

definition *defer-monotonicity* :: (′a, ′v, ′a Result) Electoral-Module ⇒ bool **where**
  defer-monotonicity m ≡
    SCF-result.electoral-module m ∧
      (∀ A V p q a.
        (a ∈ defer m V A p ∧ lifted V A p q a) ⟶ a ∈ defer m V A q)

An electoral module is defer-lift-invariant iff lifting a deferred alternative does not affect the outcome.

definition *defer-lift-invariance* :: (′a, ′v, ′a Result) Electoral-Module ⇒ bool **where**
  defer-lift-invariance m ≡
    SCF-result.electoral-module m ∧
      (∀ A V p q a. (a ∈ (defer m V A p) ∧ lifted V A p q a)
              ⟶ m V A p = m V A q)

fun *dli-rel* :: (′a, ′v, ′a Result) Electoral-Module ⇒ (′a, ′v) Election rel **where**

dli-rel m = {((A, V, p), (A, V, q)) |A V p q. (∃ a ∈ defer m V A p. lifted V A p q a)}

lemma *rewrite-dli-as-invariance*:
  fixes *m* :: (*′a, ′v, ′a Result*) *Electoral-Module*
  shows
    defer-lift-invariance m =
      (*SCF-result.electoral-module m*
          ∧ (*is-symmetry* (*fun$_\mathcal{E}$ m*) (*Invariance* (*dli-rel m*)))))
⟨*proof*⟩

Two electoral modules are disjoint-compatible if they only make decisions over disjoint sets of alternatives. Electoral modules reject alternatives for which they make no decision.

definition *disjoint-compatibility* :: (*′a, ′v, ′a Result*) *Electoral-Module* ⇒
      (*′a, ′v, ′a Result*) *Electoral-Module* ⇒ *bool* **where**
  disjoint-compatibility m n ≡
    SCF-result.electoral-module m ∧ *SCF-result.electoral-module n* ∧
      (∀ *V*.
        (∀ *A*.
          (∃ *B* ⊆ *A*.
            (∀ *a* ∈ *B. indep-of-alt m V A a* ∧
              (∀ *p. profile V A p* ⟶ *a* ∈ *reject m V A p*)) ∧
            (∀ *a* ∈ *A* − *B. indep-of-alt n V A a* ∧
              (∀ *p. profile V A p* ⟶ *a* ∈ *reject n V A p*)))))

Lifting an elected alternative a from an invariant-monotone electoral module either does not change the elect set, or makes a the only elected alternative.

definition *invariant-monotonicity* :: (*′a, ′v, ′a Result*)
      Electoral-Module ⇒ *bool* **where**
  invariant-monotonicity m ≡
    SCF-result.electoral-module m ∧
      (∀ *A V p q a.* (*a* ∈ *elect m V A p* ∧ *lifted V A p q a*) ⟶
        (*elect m V A q = elect m V A p* ∨ *elect m V A q = {a}*))

Lifting a deferred alternative a from a defer-invariant-monotone electoral module either does not change the defer set, or makes a the only deferred alternative.

definition *defer-invariant-monotonicity* :: (*′a, ′v, ′a Result*)
      Electoral-Module ⇒ *bool* **where**
  defer-invariant-monotonicity m ≡
    SCF-result.electoral-module m ∧ *non-electing m* ∧
      (∀ *A V p q a.* (*a* ∈ *defer m V A p* ∧ *lifted V A p q a*) ⟶
        (*defer m V A q = defer m V A p* ∨ *defer m V A q = {a}*))

4.4.11 Inference Rules

lemma *ccomp-and-dd-imp-def-only-winner*:

fixes
  $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
  $A :: 'a\ set$ **and**
  $V :: 'v\ set$ **and**
  $p :: ('a, 'v)\ Profile$ **and**
  $a :: 'a$
assumes
  ccomp: *condorcet-compatibility m* **and**
  dd: *defer-deciding m* **and**
  winner: *condorcet-winner V A p a*
shows *defer m V A p* = {*a*}
⟨*proof*⟩

theorem *ccomp-and-dd-imp-dcc*[*simp*]:
  fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  assumes
    ccomp: *condorcet-compatibility m* **and**
    dd: *defer-deciding m*
  shows *defer-condorcet-consistency m*
⟨*proof*⟩

If m and n are disjoint compatible, so are n and m.

theorem *disj-compat-comm*[*simp*]:
  fixes $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  assumes *disjoint-compatibility m n*
  shows *disjoint-compatibility n m*
⟨*proof*⟩

Every electoral module which is defer-lift-invariant is also defer-monotone.

theorem *dl-inv-imp-def-mono*[*simp*]:
  fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  assumes *defer-lift-invariance m*
  shows *defer-monotonicity m*
  ⟨*proof*⟩

4.4.12 Social-Choice Properties

Condorcet Consistency

definition *condorcet-consistency* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
      bool **where**
  condorcet-consistency m ≡
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \wedge$
    $(\forall\ A\ V\ p\ a.\ condorcet\text{-}winner\ V\ A\ p\ a \longrightarrow$
    $(m\ V\ A\ p = (\{e \in A.\ condorcet\text{-}winner\ V\ A\ p\ e\},\ A - (elect\ m\ V\ A\ p),\ \{\}))))$

lemma *condorcet-consistency-equiv*:
  fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  shows *condorcet-consistency m* =

$(\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \wedge$
$\qquad (\forall\ A\ V\ p\ a.\ condorcet\text{-}winner\ V\ A\ p\ a\ \longrightarrow$
$\qquad\quad (m\ V\ A\ p = (\{a\},\ A - (elect\ m\ V\ A\ p),\ \{\})))))$

$\langle proof \rangle$

lemma *condorcet-consistency-equiv′*:
  fixes $m :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  shows *condorcet-consistency* $m =$
          $(\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \wedge$
            $(\forall\ A\ V\ p\ a.$
              $condorcet\text{-}winner\ V\ A\ p\ a\ \longrightarrow\ m\ V\ A\ p = (\{a\},\ A - \{a\},\ \{\})))$

$\langle proof \rangle$

(Weak) Monotonicity

An electoral module is monotone iff when an elected alternative is lifted, this alternative remains elected.

definition *monotonicity* $::\ ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module \Rightarrow bool$ **where**
  $monotonicity\ m \equiv$
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m\ \wedge$
      $(\forall\ A\ V\ p\ q\ a.\ a \in elect\ m\ V\ A\ p\ \wedge\ lifted\ V\ A\ p\ q\ a\ \longrightarrow\ a \in elect\ m\ V\ A\ q)$

end

4.5 Electoral Module on Election Quotients

theory *Quotient-Module*
  imports *Quotients/Relation-Quotients*
          Electoral-Module
begin

lemma *invariance-is-congruence*:
  fixes
    $m :: ('a,\ 'v,\ 'r)\ Electoral\text{-}Module$ **and**
    $r :: ('a,\ 'v)\ Election\ rel$
  shows *is-symmetry* $(fun_\mathcal{E}\ m)\ (Invariance\ r) = fun_\mathcal{E}\ m\ respects\ r$
  $\langle proof \rangle$

lemma *invariance-is-congruence′*:
  fixes
    $f :: 'x \Rightarrow 'y$ **and**
    $r :: 'x\ rel$
  shows *is-symmetry* $f\ (Invariance\ r) = f\ respects\ r$
  $\langle proof \rangle$

theorem *pass-to-election-quotient*:

fixes
  m :: $('a, 'v, 'r)$ *Electoral-Module* **and**
  r :: $('a, 'v)$ *Election rel* **and**
  X :: $('a, 'v)$ *Election set*
assumes
  equiv X r **and**
  is-symmetry $(fun_{\mathcal{E}}\ m)$ $(Invariance\ r)$
shows $\forall\ A \in X\ //\ r.\ \forall\ E \in A.\ \pi_{\mathcal{Q}}\ (fun_{\mathcal{E}}\ m)\ A = fun_{\mathcal{E}}\ m\ E$
$\langle proof \rangle$

end

4.6 Evaluation Function

theory *Evaluation-Function*
  imports *Social-Choice-Types/Profile*
begin

This is the evaluation function. From a set of currently eligible alternatives, the evaluation function computes a numerical value that is then to be used for further (s)election, e.g., by the elimination module.

4.6.1 Definition

type-synonym $('a, 'v)$ *Evaluation-Function* =
  $'v\ set \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow ('a, 'v)\ Profile \Rightarrow enat$

4.6.2 Property

An Evaluation function is a Condorcet-rating iff the following holds: If a Condorcet Winner w exists, w and only w has the highest value.

definition *condorcet-rating* :: $('a, 'v)$ *Evaluation-Function* $\Rightarrow$ *bool* **where**
  condorcet-rating f $\equiv$
    $\forall\ A\ V\ p\ w$. *condorcet-winner* $V\ A\ p\ w \longrightarrow$
      $(\forall\ l \in A\ .\ l \neq w \longrightarrow f\ V\ l\ A\ p < f\ V\ w\ A\ p)$

An Evaluation function is dependent only on the participating voters iff it is invariant under profile changes that only impact non-voters.

fun *voters-determine-evaluation* :: $('a, 'v)$ *Evaluation-Function* $\Rightarrow$ *bool* **where**
  voters-determine-evaluation f =
    $(\forall\ A\ V\ p\ p'.\ (\forall\ v \in V.\ p\ v = p'\ v) \longrightarrow (\forall\ a \in A.\ f\ V\ a\ A\ p = f\ V\ a\ A\ p'))$

4.6.3 Theorems

If e is Condorcet-rating, the following holds: If a Condorcet winner w exists, w has the maximum evaluation value.

theorem *cond-winner-imp-max-eval-val*:
  fixes
    e :: *($'a$, $'v$) Evaluation-Function* **and**
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: *($'a$, $'v$) Profile* **and**
    a :: $'a$
  assumes
    rating: *condorcet-rating e* **and**
    f-prof: *finite-profile V A p* **and**
    winner: *condorcet-winner V A p a*
  shows *e V a A p = Max {e V b A p | b. b ∈ A}*
⟨*proof*⟩

If e is Condorcet-rating, the following holds: If a Condorcet Winner w exists, a non-Condorcet winner has a value lower than the maximum evaluation value.

theorem *non-cond-winner-not-max-eval*:
  fixes
    e :: *($'a$, $'v$) Evaluation-Function* **and**
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: *($'a$, $'v$) Profile* **and**
    a b :: $'a$
  assumes
    rating: *condorcet-rating e* **and**
    f-prof: *finite-profile V A p* **and**
    winner: *condorcet-winner V A p a* **and**
    lin-A: *b ∈ A* **and**
    loser: *a ≠ b*
  shows *e V b A p < Max {e V c A p | c. c ∈ A}*
⟨*proof*⟩

end

4.7 Elimination Module

theory *Elimination-Module*
  imports *Evaluation-Function*
      Electoral-Module
begin

This is the elimination module. It rejects a set of alternatives only if these are not all alternatives. The alternatives potentially to be rejected are put in a so-called elimination set. These are all alternatives that score below a preset threshold value that depends on the specific voting rule.

4.7.1   General Definitions

type-synonym *Threshold-Value = enat*

type-synonym *Threshold-Relation = enat ⇒ enat ⇒ bool*

type-synonym *('a, 'v) Electoral-Set = 'v set ⇒ 'a set ⇒ ('a, 'v) Profile ⇒ 'a set*

fun *elimination-set :: ('a, 'v) Evaluation-Function ⇒ Threshold-Value ⇒*
     Threshold-Relation ⇒ ('a, 'v) Electoral-Set **where**
  elimination-set e t r V A p = (if finite A then {a ∈ A . r (e V a A p) t} else {})

fun *average :: ('a, 'v) Evaluation-Function ⇒ 'v set ⇒ 'a set ⇒ ('a, 'v) Profile ⇒*
     Threshold-Value **where**
  average e V A p = (let sum-eval = ($\sum$ x ∈ A. e V x A p) in
                  if sum-eval = ∞ then ∞ else the-enat sum-eval div card A)

4.7.2   Social-Choice Definitions

fun *elimination-module :: ('a, 'v) Evaluation-Function ⇒ Threshold-Value ⇒*
     Threshold-Relation ⇒ ('a, 'v, 'a Result) Electoral-Module **where**
  elimination-module e t r V A p =
     (if elimination-set e t r V A p ≠ A
       then ({}, elimination-set e t r V A p, A − elimination-set e t r V A p)
       else ({}, {}, A))

4.7.3   Social-Choice Eliminators

fun *less-eliminator :: ('a, 'v) Evaluation-Function ⇒ Threshold-Value ⇒*
     ('a, 'v, 'a Result) Electoral-Module **where**
  less-eliminator e t V A p = elimination-module e t (<) V A p

fun *max-eliminator :: ('a, 'v) Evaluation-Function ⇒*
     ('a, 'v, 'a Result) Electoral-Module **where**
  max-eliminator e V A p =
   less-eliminator e (Max {e V x A p | x. x ∈ A}) V A p

fun *leq-eliminator :: ('a, 'v) Evaluation-Function ⇒ Threshold-Value ⇒*
     ('a, 'v, 'a Result) Electoral-Module **where**
  leq-eliminator e t V A p = elimination-module e t (≤) V A p

fun *min-eliminator :: ('a, 'v) Evaluation-Function ⇒*
     ('a, 'v, 'a Result) Electoral-Module **where**
  min-eliminator e V A p =

leq-eliminator e (Min {e V x A p | x. x ∈ A}) V A p

fun *less-average-eliminator* :: (*'a*, *'v*) *Evaluation-Function* ⇒
     (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
 less-average-eliminator e V A p = less-eliminator e (average e V A p) V A p

fun *leq-average-eliminator* :: (*'a*, *'v*) *Evaluation-Function* ⇒
     (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
 leq-average-eliminator e V A p = leq-eliminator e (average e V A p) V A p

4.7.4 Soundness

lemma *elim-mod-sound*[*simp*]:
  fixes
    e :: (*'a*, *'v*) *Evaluation-Function* **and**
    t :: *Threshold-Value* **and**
    r :: *Threshold-Relation*
  shows $\mathcal{SCF}$*-result.electoral-module (elimination-module e t r)*
  ⟨*proof*⟩

lemma *less-elim-sound*[*simp*]:
  fixes
    e :: (*'a*, *'v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows $\mathcal{SCF}$*-result.electoral-module (less-eliminator e t)*
  ⟨*proof*⟩

lemma *leq-elim-sound*[*simp*]:
  fixes
    e :: (*'a*, *'v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows $\mathcal{SCF}$*-result.electoral-module (leq-eliminator e t)*
  ⟨*proof*⟩

lemma *max-elim-sound*[*simp*]:
  fixes *e* :: (*'a*, *'v*) *Evaluation-Function*
  shows $\mathcal{SCF}$*-result.electoral-module (max-eliminator e)*
  ⟨*proof*⟩

lemma *min-elim-sound*[*simp*]:
  fixes *e* :: (*'a*, *'v*) *Evaluation-Function*
  shows $\mathcal{SCF}$*-result.electoral-module (min-eliminator e)*
  ⟨*proof*⟩

lemma *less-avg-elim-sound*[*simp*]:
  fixes *e* :: (*'a*, *'v*) *Evaluation-Function*
  shows $\mathcal{SCF}$*-result.electoral-module (less-average-eliminator e)*
  ⟨*proof*⟩

lemma *leq-avg-elim-sound*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows $\mathcal{SCF}$-*result.electoral-module* (*leq-average-eliminator e*)
  ⟨*proof*⟩

4.7.5 Independence of Non-Voters

lemma *voters-determine-elim-mod*[*simp*]:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value* **and**
    r :: *Threshold-Relation*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*elimination-module e t r*)
⟨*proof*⟩

lemma *voters-determine-less-elim*[*simp*]:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*less-eliminator e t*)
  ⟨*proof*⟩

lemma *voters-determine-leq-elim*[*simp*]:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*leq-eliminator e t*)
  ⟨*proof*⟩

lemma *voters-determine-max-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*max-eliminator e*)
⟨*proof*⟩

lemma *voters-determine-min-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*min-eliminator e*)
⟨*proof*⟩

lemma *voters-determine-less-avg-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*less-average-eliminator e*)
⟨*proof*⟩

lemma *voters-determine-leq-avg-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *voters-determine-evaluation e*
  shows *voters-determine-election* (*leq-average-eliminator e*)
⟨*proof*⟩

4.7.6   Non-Blocking

lemma *elim-mod-non-blocking*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value* **and**
    r :: *Threshold-Relation*
  shows *non-blocking* (*elimination-module e t r*)
  ⟨*proof*⟩

lemma *less-elim-non-blocking*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows *non-blocking* (*less-eliminator e t*)
  ⟨*proof*⟩

lemma *leq-elim-non-blocking*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows *non-blocking* (*leq-eliminator e t*)
  ⟨*proof*⟩

lemma *max-elim-non-blocking*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-blocking* (*max-eliminator e*)
  ⟨*proof*⟩

lemma *min-elim-non-blocking*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-blocking* (*min-eliminator e*)
  ⟨*proof*⟩

lemma *less-avg-elim-non-blocking*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-blocking* (*less-average-eliminator e*)
  ⟨*proof*⟩

lemma *leq-avg-elim-non-blocking*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-blocking* (*leq-average-eliminator e*)

⟨*proof*⟩

4.7.7 Non-Electing

lemma *elim-mod-non-electing*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value* **and**
    r :: *Threshold-Relation*
  shows *non-electing* (*elimination-module e t r*)
  ⟨*proof*⟩

lemma *less-elim-non-electing*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows *non-electing* (*less-eliminator e t*)
  ⟨*proof*⟩

lemma *leq-elim-non-electing*:
  fixes
    e :: (′*a*, ′*v*) *Evaluation-Function* **and**
    t :: *Threshold-Value*
  shows *non-electing* (*leq-eliminator e t*)
  ⟨*proof*⟩

lemma *max-elim-non-electing*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-electing* (*max-eliminator e*)
  ⟨*proof*⟩

lemma *min-elim-non-electing*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-electing* (*min-eliminator e*)
  ⟨*proof*⟩

lemma *less-avg-elim-non-electing*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-electing* (*less-average-eliminator e*)
  ⟨*proof*⟩

lemma *leq-avg-elim-non-electing*:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  shows *non-electing* (*leq-average-eliminator e*)
  ⟨*proof*⟩

4.7.8   Inference Rules

If the used evaluation function is Condorcet rating, max-eliminator is Condorcet compatible.

theorem *cr-eval-imp-ccomp-max-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *condorcet-rating e*
  shows *condorcet-compatibility* (*max-eliminator e*)
⟨*proof*⟩

If the used evaluation function is Condorcet rating, max-eliminator is defer-Condorcet-consistent.

theorem *cr-eval-imp-dcc-max-elim*[*simp*]:
  fixes *e* :: (′*a*, ′*v*) *Evaluation-Function*
  assumes *condorcet-rating e*
  shows *defer-condorcet-consistency* (*max-eliminator e*)
⟨*proof*⟩

end

4.8   Aggregator

theory *Aggregator*
  imports *Social-Choice-Types/Social-Choice-Result*
begin

An aggregator gets two partitions (results of electoral modules) as input and output another partition. They are used to aggregate results of parallel composed electoral modules. They are commutative, i.e., the order of the aggregated modules does not affect the resulting aggregation. Moreover, they are conservative in the sense that the resulting decisions are subsets of the two given partitions' decisions.

4.8.1   Definition

type-synonym ′*a Aggregator* = ′*a set* ⇒ ′*a Result* ⇒ ′*a Result* ⇒ ′*a Result*

definition *aggregator* :: ′*a Aggregator* ⇒ *bool* **where**
  aggregator agg ≡
    ∀ *A e e*′ *d d*′ *r r*′.
      (*well-formed-SCF A* (*e*, *r*, *d*) ∧ *well-formed-SCF A* (*e*′, *r*′, *d*′)) ⟶
      well-formed-SCF A (*agg A* (*e*, *r*, *d*) (*e*′, *r*′, *d*′))

4.8.2 Properties

definition *agg-commutative* :: *$'a$ Aggregator $\Rightarrow$ bool* **where**
  agg-commutative agg $\equiv$
    aggregator agg $\wedge$ ($\forall$ A e e$'$ d d$'$ r r$'$.
      agg A (e, r, d) (e$'$, r$'$, d$'$) = agg A (e$'$, r$'$, d$'$) (e, r, d))

definition *agg-conservative* :: *$'a$ Aggregator $\Rightarrow$ bool* **where**
  agg-conservative agg $\equiv$
    aggregator agg $\wedge$
    ($\forall$ A e e$'$ d d$'$ r r$'$.
      ((well-formed-$\mathcal{SCF}$ A (e, r, d) $\wedge$ well-formed-$\mathcal{SCF}$ A (e$'$, r$'$, d$'$)) $\longrightarrow$
        elect-r (agg A (e, r, d) (e$'$, r$'$, d$'$)) $\subseteq$ (e $\cup$ e$'$) $\wedge$
        reject-r (agg A (e, r, d) (e$'$, r$'$, d$'$)) $\subseteq$ (r $\cup$ r$'$) $\wedge$
        defer-r (agg A (e, r, d) (e$'$, r$'$, d$'$)) $\subseteq$ (d $\cup$ d$'$)))

end

4.9 Maximum Aggregator

theory *Maximum-Aggregator*
  imports *Aggregator*
begin

The max(imum) aggregator takes two partitions of an alternative set A as input. It returns a partition where every alternative receives the maximum result of the two input partitions.

4.9.1 Definition

fun *max-aggregator* :: *$'a$ Aggregator* **where**
  max-aggregator A (e, r, d) (e$'$, r$'$, d$'$) =
    (e $\cup$ e$'$,
      A $-$ (e $\cup$ e$'$ $\cup$ d $\cup$ d$'$),
      (d $\cup$ d$'$) $-$ (e $\cup$ e$'$))

4.9.2 Auxiliary Lemma

lemma *max-agg-rej-set*:
  fixes
    A e e$'$ d d$'$ r r$'$ :: $'a$ set **and**
    a :: $'a$
  assumes
    wf-first-mod: *well-formed-$\mathcal{SCF}$ A (e, r, d)* **and**
    wf-second-mod: *well-formed-$\mathcal{SCF}$ A (e$'$, r$'$, d$'$)*

shows *reject-r (max-aggregator A (e, r, d) (e′, r′, d′)) = r ∩ r′*
⟨*proof*⟩

4.9.3 Soundness

theorem *max-agg-sound[simp]: aggregator max-aggregator*
⟨*proof*⟩

4.9.4 Properties

The max-aggregator is conservative.

theorem *max-agg-consv[simp]: agg-conservative max-aggregator*
⟨*proof*⟩

The max-aggregator is commutative.

theorem *max-agg-comm[simp]: agg-commutative max-aggregator*
  ⟨*proof*⟩

end

4.10   Termination Condition

theory *Termination-Condition*
  imports *Social-Choice-Types/Result*
begin

The termination condition is used in loops. It decides whether or not to terminate the loop after each iteration, depending on the current state of the loop.

type-synonym *′r Termination-Condition = ′r Result ⇒ bool*

end

4.11   Defer Equal Condition

theory *Defer-Equal-Condition*
  imports *Termination-Condition*
begin

This is a family of termination conditions. For a natural number n, the according defer-equal condition is true if and only if the given result's defer-set contains exactly n elements.

fun *defer-equal-condition* :: *nat* $\Rightarrow$ $'a$ *Termination-Condition* **where**
  defer-equal-condition n $(e,\ r,\ d) = (card\ d = n)$

end

Chapter 5

Basic Modules

5.1 Defer Module

theory *Defer-Module*
  imports *Component-Types/Electoral-Module*
begin

The defer module is not concerned about the voter's ballots, and simply
defers all alternatives. It is primarily used for defining an empty loop.

5.1.1 Definition
fun *defer-module* :: ($'a$, $'v$, $'a$ Result) *Electoral-Module* **where**
  defer-module V A p = ({}, {}, *A*)

5.1.2 Soundness
theorem *def-mod-sound*[*simp*]: $\mathcal{SCF}$-*result.electoral-module defer-module*
  ⟨*proof*⟩

5.1.3 Properties
theorem *def-mod-non-electing*: *non-electing defer-module*
  ⟨*proof*⟩

theorem *def-mod-def-lift-inv*: *defer-lift-invariance defer-module*
  ⟨*proof*⟩

end

5.2 Elect-First Module

theory *Elect-First-Module*

imports *Component-Types/Electoral-Module*
begin

The elect first module elects the alternative that is most preferred on the first ballot and rejects all other alternatives.

5.2.1 Definition

fun *least* :: $'v$ :: *wellorder set* $\Rightarrow$ $'v$ **where**
  least V = (*Least* (λ *v. v* $\in$ *V*))

fun *elect-first-module* :: ($'a$, $'v$ :: *wellorder*, $'a$ *Result*) *Electoral-Module* **where**
  elect-first-module V A p =
    ({$a \in A$. *above* (*p* (*least V*)) $a = \{a\}$},
     {$a \in A$. *above* (*p* (*least V*)) $a \neq \{a\}$},
     {})

5.2.2 Soundness

theorem *elect-first-mod-sound*: $\mathcal{SCF}$-*result.electoral-module elect-first-module*
⟨*proof*⟩

end

5.3 Consensus Class

theory *Consensus-Class*
  imports *Consensus*
        ../Defer-Module
        ../Elect-First-Module
begin

A consensus class is a pair of a set of elections and a mapping that assigns a unique alternative to each election in that set (of elections). This alternative is then called the consensus alternative (winner). Here, we model the mapping by an electoral module that defers alternatives which are not in the consensus.

5.3.1 Definition

type-synonym ($'a$, $'v$, $'r$) *Consensus-Class* =
        ($'a$, $'v$) *Consensus* $\times$ ($'a$, $'v$, $'r$) *Electoral-Module*

fun *consensus-$\mathcal{K}$* :: ($'a$, $'v$, $'r$) *Consensus-Class* $\Rightarrow$ ($'a$, $'v$) *Consensus* **where**
  consensus-$\mathcal{K}$ K = fst K

fun *rule-$\mathcal{K}$* :: *($'a$, $'v$, $'r$) Consensus-Class $\Rightarrow$ ($'a$, $'v$, $'r$) Electoral-Module* **where**
  rule-$\mathcal{K}$ K = snd K

5.3.2 Consensus Choice

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

fun $\mathcal{K}_\mathcal{E}$ :: *($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$ $'r$ $\Rightarrow$ ($'a$, $'v$) Election set* **where**
  $\mathcal{K}_\mathcal{E}$ *K w =*
    {(A, V, p) | A V p. (consensus-$\mathcal{K}$ K) (A, V, p) $\wedge$ finite-profile V A p
          $\wedge$ elect (rule-$\mathcal{K}$ K) V A p = {w}}

fun *elections-$\mathcal{K}$* :: *($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$ ($'a$, $'v$) Election set* **where**
  elections-$\mathcal{K}$ K = $\bigcup$ (($\mathcal{K}_\mathcal{E}$ K) ' UNIV)

A consensus class is deemed well-formed if the result of its mapping is completely determined by its consensus, the elected set of the electoral module's result.

definition *well-formed* :: *($'a$, $'v$) Consensus $\Rightarrow$ ($'a$, $'v$, $'r$) Electoral-Module $\Rightarrow$*
      bool **where**
  well-formed c m $\equiv$
    $\forall$ A V V$'$ p p$'$.
      profile V A p $\wedge$ profile V$'$ A p$'$ $\wedge$ c (A, V, p) $\wedge$ c (A, V$'$, p$'$)
          $\longrightarrow$ m V A p = m V$'$ A p$'$

A sensible social choice rule for a given arbitrary consensus and social choice rule r is the one that chooses the result of r for all consensus elections and defers all candidates otherwise.

fun *consensus-choice* :: *($'a$, $'v$) Consensus $\Rightarrow$*
      ($'a$, $'v$, $'a$ Result) Electoral-Module $\Rightarrow$
      ($'a$, $'v$, $'a$ Result) Consensus-Class **where**
  consensus-choice c m =
    (let
      w = (λ V A p. if c (A, V, p) then m V A p else defer-module V A p)
      in (c, w))

5.3.3 Auxiliary Lemmas

lemma *unanimity$'$-consensus-imp-elect-fst-mod-well-formed*:
  fixes *a* :: $'a$
  shows *well-formed*
    (λ c. nonempty-set$_\mathcal{C}$ c $\wedge$ nonempty-profile$_\mathcal{C}$ c
          $\wedge$ equal-top$_\mathcal{C}$$'$ a c) elect-first-module
$\langle proof \rangle$

lemma *strong-unanimity$'$consensus-imp-elect-fst-mod-completely-determined*:
  fixes *r* :: $'a$ *Preference-Relation*

shows *well-formed*
$(\lambda\ c.\ nonempty\text{-}set_{\mathcal{C}}\ c \wedge nonempty\text{-}profile_{\mathcal{C}}\ c \wedge equal\text{-}vote_{\mathcal{C}}{}'\ r\ c)\ elect\text{-}first\text{-}module$
$\langle proof \rangle$

lemma *strong-unanimity$'$consensus-imp-elect-fst-mod-well-formed*:
  fixes $r :: {}'a\ Preference\text{-}Relation$
  shows *well-formed*
    $(\lambda\ c.\ nonempty\text{-}set_{\mathcal{C}}\ c \wedge nonempty\text{-}profile_{\mathcal{C}}\ c$
        $\wedge\ equal\text{-}vote_{\mathcal{C}}{}'\ r\ c)\ elect\text{-}first\text{-}module$
  $\langle proof \rangle$

lemma *cons-domain-valid*:
  fixes $C :: ({}'a,\ {}'v,\ {}'r\ Result)\ Consensus\text{-}Class$
  shows *elections-$\mathcal{K}$ $C \subseteq$ well-formed-elections*
$\langle proof \rangle$

lemma *cons-domain-finite*:
  fixes $C :: ({}'a,\ {}'v,\ {}'r\ Result)\ Consensus\text{-}Class$
  shows
    finite: *elections-$\mathcal{K}$ $C \subseteq$ finite-elections* **and**
    finite-voters: *elections-$\mathcal{K}$ $C \subseteq$ finite-elections-$\mathcal{V}$*
$\langle proof \rangle$

5.3.4   Consensus Rules

definition *non-empty-set* $:: ({}'a,\ {}'v,\ {}'r)\ Consensus\text{-}Class \Rightarrow bool$ **where**
  non-empty-set $c \equiv \exists\ K.\ consensus\text{-}\mathcal{K}\ c\ K$

Unanimity condition.

definition *unanimity* $:: ({}'a,\ {}'v :: wellorder,\ {}'a\ Result)\ Consensus\text{-}Class$ **where**
  unanimity $\equiv consensus\text{-}choice\ unanimity_{\mathcal{C}}\ elect\text{-}first\text{-}module$

Strong unanimity condition.

definition *strong-unanimity* $:: ({}'a,\ {}'v :: wellorder,\ {}'a\ Result)\ Consensus\text{-}Class$
where
  strong-unanimity $\equiv consensus\text{-}choice\ strong\text{-}unanimity_{\mathcal{C}}\ elect\text{-}first\text{-}module$

5.3.5   Properties

definition *consensus-rule-anonymity* $:: ({}'a,\ {}'v,\ {}'r)\ Consensus\text{-}Class \Rightarrow bool$ **where**
  consensus-rule-anonymity $c \equiv$
    $(\forall\ A\ V\ p\ \pi :: ({}'v \Rightarrow {}'v).$
      $bij\ \pi \longrightarrow$
        $(let\ (A',\ V',\ q) = (rename\ \pi\ (A,\ V,\ p))\ in$
          $profile\ V\ A\ p \longrightarrow profile\ V'\ A'\ q$
          $\longrightarrow consensus\text{-}\mathcal{K}\ c\ (A,\ V,\ p)$
          $\longrightarrow (consensus\text{-}\mathcal{K}\ c\ (A',\ V',\ q) \wedge (rule\text{-}\mathcal{K}\ c\ V\ A\ p = rule\text{-}\mathcal{K}\ c\ V'\ A'\ q))))$

fun *consensus-rule-anonymity$'$* $:: ({}'a,\ {}'v)\ Election\ set \Rightarrow$

$('a, 'v, 'r\ Result)$ *Consensus-Class* $\Rightarrow$ *bool* **where**
  consensus-rule-anonymity′ X C =
    is-symmetry $(elect\text{-}r \circ fun_{\mathcal{E}}\ (rule\text{-}\mathcal{K}\ C))$ $(Invariance\ (anonymity_{\mathcal{R}}\ X))$

fun (**in** *result-properties*) *consensus-rule-neutrality* :: $('a, 'v)$ *Election set* $\Rightarrow$
    $('a, 'v, 'b\ Result)$ *Consensus-Class* $\Rightarrow$ *bool* **where**
  consensus-rule-neutrality X C =
    is-symmetry $(elect\text{-}r \circ fun_{\mathcal{E}}\ (rule\text{-}\mathcal{K}\ C))$
      $(action\text{-}induced\text{-}equivariance\ (carrier\ bijection_{\mathcal{AG}})\ X\ (\varphi\text{-}neutral\ X)\ (set\text{-}action$
$\psi))$

fun *consensus-rule-reversal-symmetry* :: $('a, 'v)$ *Election set* $\Rightarrow$
    $('a, 'v, 'a\ rel\ Result)$ *Consensus-Class* $\Rightarrow$ *bool* **where**
  consensus-rule-reversal-symmetry X C = is-symmetry $(elect\text{-}r \circ fun_{\mathcal{E}}\ (rule\text{-}\mathcal{K}\ C))$
      $(action\text{-}induced\text{-}equivariance\ (carrier\ reversal_{\mathcal{G}})\ X\ (\varphi\text{-}reverse\ X)\ (set\text{-}action$
$\psi\text{-}reverse))$

5.3.6   Inference Rules

lemma *if-else-cons-equivar*:
  fixes
    $m\ n :: ('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    $c :: ('a, 'v)$ *Consensus* **and**
    $G :: 'b\ set$ **and**
    $X :: ('a, 'v)$ *Election set* **and**
    $\varphi :: ('b, ('a, 'v)\ Election)$ *binary-fun* **and**
    $\psi :: ('b, 'a)$ *binary-fun* **and**
    $f :: 'a\ Result \Rightarrow 'a\ set$
  defines
    equivar $\equiv$ *action-induced-equivariance* $G\ X\ \varphi\ (set\text{-}action\ \psi)$ **and**
    if-else-cons $\equiv (c, (\lambda\ V\ A\ p.\ if\ c\ (A, V, p)\ then\ m\ V\ A\ p\ else\ n\ V\ A\ p))$
  assumes
    equivar-m: *is-symmetry* $(f \circ fun_{\mathcal{E}}\ m)$ *equivar* **and**
    equivar-n: *is-symmetry* $(f \circ fun_{\mathcal{E}}\ n)$ *equivar* **and**
    invar-cons: *is-symmetry* c $(Invariance\ (action\text{-}induced\text{-}rel\ G\ X\ \varphi))$
  shows *is-symmetry* $(f \circ fun_{\mathcal{E}}\ (rule\text{-}\mathcal{K}\ if\text{-}else\text{-}cons))$
          $(action\text{-}induced\text{-}equivariance\ G\ X\ \varphi\ (set\text{-}action\ \psi))$
$\langle proof \rangle$

lemma *consensus-choice-anonymous*:
  fixes
    $\alpha\ \beta :: ('a, 'v)$ *Consensus* **and**
    $m :: ('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    $\beta' :: 'b \Rightarrow ('a, 'v)$ *Consensus*
  assumes
    beta-sat: $\beta = (\lambda\ E.\ \exists\ a.\ \beta'\ a\ E)$ **and**
    beta′-anon: $\forall\ x.\ consensus\text{-}anonymity\ (\beta'\ x)$ **and**
    anon-cons-cond: *consensus-anonymity* α **and**
    conditions-univ: $\forall\ x.\ well\text{-}formed\ (\lambda\ E.\ \alpha\ E \wedge \beta'\ x\ E)\ m$

shows *consensus-rule-anonymity* (*consensus-choice* (λ *E*. α *E* $\wedge$ β *E*) *m*)
⟨*proof*⟩

5.3.7 Theorems

Anonymity

lemma *unanimity-anonymous*: *consensus-rule-anonymity unanimity*
⟨*proof*⟩

lemma *strong-unanimity-anonymous*: *consensus-rule-anonymity strong-unanimity*
⟨*proof*⟩

Neutrality

lemma *defer-winners-equivariant*:
  fixes
    G :: ′*b set* **and**
    E :: (′*a*, ′*v*) *Election set* **and**
    φ :: (′*b*, (′*a*, ′*v*) *Election*) *binary-fun* **and**
    ψ :: (′*b*, ′*a*) *binary-fun*
  shows *is-symmetry* (*elect-r* ∘ *fun$_\mathcal{E}$ defer-module*)
            (*action-induced-equivariance G E* φ (*set-action* ψ))
  ⟨*proof*⟩

lemma *elect-first-winners-neutral*: *is-symmetry* (*elect-r* ∘ *fun$_\mathcal{E}$ elect-first-module*)
            (*action-induced-equivariance* (*carrier bijection$_{\mathcal{AG}}$*)
              well-formed-elections (φ-*neutral well-formed-elections*)
                (*set-action* ψ-*neutral$_c$*))
⟨*proof*⟩

lemma *strong-unanimity-neutral*:
  defines *domain* ≡ *well-formed-elections* ∩ *Collect strong-unanimity$_\mathcal{C}$*
  — We want to show neutrality on a set as general as possible, as this implies
subset neutrality.
  shows $\mathcal{SCF}$-*properties.consensus-rule-neutrality domain strong-unanimity*
⟨*proof*⟩

lemma *strong-unanimity-neutral′*: $\mathcal{SCF}$-*properties.consensus-rule-neutrality*
    (*elections-$\mathcal{K}$ strong-unanimity*) *strong-unanimity*
⟨*proof*⟩

lemma *strong-unanimity-closed-under-neutrality*: *closed-restricted-rel*
        (*neutrality$_\mathcal{R}$ well-formed-elections*) *well-formed-elections*
          (*elections-$\mathcal{K}$ strong-unanimity*)
⟨*proof*⟩

end

5.4 Distance Rationalization

theory *Distance-Rationalization*
  imports *Social-Choice-Types/Refined-Types/Preference-List*
      Consensus-Class
      Distance
begin

A distance rationalization of a voting rule is its interpretation as a procedure that elects an uncontroversial winner if there is one, and otherwise elects the alternatives that are as close to becoming an uncontroversial winner as possible. Within general distance rationalization, a voting rule is characterized by a distance on profiles and a consensus class.

5.4.1 Definitions

Returns the distance of an election to the preimage of a unique winner under the given consensus elections and consensus rule.

fun *score* :: $('a, 'v)$ *Election Distance* $\Rightarrow$ $('a, 'v, 'r$ *Result*$)$ *Consensus-Class* $\Rightarrow$
    $('a, 'v)$ *Election* $\Rightarrow$ $'r$ $\Rightarrow$ *ereal* **where**
 score d K E w = Inf $(d\ E\ `\ (\mathcal{K}_{\mathcal{E}}\ K\ w))$

fun (**in** *result*) $\mathcal{R}_{\mathcal{W}}$ :: $('a, 'v)$ *Election Distance* $\Rightarrow$
    $('a, 'v, 'r$ *Result*$)$ *Consensus-Class* $\Rightarrow$ $'v$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$
    $'r$ *set* **where**
 $\mathcal{R}_{\mathcal{W}}$ *d K V A p = arg-min-set* $(score\ d\ K\ (A,\ V,\ p))$ $(limit\ A\ UNIV)$

fun (**in** *result*) *distance-$\mathcal{R}$* :: $('a, 'v)$ *Election Distance* $\Rightarrow$
    $('a, 'v, 'r$ *Result*$)$ *Consensus-Class* $\Rightarrow$
    $('a, 'v, 'r$ *Result*$)$ *Electoral-Module* **where**
 distance-$\mathcal{R}$ d K V A p =
  $(\mathcal{R}_{\mathcal{W}}\ d\ K\ V\ A\ p,\ (limit\ A\ UNIV) - \mathcal{R}_{\mathcal{W}}\ d\ K\ V\ A\ p,\ \{\})$

5.4.2 Standard Definitions

definition *standard* :: $('a, 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
 standard d $\equiv$
  $\forall\ A\ A'\ V\ V'\ p\ p'.\ (V \neq V' \vee A \neq A') \longrightarrow d\ (A,\ V,\ p)\ (A',\ V',\ p') = \infty$

definition *voters-determine-distance* :: $('a, 'v)$ *Election Distance* $\Rightarrow$ *bool* **where**
 voters-determine-distance d $\equiv$
  $\forall\ A\ A'\ V\ V'\ p\ q\ p'.$
   $(\forall\ v \in V.\ p\ v = q\ v)$
    $\longrightarrow (d\ (A,\ V,\ p)\ (A',\ V',\ p') = d\ (A,\ V,\ q)\ (A',\ V',\ p')$
     $\wedge\ (d\ (A',\ V',\ p')\ (A,\ V,\ p) = d\ (A',\ V',\ p')\ (A,\ V,\ q)))$

Creates a set of all possible profiles on a finite alternative set that are empty everywhere outside of a given finite voter set.

fun *profiles* :: *$'v$ set $\Rightarrow$ $'a$ set $\Rightarrow$ (($'a$, $'v$) Profile) set* **where**
  profiles V A =
    (if infinite A $\vee$ infinite V
      then {} else {p. p ' V $\subseteq$ pl-α ' permutations-of-set A})

fun $\mathcal{K_E}$*-std* :: *($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$ $'r$ $\Rightarrow$ $'a$ set $\Rightarrow$ $'v$ set $\Rightarrow$*
    ($'a$, $'v$) Election set **where**
  $\mathcal{K_E}$*-std K w A V =*
    (λ p. (A, V, p)) ' Set.filter
      (λ p. consensus-$\mathcal{K}$ K (A, V, p) $\wedge$ elect (rule-$\mathcal{K}$ K) V A p = {w})
      (profiles V A)

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

fun *score-std* :: *($'a$, $'v$) Election Distance $\Rightarrow$ ($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$*
    ($'a$, $'v$) Election $\Rightarrow$ $'r$ $\Rightarrow$ ereal **where**
  score-std d K E w =
    (if $\mathcal{K_E}$-std K w (alternatives-$\mathcal{E}$ E) (voters-$\mathcal{E}$ E) = {}
      then ∞ else Min (d E ' ($\mathcal{K_E}$-std K w (alternatives-$\mathcal{E}$ E) (voters-$\mathcal{E}$ E))))

fun (**in** *result*) $\mathcal{R_W}$*-std* :: *($'a$, $'v$) Election Distance $\Rightarrow$*
    ($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$ $'v$ set $\Rightarrow$ $'a$ set $\Rightarrow$ ($'a$, $'v$) Profile $\Rightarrow$
    $'r$ set **where**
  $\mathcal{R_W}$*-std d K V A p = arg-min-set (score-std d K (A, V, p)) (limit A UNIV)*

fun (**in** *result*) *distance-$\mathcal{R}$-std* :: *($'a$, $'v$) Election Distance $\Rightarrow$*
    ($'a$, $'v$, $'r$ Result) Consensus-Class $\Rightarrow$
    ($'a$, $'v$, $'r$ Result) Electoral-Module **where**
  distance-$\mathcal{R}$-std d K V A p =
    ($\mathcal{R_W}$-std d K V A p, (limit A UNIV) $-$ $\mathcal{R_W}$-std d K V A p, {})

5.4.3 Auxiliary Lemmas

lemma *fin-$\mathcal{K_E}$*:
  fixes *C* :: *($'a$, $'v$, $'r$ Result) Consensus-Class*
  shows *elections-$\mathcal{K}$ C $\subseteq$ finite-elections*
  $\langle$*proof*$\rangle$

lemma *univ-$\mathcal{K_E}$*:
  fixes *C* :: *($'a$, $'v$, $'r$ Result) Consensus-Class*
  shows *elections-$\mathcal{K}$ C $\subseteq$ UNIV*
  $\langle$*proof*$\rangle$

lemma *listset-finiteness*:
  fixes *l* :: *$'a$ set list*
  assumes *$\forall$ i :: nat. i < length l $\longrightarrow$ finite (l!i)*
  shows *finite (listset l)*
  $\langle$*proof*$\rangle$

lemma *ls-entries-empty-imp-ls-set-empty*:
  fixes $l :: \,'a\ set\ list$
  assumes
    $0 < length\ l$ **and**
    $\forall\ i :: nat.\ i < length\ l \longrightarrow l!i = \{\}$
  shows $listset\ l = \{\}$
  $\langle proof \rangle$

lemma *all-ls-elems-same-len*:
  fixes $l :: \,'a\ set\ list$
  shows $\forall\ l' :: \,'a\ list.\ l' \in listset\ l \longrightarrow length\ l' = length\ l$
$\langle proof \rangle$

lemma *fin-all-profs*:
  fixes
    $A :: \,'a\ set$ **and**
    $V :: \,'v\ set$ **and**
    $x :: \,'a\ Preference\text{-}Relation$
  assumes
    fin-A: $finite\ A$ **and**
    fin-V: $finite\ V$
  shows $finite\ (profiles\ V\ A \cap \{p.\ \forall\ v.\ v \notin V \longrightarrow p\ v = x\})$
$\langle proof \rangle$

lemma *profile-permutation-set*:
  fixes
    $A :: \,'a\ set$ **and**
    $V :: \,'v\ set$
  shows $profiles\ V\ A = \{p :: ('a,\ 'v)\ Profile.\ finite\text{-}profile\ V\ A\ p\}$
$\langle proof \rangle$

5.4.4   Soundness

lemma (**in** *result*) $\mathcal{R}$*-sound*:
  fixes
    $K :: ('a,\ 'v,\ 'r\ Result)\ Consensus\text{-}Class$ **and**
    $d :: ('a,\ 'v)\ Election\ Distance$
  shows $electoral\text{-}module\ (distance\text{-}\mathcal{R}\ d\ K)$
$\langle proof \rangle$

5.4.5   Properties

fun *distance-decisiveness* $:: ('a,\ 'v)\ Election\ set \Rightarrow ('a,\ 'v)\ Election\ Distance \Rightarrow$
    $('a,\ 'v,\ 'r\ Result)\ Electoral\text{-}Module \Rightarrow bool$ **where**
  $distance\text{-}decisiveness\ X\ d\ m =$
    $(\nexists\ E.\ E \in X$
    $\wedge\ (\exists\ \delta > 0.\ \forall\ E' \in X.\ d\ E\ E' < \delta \longrightarrow card\ (elect\text{-}r\ (fun_{\mathcal{E}}\ m\ E')) > 1))$

5.4.6 Inference Rules

lemma (**in** *result*) *standard-distance-imp-equal-score*:
  fixes
    d :: (*'a*, *'v*) *Election Distance* **and**
    K :: (*'a*, *'v*, *'r Result*) *Consensus-Class* **and**
    A :: *'a set* **and**
    V :: *'v set* **and**
    p :: (*'a*, *'v*) *Profile* **and**
    w :: *'r*
  assumes
    irr-non-V: *voters-determine-distance d* **and**
    std: *standard d*
  shows *score d K* (*A*, *V*, *p*) *w = score-std d K* (*A*, *V*, *p*) *w*
⟨*proof*⟩

lemma (**in** *result*) *anonymous-distance-and-consensus-imp-rule-anonymity*:
  fixes
    d :: (*'a*, *'v*) *Election Distance* **and**
    K :: (*'a*, *'v*, *'r Result*) *Consensus-Class*
  assumes
    d-anon: *distance-anonymity d* **and**
    K-anon: *consensus-rule-anonymity K*
  shows *anonymity* (*distance-ℛ d K*)
⟨*proof*⟩

end

5.5 Votewise Distance Rationalization

theory *Votewise-Distance-Rationalization*
  imports *Distance-Rationalization*
        Votewise-Distance
begin

A votewise distance rationalization of a voting rule is its distance rationalization with a distance function that depends on the submitted votes in a simple and a transparent manner by using a distance on individual orders and combining the components with a norm on R to n.

5.5.1 Common Rationalizations

fun *swap-ℛ* :: (*'a*, *'v* :: *linorder*, *'a Result*) *Consensus-Class* ⇒
      (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
  swap-ℛ K = 𝒮𝒞ℱ-result.distance-ℛ (*votewise-distance swap l-one*) *K*

5.5.2 Theorems

lemma *votewise-non-voters-irrelevant*:
  fixes
    d :: $'a$ *Vote Distance* **and**
    N :: *Norm*
  shows *voters-determine-distance* (*votewise-distance d N*)
⟨*proof*⟩

lemma *swap-standard*: *standard* (*votewise-distance swap l-one*)
⟨*proof*⟩

5.5.3 Equivalence Lemmas

type-synonym ($'a$, $'v$) *score-type* = ($'a$, $'v$) *Election Distance* ⇒
  ($'a$, $'v$, $'a$ *Result*) *Consensus-Class* ⇒ ($'a$, $'v$) *Election* ⇒ $'a$ ⇒ *ereal*

type-synonym ($'a$, $'v$) *dist-rat-type* = ($'a$, $'v$) *Election Distance* ⇒
  ($'a$, $'v$, $'a$ *Result*) *Consensus-Class* ⇒ $'v$ *set* ⇒ $'a$ *set* ⇒ ($'a$, $'v$) *Profile* ⇒ $'a$ *set*

type-synonym ($'a$, $'v$) *dist-rat-std-type* = ($'a$, $'v$) *Election Distance* ⇒
  ($'a$, $'v$, $'a$ *Result*) *Consensus-Class* ⇒ ($'a$, $'v$, $'a$ *Result*) *Electoral-Module*

type-synonym ($'a$, $'v$) *dist-type* = ($'a$, $'v$) *Election Distance* ⇒
  ($'a$, $'v$, $'a$ *Result*) *Consensus-Class* ⇒ ($'a$, $'v$, $'a$ *Result*) *Electoral-Module*

lemma *equal-score-swap*: (*score* :: ($'a$, $'v$ :: *linorder*) *score-type*)
    (*votewise-distance swap l-one*) = *score-std* (*votewise-distance swap l-one*)
  ⟨*proof*⟩

lemma *swap-$\mathcal{R}$-code*[*code*]: *swap-$\mathcal{R}$* =
    ($\mathcal{SCF}$-*result.distance-$\mathcal{R}$-std* :: ($'a$, $'v$ :: *linorder*) *dist-rat-std-type*)
      (*votewise-distance swap l-one*)
⟨*proof*⟩

end

5.6 Symmetry in Distance-Rationalizable Rules

theory *Distance-Rationalization-Symmetry*
  imports *Distance-Rationalization*
begin

5.6.1 Minimizer Function

fun *distance-infimum* :: $'a$ *Distance* ⇒ $'a$ *set* ⇒ $'a$ ⇒ *ereal* **where**
  distance-infimum d A a = *Inf* (*d a* ' *A*)

fun *closest-preimg-distance* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a\ Distance \Rightarrow$
$\qquad 'a \Rightarrow 'b \Rightarrow ereal$ **where**
  closest-preimg-distance f domain$_f$ d a b = distance-infimum d (preimg f domain$_f$ b) a

fun *minimizer* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a\ Distance \Rightarrow 'b\ set \Rightarrow 'a \Rightarrow 'b\ set$ **where**
  minimizer f domain$_f$ d A a = arg-min-set (closest-preimg-distance f domain$_f$ d a) A

Auxiliary Lemmas

lemma *rewrite-arg-min-set*:
  fixes
    $f :: 'a \Rightarrow 'b :: linorder$ **and**
    $A :: 'a\ set$
  shows *arg-min-set f A* $= \bigcup$ *(preimg f A ' $\{y \in f\ '\ A.\ \forall\ z \in f\ '\ A.\ y \leq z\}$)*
⟨*proof*⟩

Equivariance

abbreviation *Restrp* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ rel$ **where**
  Restrp r A $\equiv$ *r Int (A × UNIV)*

lemma *restr-induced-rel*:
  fixes
    $A :: 'a\ set$ **and**
    $B\ B' :: 'b\ set$ **and**
    $\varphi :: ('a,\ 'b)\ binary\text{-}fun$
  assumes $B' \subseteq B$
  shows *Restrp (action-induced-rel A B φ) B' = action-induced-rel A B' φ*
⟨*proof*⟩

theorem *group-action-invar-dist-and-equivar-f-imp-equivar-minimizer*:
  fixes
    $f :: 'a \Rightarrow 'b$ **and**
    domain$_f$ $X :: 'a\ set$ **and**
    $d :: 'a\ Distance$ **and**
    well-formed-img :: $'a \Rightarrow 'b\ set$ **and**
    $G :: 'c\ monoid$ **and**
    $\varphi :: ('c,\ 'a)\ binary\text{-}fun$ **and**
    $\psi :: ('c,\ 'b)\ binary\text{-}fun$
  defines *equivar-prop-set-valued* $\equiv$
    action-induced-equivariance (carrier G) X φ (set-action ψ)
  assumes
    action-φ: *group-action G X φ* **and**
    group-action-res: *group-action G UNIV ψ* **and**
    dom-in-X: *domain$_f$ $\subseteq$ X* **and**
    closed-domain:
      closed-restricted-rel (action-induced-rel (carrier G) X φ) X domain$_f$ **and**
    equivar-img: *is-symmetry well-formed-img equivar-prop-set-valued* **and**

invar-d: *invariance*$_\mathcal{D}$ *d* (*carrier G*) *X* φ **and**
equivar-f:
is-symmetry f (*action-induced-equivariance* (*carrier G*) *domain*$_f$ φ ψ)
shows *is-symmetry* (λ *x. minimizer f domain*$_f$ *d* (*well-formed-img x*) *x*) *equivar-prop-set-valued*
⟨*proof*⟩

Invariance

lemma *closest-dist-invar-under-refl-rel-and-tot-invar-dist*:
fixes
f :: $'a \Rightarrow {'}b$ **and**
domain$_f$ :: $'a$ *set* **and**
d :: $'a$ *Distance* **and**
rel :: $'a$ *rel*
assumes
reflp-on′ domain$_f$ (*Restrp rel domain*$_f$) **and**
total-invariance$_\mathcal{D}$ *d rel*
shows *is-symmetry* (*closest-preimg-distance f domain*$_f$ *d*) (*Invariance rel*)
⟨*proof*⟩

lemma *refl-rel-and-tot-invar-dist-imp-invar-minimizer*:
fixes
f :: $'a \Rightarrow {'}b$ **and**
domain$_f$ :: $'a$ *set* **and**
d :: $'a$ *Distance* **and**
rel :: $'a$ *rel* **and**
img :: $'b$ *set*
assumes
reflp-on′ domain$_f$ (*Restrp rel domain*$_f$) **and**
total-invariance$_\mathcal{D}$ *d rel*
shows *is-symmetry* (*minimizer f domain*$_f$ *d img*) (*Invariance rel*)
⟨*proof*⟩

theorem *group-act-invar-dist-and-invar-f-imp-invar-minimizer*:
fixes
f :: $'a \Rightarrow {'}b$ **and**
domain$_f$ *A* :: $'a$ *set* **and**
d :: $'a$ *Distance* **and**
img :: $'b$ *set* **and**
G :: $'c$ *monoid* **and**
φ :: ($'c$, $'a$) *binary-fun*
defines
rel $\equiv$ *action-induced-rel* (*carrier G*) *A* φ **and**
rel′ $\equiv$ *action-induced-rel* (*carrier G*) *domain*$_f$ φ
assumes
action-φ: *group-action G A* φ **and**
dom-in-A: *domain*$_f$ $\subseteq$ *A* **and**
closed-domain: *closed-restricted-rel rel A domain*$_f$ **and**
invar-d: *invariance*$_\mathcal{D}$ *d* (*carrier G*) *A* φ **and**

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invar-f: *is-symmetry f* (*Invariance rel′*)
  shows *is-symmetry* (*minimizer f domain$_f$ d img*) (*Invariance rel*)
⟨*proof*⟩

5.6.2 Minimizer Translation

lemma $\mathcal{K}_{\mathcal{E}}$*-is-preimg*:
  fixes
    d :: (*′a*, *′v*) *Election Distance* **and**
    C :: (*′a*, *′v*, *′r Result*) *Consensus-Class* **and**
    E :: (*′a*, *′v*) *Election* **and**
    w :: *′r*
  shows *preimg* (*elect-r* ∘ *fun$_{\mathcal{E}}$* (*rule-$\mathcal{K}$ C*)) (*elections-$\mathcal{K}$ C*) {*w*} = $\mathcal{K}_{\mathcal{E}}$ *C w*
⟨*proof*⟩

lemma *score-is-closest-preimg-dist*:
  fixes
    d :: (*′a*, *′v*) *Election Distance* **and**
    C :: (*′a*, *′v*, *′r Result*) *Consensus-Class* **and**
    E :: (*′a*, *′v*) *Election* **and**
    w :: *′r*
  shows *score d C E w* =
      closest-preimg-distance (*elect-r* ∘ *fun$_{\mathcal{E}}$* (*rule-$\mathcal{K}$ C*)) (*elections-$\mathcal{K}$ C*) *d E* {*w*}
⟨*proof*⟩

lemma (**in** *result*) $\mathcal{R}_{\mathcal{W}}$*-is-minimizer*:
  fixes
    d :: (*′a*, *′v*) *Election Distance* **and**
    C :: (*′a*, *′v*, *′r Result*) *Consensus-Class*
  shows *fun$_{\mathcal{E}}$* ($\mathcal{R}_{\mathcal{W}}$ *d C*) =
    (λ *E*. $\bigcup$ (*minimizer* (*elect-r* ∘ *fun$_{\mathcal{E}}$* (*rule-$\mathcal{K}$ C*)) (*elections-$\mathcal{K}$ C*) *d*
                (*singleton-set-system* (*limit* (*alternatives-$\mathcal{E}$ E*) *UNIV*)) *E*))
⟨*proof*⟩

Invariance

theorem (**in** *result*) *tot-invar-dist-imp-invar-dr-rule*:
  fixes
    d :: (*′a*, *′v*) *Election Distance* **and**
    C :: (*′a*, *′v*, *′r Result*) *Consensus-Class* **and**
    rel :: (*′a*, *′v*) *Election rel*
  assumes
    r-refl: *reflp-on′* (*elections-$\mathcal{K}$ C*) (*Restrp rel* (*elections-$\mathcal{K}$ C*)) **and**
    tot-invar-d: *total-invariance$_{\mathcal{D}}$ d rel* **and**
    invar-res:
      is-symmetry (λ *E*. *limit* (*alternatives-$\mathcal{E}$ E*) *UNIV*) (*Invariance rel*)
  shows *is-symmetry* (*fun$_{\mathcal{E}}$* (*distance-$\mathcal{R}$ d C*)) (*Invariance rel*)
⟨*proof*⟩

theorem (**in** *result*) *invar-dist-cons-imp-invar-dr-rule*:

fixes
$d :: ('a, 'v)$ *Election Distance* **and**
$C :: ('a, 'v, 'r$ *Result*$)$ *Consensus-Class* **and**
$G :: 'b$ *monoid* **and**
$\varphi :: ('b, ('a, 'v)$ *Election*$)$ *binary-fun* **and**
$B :: ('a, 'v)$ *Election set*
defines
$rel \equiv$ *action-induced-rel* (*carrier G*) $B\ \varphi$ **and**
$rel' \equiv$ *action-induced-rel* (*carrier G*) (*elections-$\mathcal{K}$ C*) φ
assumes
action-φ: *group-action G B φ* **and**
consensus-C-in-B: *elections-$\mathcal{K}$ C $\subseteq$ B* **and**
closed-domain:
  closed-restricted-rel rel B (*elections-$\mathcal{K}$ C*) **and**
invar-res:
  is-symmetry (λ *E. limit* (*alternatives-$\mathcal{E}$ E*) *UNIV*) (*Invariance rel*) **and**
invar-d: *invariance$_\mathcal{D}$ d* (*carrier G*) *B φ* **and**
invar-C-winners: *is-symmetry* (*elect-r $\circ$ fun$_\mathcal{E}$* (*rule-$\mathcal{K}$ C*)) (*Invariance rel'*)
shows *is-symmetry* (*fun$_\mathcal{E}$* (*distance-$\mathcal{R}$ d C*)) (*Invariance rel*)
$\langle proof \rangle$

Equivariance

theorem (**in** *result*) *invar-dist-equivar-cons-imp-equivar-dr-rule*:
  fixes
  $d :: ('a, 'v)$ *Election Distance* **and**
  $C :: ('a, 'v, 'r$ *Result*$)$ *Consensus-Class* **and**
  $G :: 'b$ *monoid* **and**
  $\varphi :: ('b, ('a, 'v)$ *Election*$)$ *binary-fun* **and**
  $\psi :: ('b, 'r)$ *binary-fun* **and**
  $B :: ('a, 'v)$ *Election set*
  defines
  $rel \equiv$ *action-induced-rel* (*carrier G*) $B\ \varphi$ **and**
  $rel' \equiv$ *action-induced-rel* (*carrier G*) (*elections-$\mathcal{K}$ C*) φ **and**
  equivar-prop $\equiv$
    action-induced-equivariance (*carrier G*) (*elections-$\mathcal{K}$ C*)
      φ (*set-action ψ*) **and**
  equivar-prop-global-set-valued $\equiv$
      action-induced-equivariance (*carrier G*) $B\ \varphi$ (*set-action ψ*) **and**
  equivar-prop-global-result-valued $\equiv$
      action-induced-equivariance (*carrier G*) $B\ \varphi$ (*result-action ψ*)
  assumes
  action-φ: *group-action G B φ* **and**
  group-act-res: *group-action G UNIV ψ* **and**
  cons-elect-set: *elections-$\mathcal{K}$ C $\subseteq$ B* **and**
  closed-domain: *closed-restricted-rel rel B* (*elections-$\mathcal{K}$ C*) **and**
  equivar-res:
    is-symmetry (λ *E. limit* (*alternatives-$\mathcal{E}$ E*) *UNIV*)
        equivar-prop-global-set-valued **and**

    invar-d: *invariance_D d (carrier G) B φ* **and**
    equivar-C-winners: *is-symmetry (elect-r ∘ fun_E (rule-K C)) equivar-prop*
  shows *is-symmetry (fun_E (distance-R d C)) equivar-prop-global-result-valued*
⟨*proof*⟩

5.6.3 Inference Rules

theorem (**in** *result*) *anon-dist-and-cons-imp-anon-dr*:
  fixes
    d :: ('a, 'v) Election Distance **and**
    C :: ('a, 'v, 'r Result) Consensus-Class
  assumes
    anon-d: *distance-anonymity' well-formed-elections d* **and**
    anon-C: *consensus-rule-anonymity' (elections-K C) C* **and**
    closed-C: *closed-restricted-rel (anonymity_R well-formed-elections)*
          well-formed-elections (elections-K C)
    shows *anonymity' (distance-R d C)*
⟨*proof*⟩

theorem (**in** *result-properties*) *neutr-dist-and-cons-imp-neutr-dr*:
  fixes
    d :: ('a, 'v) Election Distance **and**
    C :: ('a, 'v, 'b Result) Consensus-Class
  assumes
    neutral-d: *distance-neutrality well-formed-elections d* **and**
    neutral-C: *consensus-rule-neutrality (elections-K C) C* **and**
    closed-C: *closed-restricted-rel (neutrality_R well-formed-elections)*
          well-formed-elections (elections-K C)
    shows *neutrality (distance-R d C)*
⟨*proof*⟩

theorem *reversal-sym-dist-and-cons-imp-reversal-sym-dr*:
  fixes
    d :: ('a, 'c) Election Distance **and**
    C :: ('a, 'c, 'a rel Result) Consensus-Class
  assumes
    reverse-sym-d: *distance-reversal-symmetry well-formed-elections d* **and**
    reverse-sym-C: *consensus-rule-reversal-symmetry (elections-K C) C* **and**
    closed-C: *closed-restricted-rel (reversal_R well-formed-elections)*
          well-formed-elections (elections-K C)
    shows *reversal-symmetry (SWF-result.distance-R d C)*
⟨*proof*⟩

theorem (**in** *result*) *tot-hom-dist-imp-hom-dr*:
  fixes
    d :: ('a, nat) Election Distance **and**
    C :: ('a, nat, 'r Result) Consensus-Class
  assumes *distance-homogeneity finite-elections-V d*
  shows *homogeneity (distance-R d C)*

⟨*proof*⟩

theorem (**in** *result*) *tot-hom-dist-imp-hom-dr′*:
  fixes
    d :: (′*a*, ′*v* :: *linorder*) *Election Distance* **and**
    C :: (′*a*, ′*v*, ′*r Result*) *Consensus-Class*
  assumes *distance-homogeneity′ finite-elections-V d*
  shows *homogeneity′* (*distance-R d C*)
⟨*proof*⟩

  end

5.7   Distance Rationalization on Election Quotients

theory *Quotient-Distance-Rationalization*
  imports *Quotient-Module*
        Distance-Rationalization-Symmetry
begin

5.7.1   Distances

fun *distance*$_Q$ :: ′*x Distance* $\Rightarrow$ ′*x set Distance* **where**
  distance$_Q$ *d A B* = (*if A* = {} $\wedge$ *B* = {} *then 0 else*
        (*if A* = {} $\vee$ *B* = {} *then* ∞ *else*
        π_Q (*tup d*) (*A* $\times$ *B*)))

fun *relation-paths* :: ′*x rel* $\Rightarrow$ ′*x list set* **where**
  relation-paths r =
      {*p*. $\exists$ *k*. *length p* = *2* $*$ *k* $\wedge$ ($\forall$ *i* < *k*. (*p*!(*2* $*$ *i*), *p*!(*2* $*$ *i* + *1*)) $\in$ *r*)}

fun *admissible-paths* :: ′*x rel* $\Rightarrow$ ′*x set* $\Rightarrow$ ′*x set* $\Rightarrow$ ′*x list set* **where**
  admissible-paths r X Y =
      {*x*#*p*@[*y*] | *x y p*. *x* $\in$ *X* $\wedge$ *y* $\in$ *Y* $\wedge$ *p* $\in$ *relation-paths r*}

fun *path-length* :: ′*x list* $\Rightarrow$ ′*x Distance* $\Rightarrow$ *ereal* **where**
  path-length [] *d* = *0* |
  path-length [*x*] *d* = *0* |
  path-length (*x*#*y*#*xs*) *d* = *d x y* + *path-length xs d*

fun *quotient-dist* :: ′*x rel* $\Rightarrow$ ′*x Distance* $\Rightarrow$ ′*x set Distance* **where**
  quotient-dist r d A B =
    Inf ($\bigcup$ {{*path-length p d* | *p*. *p* $\in$ *admissible-paths r A B*}})

fun *distance-infimum*$_Q$ :: ′*x Distance* $\Rightarrow$ ′*x set Distance* **where**
  distance-infimum$_Q$ *d A B* = *Inf* {*d a b* | *a b*. *a* $\in$ *A* $\wedge$ *b* $\in$ *B*}

fun *simple* :: ′*x rel* $\Rightarrow$ ′*x set* $\Rightarrow$ ′*x Distance* $\Rightarrow$ *bool* **where**

simple r X d =
  (∀ *A* ∈ *X // r*.
    ∃ *a* ∈ *A*. ∀ *B* ∈ *X // r*.
      distance-infimum$_\mathcal{Q}$ d A B = Inf {d a b | b. b ∈ B})
— We call a distance simple with respect to a relation if for all relation classes, there is an *a* in *A* that minimizes the infimum distance between *A* and all *B* such that the infimum distance between these sets coincides with the infimum distance over all *b* in *B* for a fixed *a*.

fun *product′ :: ′x rel ⇒ (′x ∗ ′x) rel* **where**
  product′ r = {(p$_1$, p$_2$). ((fst p$_1$, fst p$_2$) ∈ r ∧ snd p$_1$ = snd p$_2$)
                  ∨ *((snd p$_1$, snd p$_2$) ∈ r ∧ fst p$_1$ = fst p$_2$)}*

Auxiliary Lemmas

lemma *tot-dist-invariance-is-congruence*:
  fixes
    d :: ′x Distance **and**
    r :: ′x rel
  shows *(total-invariance$_\mathcal{D}$ d r) = (tup d respects (product r))*
  ⟨*proof*⟩

lemma *product-helper*:
  fixes
    r :: ′x rel **and**
    X :: ′x set
  shows
    trans-imp: *Relation.trans r ⟶ Relation.trans (product r)* **and**
    refl-imp: *refl-on X r ⟶ refl-on (X × X) (product r)* **and**
    sym: *sym-on X r ⟶ sym-on (X × X) (product r)*
  ⟨*proof*⟩

theorem *dist-pass-to-quotient*:
  fixes
    d :: ′x Distance **and**
    r :: ′x rel **and**
    X :: ′x set
  assumes
    equiv-X-r: *equiv X r* **and**
    tot-inv-dist-d-r: *total-invariance$_\mathcal{D}$ d r*
  shows ∀ *A B. A ∈ X // r ∧ B ∈ X // r*
        ⟶ (∀ *a b. a ∈ A ∧ b ∈ B ⟶ distance$_\mathcal{Q}$ d A B = d a b)*
⟨*proof*⟩

lemma *relation-paths-subset*:
  fixes
    n :: nat **and**
    p :: ′x list **and**
    r :: ′x rel **and**

$X :: {}'x\ set$
  assumes $r \subseteq X \times X$
  shows $\forall\ p.\ p \in relation\text{-}paths\ r \longrightarrow (\forall\ i < length\ p.\ p!i \in X)$
$\langle proof \rangle$

lemma *admissible-path-len*:
  fixes
    $d :: {}'x\ Distance$ **and**
    $r :: {}'x\ rel$ **and**
    $X :: {}'x\ set$ **and**
    $a\ b :: {}'x$ **and**
    $p :: {}'x\ list$
  assumes *refl-on* $X\ r$
  shows *triangle-ineq* $X\ d \wedge p \in relation\text{-}paths\ r \wedge total\text{-}invariance_{\mathcal{D}}\ d\ r$
        $\wedge\ a \in X \wedge b \in X \longrightarrow path\text{-}length\ (a\#p@[b])\ d \geq d\ a\ b$
$\langle proof \rangle$

lemma *quotient-dist-coincides-with-dist$_{\mathcal{Q}}$*:
  fixes
    $d :: {}'x\ Distance$ **and**
    $r :: {}'x\ rel$ **and**
    $X :: {}'x\ set$
  assumes
    equiv: *equiv* $X\ r$ **and**
    tri: *triangle-ineq* $X\ d$ **and**
    invar: *total-invariance$_{\mathcal{D}}$* $d\ r$
  shows $\forall\ A \in X\ //\ r.\ \forall\ B \in X\ //\ r.\ quotient\text{-}dist\ r\ d\ A\ B = distance_{\mathcal{Q}}\ d\ A\ B$
$\langle proof \rangle$

lemma *inf-dist-coincides-with-dist$_{\mathcal{Q}}$*:
  fixes
    $d :: {}'x\ Distance$ **and**
    $r :: {}'x\ rel$ **and**
    $X :: {}'x\ set$
  assumes
    equiv-X-r: *equiv* $X\ r$ **and**
    tot-inv-d-r: *total-invariance$_{\mathcal{D}}$* $d\ r$
  shows $\forall\ A \in X\ //\ r.\ \forall\ B \in X\ //\ r.$
        $distance\text{-}infimum_{\mathcal{Q}}\ d\ A\ B = distance_{\mathcal{Q}}\ d\ A\ B$
$\langle proof \rangle$

lemma *inf-helper*:
  fixes
    $A\ B :: {}'x\ set$ **and**
    $d :: {}'x\ Distance$
  shows $Inf\ \{d\ a\ b\ |\ a\ b.\ a \in A \wedge b \in B\} =$
        $Inf\ \{Inf\ \{d\ a\ b\ |\ b.\ b \in B\}\ |\ a.\ a \in A\}$
$\langle proof \rangle$

lemma *invar-dist-simple*:
  fixes
    $d :: {}'y\ Distance$ **and**
    $G :: {}'x\ monoid$ **and**
    $Y :: {}'y\ set$ **and**
    $\varphi :: ({}'x,\ {}'y)\ binary\text{-}fun$
  assumes
    action-φ: *group-action* $G\ Y\ \varphi$ **and**
    invar: *invariance$_\mathcal{D}$* d (*carrier* G) $Y\ \varphi$
  shows *simple* (*action-induced-rel* (*carrier* G) $Y\ \varphi$) $Y\ d$
⟨*proof*⟩

lemma *tot-invar-dist-simple*:
  fixes
    $d :: {}'x\ Distance$ **and**
    $r :: {}'x\ rel$ **and**
    $X :: {}'x\ set$
  assumes
    equiv-on-X: *equiv* $X\ r$ **and**
    invar: *total-invariance$_\mathcal{D}$* $d\ r$
  shows *simple* $r\ X\ d$
⟨*proof*⟩

5.7.2 Consensus and Results

fun *elections-$\mathcal{K}_\mathcal{Q}$* $:: ({}'a,\ {}'v)\ Election\ rel \Rightarrow ({}'a,\ {}'v,\ {}'r\ Result)\ Consensus\text{-}Class \Rightarrow$
    $({}'a,\ {}'v)\ Election\ set\ set$ **where**
  elections-$\mathcal{K}_\mathcal{Q}$ $r\ C = (elections\text{-}\mathcal{K}\ C)\ /\!/\ r$

fun (**in** *result*) *limit$_\mathcal{Q}$* $:: ({}'a,\ {}'v)\ Election\ set \Rightarrow {}'r\ set \Rightarrow {}'r\ set$ **where**
  limit$_\mathcal{Q}$ $X\ res = \bigcap\ \{limit\ (alternatives\text{-}\mathcal{E}\ E)\ res \mid E.\ E \in X\}$

Auxiliary Lemmas

lemma *closed-under-equiv-rel-subset*:
  fixes
    $X\ Y\ Z :: {}'x\ set$ **and**
    $r :: {}'x\ rel$
  assumes
    equiv $X\ r$ **and**
    $Y \subseteq X$ **and**
    $Z \subseteq X$ **and**
    $Z \in Y\ /\!/\ r$ **and**
    closed-restricted-rel $r\ X\ Y$
  shows $Z \subseteq Y$
⟨*proof*⟩

lemma (**in** *result*) *limit-invar*:
  fixes
    $d :: ({}'a,\ {}'v)\ Election\ Distance$ **and**

    r :: (*'a*, *'v*) *Election rel* **and**
    C :: (*'a*, *'v*, *'r Result*) *Consensus-Class* **and**
    X A :: (*'a*, *'v*) *Election set*
  assumes
    quot-class: $A \in X \; // \; r$ **and**
    equiv-rel: *equiv X r* **and**
    cons-subset: *elections-*$\mathcal{K}$ *C* $\subseteq$ *X* **and**
    invar-res: *is-symmetry* (λ *E. limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*) (*Invariance r*)
  shows $\forall \; a \in A.$ *limit* (*alternatives-*$\mathcal{E}$ *a*) *UNIV* = *limit*$_{\mathcal{Q}}$ *A UNIV*
⟨*proof*⟩

lemma (**in** *result*) *preimg-invar*:
  fixes
    f :: *'x* $\Rightarrow$ *'y* **and**
    domain$_f$ *X* :: *'x set* **and**
    d :: *'x Distance* **and**
    r :: *'x rel*
  assumes
    equiv-rel: *equiv X r* **and**
    cons-subset: *domain*$_f$ $\subseteq$ *X* **and**
    closed-domain: *closed-restricted-rel r X domain*$_f$ **and**
    invar-f: *is-symmetry f* (*Invariance* (*Restr r domain*$_f$))
  shows $\forall \; y.$ (*preimg f domain*$_f$ *y*) $// \; r$ = *preimg* ($\pi_{\mathcal{Q}}$ *f*) (*domain*$_f$ $// \; r$) *y*
⟨*proof*⟩

lemma *minimizer-helper*:
  fixes
    f :: *'x* $\Rightarrow$ *'y* **and**
    domain$_f$ :: *'x set* **and**
    d :: *'x Distance* **and**
    Y :: *'y set* **and**
    x :: *'x* **and**
    y :: *'y*
  shows *y* $\in$ *minimizer f domain*$_f$ *d Y x* =
    (*y* $\in$ *Y* $\wedge$ ($\forall \; y' \in Y.$
      Inf (*d x* ' (*preimg f domain*$_f$ *y*)) $\leq$ *Inf* (*d x* ' (*preimg f domain*$_f$ *y'*))))
  ⟨*proof*⟩

lemma *rewr-singleton-set-system-union*:
  fixes
    Y :: *'x set set* **and**
    X :: *'x set*
  assumes *Y* $\subseteq$ *singleton-set-system X*
  shows
    singleton-set-union: $x \in \bigcup Y \longleftrightarrow \{x\} \in Y$ **and**
    obtain-singleton: *A* $\in$ *singleton-set-system X* $\longleftrightarrow$ ($\exists \; x \in X.$ *A* = $\{x\}$)
  ⟨*proof*⟩

lemma *union-inf*:

fixes X :: *ereal set set*
  shows *Inf* $\{$*Inf A* $\mid$ *A.* $A \in X\}$ = *Inf* $(\bigcup X)$
⟨*proof*⟩

5.7.3 Distance Rationalization

fun (**in** *result*) $\mathcal{R}_\mathcal{Q}$ :: $('a,\ 'v)$ *Election rel* $\Rightarrow$ $('a,\ 'v)$ *Election Distance* $\Rightarrow$
      $('a,\ 'v,\ 'r\ Result)$ *Consensus-Class* $\Rightarrow$ $('a,\ 'v)$ *Election set* $\Rightarrow$ $'r\ set$ **where**
  $\mathcal{R}_\mathcal{Q}\ r\ d\ C\ A$ =
    $\bigcup$ (*minimizer* $(\pi_\mathcal{Q}$ (*elect-r* $\circ$ *fun*$_\mathcal{E}$ (*rule-K* C))) (*elections-K*$_\mathcal{Q}$ $r\ C$)
        (*distance-infimum*$_\mathcal{Q}$ d) (*singleton-set-system* (*limit*$_\mathcal{Q}$ $A\ UNIV$)) A)

fun (**in** *result*) *distance-*$\mathcal{R}_\mathcal{Q}$ :: $('a,\ 'v)$ *Election rel* $\Rightarrow$ $('a,\ 'v)$ *Election Distance* $\Rightarrow$
      $('a,\ 'v,\ 'r\ Result)$ *Consensus-Class* $\Rightarrow$ $('a,\ 'v)$ *Election set* $\Rightarrow$ $'r\ Result$ **where**
  distance-$\mathcal{R}_\mathcal{Q}$ $r\ d\ C\ A$ =
    $(\mathcal{R}_\mathcal{Q}\ r\ d\ C\ A,$
      $\pi_\mathcal{Q}$ (λ $E.\ limit$ (*alternatives-*$\mathcal{E}$ E) $UNIV$) $A - \mathcal{R}_\mathcal{Q}\ r\ d\ C\ A,$
      $\{\})$

Proposition 4.17 by Hadjibeyli and Wilson [3].

theorem (**in** *result*) *invar-dr-simple-dist-imp-quotient-dr-winners*:
  fixes
    d :: $('a,\ 'v)$ *Election Distance* **and**
    C :: $('a,\ 'v,\ 'r\ Result)$ *Consensus-Class* **and**
    r :: $('a,\ 'v)$ *Election rel* **and**
    $X\ A$ :: $('a,\ 'v)$ *Election set*
  assumes
    simple: *simple* $r\ X\ d$ **and**
    closed-domain: *closed-restricted-rel* $r\ X$ (*elections-K* C) **and**
    invar-res:
      is-symmetry (λ $E.\ limit$ (*alternatives-*$\mathcal{E}$ E) $UNIV$) (*Invariance* r) **and**
    invar-C: *is-symmetry* (*elect-r* $\circ$ *fun*$_\mathcal{E}$ (*rule-K* C))
                (*Invariance* (*Restr* r (*elections-K* C))) **and**
    invar-dr: *is-symmetry* (*fun*$_\mathcal{E}$ ($\mathcal{R}_\mathcal{W}$ $d\ C$)) (*Invariance* r) **and**
    quot-class: $A \in X\ //\ r$ **and**
    equiv-rel: *equiv* $X\ r$ **and**
    cons-subset: *elections-K* $C \subseteq X$
  shows $\pi_\mathcal{Q}$ (*fun*$_\mathcal{E}$ ($\mathcal{R}_\mathcal{W}$ $d\ C$)) $A = \mathcal{R}_\mathcal{Q}\ r\ d\ C\ A$
⟨*proof*⟩

theorem (**in** *result*) *invar-dr-simple-dist-imp-quotient-dr*:
  fixes
    d :: $('a,\ 'v)$ *Election Distance* **and**
    C :: $('a,\ 'v,\ 'r\ Result)$ *Consensus-Class* **and**
    r :: $('a,\ 'v)$ *Election rel* **and**
    $X\ A$ :: $('a,\ 'v)$ *Election set*
  assumes
    simple: *simple* $r\ X\ d$ **and**
    closed-domain: *closed-restricted-rel* $r\ X$ (*elections-K* C) **and**

invar-res:
  is-symmetry (λ *E. limit* (*alternatives-$\mathcal{E}$ E*) *UNIV*) (*Invariance r*) **and**
invar-C: *is-symmetry* (*elect-r* $\circ$ *fun$_{\mathcal{E}}$* (*rule-$\mathcal{K}$ C*))
        (*Invariance* (*Restr r* (*elections-$\mathcal{K}$ C*))) **and**
invar-dr: *is-symmetry* (*fun$_{\mathcal{E}}$* ($\mathcal{R_W}$ *d C*)) (*Invariance r*) **and**
quot-class: $A \in X \;//\; r$ **and**
equiv-rel: *equiv X r* **and**
cons-subset: *elections-$\mathcal{K}$ C* $\subseteq X$
  shows $\pi_{\mathcal{Q}}$ (*fun$_{\mathcal{E}}$* (*distance-$\mathcal{R}$ d C*)) A = *distance-$\mathcal{R_Q}$ r d C A*
$\langle proof \rangle$

end

5.8 Code Generation Interpretations for Results and Properties

theory *Interpretation-Code*
  imports *Electoral-Module*
        Distance-Rationalization
begin
$\langle ML \rangle$

5.8.1 Code Lemmas

Lemmas stating the explicit instantiations of interpreted abstract functions from locales.

lemma *electoral-module-$\mathcal{SCF}$-code-lemma*:
  fixes m :: ($'a$, $'v$, $'a$ Result) *Electoral-Module*
  shows $\mathcal{SCF}$-*result.electoral-module m* =
        ($\forall$ *A V p. profile V A p* $\longrightarrow$ *well-formed-$\mathcal{SCF}$ A* (*m V A p*))
  $\langle proof \rangle$

lemma $\mathcal{R_W}$-$\mathcal{SCF}$-code-lemma:
  fixes
    d :: ($'a$, $'v$) *Election Distance* **and**
    K :: ($'a$, $'v$, $'a$ Result) *Consensus-Class* **and**
    V :: $'v$ *set* **and**
    A :: $'a$ *set* **and**
    p :: ($'a$, $'v$) *Profile*
  shows $\mathcal{SCF}$-*result.$\mathcal{R_W}$ d K V A p* =
        arg-min-set (*score d K* (A, V, p)) (*limit-$\mathcal{SCF}$ A UNIV*)
  $\langle proof \rangle$

lemma *distance-$\mathcal{R}$-$\mathcal{SCF}$-code-lemma*:
  fixes
    d :: ($'a$, $'v$) *Election Distance* **and**

$K :: ('a, 'v, 'a$ $Result)$ $Consensus\text{-}Class$ **and**
$V :: 'v$ set **and**
$A :: 'a$ set **and**
$p :: ('a, 'v)$ $Profile$
shows $\mathcal{SCF}\text{-}result.distance\text{-}\mathcal{R}$ d K V A $p =$
$\quad (\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}$ d K V A $p,$
$\qquad (limit\text{-}\mathcal{SCF}$ A $UNIV) - \mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}$ d K V A $p,$
$\qquad \{\})$
$\langle proof \rangle$

lemma $\mathcal{R}_{\mathcal{W}}\text{-}std\text{-}\mathcal{SCF}\text{-}code\text{-}lemma$:
fixes
$d :: ('a, 'v)$ $Election$ $Distance$ **and**
$K :: ('a, 'v, 'a$ $Result)$ $Consensus\text{-}Class$ **and**
$V :: 'v$ set **and**
$A :: 'a$ set **and**
$p :: ('a, 'v)$ $Profile$
shows $\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std$ d K V A $p =$
$\quad arg\text{-}min\text{-}set$ $(score\text{-}std$ d K $(A, V, p))$ $(limit\text{-}\mathcal{SCF}$ A $UNIV)$
$\langle proof \rangle$

lemma $distance\text{-}\mathcal{R}\text{-}std\text{-}\mathcal{SCF}\text{-}code\text{-}lemma$:
fixes
$d :: ('a, 'v)$ $Election$ $Distance$ **and**
$K :: ('a, 'v, 'a$ $Result)$ $Consensus\text{-}Class$ **and**
$V :: 'v$ set **and**
$A :: 'a$ set **and**
$p :: ('a, 'v)$ $Profile$
shows $\mathcal{SCF}\text{-}result.distance\text{-}\mathcal{R}\text{-}std$ d K V A $p =$
$\quad (\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std$ d K V A $p,$
$\qquad (limit\text{-}\mathcal{SCF}$ A $UNIV) - \mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std$ d K V A $p,$
$\qquad \{\})$
$\langle proof \rangle$

lemma $anonymity\text{-}\mathcal{SCF}\text{-}code\text{-}lemma$: $\mathcal{SCF}\text{-}result.anonymity =$
$\quad (\lambda$ $m :: ('a, 'v, 'a$ $Result)$ $Electoral\text{-}Module.$
$\quad\quad \mathcal{SCF}\text{-}result.electoral\text{-}module$ m $\wedge$
$\quad\quad\quad (\forall$ A V p $\pi :: ('v \Rightarrow 'v).$
$\quad\quad\quad\quad bij$ $\pi \longrightarrow (let$ $(A', V', q) = (rename$ π $(A, V, p))$ in
$\quad\quad\quad profile$ V A p $\wedge$ $profile$ V' A' $q \longrightarrow m$ V A $p = m$ V' A' $q)))$
$\langle proof \rangle$

5.8.2 Interpretation Declarations and Constants

Declarations for replacing interpreted abstract functions from locales by
their explicit instantiations.

declare $[[lc\text{-}add$ $\mathcal{SCF}\text{-}result.electoral\text{-}module$ $electoral\text{-}module\text{-}\mathcal{SCF}\text{-}code\text{-}lemma]]$
declare $[[lc\text{-}add$ $\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}$ $\mathcal{R}_{\mathcal{W}}\text{-}\mathcal{SCF}\text{-}code\text{-}lemma]]$
declare $[[lc\text{-}add$ $\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std$ $\mathcal{R}_{\mathcal{W}}\text{-}std\text{-}\mathcal{SCF}\text{-}code\text{-}lemma]]$

declare [[*lc-add $\mathcal{SCF}$-result.distance-$\mathcal{R}$ distance-$\mathcal{R}$-$\mathcal{SCF}$-code-lemma*]]
declare [[*lc-add $\mathcal{SCF}$-result.distance-$\mathcal{R}$-std distance-$\mathcal{R}$-std-$\mathcal{SCF}$-code-lemma*]]
declare [[*lc-add $\mathcal{SCF}$-result.anonymity anonymity-$\mathcal{SCF}$-code-lemma*]]

Constant aliases to use instead of the interpreted functions.

definition *$\mathcal{R}_{\mathcal{W}}$-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.$\mathcal{R}_{\mathcal{W}}$*
definition *$\mathcal{R}_{\mathcal{W}}$-std-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.$\mathcal{R}_{\mathcal{W}}$-std*
definition *distance-$\mathcal{R}$-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.distance-$\mathcal{R}$*
definition *distance-$\mathcal{R}$-std-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.distance-$\mathcal{R}$-std*
definition *electoral-module-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.electoral-module*
definition *anonymity-$\mathcal{SCF}$-code $\equiv$ $\mathcal{SCF}$-result.anonymity*

⟨*ML*⟩

end

5.9 Drop Module

theory *Drop-Module*
  imports *Component-Types/Electoral-Module*
        Component-Types/Social-Choice-Types/Result
begin

This is a family of electoral modules. For a natural number n and a lexicon
(linear order) r of all alternatives, the according drop module rejects the
lexicographically first n alternatives (from A) and defers the rest. It is
primarily used as counterpart to the pass module in a parallel composition,
in order to segment the alternatives into two groups.

5.9.1 Definition

fun *drop-module* :: *nat $\Rightarrow$ 'a Preference-Relation $\Rightarrow$*
      ('a, 'v, 'a Result) Electoral-Module **where**
  drop-module n r V A p =
    ({},
    {a $\in$ A. rank (limit A r) a $\leq$ n},
    {a $\in$ A. rank (limit A r) a > n})

5.9.2 Soundness

theorem *drop-mod-sound*[*simp*]:
  fixes
    r :: 'a Preference-Relation **and**
    n :: nat

shows $\mathcal{SCF}$-result.electoral-module (drop-module n r)
⟨proof⟩

lemma voters-determine-drop-mod:
  fixes
    r :: 'a Preference-Relation **and**
    n :: nat
  shows voters-determine-election (drop-module n r)
  ⟨proof⟩

5.9.3 Non-Electing

The drop module is non-electing.

theorem drop-mod-non-electing[simp]:
  fixes
    r :: 'a Preference-Relation **and**
    n :: nat
  shows non-electing (drop-module n r)
  ⟨proof⟩

5.9.4 Properties

The drop module is strictly defer-monotone.

theorem drop-mod-def-lift-inv[simp]:
  fixes
    r :: 'a Preference-Relation **and**
    n :: nat
  shows defer-lift-invariance (drop-module n r)
  ⟨proof⟩

end

5.10 Pass Module

theory Pass-Module
  imports Component-Types/Electoral-Module
begin

This is a family of electoral modules. For a natural number n and a lexicon (linear order) r of all alternatives, the according pass module defers the lexicographically first n alternatives (from A) and rejects the rest. It is primarily used as counterpart to the drop module in a parallel composition in order to segment the alternatives into two groups.

5.10.1 Definition

fun *pass-module* :: *nat* $\Rightarrow$ $'a$ *Preference-Relation* $\Rightarrow$
  $('a, 'v, 'a$ *Result) Electoral-Module* **where**
 pass-module n r V A p =
  $(\{\},$
  $\{a \in A.\ rank\ (limit\ A\ r)\ a > n\},$
  $\{a \in A.\ rank\ (limit\ A\ r)\ a \leq n\})$

5.10.2 Soundness

theorem *pass-mod-sound*[*simp*]:
 fixes
  r :: $'a$ *Preference-Relation* **and**
  n :: *nat*
 shows $\mathcal{SCF}$-*result.electoral-module* (*pass-module n r*)
$\langle proof \rangle$

lemma *voters-determine-pass-mod*:
 fixes
  r :: $'a$ *Preference-Relation* **and**
  n :: *nat*
 shows *voters-determine-election* (*pass-module n r*)
 $\langle proof \rangle$

5.10.3 Non-Blocking

The pass module is non-blocking.

theorem *pass-mod-non-blocking*[*simp*]:
 fixes
  r :: $'a$ *Preference-Relation* **and**
  n :: *nat*
 assumes
  order: *linear-order r* **and**
  greater-zero: $n > 0$
 shows *non-blocking* (*pass-module n r*)
$\langle proof \rangle$

5.10.4 Non-Electing

The pass module is non-electing.

theorem *pass-mod-non-electing*[*simp*]:
 fixes
  r :: $'a$ *Preference-Relation* **and**
  n :: *nat*
 assumes *linear-order r*
 shows *non-electing* (*pass-module n r*)
 $\langle proof \rangle$

5.10.5   Properties

The pass module is strictly defer-monotone.

theorem *pass-mod-dl-inv*[*simp*]:
  fixes
    r :: *'a Preference-Relation* **and**
    n :: *nat*
  assumes *linear-order r*
  shows *defer-lift-invariance* (*pass-module n r*)
  ⟨*proof*⟩

theorem *pass-zero-mod-def-zero*[*simp*]:
  fixes *r* :: *'a Preference-Relation*
  assumes *linear-order r*
  shows *defers 0* (*pass-module 0 r*)
⟨*proof*⟩

For any natural number n and any linear order, the according pass module
defers n alternatives (if there are n alternatives). NOTE: The induction
proof is still missing. The following are the proofs for n=1 and n=2.

theorem *pass-one-mod-def-one*[*simp*]:
  fixes *r* :: *'a Preference-Relation*
  assumes *linear-order r*
  shows *defers 1* (*pass-module 1 r*)
⟨*proof*⟩

theorem *pass-two-mod-def-two*:
  fixes *r* :: *'a Preference-Relation*
  assumes *linear-order r*
  shows *defers 2* (*pass-module 2 r*)
⟨*proof*⟩

end

5.11   Elect Module

theory *Elect-Module*
  imports *Component-Types/Electoral-Module*
begin

The elect module is not concerned about the voter's ballots, and just elects
all alternatives. It is primarily used in sequence after an electoral mod-
ule that only defers alternatives to finalize the decision, thereby inducing a
proper voting rule in the social choice sense.

5.11.1 Definition

fun *elect-module* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  elect-module V A p = $(A, \{\}, \{\})$

5.11.2 Soundness

theorem *elect-mod-sound*[*simp*]: $\mathcal{SCF}$-*result.electoral-module elect-module*
  ⟨*proof*⟩

lemma *elect-mod-only-voters*: *voters-determine-election elect-module*
  ⟨*proof*⟩

5.11.3 Electing

theorem *elect-mod-electing*[*simp*]: *electing elect-module*
  ⟨*proof*⟩

end

5.12 Plurality Module

theory *Plurality-Module*
  imports *Component-Types/Elimination-Module*
begin

The plurality module implements the plurality voting rule. The plurality rule elects all modules with the maximum amount of top preferences among all alternatives, and rejects all the other alternatives. It is electing and induces the classical plurality (voting) rule from social-choice theory.

5.12.1 Definition

fun *plurality-score* :: $('a, 'v)$ *Evaluation-Function* **where**
  plurality-score V x A p = *win-count V p x*

fun *plurality* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  plurality V A p = *max-eliminator plurality-score V A p*

fun *plurality′* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  plurality′ V A p =
    (*if finite A*
      then $(\{\},$
          $\{a \in A.\ \exists\ x \in A.\ win\text{-}count\ V\ p\ x > win\text{-}count\ V\ p\ a\},$
          $\{a \in A.\ \forall\ x \in A.\ win\text{-}count\ V\ p\ x \leq win\text{-}count\ V\ p\ a\})$

else ({}, {}, *A*))

lemma *enat-leq-enat-set-max*:
 fixes
  x :: *enat* **and**
  X :: *enat set*
 assumes
  x ∈ *X* **and**
  finite X
 shows $x \leq Max\ X$
 ⟨*proof*⟩

lemma *plurality-mod-equiv*:
 fixes
  A :: ′*a set* **and**
  V :: ′*v set* **and**
  p :: (′*a*, ′*v*) *Profile*
 shows *plurality V A p = plurality*′ *V A p*
⟨*proof*⟩

5.12.2　Soundness

theorem *plurality-sound*[*simp*]: $\mathcal{SCF}$-*result.electoral-module plurality*
 ⟨*proof*⟩

theorem *plurality*′-*sound*[*simp*]: $\mathcal{SCF}$-*result.electoral-module plurality*′
⟨*proof*⟩

lemma *voters-determine-plurality-score*: *voters-determine-evaluation plurality-score*
⟨*proof*⟩

lemma *voters-determine-plurality*: *voters-determine-election plurality*
 ⟨*proof*⟩

lemma *voters-determine-plurality*′: *voters-determine-election plurality*′
⟨*proof*⟩

5.12.3　Non-Blocking

The plurality module is non-blocking.

theorem *plurality-mod-non-blocking*[*simp*]: *non-blocking plurality*
 ⟨*proof*⟩

theorem *plurality*′-*mod-non-blocking*[*simp*]: *non-blocking plurality*′
 ⟨*proof*⟩

5.12.4　Non-Electing

The plurality module is non-electing.

theorem *plurality-non-electing*[*simp*]: *non-electing plurality*
  ⟨*proof*⟩

theorem *plurality'-non-electing*[*simp*]: *non-electing plurality'*
  ⟨*proof*⟩

5.12.5 Property

lemma *plurality-def-inv-mono-alts*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p q :: $('a, 'v)$ *Profile* **and**
    a :: $'a$
  assumes
    defer-a: $a \in$ *defer plurality V A p* **and**
    lift-a: *lifted V A p q a*
  shows *defer plurality V A q = defer plurality V A p*
        $\lor$ *defer plurality V A q = {a}*
⟨*proof*⟩

lemma *plurality'-def-inv-mono-alts*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p q :: $('a, 'v)$ *Profile* **and**
    a :: $'a$
  assumes
    $a \in$ *defer plurality' V A p* **and**
    lifted V A p q a
  shows *defer plurality' V A q = defer plurality' V A p*
        $\lor$ *defer plurality' V A q = {a}*
  ⟨*proof*⟩

The plurality rule is invariant-monotone.

theorem *plurality-mod-def-inv-mono*[*simp*]: *defer-invariant-monotonicity plurality*
⟨*proof*⟩

theorem *plurality'-mod-def-inv-mono*[*simp*]: *defer-invariant-monotonicity plurality'*
  ⟨*proof*⟩

end

5.13 Borda Module

theory *Borda-Module*
  imports *Component-Types/Elimination-Module*
begin

This is the Borda module used by the Borda rule. The Borda rule is a voting rule, where on each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.13.1 Definition

fun *borda-score* :: $('a, 'v)$ *Evaluation-Function* **where**
  borda-score V x A p = $(\sum y \in A. \ (prefer\text{-}count \ V \ p \ x \ y))$

fun *borda* :: $('a, 'v, 'a \ Result)$ *Electoral-Module* **where**
  borda V A p = max-eliminator borda-score V A p

5.13.2 Soundness

theorem *borda-sound*: $\mathcal{SCF}$*-result.electoral-module borda*
  $\langle proof \rangle$

5.13.3 Non-Blocking

The Borda module is non-blocking.

theorem *borda-mod-non-blocking*[*simp*]: *non-blocking borda*
  $\langle proof \rangle$

5.13.4 Non-Electing

The Borda module is non-electing.

theorem *borda-mod-non-electing*[*simp*]: *non-electing borda*
  $\langle proof \rangle$

end

5.14 Condorcet Module

theory *Condorcet-Module*

imports *Component-Types/Elimination-Module*
begin

This is the Condorcet module used by the Condorcet (voting) rule. The Condorcet rule is a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.14.1 Definition

fun *condorcet-score* :: (*′a, ′v*) *Evaluation-Function* **where**
  condorcet-score V x A p = (if condorcet-winner V A p x then 1 else 0)

fun *condorcet* :: (*′a, ′v, ′a Result*) *Electoral-Module* **where**
  condorcet V A p = (max-eliminator condorcet-score) V A p

5.14.2 Soundness

theorem *condorcet-sound*: $\mathcal{SCF}$-*result.electoral-module condorcet*
  ⟨*proof*⟩

5.14.3 Property

theorem *condorcet-score-is-condorcet-rating*: *condorcet-rating condorcet-score*
⟨*proof*⟩

theorem *condorcet-is-dcc*: *defer-condorcet-consistency condorcet*
⟨*proof*⟩

end

5.15 Copeland Module

theory *Copeland-Module*
  imports *Component-Types/Elimination-Module*
begin

This is the Copeland module used by the Copeland voting rule. The Copeland rule elects the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.15.1 Definition

fun *copeland-score* :: ($'a$, $'v$) *Evaluation-Function* **where**
  copeland-score V x A p =
    card {y ∈ A . wins V x p y} − card {y ∈ A . wins V y p x}

fun *copeland* :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* **where**
  copeland V A p = max-eliminator copeland-score V A p

5.15.2 Soundness

theorem *copeland-sound*: $\mathcal{SCF}$*-result.electoral-module copeland*
  ⟨*proof*⟩

5.15.3 Lemmas

lemma *voters-determine-copeland-score*: *voters-determine-evaluation copeland-score*
⟨*proof*⟩

theorem *voters-determine-copeland*: *voters-determine-election copeland*
  ⟨*proof*⟩

For a Condorcet winner w, we have: "$|\{y ∈ A . wins\ V\ w\ p\ y\}| = |A| − 1$".

lemma *cond-winner-imp-win-count*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: ($'a$, $'v$) *Profile* **and**
    w :: $'a$
  assumes *condorcet-winner V A p w*
  shows *card {a ∈ A. wins V w p a} = card A − 1*
⟨*proof*⟩

For a Condorcet winner w, we have: "$|\{y ∈ A . wins\ V\ y\ p\ w\}| = 0$".

lemma *cond-winner-imp-loss-count*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: ($'a$, $'v$) *Profile* **and**
    w :: $'a$
  assumes *condorcet-winner V A p w*
  shows *card {a ∈ A. wins V a p w} = 0*
  ⟨*proof*⟩

Copeland score of a Condorcet winner.

lemma *cond-winner-imp-copeland-score*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**

p :: (*′a*, *′v*) *Profile* **and**
  w :: *′a*
 assumes *condorcet-winner V A p w*
 shows *copeland-score V w A p = card A − 1*
⟨*proof*⟩

For a non-Condorcet winner l, we have: "|{*y* ∈ *A* . *wins V l p y*}| = |*A*| − *2*".

lemma *non-cond-winner-imp-win-count*:
 fixes
  A :: *′a set* **and**
  V :: *′v set* **and**
  p :: (*′a*, *′v*) *Profile* **and**
  w l :: *′a*
 assumes
  winner: *condorcet-winner V A p w* **and**
  loser: *l ≠ w* **and**
  l-in-A: *l ∈ A*
 shows *card {a ∈ A . wins V l p a} ≤ card A − 2*
⟨*proof*⟩

5.15.4 Property

The Copeland score is Condorcet rating.

theorem *copeland-score-is-cr*: *condorcet-rating copeland-score*
⟨*proof*⟩

theorem *copeland-is-dcc*: *defer-condorcet-consistency copeland*
⟨*proof*⟩

end

5.16 Minimax Module

theory *Minimax-Module*
 imports *Component-Types/Elimination-Module*
begin

This is the Minimax module used by the Minimax voting rule. The Minimax rule elects the alternatives with the highest Minimax score. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

5.16.1 Definition

fun *minimax-score* :: $('a, 'v)$ *Evaluation-Function* **where**
  minimax-score V x A p =
    Min {*prefer-count V p x y* | *y* . *y* ∈ *A* − {*x*}}

fun *minimax* :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
  minimax A p = max-eliminator minimax-score A p

5.16.2 Soundness

theorem *minimax-sound*: $\mathcal{SCF}$-*result.electoral-module minimax*
  ⟨*proof*⟩

5.16.3 Lemma

lemma *non-cond-winner-minimax-score*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile* **and**
    w l :: $'a$
  assumes
    prof: *profile V A p* **and**
    winner: *condorcet-winner V A p w* **and**
    l-in-A: *l* ∈ *A* **and**
    l-neq-w: *l* ≠ *w*
  shows *minimax-score V l A p* ≤ *prefer-count V p l w*
⟨*proof*⟩

5.16.4 Property

theorem *minimax-score-cond-rating*: *condorcet-rating minimax-score*
⟨*proof*⟩

theorem *minimax-is-dcc*: *defer-condorcet-consistency minimax*
⟨*proof*⟩

end

Chapter 6

Compositional Structures

6.1 Drop- and Pass-Compatibility

theory *Drop-And-Pass-Compatibility*
  imports *Basic-Modules/Drop-Module*
       Basic-Modules/Pass-Module
begin

This is a collection of properties about the interplay and compatibility of both the drop module and the pass module.

theorem *drop-zero-mod-rej-zero*[*simp*]:
  fixes r :: $'a$ *Preference-Relation*
  assumes *linear-order r*
  shows *rejects 0* (*drop-module 0 r*)
⟨*proof*⟩

The drop module rejects n alternatives (if there are at least n alternatives).

theorem *drop-two-mod-rej-n*[*simp*]:
  fixes r :: $'a$ *Preference-Relation*
  assumes *linear-order r*
  shows *rejects n* (*drop-module n r*)
⟨*proof*⟩

The pass and drop module are (disjoint-)compatible.

theorem *drop-pass-disj-compat*[*simp*]:
  fixes
    r :: $'a$ *Preference-Relation* **and**
    n :: *nat*
  assumes *linear-order r*
  shows *disjoint-compatibility* (*drop-module n r*) (*pass-module n r*)
⟨*proof*⟩

end

6.2 Revision Composition

theory *Revision-Composition*
  imports *Basic-Modules/Component-Types/Electoral-Module*
begin

A revised electoral module rejects all originally rejected or deferred alternatives, and defers the originally elected alternatives. It does not elect any alternatives.

6.2.1 Definition

fun *revision-composition* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* $\Rightarrow$
      (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
  revision-composition m V A p = ({}, *A* − *elect m V A p*, *elect m V A p*)

abbreviation *rev* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* $\Rightarrow$
      (*'a*, *'v*, *'a Result*) *Electoral-Module* (-$\downarrow$ *50*) **where**
  m$\downarrow$ $\equiv$ *revision-composition m*

6.2.2 Soundness

theorem *rev-comp-sound*[*simp*]:
  fixes *m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes $\mathcal{SCF}$-*result.electoral-module m*
  shows $\mathcal{SCF}$-*result.electoral-module* (*revision-composition m*)
$\langle proof \rangle$

lemma *voters-determine-rev-comp*:
  fixes *m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes *voters-determine-election m*
  shows *voters-determine-election* (*revision-composition m*)
  $\langle proof \rangle$

6.2.3 Composition Rules

An electoral module received by revision is never electing.

theorem *rev-comp-non-electing*[*simp*]:
  fixes *m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes $\mathcal{SCF}$-*result.electoral-module m*
  shows *non-electing* (*m*$\downarrow$)
  $\langle proof \rangle$

Revising an electing electoral module results in a non-blocking electoral module.

theorem *rev-comp-non-blocking*[*simp*]:
  fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  assumes *electing m*
  shows *non-blocking* ($m\downarrow$)
⟨*proof*⟩

Revising an invariant monotone electoral module results in a defer-invariant-monotone electoral module.

theorem *rev-comp-def-inv-mono*[*simp*]:
  fixes $m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
  assumes *invariant-monotonicity m*
  shows *defer-invariant-monotonicity* ($m\downarrow$)
⟨*proof*⟩

end

6.3   Sequential Composition

theory *Sequential-Composition*
  imports *Basic-Modules/Component-Types/Electoral-Module*
begin

The sequential composition creates a new electoral module from two electoral modules. In a sequential composition, the second electoral module makes decisions over alternatives deferred by the first electoral module.

6.3.1   Definition

fun *sequential-composition* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
      $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
      $('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **where**
  sequential-composition m n V A p =
    (**let** *new-A* = *defer m V A p*;
        new-p = *limit-profile new-A p* **in** (
            (*elect m V A p*) $\cup$ (*elect n V new-A new-p*),
            (*reject m V A p*) $\cup$ (*reject n V new-A new-p*),
            defer n V new-A new-p))

abbreviation *sequence* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
      $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
      (**infix** $\triangleright$ *50*) **where**
  $m \triangleright n \equiv$ *sequential-composition m n*

fun *sequential-composition′* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$

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$('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
$('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **where**
$sequential\text{-}composition'\ m\ n\ V\ A\ p =$
  $(\mathbf{let}\ (m\text{-}e,\ m\text{-}r,\ m\text{-}d) = m\ V\ A\ p;\ new\text{-}A = m\text{-}d;$
    $new\text{-}p = limit\text{-}profile\ new\text{-}A\ p;$
    $(n\text{-}e,\ n\text{-}r,\ n\text{-}d) = n\ V\ new\text{-}A\ new\text{-}p\ \mathbf{in}$
      $(m\text{-}e \cup n\text{-}e,\ m\text{-}r \cup n\text{-}r,\ n\text{-}d))$

lemma *voters-determine-seq-comp*:
  fixes $m\ n ::\ ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  assumes *voters-determine-election* $m \wedge$ *voters-determine-election* n
  shows *voters-determine-election* $(m \triangleright n)$
⟨*proof*⟩

lemma *seq-comp-presv-disj*:
  fixes
    $m\ n ::\ ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A ::\ 'a\ set$ **and**
    $V ::\ 'v\ set$ **and**
    $p ::\ ('a,\ 'v)\ Profile$
  assumes
    module-m: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
    module-n: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ n$ **and**
    prof: *profile* $V\ A\ p$
  shows *disjoint3* $((m \triangleright n)\ V\ A\ p)$
⟨*proof*⟩

lemma *seq-comp-presv-alts*:
  fixes
    $m\ n ::\ ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A ::\ 'a\ set$ **and**
    $V ::\ 'v\ set$ **and**
    $p ::\ ('a,\ 'v)\ Profile$
  assumes
    module-m: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
    module-n: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ n$ **and**
    prof: *profile* $V\ A\ p$
  shows *set-equals-partition* $A\ ((m \triangleright n)\ V\ A\ p)$
⟨*proof*⟩

lemma *seq-comp-alt-eq*[*fundef-cong*, *code*]: *sequential-composition* = *sequential-composition'*
⟨*proof*⟩

6.3.2 Soundness

theorem *seq-comp-sound*[*simp*]:
  fixes $m\ n ::\ ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  assumes
    $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**

$\mathcal{SCF}$-*result.electoral-module n*
  shows $\mathcal{SCF}$-*result.electoral-module* $(m \rhd n)$
$\langle proof \rangle$

6.3.3   Lemmas

lemma *seq-comp-decrease-only-defer*:
  fixes
    $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A :: 'a\ set$ **and**
    $V :: 'v\ set$ **and**
    $p :: ('a,\ 'v)\ Profile$
  assumes
    module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
    module-n: $\mathcal{SCF}$-*result.electoral-module n* **and**
    prof: *profile V A p* **and**
    empty-defer: *defer m V A p* $= \{\}$
  shows $(m \rhd n)\ V\ A\ p = m\ V\ A\ p$
$\langle proof \rangle$

lemma *seq-comp-def-then-elect*:
  fixes
    $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A :: 'a\ set$ **and**
    $V :: 'v\ set$ **and**
    $p :: ('a,\ 'v)\ Profile$
  assumes
    n-electing-m: *non-electing m* **and**
    def-one-m: *defers 1 m* **and**
    electing-n: *electing n* **and**
    f-prof: *finite-profile V A p*
  shows *elect* $(m \rhd n)\ V\ A\ p = defer\ m\ V\ A\ p$
$\langle proof \rangle$

lemma *seq-comp-def-card-bounded*:
  fixes
    $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A :: 'a\ set$ **and**
    $V :: 'v\ set$ **and**
    $p :: ('a,\ 'v)\ Profile$
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    $\mathcal{SCF}$-*result.electoral-module n* **and**
    finite-profile V A p
  shows *card* $(defer\ (m \rhd n)\ V\ A\ p) \leq card\ (defer\ m\ V\ A\ p)$
  $\langle proof \rangle$

lemma *seq-comp-def-set-bounded*:
  fixes

m n :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
  A :: $'a\ set$ **and**
  V :: $'v\ set$ **and**
  p :: $('a, 'v)$ *Profile*
assumes
  $\mathcal{SCF}$*-result.electoral-module* m **and**
  $\mathcal{SCF}$*-result.electoral-module* n **and**
  profile V A p
shows *defer* $(m \triangleright n)$ V A $p \subseteq$ *defer* m V A p
$\langle proof \rangle$

lemma *seq-comp-defers-def-set*:
  fixes
    m n :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)$ *Profile*
  shows *defer* $(m \triangleright n)$ V A $p =$
        defer n V (*defer* m V A p) (*limit-profile* (*defer* m V A p) p)
  $\langle proof \rangle$

lemma *seq-comp-def-then-elect-elec-set*:
  fixes
    m n :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)$ *Profile*
  shows *elect* $(m \triangleright n)$ V A $p =$
        elect n V (*defer* m V A p)
          (*limit-profile* (*defer* m V A p) p) $\cup$ (*elect* m V A p)
  $\langle proof \rangle$

lemma *seq-comp-elim-one-red-def-set*:
  fixes
    m n :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    A :: $'a\ set$ **and**
    V :: $'v\ set$ **and**
    p :: $('a, 'v)$ *Profile*
  assumes
    $\mathcal{SCF}$*-result.electoral-module* m **and**
    eliminates 1 n **and**
    profile V A p **and**
    card (*defer* m V A p) > 1
  shows *defer* $(m \triangleright n)$ V A $p \subset$ *defer* m V A p
  $\langle proof \rangle$

lemma *seq-comp-def-set-trans*:
  fixes
    m n :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**

$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a,\ 'v)\ Profile$ **and**
$a :: 'a$
assumes
  $a \in (defer\ (m \rhd n)\ V\ A\ p)$ **and**
  $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m \wedge \mathcal{SCF}\text{-}result.electoral\text{-}module\ n$ **and**
  profile V A p
shows $a \in defer\ n\ V\ (defer\ m\ V\ A\ p)\ (limit\text{-}profile\ (defer\ m\ V\ A\ p)\ p)\ \wedge$
      $a \in defer\ m\ V\ A\ p$
⟨*proof*⟩

6.3.4   Composition Rules

The sequential composition preserves the non-blocking property.

theorem *seq-comp-presv-non-blocking*[*simp*]:
  fixes $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  assumes
    non-blocking-m: *non-blocking m* **and**
    non-blocking-n: *non-blocking n*
  shows *non-blocking* $(m \rhd n)$
⟨*proof*⟩

Sequential composition preserves the non-electing property.

theorem *seq-comp-presv-non-electing*[*simp*]:
  fixes $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  assumes
    non-electing m **and**
    non-electing n
  shows *non-electing* $(m \rhd n)$
⟨*proof*⟩

Composing an electoral module that defers exactly 1 alternative in sequence after an electoral module that is electing results (still) in an electing electoral module.

theorem *seq-comp-electing*[*simp*]:
  fixes $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$
  assumes
    def-one-m: *defers 1 m* **and**
    electing-n: *electing n*
  shows *electing* $(m \rhd n)$
⟨*proof*⟩

lemma *def-lift-inv-seq-comp-help*:
  fixes
    $m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
    $A :: 'a\ set$ **and**
    $V :: 'v\ set$ **and**

   p q :: (*'a*, *'v*) *Profile* **and**
   a :: *'a*
  assumes
   monotone-m: *defer-lift-invariance m* **and**
   monotone-n: *defer-lift-invariance n* **and**
   voters-determine-n: *voters-determine-election n* **and**
   def-and-lifted: *a* ∈ (*defer* (*m* ▷ *n*) *V A p*) ∧ *lifted V A p q a*
  shows (*m* ▷ *n*) *V A p* = (*m* ▷ *n*) *V A q*
⟨*proof*⟩

Sequential composition preserves the property defer-lift-invariance.

theorem *seq-comp-presv-def-lift-inv*[*simp*]:
  fixes *m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes
   defer-lift-invariance m **and**
   defer-lift-invariance n **and**
   voters-determine-election n
  shows *defer-lift-invariance* (*m* ▷ *n*)
⟨*proof*⟩

Composing a non-blocking, non-electing electoral module in sequence with an electoral module that defers exactly one alternative results in an electoral module that defers exactly one alternative.

theorem *seq-comp-def-one*[*simp*]:
  fixes *m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes
   non-blocking-m: *non-blocking m* **and**
   non-electing-m: *non-electing m* **and**
   def-one-n: *defers 1 n*
  shows *defers 1* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

theorem *disj-compat-seq*[*simp*]:
  fixes *m m' n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes
   compatible: *disjoint-compatibility m n* **and**
   module-m': $\mathcal{SCF}$-*result.electoral-module m'* **and**
   voters-determine-m': *voters-determine-election m'*
  shows *disjoint-compatibility* (*m* ▷ *m'*) *n*
⟨*proof*⟩

theorem *seq-comp-cond-compat*[*simp*]:
  fixes *m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module*
  assumes
   dcc-m: *defer-condorcet-consistency m* **and**

nb-n: *non-blocking n* **and**
ne-n: *non-electing n*
shows *condorcet-compatibility* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-condorcet-consistent electoral module in sequence with a non-blocking and non-electing electoral module results in a defer-condorcet-consistent module.

theorem *seq-comp-dcc*[*simp*]:
  fixes *m n* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module*
  assumes
    dcc-m: *defer-condorcet-consistency m* **and**
    nb-n: *non-blocking n* **and**
    ne-n: *non-electing n*
  shows *defer-condorcet-consistency* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

theorem *seq-comp-mono*[*simp*]:
  fixes *m n* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module*
  assumes
    def-monotone-m: *defer-lift-invariance m* **and**
    non-ele-m: *non-electing m* **and**
    def-one-m: *defers 1 m* **and**
    electing-n: *electing n*
  shows *monotonicity* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-invariant-monotone electoral module in sequence before a non-electing, defer-monotone electoral module that defers exactly 1 alternative results in a defer-lift-invariant electoral module.

theorem *def-inv-mono-imp-def-lift-inv*[*simp*]:
  fixes *m n* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module*
  assumes
    strong-def-mon-m: *defer-invariant-monotonicity m* **and**
    non-electing-n: *non-electing n* **and**
    defers-one: *defers 1 n* **and**
    defer-monotone-n: *defer-monotonicity n* **and**
    voters-determine-n: *voters-determine-election n*
  shows *defer-lift-invariance* (*m* ▷ *n*)
⟨*proof*⟩

end

6.4 Parallel Composition

theory *Parallel-Composition*
  imports *Basic-Modules/Component-Types/Aggregator*
        Basic-Modules/Component-Types/Electoral-Module
begin

The parallel composition composes a new electoral module from two electoral modules combined with an aggregator. Therein, the two modules each make a decision and the aggregator combines them to a single (aggregated) result.

6.4.1 Definition

fun *parallel-composition* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$
        $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$ $'a$ *Aggregator* $\Rightarrow$
        $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  parallel-composition m n agg V A p = agg A $(m\ V\ A\ p)\ (n\ V\ A\ p)$

abbreviation *parallel* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$ $'a$ *Aggregator* $\Rightarrow$
        $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$ $('a, 'v, 'a$ *Result*$)$ *Electoral-Module*
     $(\text{-} \parallel_{\text{-}} \text{-} [50,\ 1000,\ 51]\ 50)$ **where**
  $m \parallel_a n \equiv$ *parallel-composition m n a*

6.4.2 Soundness

theorem *par-comp-sound*[*simp*]:
  fixes
    m n :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **and**
    a :: $'a$ *Aggregator*
  assumes
    $\mathcal{SCF}$-*result.electoral-module m* **and**
    $\mathcal{SCF}$-*result.electoral-module n* **and**
    aggregator a
  shows $\mathcal{SCF}$-*result.electoral-module* $(m \parallel_a n)$
⟨*proof*⟩

6.4.3 Composition Rule

Using a conservative aggregator, the parallel composition preserves the property non-electing.

theorem *conserv-agg-presv-non-electing*[*simp*]:
  fixes
    m n :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **and**
    a :: $'a$ *Aggregator*
  assumes
    non-electing-m: *non-electing m* **and**

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    non-electing-n: *non-electing n* **and**
    conservative: *agg-conservative a*
  shows *non-electing* $(m \parallel_a n)$
$\langle proof \rangle$

end

6.5   Loop Composition

theory *Loop-Composition*
  imports *Basic-Modules/Component-Types/Termination-Condition*
       Basic-Modules/Defer-Module
       Sequential-Composition
begin

The loop composition uses the same module in sequence, combined with a termination condition, until either

- the termination condition is met or

- no new decisions are made (i.e., a fixed point is reached).

6.5.1   Definition

lemma *loop-termination-helper*:
  fixes
    m acc :: $('a, \, 'v, \, 'a \; Result) \; Electoral\text{-}Module$ **and**
    t :: $'a \; Termination\text{-}Condition$ **and**
    A :: $'a \; set$ **and**
    V :: $'v \; set$ **and**
    p :: $('a, \, 'v) \; Profile$
  assumes
    $\neg \; t \; (acc \; V \; A \; p)$ **and**
    defer $(acc \rhd m) \; V \; A \; p \subset$ *defer acc V A p* **and**
    finite (*defer acc V A p*)
  shows $((acc \rhd m, \, m, \, t, \, V, \, A, \, p), \, (acc, \, m, \, t, \, V, \, A, \, p)) \in$
       measure $(\lambda \; (acc, \, m, \, t, \, V, \, A, \, p). \; card \; (defer \; acc \; V \; A \; p))$
  $\langle proof \rangle$

This function handles the accumulator for the following loop composition function.

function *loop-comp-helper* :: $('a, \, 'v, \, 'a \; Result) \; Electoral\text{-}Module \Rightarrow$
     $('a, \, 'v, \, 'a \; Result) \; Electoral\text{-}Module \Rightarrow 'a \; Termination\text{-}Condition \Rightarrow$

$('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
loop-comp-helper-finite:
finite $($*defer acc V A p*$) \wedge ($*defer* $($*acc* $\triangleright$ *m*$)$ *V A p*$) \subset ($*defer acc V A p*$)$
$\longrightarrow$ *t* $($*acc V A p*$) \Longrightarrow$
loop-comp-helper acc m t V A p $=$ *acc V A p* $\mid$
loop-comp-helper-infinite:
$\neg$ $($*finite* $($*defer acc V A p*$) \wedge ($*defer* $($*acc* $\triangleright$ *m*$)$ *V A p*$) \subset ($*defer acc V A p*$)$
$\longrightarrow$ *t* $($*acc V A p*$)) \Longrightarrow$
loop-comp-helper acc m t V A p $=$ *loop-comp-helper* $($*acc* $\triangleright$ *m*$)$ *m t V A p*
$\langle$*proof*$\rangle$
termination
$\langle$*proof*$\rangle$

lemma *loop-comp-code-helper*[*code*]:
  fixes
    m acc :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **and**
    t :: $'a$ *Termination-Condition* **and**
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile*
  shows
    loop-comp-helper acc m t V A p $=$
      $($*if t* $($*acc V A p*$) \vee \neg$ *defer* $($*acc* $\triangleright$ *m*$)$ *V A p* $\subset$ *defer acc V A p*
      $\vee$ *infinite* $($*defer acc V A p*$)$
      then acc V A p else loop-comp-helper $($*acc* $\triangleright$ *m*$)$ *m t V A p*$)$
  $\langle$*proof*$\rangle$

function *loop-composition* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$
      $'a$ *Termination-Condition* $\Rightarrow$ $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  t $(\{\}, \{\}, A)$
    $\Longrightarrow$ *loop-composition m t V A p* $=$ *defer-module V A p* $\mid$
  $\neg(t \ (\{\}, \{\}, A))$
    $\Longrightarrow$ *loop-composition m t V A p* $=$ $($*loop-comp-helper m m t*$)$ *V A p*
  $\langle$*proof*$\rangle$
termination
  $\langle$*proof*$\rangle$

abbreviation *loop* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* $\Rightarrow$
      $'a$ *Termination-Condition* $\Rightarrow$ $('a, 'v, 'a$ *Result*$)$ *Electoral-Module*
      $(- \circlearrowleft_- 50)$ **where**
  m $\circlearrowleft_t \equiv$ *loop-composition m t*

lemma *loop-comp-code*[*code*]:
  fixes
    m :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **and**
    t :: $'a$ *Termination-Condition* **and**
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile*

shows *loop-composition m t V A p =*
      (if t ({}, {}, A)
          then defer-module V A p else (loop-comp-helper m m t) V A p)
⟨*proof*⟩

lemma *loop-comp-helper-imp-partit*:
  fixes
    m acc :: (*′a, ′v, ′a Result*) *Electoral-Module* **and**
    t :: *′a Termination-Condition* **and**
    A :: *′a set* **and**
    V :: *′v set* **and**
    p :: (*′a, ′v*) *Profile* **and**
    n :: *nat*
  assumes
    module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile: *profile V A p* **and**
    module-acc: $\mathcal{SCF}$-*result.electoral-module acc* **and**
    defer-card-n: *n = card (defer acc V A p)*
  shows *well-formed-*$\mathcal{SCF}$ *A (loop-comp-helper acc m t V A p)*
⟨*proof*⟩

6.5.2 Soundness

theorem *loop-comp-sound*:
  fixes
    m :: (*′a, ′v, ′a Result*) *Electoral-Module* **and**
    t :: *′a Termination-Condition*
  assumes $\mathcal{SCF}$-*result.electoral-module m*
  shows $\mathcal{SCF}$-*result.electoral-module (m ↻$_t$)*
⟨*proof*⟩

lemma *loop-comp-helper-imp-no-def-incr*:
  fixes
    m acc :: (*′a, ′v, ′a Result*) *Electoral-Module* **and**
    t :: *′a Termination-Condition* **and**
    A :: *′a set* **and**
    V :: *′v set* **and**
    p :: (*′a, ′v*) *Profile* **and**
    n :: *nat*
  assumes
    module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
    profile: *profile V A p* **and**
    mod-acc: $\mathcal{SCF}$-*result.electoral-module acc* **and**
    card-n-defer-acc: *n = card (defer acc V A p)*
  shows *defer (loop-comp-helper acc m t) V A p ⊆ defer acc V A p*
⟨*proof*⟩

6.5.3 Lemmas

lemma *loop-comp-helper-def-lift-inv-helper*:

fixes
   m acc :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **and**
   t :: *′a Termination-Condition* **and**
   A :: *′a set* **and**
   V :: *′v set* **and**
   p :: (*′a*, *′v*) *Profile* **and**
   n :: *nat*
assumes
   monotone-m: *defer-lift-invariance m* **and**
   prof: *profile V A p* **and**
   dli-acc: *defer-lift-invariance acc* **and**
   card-n-defer: $n = card$ (*defer acc V A p*) **and**
   defer-finite: *finite* (*defer acc V A p*) **and**
   voters-determine-m: *voters-determine-election m*
shows
   $\forall$ *q a*. $a \in$ (*defer* (*loop-comp-helper acc m t*) *V A p*) $\wedge$ *lifted V A p q a* $\longrightarrow$
     (*loop-comp-helper acc m t*) *V A p* = (*loop-comp-helper acc m t*) *V A q*
⟨*proof*⟩

lemma *loop-comp-helper-def-lift-inv*:
  fixes
   m acc :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **and**
   t :: *′a Termination-Condition* **and**
   A :: *′a set* **and**
   V :: *′v set* **and**
   p q :: (*′a*, *′v*) *Profile* **and**
   a :: *′a*
  assumes
   defer-lift-invariance m **and**
   voters-determine-election m **and**
   defer-lift-invariance acc **and**
   profile V A p **and**
   lifted V A p q a **and**
   $a \in$ *defer* (*loop-comp-helper acc m t*) *V A p*
  shows (*loop-comp-helper acc m t*) *V A p* = (*loop-comp-helper acc m t*) *V A q*
  ⟨*proof*⟩

lemma *lifted-imp-fin-prof*:
  fixes
   A :: *′a set* **and**
   V :: *′v set* **and**
   p q :: (*′a*, *′v*) *Profile* **and**
   a :: *′a*
  assumes *lifted V A p q a*
  shows *finite-profile V A p*
  ⟨*proof*⟩

lemma *loop-comp-helper-presv-def-lift-inv*:
  fixes

m acc :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **and**
t :: *′a Termination-Condition*
assumes
defer-lift-invariance m **and**
voters-determine-election m **and**
defer-lift-invariance acc
shows *defer-lift-invariance* (*loop-comp-helper acc m t*)
⟨*proof*⟩

lemma *loop-comp-presv-non-electing-helper*:
fixes
m acc :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **and**
t :: *′a Termination-Condition* **and**
A :: *′a set* **and**
V :: *′v set* **and**
p :: (*′a*, *′v*) *Profile* **and**
n :: *nat*
assumes
non-electing-m: *non-electing m* **and**
non-electing-acc: *non-electing acc* **and**
prof: *profile V A p* **and**
acc-defer-card: *n = card* (*defer acc V A p*)
shows *elect* (*loop-comp-helper acc m t*) *V A p = {}*
⟨*proof*⟩

lemma *loop-comp-helper-iter-elim-def-n-helper*:
fixes
m acc :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **and**
t :: *′a Termination-Condition* **and**
A :: *′a set* **and**
V :: *′v set* **and**
p :: (*′a*, *′v*) *Profile* **and**
n x :: *nat*
assumes
non-electing-m: *non-electing m* **and**
single-elimination: *eliminates 1 m* **and**
terminate-if-n-left: ∀ *r*. *t r = (card* (*defer-r r*) *= x*) **and**
x-greater-zero: *x > 0* **and**
prof: *profile V A p* **and**
n-acc-defer-card: *n = card* (*defer acc V A p*) **and**
n-ge-x: *n ≥ x* **and**
def-card-gt-one: *card* (*defer acc V A p*) *> 1* **and**
acc-nonelect: *non-electing acc*
shows *card* (*defer* (*loop-comp-helper acc m t*) *V A p*) *= x*
⟨*proof*⟩

lemma *loop-comp-helper-iter-elim-def-n*:
fixes

   m acc :: $('a, 'v, 'a$ $Result)$ $Electoral$-$Module$ **and**
   t :: $'a$ $Termination$-$Condition$ **and**
   A :: $'a$ set **and**
   V :: $'v$ set **and**
   p :: $('a, 'v)$ $Profile$ **and**
   x :: nat
 assumes
   non-$electing$ m **and**
   $eliminates$ 1 m **and**
   $\forall$ $r.$ $(t$ $r)$ $=$ $(card$ $(defer$-r $r)$ $=$ $x)$ **and**
   $x > 0$ **and**
   $profile$ V A p **and**
   $card$ $(defer$ acc V A $p) \geq x$ **and**
   non-$electing$ acc
 shows $card$ $(defer$ $(loop$-$comp$-$helper$ acc m $t)$ V A $p) = x$
 ⟨$proof$⟩

lemma $iter$-$elim$-def-n-$helper$:
 fixes
   m :: $('a, 'v, 'a$ $Result)$ $Electoral$-$Module$ **and**
   t :: $'a$ $Termination$-$Condition$ **and**
   A :: $'a$ set **and**
   V :: $'v$ set **and**
   p :: $('a, 'v)$ $Profile$ **and**
   x :: nat
 assumes
   non-$electing$-m: non-$electing$ m **and**
   $single$-$elimination$: $eliminates$ 1 m **and**
   $terminate$-if-n-$left$: $\forall$ $r.$ $(t$ $r)$ $=$ $(card$ $(defer$-r $r)$ $=$ $x)$ **and**
   x-$greater$-$zero$: $x > 0$ **and**
   $prof$: $profile$ V A p **and**
   $enough$-$alternatives$: $card$ $A \geq x$
 shows $card$ $(defer$ $(m$ $\circlearrowleft_t)$ V A $p) = x$
⟨$proof$⟩

6.5.4 Composition Rules

The loop composition preserves defer-lift-invariance.

theorem $loop$-$comp$-$presv$-def-$lift$-inv[$simp$]:
 fixes
   m :: $('a, 'v, 'a$ $Result)$ $Electoral$-$Module$ **and**
   t :: $'a$ $Termination$-$Condition$
 assumes
   $defer$-$lift$-$invariance$ m **and**
   $voters$-$determine$-$election$ m
 shows $defer$-$lift$-$invariance$ $(m$ $\circlearrowleft_t)$
⟨$proof$⟩

The loop composition preserves the property non-electing.

theorem *loop-comp-presv-non-electing*[*simp*]:
  fixes
    $m :: ('a, \,'v, \,'a\ Result)\ Electoral\text{-}Module$ **and**
    $t :: 'a\ Termination\text{-}Condition$
  assumes *non-electing m*
  shows *non-electing* $(m \circlearrowleft_t)$
⟨*proof*⟩

theorem *iter-elim-def-n*[*simp*]:
  fixes
    $m :: ('a, \,'v, \,'a\ Result)\ Electoral\text{-}Module$ **and**
    $t :: 'a\ Termination\text{-}Condition$ **and**
    $n :: nat$
  assumes
    non-electing-m: *non-electing m* **and**
    single-elimination: *eliminates 1 m* **and**
    terminate-if-n-left: $\forall\ r.\ t\ r = (card\ (defer\text{-}r\ r) = n)$ **and**
    x-greater-zero: $n > 0$
  shows *defers n* $(m \circlearrowleft_t)$
⟨*proof*⟩

end

6.6 Maximum Parallel Composition

theory *Maximum-Parallel-Composition*
  imports *Basic-Modules/Component-Types/Maximum-Aggregator*
        Parallel-Composition
begin

This is a family of parallel compositions. It composes a new electoral module from two electoral modules combined with the maximum aggregator. Therein, the two modules each make a decision and then a partition is returned where every alternative receives the maximum result of the two input partitions. This means that, if any alternative is elected by at least one of the modules, then it gets elected, if any non-elected alternative is deferred by at least one of the modules, then it gets deferred, only alternatives rejected by both modules get rejected.

6.6.1 Definition

fun *maximum-parallel-composition* :: $('a, \,'v, \,'a\ Result)\ Electoral\text{-}Module \Rightarrow$
        $('a, \,'v, \,'a\ Result)\ Electoral\text{-}Module \Rightarrow$
        $('a, \,'v, \,'a\ Result)\ Electoral\text{-}Module$ **where**

maximum-parallel-composition m n =
  (let a = max-aggregator in (m $\parallel_a$ n))

abbreviation *max-parallel* :: *($'a$, $'v$, $'a$ Result) Electoral-Module $\Rightarrow$*
    ($'a$, $'v$, $'a$ Result) Electoral-Module $\Rightarrow$
    ($'a$, $'v$, $'a$ Result) Electoral-Module (**infix** $\parallel_\uparrow$ *50*) **where**
  m $\parallel_\uparrow$ n $\equiv$ maximum-parallel-composition m n

6.6.2   Soundness

theorem *max-par-comp-sound*:
  fixes *m n* :: *($'a$, $'v$, $'a$ Result) Electoral-Module*
  assumes
    $\mathcal{SCF}$-result.electoral-module m **and**
    $\mathcal{SCF}$-result.electoral-module n
  shows *$\mathcal{SCF}$-result.electoral-module (m $\parallel_\uparrow$ n)*
  $\langle proof \rangle$

lemma *voters-determine-max-par-comp*:
  fixes *m n* :: *($'a$, $'v$, $'a$ Result) Electoral-Module*
  assumes
    voters-determine-election m **and**
    voters-determine-election n
  shows *voters-determine-election (m $\parallel_\uparrow$ n)*
  $\langle proof \rangle$

6.6.3   Lemmas

lemma *max-agg-eq-result*:
  fixes
    m n :: *($'a$, $'v$, $'a$ Result) Electoral-Module* **and**
    A :: *$'a$ set* **and**
    V :: *$'v$ set* **and**
    p :: *($'a$, $'v$) Profile* **and**
    a :: *$'a$*
  assumes
    module-m: *$\mathcal{SCF}$-result.electoral-module m* **and**
    module-n: *$\mathcal{SCF}$-result.electoral-module n* **and**
    prof-p: *profile V A p* **and**
    a-in-A: *a $\in$ A*
  shows *mod-contains-result (m $\parallel_\uparrow$ n) m V A p a $\vee$*
      mod-contains-result (m $\parallel_\uparrow$ n) n V A p a
$\langle proof \rangle$

lemma *max-agg-rej-iff-both-reject*:
  fixes
    m n :: *($'a$, $'v$, $'a$ Result) Electoral-Module* **and**
    A :: *$'a$ set* **and**
    V :: *$'v$ set* **and**
    p :: *($'a$, $'v$) Profile* **and**

$a :: \, 'a$
assumes
  finite-profile V A p **and**
  $\mathcal{SCF}$-*result.electoral-module m* **and**
  $\mathcal{SCF}$-*result.electoral-module n*
 shows $(a \in reject \,\, (m \parallel_\uparrow n) \,\, V \, A \, p) =$
      $(a \in reject \,\, m \,\, V \, A \, p \land a \in reject \,\, n \,\, V \, A \, p)$
$\langle proof \rangle$

lemma *max-agg-rej-fst-imp-seq-contained*:
 fixes
  $m \,\, n :: \, ('a, \, 'v, \, 'a \,\, Result) \,\, Electoral\text{-}Module$ **and**
  $A :: \, 'a \,\, set$ **and**
  $V :: \, 'v \,\, set$ **and**
  $p :: \, ('a, \, 'v) \,\, Profile$ **and**
  $a :: \, 'a$
 assumes
  f-prof: *finite-profile V A p* **and**
  module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
  module-n: $\mathcal{SCF}$-*result.electoral-module n* **and**
  rejected: $a \in reject \,\, n \,\, V \, A \, p$
 shows *mod-contains-result m* $(m \parallel_\uparrow n) \,\, V \, A \, p \, a$
 $\langle proof \rangle$

lemma *max-agg-rej-fst-equiv-seq-contained*:
 fixes
  $m \,\, n :: \, ('a, \, 'v, \, 'a \,\, Result) \,\, Electoral\text{-}Module$ **and**
  $A :: \, 'a \,\, set$ **and**
  $V :: \, 'v \,\, set$ **and**
  $p :: \, ('a, \, 'v) \,\, Profile$ **and**
  $a :: \, 'a$
 assumes
  finite-profile V A p **and**
  $\mathcal{SCF}$-*result.electoral-module m* **and**
  $\mathcal{SCF}$-*result.electoral-module n* **and**
  $a \in reject \,\, n \,\, V \, A \, p$
 shows *mod-contains-result-sym* $(m \parallel_\uparrow n) \,\, m \,\, V \, A \, p \, a$
 $\langle proof \rangle$

lemma *max-agg-rej-snd-imp-seq-contained*:
 fixes
  $m \,\, n :: \, ('a, \, 'v, \, 'a \,\, Result) \,\, Electoral\text{-}Module$ **and**
  $A :: \, 'a \,\, set$ **and**
  $V :: \, 'v \,\, set$ **and**
  $p :: \, ('a, \, 'v) \,\, Profile$ **and**
  $a :: \, 'a$
 assumes
  f-prof: *finite-profile V A p* **and**
  module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**

module-n: $\mathcal{SCF}$-*result.electoral-module n* **and**
rejected: $a \in reject\ m\ V\ A\ p$
shows *mod-contains-result n* $(m \parallel_\uparrow n)\ V\ A\ p\ a$
⟨*proof*⟩

lemma *max-agg-rej-snd-equiv-seq-contained*:
fixes
$m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a,\ 'v)\ Profile$ **and**
$a :: 'a$
assumes
finite-profile V A p **and**
$\mathcal{SCF}$-*result.electoral-module m* **and**
$\mathcal{SCF}$-*result.electoral-module n* **and**
$a \in reject\ m\ V\ A\ p$
shows *mod-contains-result-sym* $(m \parallel_\uparrow n)\ n\ V\ A\ p\ a$
⟨*proof*⟩

lemma *max-agg-rej-intersect*:
fixes
$m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a,\ 'v)\ Profile$
assumes
$\mathcal{SCF}$-*result.electoral-module m* **and**
$\mathcal{SCF}$-*result.electoral-module n* **and**
profile V A p **and**
finite A
shows *reject* $(m \parallel_\uparrow n)\ V\ A\ p = (reject\ m\ V\ A\ p) \cap (reject\ n\ V\ A\ p)$
⟨*proof*⟩

lemma *dcompat-dec-by-one-mod*:
fixes
$m\ n :: ('a,\ 'v,\ 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$a :: 'a$
assumes
disjoint-compatibility m n **and**
$a \in A$
shows
$(\forall\ p.\ finite\text{-}profile\ V\ A\ p \longrightarrow mod\text{-}contains\text{-}result\ m\ (m \parallel_\uparrow n)\ V\ A\ p\ a)$
$\lor\ (\forall\ p.\ finite\text{-}profile\ V\ A\ p \longrightarrow mod\text{-}contains\text{-}result\ n\ (m \parallel_\uparrow n)\ V\ A\ p\ a)$
⟨*proof*⟩

6.6.4 Composition Rules

Using a conservative aggregator, the parallel composition preserves the property non-electing.

theorem *conserv-max-agg-presv-non-electing*[*simp*]:
  fixes m n :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module*
  assumes
    non-electing m **and**
    non-electing n
  shows *non-electing* ($m \parallel_\uparrow n$)
  ⟨*proof*⟩

Using the max aggregator, composing two compatible electoral modules in parallel preserves defer-lift-invariance.

theorem *par-comp-def-lift-inv*[*simp*]:
  fixes m n :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module*
  assumes
    compatible: *disjoint-compatibility* m n **and**
    monotone-m: *defer-lift-invariance* m **and**
    monotone-n: *defer-lift-invariance* n
  shows *defer-lift-invariance* ($m \parallel_\uparrow n$)
⟨*proof*⟩

lemma *par-comp-rej-card*:
  fixes
    m n :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module* **and**
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: ($'a$, $'v$) *Profile* **and**
    c :: *nat*
  assumes
    compatible: *disjoint-compatibility* m n **and**
    prof: *profile* V A p **and**
    fin-A: *finite* A **and**
    reject-sum: *card* (*reject* m V A p) + *card* (*reject* n V A p) = *card* A + c
  shows *card* (*reject* ($m \parallel_\uparrow n$) V A p) = c
⟨*proof*⟩

Using the max-aggregator for composing two compatible modules in parallel, whereof the first one is non-electing and defers exactly one alternative, and the second one rejects exactly two alternatives, the composition results in an electoral module that eliminates exactly one alternative.

theorem *par-comp-elim-one*[*simp*]:
  fixes m n :: ($'a$, $'v$, $'a$ *Result*) *Electoral-Module*
  assumes
    defers-m-one: *defers 1 m* **and**
    non-elec-m: *non-electing m* **and**
    rejec-n-two: *rejects 2 n* **and**

  disj-comp: *disjoint-compatibility m n*
 shows *eliminates 1 (m ∥↑ n)*
⟨*proof*⟩

end

6.7 Elect Composition

theory *Elect-Composition*
 imports *Basic-Modules/Elect-Module*
   Sequential-Composition
begin

The elect composition sequences an electoral module and the elect module.
It finalizes the module's decision as it simply elects all their non-rejected
alternatives. Thereby, any such elect-composed module induces a proper
voting rule in the social choice sense, as all alternatives are either rejected
or elected.

6.7.1 Definition

fun *elector* :: *($'a$, $'v$, $'a$ Result) Electoral-Module ⇒*
  ($'a$, $'v$, $'a$ Result) Electoral-Module **where**
 elector m = (m ▷ elect-module)

6.7.2 Auxiliary Lemmas

lemma *elector-seqcomp-assoc*:
 fixes *a b* :: *($'a$, $'v$, $'a$ Result) Electoral-Module*
 shows *(a ▷ (elector b)) = (elector (a ▷ b))*
 ⟨*proof*⟩

6.7.3 Soundness

theorem *elector-sound*[*simp*]:
 fixes *m* :: *($'a$, $'v$, $'a$ Result) Electoral-Module*
 assumes $\mathcal{SCF}$-*result.electoral-module m*
 shows $\mathcal{SCF}$-*result.electoral-module (elector m)*
 ⟨*proof*⟩

lemma *voters-determine-elector*:
 fixes *m* :: *($'a$, $'v$, $'a$ Result) Electoral-Module*
 assumes *voters-determine-election m*
 shows *voters-determine-election (elector m)*
 ⟨*proof*⟩

6.7.4 Electing

theorem *elector-electing*[*simp*]:
  fixes m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module*
  assumes
    module-m: $\mathcal{SCF}$-*result.electoral-module m* **and**
    non-block-m: *non-blocking m*
  shows *electing* (*elector m*)
⟨*proof*⟩

6.7.5 Composition Rule

If m is defer-Condorcet-consistent, then elector(m) is Condorcet consistent.

lemma *dcc-imp-cc-elector*:
  fixes m :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module*
  assumes *defer-condorcet-consistency m*
  shows *condorcet-consistency* (*elector m*)
⟨*proof*⟩

end

6.8 Defer-One Loop Composition

theory *Defer-One-Loop-Composition*
  imports *Basic-Modules/Component-Types/Defer-Equal-Condition*
        Loop-Composition
        Elect-Composition
begin

This is a family of loop compositions. It uses the same module in sequence until either no new decisions are made or only one alternative is remaining in the defer-set. The second family herein uses the above family and subsequently elects the remaining alternative.

fun *iter* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒
      (′*a*, ′*v*, ′*a Result*) *Electoral-Module* **where**
  iter m =
    (*let t = defer-equal-condition 1 in*
      ($m \circlearrowleft_t$))

abbreviation *defer-one-loop* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒
      (′*a*, ′*v*, ′*a Result*) *Electoral-Module* (-$\circlearrowleft_{\exists\,!d}$ 50) **where**
  $m \circlearrowleft_{\exists\,!d} \equiv iter\ m$

fun *iter-elect* :: (′*a*, ′*v*, ′*a Result*) *Electoral-Module* ⇒

$('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
iter-elect m = elector $(m \circlearrowleft_{\exists\,!d})$

6.8.1 Soundness

theorem *defer-one-loop-comp-sound*:
  fixes
    $m :: ('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
    $t :: 'a$ *Termination-Condition*
  assumes $\mathcal{SCF}$*-result.electoral-module m*
  shows $\mathcal{SCF}$*-result.electoral-module* $(m \circlearrowleft_{\exists\,!d})$
  $\langle proof \rangle$

end

Chapter 7

Voting Rules

7.1 Plurality Rule

theory *Plurality-Rule*
  imports *Compositional-Structures/Basic-Modules/Plurality-Module*
      Compositional-Structures/Revision-Composition
      Compositional-Structures/Elect-Composition
begin

This is a definition of the plurality voting rule as elimination module as well as directly. In the former one, the max operator of the set of the scores of all alternatives is evaluated and is used as the threshold value.

7.1.1 Definition

fun *plurality-rule* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  plurality-rule V A p = elector plurality V A p

fun *plurality-rule′* :: $('a, 'v, 'a$ *Result*$)$ *Electoral-Module* **where**
  plurality-rule′ V A p =
    $($*if finite A*
      then $(\{a \in A. \ \forall \ x \in A.$ *win-count V p x* $\leq$ *win-count V p a*$\},$
        $\{a \in A. \ \exists \ x \in A.$ *win-count V p x* $>$ *win-count V p a*$\},$
        $\{\})$
     else $(A, \{\}, \{\}))$

lemma *plurality-revision-equiv*:
  fixes
    A :: $'a$ *set* **and**
    V :: $'v$ *set* **and**
    p :: $('a, 'v)$ *Profile*
  shows *plurality V A p =* $($*plurality-rule$\downarrow$*$)$ *V A p*
  $\langle proof \rangle$

lemma *plurality′-revision-equiv*:

fixes
   $A :: \,'a\ set$ **and**
   $V :: \,'v\ set$ **and**
   $p :: (\,'a,\ 'v)\ Profile$
   shows $plurality'\ V\ A\ p = (plurality\text{-}rule'\downarrow)\ V\ A\ p$
⟨*proof*⟩

lemma *plurality-rule-equiv*:
  fixes
   $A :: \,'a\ set$ **and**
   $V :: \,'v\ set$ **and**
   $p :: (\,'a,\ 'v)\ Profile$
   shows $plurality\text{-}rule\ V\ A\ p = plurality\text{-}rule'\ V\ A\ p$
⟨*proof*⟩

7.1.2   Soundness

theorem *plurality-rule-sound*[*simp*]: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ plurality\text{-}rule$
  ⟨*proof*⟩

theorem *plurality-rule'-sound*[*simp*]: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ plurality\text{-}rule'$
⟨*proof*⟩

lemma *voters-determine-plurality-rule*: *voters-determine-election plurality-rule*
  ⟨*proof*⟩

lemma *voters-determine-plurality-rule'*: *voters-determine-election plurality-rule'*
⟨*proof*⟩

7.1.3   Electing

lemma *plurality-rule-elect-non-empty*:
  fixes
   $A :: \,'a\ set$ **and**
   $V :: \,'v\ set$ **and**
   $p :: (\,'a,\ 'v)\ Profile$
  assumes
   $A \neq \{\}$ **and**
   finite A
  shows $elect\ plurality\text{-}rule\ V\ A\ p \neq \{\}$
⟨*proof*⟩

lemma *plurality-rule'-elect-non-empty*:
  fixes
   $A :: \,'a\ set$ **and**
   $V :: \,'v\ set$ **and**
   $p :: (\,'a,\ 'v)\ Profile$
  assumes
   $A \neq \{\}$ **and**
   profile V A p **and**

finite A
shows *elect plurality-rule′ V A p ≠ {}*
⟨*proof*⟩

The plurality module is electing.

theorem *plurality-rule-electing*[*simp*]: *electing plurality-rule*
⟨*proof*⟩

theorem *plurality-rule′-electing*[*simp*]: *electing plurality-rule′*
⟨*proof*⟩

7.1.4 Properties

lemma *plurality-rule-inv-mono-eq*:
  fixes
    A :: *′a set* **and**
    V :: *′v set* **and**
    p q :: (*′a*, *′v*) *Profile* **and**
    a :: *′a*
  assumes
    elect-a: *a ∈ elect plurality-rule V A p* **and**
    lift-a: *lifted V A p q a*
  shows *elect plurality-rule V A q = elect plurality-rule V A p*
      ∨ elect plurality-rule V A q = {a}
⟨*proof*⟩

lemma *plurality-rule′-inv-mono-eq*:
  fixes
    A :: *′a set* **and**
    V :: *′v set* **and**
    p q :: (*′a*, *′v*) *Profile* **and**
    a :: *′a*
  assumes
    a ∈ elect plurality-rule′ V A p **and**
    lifted V A p q a
  shows *elect plurality-rule′ V A q = elect plurality-rule′ V A p*
      ∨ elect plurality-rule′ V A q = {a}
  ⟨*proof*⟩

The plurality rule is invariant-monotone.

theorem *plurality-rule-inv-mono*[*simp*]: *invariant-monotonicity plurality-rule*
⟨*proof*⟩

theorem *plurality-rule′-inv-mono*[*simp*]: *invariant-monotonicity plurality-rule′*
⟨*proof*⟩

(Weak) Monotonicity

theorem *plurality-rule-monotone*: *monotonicity plurality-rule*

⟨proof⟩

end

7.2 Borda Rule

theory *Borda-Rule*
  imports *Compositional-Structures/Basic-Modules/Borda-Module*
      Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization
        Compositional-Structures/Elect-Composition
begin

This is the Borda rule. On each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected.

7.2.1 Definition

fun *borda-rule* :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
  borda-rule V A p = elector borda V A p

fun *borda-rule$_\mathcal{R}$* :: $('a, 'v :: wellorder, 'a\ Result)$ *Electoral-Module* **where**
  borda-rule$_\mathcal{R}$ V A p = swap-$\mathcal{R}$ unanimity V A p

7.2.2 Soundness

theorem *borda-rule-sound*: $\mathcal{SCF}$-*result.electoral-module borda-rule*
  ⟨proof⟩

theorem *borda-rule$_\mathcal{R}$-sound*: $\mathcal{SCF}$-*result.electoral-module borda-rule$_\mathcal{R}$*
  ⟨proof⟩

7.2.3 Anonymity

theorem *borda-rule$_\mathcal{R}$-anonymous*: $\mathcal{SCF}$-*result.anonymity borda-rule$_\mathcal{R}$*
⟨proof⟩

end

7.3 Pairwise Majority Rule

theory *Pairwise-Majority-Rule*
  imports *Compositional-Structures/Basic-Modules/Condorcet-Module*
       Compositional-Structures/Defer-One-Loop-Composition
begin

This is the pairwise majority rule, a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives.

7.3.1 Definition

fun *pairwise-majority-rule* :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
  pairwise-majority-rule V A p = elector condorcet V A p

fun *condorcet$'$* :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
  condorcet$'$ V A p = ((min-eliminator condorcet-score) $\circlearrowleft_{\exists\,!d}$) V A p

fun *pairwise-majority-rule$'$* :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **where**
  pairwise-majority-rule$'$ V A p = iter-elect condorcet$'$ V A p

7.3.2 Soundness

theorem *pairwise-majority-rule-sound*: $\mathcal{SCF}$-*result.electoral-module pairwise-majority-rule*
  ⟨*proof*⟩

theorem *condorcet$'$-sound*: $\mathcal{SCF}$-*result.electoral-module condorcet$'$*
  ⟨*proof*⟩

theorem *pairwise-majority-rule$'$-sound*: $\mathcal{SCF}$-*result.electoral-module pairwise-majority-rule$'$*
  ⟨*proof*⟩

7.3.3 Condorcet Consistency

theorem *condorcet-condorcet*: *condorcet-consistency pairwise-majority-rule*
⟨*proof*⟩

end

7.4 Copeland Rule

theory *Copeland-Rule*
  imports *Compositional-Structures/Basic-Modules/Copeland-Module*
       Compositional-Structures/Elect-Composition
begin

This is the Copeland voting rule. The idea is to elect the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses.

7.4.1   Definition

fun *copeland-rule* :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **where**
  copeland-rule V A p = elector copeland V A p

7.4.2   Soundness

theorem *copeland-rule-sound*: $\mathcal{SCF}$*-result.electoral-module copeland-rule*
  ⟨*proof*⟩

7.4.3   Condorcet Consistency

theorem *copeland-condorcet*: *condorcet-consistency copeland-rule*
⟨*proof*⟩

end

7.5   Minimax Rule

theory *Minimax-Rule*
  imports *Compositional-Structures/Basic-Modules/Minimax-Module*
        Compositional-Structures/Elect-Composition
begin

This is the Minimax voting rule. It elects the alternatives with the highest Minimax score.

7.5.1   Definition

fun *minimax-rule* :: (*′a*, *′v*, *′a Result*) *Electoral-Module* **where**
  minimax-rule V A p = elector minimax V A p

7.5.2   Soundness

theorem *minimax-rule-sound*: $\mathcal{SCF}$*-result.electoral-module minimax-rule*
  ⟨*proof*⟩

7.5.3   Condorcet Consistency

theorem *minimax-condorcet*: *condorcet-consistency minimax-rule*
⟨*proof*⟩

end

7.6 Black's Rule

theory *Blacks-Rule*
  imports *Pairwise-Majority-Rule*
        Borda-Rule
begin

This is Black's voting rule. It is composed of a function that determines the Condorcet winner, i.e., the Pairwise Majority rule, and the Borda rule. Whenever there exists no Condorcet winner, it elects the choice made by the Borda rule, otherwise the Condorcet winner is elected.

7.6.1 Definition

fun *black* :: $(\prime a, \prime v, \prime a\ Result)\ Electoral\text{-}Module$ **where**
  black A p = (condorcet ▷ borda) A p

fun *blacks-rule* :: $(\prime a, \prime v, \prime a\ Result)\ Electoral\text{-}Module$ **where**
  blacks-rule A p = elector black A p

7.6.2 Soundness

theorem *blacks-sound*: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ black$
  ⟨*proof*⟩

theorem *blacks-rule-sound*: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ blacks\text{-}rule$
  ⟨*proof*⟩

7.6.3 Condorcet Consistency

theorem *black-is-dcc*: *defer-condorcet-consistency black*
  ⟨*proof*⟩

theorem *black-condorcet*: *condorcet-consistency blacks-rule*
  ⟨*proof*⟩

end

7.7 Nanson-Baldwin Rule

theory *Nanson-Baldwin-Rule*
  imports *Compositional-Structures/Basic-Modules/Borda-Module*
        Compositional-Structures/Defer-One-Loop-Composition
begin

This is the Nanson-Baldwin voting rule. It excludes alternatives with the lowest Borda score from the set of possible winners and then adjusts the Borda score to the new (remaining) set of still eligible alternatives.

7.7.1 Definition

fun *nanson-baldwin-rule* :: $('a, 'v, 'a$ *Result$)$ Electoral-Module* **where**
  nanson-baldwin-rule A p =
    $((\textit{min-eliminator borda-score})\ \circlearrowleft_{\exists\,!d})\ A\ p$

7.7.2 Soundness

theorem *nanson-baldwin-rule-sound*: $\mathcal{SCF}$-*result.electoral-module nanson-baldwin-rule*
  $\langle proof \rangle$

end

7.8 Classic Nanson Rule

theory *Classic-Nanson-Rule*
  imports *Compositional-Structures/Basic-Modules/Borda-Module*
        Compositional-Structures/Defer-One-Loop-Composition
begin

This is the classic Nanson's voting rule, i.e., the rule that was originally invented by Nanson, but not the Nanson-Baldwin rule. The idea is similar, however, as alternatives with a Borda score less or equal than the average Borda score are excluded. The Borda scores of the remaining alternatives are hence adjusted to the new set of (still) eligible alternatives.

7.8.1 Definition

fun *classic-nanson-rule* :: $('a, 'v, 'a$ *Result$)$ Electoral-Module* **where**
  classic-nanson-rule V A p =
    $((\textit{leq-average-eliminator borda-score})\ \circlearrowleft_{\exists\,!d})\ V\ A\ p$

7.8.2 Soundness

theorem *classic-nanson-rule-sound*: $\mathcal{SCF}$*-result.electoral-module classic-nanson-rule*
  ⟨*proof*⟩

end

7.9 Schwartz Rule

theory *Schwartz-Rule*
  imports *Compositional-Structures/Basic-Modules/Borda-Module*
       Compositional-Structures/Defer-One-Loop-Composition
begin

This is the Schwartz voting rule. Confusingly, it is sometimes also referred as Nanson's rule. The Schwartz rule proceeds as in the classic Nanson's rule, but excludes alternatives with a Borda score that is strictly less than the average Borda score.

7.9.1 Definition

fun *schwartz-rule* :: $('a,\ 'v,\ 'a\ Result)$ *Electoral-Module* **where**
  schwartz-rule V A p =
    $((\textit{less-average-eliminator borda-score})\ \circlearrowleft_{\exists\,!d})\ V\ A\ p$

7.9.2 Soundness

theorem *schwartz-rule-sound*: $\mathcal{SCF}$*-result.electoral-module schwartz-rule*
  ⟨*proof*⟩

end

7.10 Sequential Majority Comparison

theory *Sequential-Majority-Comparison*
  imports *Plurality-Rule*
       Compositional-Structures/Drop-And-Pass-Compatibility
       Compositional-Structures/Revision-Composition
       Compositional-Structures/Maximum-Parallel-Composition
       Compositional-Structures/Defer-One-Loop-Composition
begin

Sequential majority comparison compares two alternatives by plurality voting. The loser gets rejected, and the winner is compared to the next alternative. This process is repeated until only a single alternative is left, which is then elected.

7.10.1 Definition

fun *smc* :: *$'a$ Preference-Relation* $\Rightarrow$ (*$'a$, $'v$, $'a$ Result*) *Electoral-Module* **where**
  smc x V A p =
    ((*elector* ((((*pass-module 2 x*) $\triangleright$ ((*plurality-rule$\downarrow$*) $\triangleright$ (*pass-module 1 x*))) $\parallel_\uparrow$
    (*drop-module 2 x*)) $\circlearrowleft_{\exists\,!d}$)) *V A p*)

7.10.2 Soundness

As all base components are electoral modules (, aggregators, or termination conditions), and all used compositional structures create electoral modules, sequential majority comparison unsurprisingly is an electoral module.

theorem *smc-sound*:
  fixes *x* :: *$'a$ Preference-Relation*
  shows $\mathcal{SCF}$-*result.electoral-module* (*smc x*)
⟨*proof*⟩

7.10.3 Electing

The sequential majority comparison electoral module is electing. This property is needed to convert electoral modules to a social choice function. Apart from the very last proof step, it is a part of the monotonicity proof below.

theorem *smc-electing*:
  fixes *x* :: *$'a$ Preference-Relation*
  assumes *linear-order x*
  shows *electing* (*smc x*)
⟨*proof*⟩

7.10.4 (Weak) Monotonicity

The following proof is a fully modular proof for weak monotonicity of sequential majority comparison. It is composed of many small steps.

theorem *smc-monotone*:
  fixes *x* :: *$'a$ Preference-Relation*
  assumes *linear-order x*
  shows *monotonicity* (*smc x*)
⟨*proof*⟩

end

7.11 Kemeny Rule

theory *Kemeny-Rule*
  imports
   Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization
   Compositional-Structures/Basic-Modules/Component-Types/Distance-Rationalization-Symmetry
begin

This is the Kemeny rule. It creates a complete ordering of alternatives and evaluates each ordering of the alternatives in terms of the sum of preference reversals on each ballot that would have to be performed in order to produce that transitive ordering. The complete ordering which requires the fewest preference reversals is the final result of the method.

7.11.1 Definition

fun *kemeny-rule* :: $('a, 'v :: wellorder, 'a\ Result)$ *Electoral-Module* **where**
  kemeny-rule V A p = swap-$\mathcal{R}$ strong-unanimity V A p

7.11.2 Soundness

theorem *kemeny-rule-sound*: $\mathcal{SCF}$-*result.electoral-module kemeny-rule*
  ⟨*proof*⟩

7.11.3 Anonymity

theorem *kemeny-rule-anonymous*: $\mathcal{SCF}$-*result.anonymity kemeny-rule*
⟨*proof*⟩

7.11.4 Neutrality

lemma *swap-dist-neutral*: *distance-neutrality well-formed-elections*
                         (*votewise-distance swap l-one*)
  ⟨*proof*⟩

theorem *kemeny-rule-neutral*: $\mathcal{SCF}$-*properties.neutrality kemeny-rule*
  ⟨*proof*⟩

end

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