# Verified Construction of Fair Voting Rules

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**Abstract**

Voting rules aggregate multiple individual preferences in order to make a collective decision. Commonly, these mechanisms are expected to respect a multitude of different notions of fairness and reliability, which must be carefully balanced to avoid inconsistencies.

This article contains a formalisation of a framework for the construction of such fair voting rules using composable modules [1, 2]. The framework is a formal and systematic approach for the flexible and verified construction of voting rules from individual composable modules to respect such social-choice properties by construction. Formal composition rules guarantee resulting social-choice properties from properties of the individual components which are of generic nature to be reused for various voting rules. We provide proofs for a selected set of structures and composition rules. The approach can be readily extended in order to support more voting rules, e.g., from the literature by extending the sets of modules and composition rules.

# Contents

# Chapter 1

# Social-Choice Types

## 1.1 Auxiliary Lemmas

**theory** *Auxiliary-Lemmas*
**imports** *Main*
**begin**

**lemma** *sum-comp*:
**fixes**
$f :: 'x \Rightarrow 'z :: comm\text{-}monoid\text{-}add$ **and**
$g :: 'y \Rightarrow 'x$ **and**
$X :: 'x\ set$ **and**
$Y :: 'y\ set$
**assumes** *bij-betw g Y X*
**shows** $sum\ f\ X = sum\ (f \circ g)\ Y$
⟨*proof*⟩

**lemma** *the-inv-comp*:
**fixes**
$f :: 'y \Rightarrow 'z$ **and**
$g :: 'x \Rightarrow 'y$ **and**
$s :: 'x\ set$ **and**
$t :: 'y\ set$ **and**
$u :: 'z\ set$ **and**
$x :: 'z$
**assumes**
*bij-betw f t u* **and**
*bij-betw g s t* **and**
$x \in u$
**shows** $the\text{-}inv\text{-}into\ s\ (f \circ g)\ x = ((the\text{-}inv\text{-}into\ s\ g) \circ (the\text{-}inv\text{-}into\ t\ f))\ x$
⟨*proof*⟩

**end**

## 1.2 Preference Relation

**theory** *Preference-Relation*
  **imports** *Main*
**begin**

The very core of the composable modules voting framework: types and functions, derivations, lemmas, operations on preference relations, etc.

### 1.2.1 Definition

Each voter expresses pairwise relations between all alternatives, thereby inducing a linear order.

**type-synonym** *'a Preference-Relation = 'a rel*

**type-synonym** *'a Vote = 'a set × 'a Preference-Relation*

**fun** *is-less-preferred-than* :: *'a ⇒ 'a Preference-Relation ⇒ 'a ⇒ bool*
      *(- ≼- - [50, 1000, 51] 50)* **where**
    *a ≼r b = ((a, b) ∈ r)*

**fun** *alts-𝒱* :: *'a Vote ⇒ 'a set* **where**
  *alts-𝒱 V = fst V*

**fun** *pref-𝒱* :: *'a Vote ⇒ 'a Preference-Relation* **where**
  *pref-𝒱 V = snd V*

**lemma** *lin-imp-antisym*:
  **fixes**
    *A* :: *'a set* **and**
    *r* :: *'a Preference-Relation*
  **assumes** *linear-order-on A r*
  **shows** *antisym r*
  ⟨*proof*⟩

**lemma** *lin-imp-trans*:
  **fixes**
    *A* :: *'a set* **and**
    *r* :: *'a Preference-Relation*
  **assumes** *linear-order-on A r*
  **shows** *trans r*
  ⟨*proof*⟩

### 1.2.2 Ranking

**fun** *rank* :: *'a Preference-Relation* ⇒ *'a* ⇒ *nat* **where**
*rank r a* = *card* (*above r a*)

**lemma** *rank-gt-zero*:
**fixes**
*r* :: *'a Preference-Relation* **and**
*a* :: *'a*
**assumes**
*refl*: *a* ⪯$_r$ *a* **and**
*fin*: *finite r*
**shows** *rank r a* ≥ *1*
⟨*proof*⟩

### 1.2.3 Limited Preference

**definition** *limited* :: *'a set* ⇒ *'a Preference-Relation* ⇒ *bool* **where**
*limited A r* ≡ *r* ⊆ *A* × *A*

**lemma** *limited-dest*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes**
*a* ⪯$_r$ *b* **and**
*limited A r*
**shows** *a* ∈ *A* ∧ *b* ∈ *A*
⟨*proof*⟩

**fun** *limit* :: *'a set* ⇒ *'a Preference-Relation* ⇒ *'a Preference-Relation* **where**
*limit A r* = {(*a*, *b*) ∈ *r*. *a* ∈ *A* ∧ *b* ∈ *A*}

**definition** *connex* :: *'a set* ⇒ *'a Preference-Relation* ⇒ *bool* **where**
*connex A r* ≡ *limited A r* ∧ (∀ *a* ∈ *A*. ∀ *b* ∈ *A*. *a* ⪯$_r$ *b* ∨ *b* ⪯$_r$ *a*)

**lemma** *connex-imp-refl*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** *connex A r*
**shows** *refl-on A r*
⟨*proof*⟩

**lemma** *lin-ord-imp-connex*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** *linear-order-on A r*

**shows** *connex A r*
⟨*proof*⟩

**lemma** *connex-antsym-and-trans-imp-lin-ord*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes**
*connex-r*: *connex A r* **and**
*antisym-r*: *antisym r* **and**
*trans-r*: *trans r*
**shows** *linear-order-on A r*
⟨*proof*⟩

**lemma** *limit-to-limits*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**shows** *limited A* (*limit A r*)
⟨*proof*⟩

**lemma** *limit-presv-connex*:
**fixes**
*A B* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes**
*connex*: *connex B r* **and**
*subset*: $A \subseteq B$
**shows** *connex A* (*limit A r*)
⟨*proof*⟩

**lemma** *limit-presv-antisym*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** *antisym r*
**shows** *antisym* (*limit A r*)
⟨*proof*⟩

**lemma** *limit-presv-trans*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** *trans r*
**shows** *trans* (*limit A r*)
⟨*proof*⟩

**lemma** *limit-presv-lin-ord*:
**fixes**

*A B* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes**
*linear-order-on B r* **and**
$A \subseteq B$
**shows** *linear-order-on A* (*limit A r*)
⟨*proof*⟩

**lemma** *limit-presv-prefs*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes**
$a \preceq_r b$ **and**
$a \in A$ **and**
$b \in A$
**shows** *let s* = *limit A r in a* $\preceq_s$ *b*
⟨*proof*⟩

**lemma** *limit-rel-presv-prefs*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes** (*a*, *b*) ∈ *limit A r*
**shows** $a \preceq_r b$
⟨*proof*⟩

**lemma** *limit-trans*:
**fixes**
*A B* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** $A \subseteq B$
**shows** *limit A r* = *limit A* (*limit B r*)
⟨*proof*⟩

**lemma** *lin-ord-not-empty*:
**fixes** *r* :: *'a Preference-Relation*
**assumes** $r \neq \{\}$
**shows** ¬ *linear-order-on* {} *r*
⟨*proof*⟩

**lemma** *lin-ord-singleton*:
**fixes** *a* :: *'a*
**shows** ∀ *r*. *linear-order-on* {*a*} *r* ⟶ *r* = {(*a*, *a*)}
⟨*proof*⟩

### 1.2.4 Auxiliary Lemmas

**lemma** *above-trans*:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$a\ b :: 'a$
**assumes**
*trans* $r$ **and**
$(a,\ b) \in r$
**shows** *above* $r\ b \subseteq$ *above* $r\ a$
⟨*proof*⟩

**lemma** *above-refl*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes**
*refl-on* $A\ r$ **and**
$a \in A$
**shows** $a \in$ *above* $r\ a$
⟨*proof*⟩

**lemma** *above-subset-geq-one*:
**fixes**
$A :: 'a$ *set* **and**
$r\ r' :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes**
*linear-order-on* $A\ r$ **and**
*linear-order-on* $A\ r'$ **and**
*above* $r\ a \subseteq$ *above* $r'\ a$ **and**
*above* $r'\ a = \{a\}$
**shows** *above* $r\ a = \{a\}$
⟨*proof*⟩

**lemma** *above-connex*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes**
*connex* $A\ r$ **and**
$a \in A$
**shows** $a \in$ *above* $r\ a$
⟨*proof*⟩

**lemma** *pref-imp-in-above*:
**fixes**
$r :: 'a$ *Preference-Relation* **and**

*a b* :: *'a*
**shows** $(a \preceq_r b) = (b \in \mathit{above}\ r\ a)$
⟨*proof*⟩

**lemma** *limit-presv-above*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes**
$b \in \mathit{above}\ r\ a$ **and**
$a \in A$ **and**
$b \in A$
**shows** $b \in \mathit{above}\ (\mathit{limit}\ A\ r)\ a$
⟨*proof*⟩

**lemma** *limit-rel-presv-above*:
**fixes**
*A B* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes** $b \in \mathit{above}\ (\mathit{limit}\ B\ r)\ a$
**shows** $b \in \mathit{above}\ r\ a$
⟨*proof*⟩

**lemma** *above-one*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes**
*lin-ord-r*: *linear-order-on A r* **and**
*fin-A*: *finite A* **and**
*non-empty-A*: $A \neq \{\}$
**shows** $\exists\ a \in A.\ \mathit{above}\ r\ a = \{a\} \wedge (\forall\ a' \in A.\ \mathit{above}\ r\ a' = \{a'\} \longrightarrow a' = a)$
⟨*proof*⟩

**lemma** *above-one-eq*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation* **and**
*a b* :: *'a*
**assumes**
*lin-ord*: *linear-order-on A r* **and**
*fin-A*: *finite A* **and**
*not-empty-A*: $A \neq \{\}$ **and**
*above-a*: $\mathit{above}\ r\ a = \{a\}$ **and**
*above-b*: $\mathit{above}\ r\ b = \{b\}$
**shows** $a = b$
⟨*proof*⟩

**lemma** *above-one-imp-rank-one*:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes** *above* $r\ a = \{a\}$
**shows** *rank* $r\ a = 1$
⟨*proof*⟩

**lemma** *rank-one-imp-above-one*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes**
*lin-ord*: *linear-order-on* $A\ r$ **and**
*rank-one*: *rank* $r\ a = 1$
**shows** *above* $r\ a = \{a\}$
⟨*proof*⟩

**theorem** *above-rank*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**assumes** *linear-order-on* $A\ r$
**shows** $(above\ r\ a = \{a\}) = (rank\ r\ a = 1)$
⟨*proof*⟩

**lemma** *rank-unique*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a\ b :: 'a$
**assumes**
*lin-ord*: *linear-order-on* $A\ r$ **and**
*fin-A*: *finite* $A$ **and**
*a-in-A*: $a \in A$ **and**
*b-in-A*: $b \in A$ **and**
*a-neq-b*: $a \neq b$
**shows** *rank* $r\ a \neq rank\ r\ b$
⟨*proof*⟩

**lemma** *above-presv-limit*:
**fixes**
$A :: 'a$ *set* **and**
$r :: 'a$ *Preference-Relation* **and**
$a :: 'a$
**shows** *above* $(limit\ A\ r)\ a \subseteq A$

⟨*proof*⟩

### 1.2.5 Lifting Property

**definition** *equiv-rel-except-a* :: *'a set* ⇒ *'a Preference-Relation* ⇒
*'a Preference-Relation* ⇒ *'a* ⇒ *bool* **where**
*equiv-rel-except-a A r r' a* ≡
*linear-order-on A r* ∧ *linear-order-on A r'* ∧ *a* ∈ *A* ∧
(∀ *a'* ∈ *A* − {*a*}. ∀ *b'* ∈ *A* − {*a*}. (*a'* ≼*r b'*) = (*a'* ≼*r'* *b'*))

**definition** *lifted* :: *'a set* ⇒ *'a Preference-Relation* ⇒ *'a Preference-Relation* ⇒
*'a* ⇒ *bool* **where**
*lifted A r r' a* ≡
*equiv-rel-except-a A r r' a* ∧ (∃ *a'* ∈ *A* − {*a*}. *a* ≼*r a'* ∧ *a'* ≼*r'* *a*)

**lemma** *trivial-equiv-rel*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a Preference-Relation*
**assumes** *linear-order-on A r*
**shows** ∀ *a* ∈ *A*. *equiv-rel-except-a A r r a*
⟨*proof*⟩

**lemma** *lifted-imp-equiv-rel-except-a*:
**fixes**
*A* :: *'a set* **and**
*r r'* :: *'a Preference-Relation* **and**
*a* :: *'a*
**assumes** *lifted A r r' a*
**shows** *equiv-rel-except-a A r r' a*
⟨*proof*⟩

**lemma** *lifted-imp-switched*:
**fixes**
*A* :: *'a set* **and**
*r r'* :: *'a Preference-Relation* **and**
*a* :: *'a*
**assumes** *lifted A r r' a*
**shows** ∀ *a'* ∈ *A* − {*a*}. ¬ (*a'* ≼*r a* ∧ *a* ≼*r'* *a'*)
⟨*proof*⟩

**lemma** *lifted-mono*:
**fixes**
*A* :: *'a set* **and**
*r r'* :: *'a Preference-Relation* **and**
*a a'* :: *'a*
**assumes**
*lifted*: *lifted A r r' a* **and**
*a'-pref-a*: *a'* ≼*r a*

**shows** $a' \preceq_r' a$
⟨*proof*⟩

**lemma** *lifted-above-subset*:
**fixes**
$A :: 'a\ set$ **and**
$r\ r' :: 'a\ Preference\text{-}Relation$ **and**
$a :: 'a$
**assumes** *lifted* $A\ r\ r'\ a$
**shows** *above* $r'\ a \subseteq$ *above* $r\ a$
⟨*proof*⟩

**lemma** *lifted-above-mono*:
**fixes**
$A :: 'a\ set$ **and**
$r\ r' :: 'a\ Preference\text{-}Relation$ **and**
$a\ a' :: 'a$
**assumes**
*lifted-a*: *lifted* $A\ r\ r'\ a$ **and**
*a'-in-A-sub-a*: $a' \in A - \{a\}$
**shows** *above* $r\ a' \subseteq$ *above* $r'\ a' \cup \{a\}$
⟨*proof*⟩

**lemma** *limit-lifted-imp-eq-or-lifted*:
**fixes**
$A\ A' :: 'a\ set$ **and**
$r\ r' :: 'a\ Preference\text{-}Relation$ **and**
$a :: 'a$
**assumes**
*lifted*: *lifted* $A'\ r\ r'\ a$ **and**
*subset*: $A \subseteq A'$
**shows** *limit* $A\ r =$ *limit* $A\ r' \vee$ *lifted* $A$ (*limit* $A\ r$) (*limit* $A\ r'$) $a$
⟨*proof*⟩

**lemma** *negl-diff-imp-eq-limit*:
**fixes**
$A\ A' :: 'a\ set$ **and**
$r\ r' :: 'a\ Preference\text{-}Relation$ **and**
$a :: 'a$
**assumes**
*change*: *equiv-rel-except-a* $A'\ r\ r'\ a$ **and**
*subset*: $A \subseteq A'$ **and**
*not-in-A*: $a \notin A$
**shows** *limit* $A\ r =$ *limit* $A\ r'$
⟨*proof*⟩

**theorem** *lifted-above-winner-alts*:
**fixes**
$A :: 'a\ set$ **and**

$r\ r'$ :: $'a$ *Preference-Relation* **and**
$a\ a'$ :: $'a$
**assumes**
*lifted-a*: *lifted* $A\ r\ r'\ a$ **and**
$a'$*-above-a'*: *above* $r\ a' = \{a'\}$ **and**
*fin-A*: *finite* $A$
**shows** *above* $r'\ a' = \{a'\} \vee$ *above* $r'\ a = \{a\}$
⟨*proof*⟩

**theorem** *lifted-above-winner-single*:
**fixes**
$A$ :: $'a$ *set* **and**
$r\ r'$ :: $'a$ *Preference-Relation* **and**
$a$ :: $'a$
**assumes**
*lifted* $A\ r\ r'\ a$ **and**
*above* $r\ a = \{a\}$ **and**
*finite* $A$
**shows** *above* $r'\ a = \{a\}$
⟨*proof*⟩

**theorem** *lifted-above-winner-other*:
**fixes**
$A$ :: $'a$ *set* **and**
$r\ r'$ :: $'a$ *Preference-Relation* **and**
$a\ a'$ :: $'a$
**assumes**
*lifted-a*: *lifted* $A\ r\ r'\ a$ **and**
$a'$*-above-a'*: *above* $r'\ a' = \{a'\}$ **and**
*fin-A*: *finite* $A$ **and**
*a-not-a'*: $a \neq a'$
**shows** *above* $r\ a' = \{a'\}$
⟨*proof*⟩

**end**

## 1.3 Norm

**theory** *Norm*
**imports** *HOL−Library.Extended-Real*
*HOL−Combinatorics.List-Permutation*
*Auxiliary-Lemmas*
**begin**

A norm on R to n is a mapping $N$: $R \mapsto n$ on R that has the following properties for all mappings $u$ (and $v$) in $R$ to $n$:

- positive scalability: $N(a * u) = |a| * N(u)$ for all $a$ in $R$.
- positive semidefiniteness: $N(u) \geq 0$ with $N(u) = 0$ if and only if $u = (0, 0, \ldots, 0)$.
- triangle inequality: $N(u + v) \leq N(u) + N(v)$.

### 1.3.1 Definition

**type-synonym** *Norm* = *ereal list* ⇒ *ereal*

**definition** *norm* :: *Norm* ⇒ *bool* **where**
*norm n* ≡ ∀ (*x*::*ereal list*). *n x* ≥ *0* ∧ (∀ *i* < *length x*. (*x*!*i* = *0*) ⟶ *n x* = *0*)

### 1.3.2 Auxiliary Lemmas

**lemma** *sum-over-image-of-bijection*:
**fixes**
*A* :: *'a set* **and**
*A'* :: *'b set* **and**
*f* :: *'a* ⇒ *'b* **and**
*g* :: *'a* ⇒ *ereal*
**assumes** *bij-betw f A A'*
**shows** (∑ *a* ∈ *A*. *g a*) = (∑ *a'* ∈ *A'*. *g* (*the-inv-into A f a'*))
⟨*proof*⟩

### 1.3.3 Common Norms

**fun** *l-one* :: *Norm* **where**
*l-one x* = (∑ *i* < *length x*. |*x*!*i*|)

### 1.3.4 Properties

**definition** *symmetry* :: *Norm* ⇒ *bool* **where**
*symmetry n* ≡ ∀ *x y*. *x* <~~> *y* ⟶ *n x* = *n y*

### 1.3.5 Theorems

**theorem** *l-one-is-sym*: *symmetry l-one*
⟨*proof*⟩

**end**

## 1.4 Electoral Result

**theory** *Result*
  **imports** *Main*
**begin**

An electoral result is the principal result type of the composable modules voting framework, as it is a generalization of the set of winning alternatives from social choice functions. Electoral results are selections of the received (possibly empty) set of alternatives into the three disjoint groups of elected, rejected and deferred alternatives. Any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives.

### 1.4.1 Auxiliary Functions

**type-synonym** *'r Result* = *'r set* ∗ *'r set* ∗ *'r set*

A partition of a set A are pairwise disjoint sets that "set equals partition" A. For this specific predicate, we have three disjoint sets in a three-tuple.

**fun** *disjoint3* :: *'r Result* ⇒ *bool* **where**
  *disjoint3* (*e*, *r*, *d*) =
    ((*e* ∩ *r* = {}) ∧
      (*e* ∩ *d* = {}) ∧
      (*r* ∩ *d* = {}))

**fun** *set-equals-partition* :: *'r set* ⇒*'r Result* ⇒ *bool* **where**
  *set-equals-partition X* (*e*, *r*, *d*) = (*e* ∪ *r* ∪ *d* = *X*)

### 1.4.2 Definition

A result generally is related to the alternative set A (of type 'a). A result should be well-formed on the alternatives. Also it should be possible to limit a well-formed result to a subset of the alternatives.

Specific result types like social choice results (sets of alternatives) can be realized via sublocales of the result locale.

**locale** *result* =
  **fixes**
    *well-formed* :: *'a set* ⇒ (*'r Result*) ⇒ *bool* **and**
    *limit* :: *'a set* ⇒ *'r set* ⇒ *'r set*
  **assumes** ∀ (*A* :: *'a set*) (*r* :: *'r Result*).
    (*set-equals-partition* (*limit A UNIV*) *r* ∧ *disjoint3 r*) ⟶ *well-formed A r*

These three functions return the elect, reject, or defer set of a result.

**fun** (**in** *result*) $limit_{\mathcal{R}}$ :: *'a set* ⇒ *'r Result* ⇒ *'r Result* **where**
  $limit_{\mathcal{R}}$ *A* (*e*, *r*, *d*) = (*limit A e*, *limit A r*, *limit A d*)

**abbreviation** *elect-r* :: *'r Result* ⇒ *'r set* **where**
*elect-r r* ≡ *fst r*

**abbreviation** *reject-r* :: *'r Result* ⇒ *'r set* **where**
*reject-r r* ≡ *fst* (*snd r*)

**abbreviation** *defer-r* :: *'r Result* ⇒ *'r set* **where**
*defer-r r* ≡ *snd* (*snd r*)

**end**

## 1.5 Preference Profile

**theory** *Profile*
**imports** *Preference-Relation*
*Auxiliary-Lemmas*
*HOL−Library.Extended-Nat*
*HOL−Combinatorics.Permutations*
**begin**

Preference profiles denote the decisions made by the individual voters on the eligible alternatives. They are represented in the form of one preference relation (e.g., selected on a ballot) per voter, collectively captured in a mapping of voters onto their respective preference relations. If there are finitely many voters, they can be enumerated and the mapping can be interpreted as a list of preference relations. Unlike the common preference profiles in the social-choice sense, the profiles described here consider only the (sub-)set of alternatives that are received.

### 1.5.1 Definition

A profile contains one ballot for each voter. An election consists of a set of participating voters, a set of eligible alternatives, and a corresponding profile.

**type-synonym** ('*a*, '*v*) *Profile* = *'v* ⇒ (*'a Preference-Relation*)

**type-synonym** ('*a*, '*v*) *Election* = *'a set* × *'v set* × ('*a*, '*v*) *Profile*

**fun** *alternatives-*$\mathcal{E}$ :: ('*a*, '*v*) *Election* ⇒ *'a set* **where**
*alternatives-*$\mathcal{E}$ *E* = *fst E*

**fun** *voters-*$\mathcal{E}$ :: ('*a*, '*v*) *Election* ⇒ *'v set* **where**

*voters-$\mathcal{E}$ E = fst (snd E)*

**fun** *profile-$\mathcal{E}$* :: *('a, 'v) Election ⇒ ('a, 'v) Profile* **where**
*profile-$\mathcal{E}$ E = snd (snd E)*

**fun** *election-equality* :: *('a, 'v) Election ⇒ ('a, 'v) Election ⇒ bool* **where**
*election-equality (A, V, p) (A', V', p') =*
*(A = A' ∧ V = V' ∧ (∀ v ∈ V. p v = p' v))*

A profile on a set of alternatives A and a voter set V consists of ballots that are linear orders on A for all voters in V. A finite profile is one with finitely many alternatives and voters.

**definition** *profile* :: *'v set ⇒ 'a set ⇒ ('a, 'v) Profile ⇒ bool* **where**
*profile V A p ≡ ∀ v ∈ V. linear-order-on A (p v)*

**abbreviation** *finite-profile* :: *'v set ⇒ 'a set ⇒ ('a, 'v) Profile ⇒ bool* **where**
*finite-profile V A p ≡ finite A ∧ finite V ∧ profile V A p*

**abbreviation** *finite-election* :: *('a, 'v) Election ⇒ bool* **where**
*finite-election E ≡ finite-profile (voters-$\mathcal{E}$ E) (alternatives-$\mathcal{E}$ E) (profile-$\mathcal{E}$ E)*

**definition** *finite-elections-$\mathcal{V}$* :: *('a, 'v) Election set* **where**
*finite-elections-$\mathcal{V}$ ≡ {E :: ('a, 'v) Election. finite (voters-$\mathcal{E}$ E)}*

**definition** *finite-elections* :: *('a, 'v) Election set* **where**
*finite-elections ≡ {E :: ('a, 'v) Election. finite-election E}*

**definition** *well-formed-elections* :: *('a, 'v) Election set* **where**
*well-formed-elections ≡ {E. profile (voters-$\mathcal{E}$ E) (alternatives-$\mathcal{E}$ E) (profile-$\mathcal{E}$ E)}*

— This function subsumes elections with fixed alternatives, finite voters, and a default value for the profile value on non-voters.
**fun** *elections-$\mathcal{A}$* :: *'a set ⇒ ('a, 'v) Election set* **where**
*elections-$\mathcal{A}$ A =*
*well-formed-elections*
*∩ {E. alternatives-$\mathcal{E}$ E = A ∧ finite (voters-$\mathcal{E}$ E)*
*∧ (∀ v. v ∉ voters-$\mathcal{E}$ E ⟶ profile-$\mathcal{E}$ E v = {})}*

— Here, we count the occurrences of a ballot in an election, i.e., how many voters specifically chose that exact ballot.
**fun** *vote-count* :: *'a Preference-Relation ⇒ ('a, 'v) Election ⇒ nat* **where**
*vote-count p E = card {v ∈ (voters-$\mathcal{E}$ E). (profile-$\mathcal{E}$ E) v = p}*

### 1.5.2 Vote Count

**lemma** *vote-count-sum*:
**fixes** *E :: ('a, 'v) Election*
**assumes**
*finite (voters-$\mathcal{E}$ E)* **and**

*finite* (*UNIV* :: (*'a* × *'a*) *set*)
**shows** *sum* (λ *p*. *vote-count p E*) *UNIV* = *card* (*voters-*$\mathcal{E}$ *E*)
⟨*proof*⟩

### 1.5.3 Voter Permutations

A common action of interest on elections is renaming the voters, e.g., when talking about anonymity.

**fun** *rename* :: (*'v* ⇒ *'v*) ⇒ (*'a*, *'v*) *Election* ⇒ (*'a*, *'v*) *Election* **where**
*rename π* (*A*, *V*, *p*) = (*A*, *π* ‘ *V*, *p* ∘ (*the-inv π*))

**lemma** *rename-sound*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*π* :: *'v* ⇒ *'v*
**assumes**
*prof*: *profile V A p* **and**
*renamed*: (*A*, *V'*, *q*) = *rename π* (*A*, *V*, *p*) **and**
*bij-perm*: *bij π*
**shows** *profile V' A q*
⟨*proof*⟩

**lemma** *rename-prof*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*π* :: *'v* ⇒ *'v*
**assumes**
*profile V A p* **and**
(*A*, *V'*, *q*) = *rename π* (*A*, *V*, *p*) **and**
*bij π*
**shows** *profile V' A q*
⟨*proof*⟩

**lemma** *rename-finite*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*π* :: *'v* ⇒ *'v*
**assumes**
*finite V* **and**
(*A*, *V'*, *q*) = *rename π* (*A*, *V*, *p*) **and**
*bij π*
**shows** *finite V'*
⟨*proof*⟩

**lemma** *rename-inv*:
**fixes**
$\pi :: 'v \Rightarrow 'v$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes** *bij* $\pi$
**shows** *rename* $\pi$ (*rename* (*the-inv* $\pi$) $(A, V, p)) = (A, V, p)$
⟨*proof*⟩

**lemma** *rename-inj*:
**fixes** $\pi :: 'v \Rightarrow 'v$
**assumes** *bij* $\pi$
**shows** *inj* (*rename* $\pi$)
⟨*proof*⟩

**lemma** *rename-surj*:
**fixes** $\pi :: 'v \Rightarrow 'v$
**assumes** *bij* $\pi$
**shows**
*rename* $\pi$ ‘ *well-formed-elections* = *well-formed-elections* **and**
*rename* $\pi$ ‘ *finite-elections* = *finite-elections*
⟨*proof*⟩

### 1.5.4 List Representation

A profile on a voter set that has a natural order can be viewed as a list of ballots.

**fun** *to-list* :: $'v ::$ *linorder set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$
$('a$ *Preference-Relation*$)$ *list* **where**
*to-list* $V\ p$ = (*if* (*finite* $V$)
*then* (*map* $p$ (*sorted-list-of-set* $V$))
*else* [])

**lemma** *map-helper*:
**fixes**
$f :: 'x \Rightarrow 'y \Rightarrow 'z$ **and**
$g :: 'x \Rightarrow 'x$ **and**
$h :: 'y \Rightarrow 'y$ **and**
$l :: 'x\ list$ **and**
$l' :: 'y\ list$
**shows** *map2* $f$ (*map* $g\ l$) (*map* $h\ l'$) = *map2* $(\lambda\ x\ y.\ f\ (g\ x)\ (h\ y))\ l\ l'$
⟨*proof*⟩

**lemma** *to-list-simp*:
**fixes**
$i ::$ *nat* **and**
$V :: 'v ::$ *linorder set* **and**

$p :: ('a, 'v)$ *Profile*
**assumes** $i < card\ V$
**shows** $(to\text{-}list\ V\ p)!i = p\ ((sorted\text{-}list\text{-}of\text{-}set\ V)!i)$
⟨*proof*⟩

**lemma** *to-list-comp*:
**fixes**
$V :: 'v :: linorder\ set$ **and**
$p :: ('a, 'v)$ *Profile* **and**
$f :: 'a\ rel \Rightarrow 'a\ rel$
**shows** $to\text{-}list\ V\ (f \circ p) = map\ f\ (to\text{-}list\ V\ p)$
⟨*proof*⟩

**lemma** *set-card-upper-bound*:
**fixes**
$i :: nat$ **and**
$V :: nat\ set$
**assumes**
*fin-V*: *finite* $V$ **and**
*bound-v*: $\forall\ v \in V.\ v < i$
**shows** $card\ V \leq i$
⟨*proof*⟩

**lemma** *sorted-list-of-set-nth-equals-card*:
**fixes**
$V :: 'v :: linorder\ set$ **and**
$x :: 'v$
**assumes**
*fin-V*: *finite* $V$ **and**
*x-V*: $x \in V$
**shows** $sorted\text{-}list\text{-}of\text{-}set\ V!(card\ \{v \in V.\ v < x\}) = x$
⟨*proof*⟩

**lemma** *to-list-permutes-under-bij*:
**fixes**
$\pi :: 'v :: linorder \Rightarrow 'v$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)$ *Profile*
**assumes** *bij* $\pi$
**shows**
*let* $\varphi = (\lambda\ i.\ card\ \{v \in \pi\ `\ V.\ v < \pi\ ((sorted\text{-}list\text{-}of\text{-}set\ V)!i)\})$
*in* $(to\text{-}list\ V\ p) = permute\text{-}list\ \varphi\ (to\text{-}list\ (\pi\ `\ V)\ (\lambda\ x.\ p\ (the\text{-}inv\ \pi\ x)))$
⟨*proof*⟩

### 1.5.5 Preference Counts

The win count for an alternative a with respect to a finite voter set V in a profile p is the amount of ballots from V in p that rank alternative a in first position. If the voter set is infinite, counting is not generally possible.

**fun** *win-count* :: $'v\ set \Rightarrow ('a, 'v)\ Profile \Rightarrow 'a \Rightarrow enat$ **where**
*win-count V p a* = (*if* (*finite V*)
*then card* $\{v \in V.\ above\ (p\ v)\ a = \{a\}\}$ *else* $\infty$)

**fun** *prefer-count* :: $'v\ set \Rightarrow ('a, 'v)\ Profile \Rightarrow 'a \Rightarrow 'a \Rightarrow enat$ **where**
*prefer-count V p x y* = (*if* (*finite V*)
*then card* $\{v \in V.\ (let\ r = (p\ v)\ in\ (y \preceq_r x))\}$ *else* $\infty$)

**lemma** *pref-count-voter-set-card*:
**fixes**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a\ b :: 'a$
**assumes** *finite V*
**shows** *prefer-count V p a b* $\leq$ *card V*
⟨*proof*⟩

**lemma** *set-compr*:
**fixes**
$A :: 'a\ set$ **and**
$f :: 'a \Rightarrow 'a\ set$
**shows** $\{f\ x \mid x.\ x \in A\} = f\ `\ A$
⟨*proof*⟩

**lemma** *pref-count-set-compr*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**shows** $\{$*prefer-count V p a a'* $\mid a'.\ a' \in A - \{a\}\} =$
(*prefer-count V p a*) ` $(A - \{a\})$
⟨*proof*⟩

**lemma** *pref-count*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a\ b :: 'a$
**assumes**
*prof*: *profile V A p* **and**
*fin*: *finite V* **and**
*a-in-A*: $a \in A$ **and**
*b-in-A*: $b \in A$ **and**
*neq*: $a \neq b$
**shows** *prefer-count V p a b* = *card V* − (*prefer-count V p b a*)
⟨*proof*⟩

**lemma** *pref-count-sym*:
**fixes**
*p* :: ('*a*, '*v*) *Profile* **and**
*V* :: '*v set* **and**
*a b c* :: '*a*
**assumes**
*pref-count-ineq*: *prefer-count V p a c* ≥ *prefer-count V p c b* **and**
*prof*: *profile V A p* **and**
*a-in-A*: *a* ∈ *A* **and**
*b-in-A*: *b* ∈ *A* **and**
*c-in-A*: *c* ∈ *A* **and**
*a-neq-c*: *a* ≠ *c* **and**
*c-neq-b*: *c* ≠ *b*
**shows** *prefer-count V p b c* ≥ *prefer-count V p c a*
⟨*proof*⟩

**lemma** *empty-prof-imp-zero-pref-count*:
**fixes**
*p* :: ('*a*, '*v*) *Profile* **and**
*V* :: '*v set* **and**
*a b* :: '*a*
**assumes** *V* = {}
**shows** *prefer-count V p a b* = *0*
⟨*proof*⟩

**fun** *wins* :: '*v set* ⇒ '*a* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*wins V a p b* =
(*prefer-count V p a b* > *prefer-count V p b a*)

**lemma** *wins-inf-voters*:
**fixes**
*p* :: ('*a*, '*v*) *Profile* **and**
*a b* :: '*a* **and**
*V* :: '*v set*
**assumes** *infinite V*
**shows** ¬ *wins V b p a*
⟨*proof*⟩

Having alternative *a* win against *b* implies that *b* does not win against *a*.

**lemma** *wins-antisym*:
**fixes**
*p* :: ('*a*, '*v*) *Profile* **and**
*a b* :: '*a* **and**
*V* :: '*v set*
**assumes** *wins V a p b* — This already implies that *V* is finite.
**shows** ¬ *wins V b p a*
⟨*proof*⟩

**lemma** *wins-irreflex*:

**fixes**
*p* :: ('*a*, '*v*) *Profile* **and**
*a* :: '*a* **and**
*V* :: '*v set*
**shows** ¬ *wins V a p a*
⟨*proof*⟩

### 1.5.6 Condorcet Winner

**fun** *condorcet-winner* :: '*v set* ⇒ '*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*condorcet-winner V A p a* =
(*finite-profile V A p* ∧ *a* ∈ *A* ∧ (∀ *x* ∈ *A* − {*a*}. *wins V a p x*))

**lemma** *cond-winner-unique-eq*:
**fixes**
*V* :: '*v set* **and**
*A* :: '*a set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a b* :: '*a*
**assumes**
*condorcet-winner V A p a* **and**
*condorcet-winner V A p b*
**shows** *b* = *a*
⟨*proof*⟩

**lemma** *cond-winner-unique*:
**fixes**
*A* :: '*a set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a* :: '*a*
**assumes** *condorcet-winner V A p a*
**shows** {*a*′ ∈ *A*. *condorcet-winner V A p a*′} = {*a*}
⟨*proof*⟩

**lemma** *cond-winner-unique*′:
**fixes**
*V* :: '*v set* **and**
*A* :: '*a set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a b* :: '*a*
**assumes**
*condorcet-winner V A p a* **and**
*b* ≠ *a*
**shows** ¬ *condorcet-winner V A p b*
⟨*proof*⟩

### 1.5.7 Limited Profile

This function restricts a profile p to a set A of alternatives and a set V of voters s.t. voters outside of V do not have any preferences or do not cast a vote. This keeps all of A's preferences.

**fun** *limit-profile* :: *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒ (*'a*, *'v*) *Profile* **where**
*limit-profile A p* = (λ *v*. *limit A* (*p v*))

**lemma** *limit-prof-trans*:
**fixes**
*A B C* :: *'a set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*B* ⊆ *A* **and**
*C* ⊆ *B*
**shows** *limit-profile C p* = *limit-profile C* (*limit-profile B p*)
⟨*proof*⟩

**lemma** *limit-profile-sound*:
**fixes**
*A B* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*profile V B p* **and**
*A* ⊆ *B*
**shows** *profile V A* (*limit-profile A p*)
⟨*proof*⟩

### 1.5.8 Lifting Property

**definition** *equiv-prof-except-a* :: *'v set* ⇒ *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒
(*'a*, *'v*) *Profile* ⇒ *'a* ⇒ *bool* **where**
*equiv-prof-except-a V A p p' a* ≡
*profile V A p* ∧ *profile V A p'* ∧ *a* ∈ *A* ∧
(∀ *v* ∈ *V*. *equiv-rel-except-a A* (*p v*) (*p' v*) *a*)

An alternative gets lifted from one profile to another iff its ranking increases in at least one ballot, and nothing else changes.

**definition** *lifted* :: *'v set* ⇒ *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒
(*'a*, *'v*) *Profile* ⇒ *'a* ⇒ *bool* **where**
*lifted V A p p' a* ≡
*finite-profile V A p* ∧ *finite-profile V A p'* ∧ *a* ∈ *A*
∧ (∀ *v* ∈ *V*. ¬ *Preference-Relation.lifted A* (*p v*) (*p' v*) *a* ⟶ (*p v*) = (*p' v*))
∧ (∃ *v* ∈ *V*. *Preference-Relation.lifted A* (*p v*) (*p' v*) *a*)

**lemma** *lifted-imp-equiv-prof-except-a*:
**fixes**
*A* :: *'a set* **and**

$V :: 'v\ set$ **and**
$p\ p' :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes** *lifted* $V\ A\ p\ p'\ a$
**shows** *equiv-prof-except-a* $V\ A\ p\ p'\ a$
⟨*proof*⟩

**lemma** *negl-diff-imp-eq-limit-prof*:
**fixes**
$A\ A' :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p\ p' :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*change*: *equiv-prof-except-a* $V\ A'\ p\ q\ a$ **and**
*subset*: $A \subseteq A'$ **and**
*not-in-A*: $a \notin A$
**shows** $\forall\ v \in V.\ (limit\text{-}profile\ A\ p)\ v = (limit\text{-}profile\ A\ q)\ v$
— With the current definitions of *equiv-prof-except-a* and *limit-prof*, we can only conclude that the limited profiles coincide on the given voter set, since *limit-prof* may change the profiles everywhere, while *equiv-prof-except-a* only makes statements about the voter set.
⟨*proof*⟩

**lemma** *limit-prof-eq-or-lifted*:
**fixes**
$A\ A' :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p\ p' :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*lifted-a*: *lifted* $V\ A'\ p\ p'\ a$ **and**
*subset*: $A \subseteq A'$
**shows** $(\forall\ v \in V.\ limit\text{-}profile\ A\ p\ v = limit\text{-}profile\ A\ p'\ v)$
$\vee$ *lifted* $V\ A\ (limit\text{-}profile\ A\ p)\ (limit\text{-}profile\ A\ p')\ a$
⟨*proof*⟩

**end**

## 1.6 Social Choice Result

**theory** *Social-Choice-Result*
**imports** *Result*
**begin**

### 1.6.1 Definition

A social choice result contains three sets of alternatives: elected, rejected, and deferred alternatives.

**fun** *well-formed-$\mathcal{SCF}$* :: *'a set* ⇒ *'a Result* ⇒ *bool* **where**
  *well-formed-$\mathcal{SCF}$ A res* = (*disjoint3 res* ∧ *set-equals-partition A res*)

**fun** *limit-$\mathcal{SCF}$* :: *'a set* ⇒ *'a set* ⇒ *'a set* **where**
  *limit-$\mathcal{SCF}$ A r* = *A* ∩ *r*

### 1.6.2 Auxiliary Lemmas

**lemma** *result-imp-rej*:
  **fixes** *A e r d* :: *'a set*
  **assumes** *well-formed-$\mathcal{SCF}$ A* (*e*, *r*, *d*)
  **shows** *A* − (*e* ∪ *d*) = *r*
⟨*proof*⟩

**lemma** *result-count*:
  **fixes** *A e r d* :: *'a set*
  **assumes**
    *wf-result*: *well-formed-$\mathcal{SCF}$ A* (*e*, *r*, *d*) **and**
    *fin-A*: *finite A*
  **shows** *card A* = *card e* + *card r* + *card d*
⟨*proof*⟩

**lemma** *defer-subset*:
  **fixes**
    *A* :: *'a set* **and**
    *r* :: *'a Result*
  **assumes** *well-formed-$\mathcal{SCF}$ A r*
  **shows** *defer-r r* ⊆ *A*
⟨*proof*⟩

**lemma** *elect-subset*:
  **fixes**
    *A* :: *'a set* **and**
    *r* :: *'a Result*
  **assumes** *well-formed-$\mathcal{SCF}$ A r*
  **shows** *elect-r r* ⊆ *A*
⟨*proof*⟩

**lemma** *reject-subset*:
  **fixes**
    *A* :: *'a set* **and**
    *r* :: *'a Result*
  **assumes** *well-formed-$\mathcal{SCF}$ A r*
  **shows** *reject-r r* ⊆ *A*
⟨*proof*⟩

**end**

## 1.7 Social Welfare Result

**theory** *Social-Welfare-Result*
**imports** *Result*
*Preference-Relation*
**begin**

A social welfare result contains three sets of relations: elected, rejected, and deferred. A well-formed social welfare result consists only of linear orders on the alternatives.

**fun** *well-formed-$\mathcal{SWF}$* :: *'a set* ⇒ (*'a Preference-Relation*) *Result* ⇒ *bool* **where**
*well-formed-$\mathcal{SWF}$ A res* = (*disjoint3 res* ∧
*set-equals-partition* {*r*. *linear-order-on A r*} *res*)

**fun** *limit-$\mathcal{SWF}$* :: *'a set* ⇒ (*'a Preference-Relation*) *set* ⇒
(*'a Preference-Relation*) *set* **where**
*limit-$\mathcal{SWF}$ A res* = {*limit A r* | *r*. *r* ∈ *res* ∧ *linear-order-on A* (*limit A r*)}

**end**

## 1.8 Electoral Result Types

**theory** *Result-Interpretations*
**imports** *Social-Choice-Result*
*Social-Welfare-Result*
*Collections.Locale-Code*
**begin**

Interpretations of the result locale are placed inside a Locale-Code block in order to enable code generation of later definitions in the locale. Those definitions need to be added via a Locale-Code block as well.

⟨*ML*⟩
Results from social choice functions ($\mathcal{SCF}s$), for the purpose of composability and modularity given as three sets of (potentially tied) alternatives. See `Social_Choice_Result.thy` for details.

**global-interpretation** *$\mathcal{SCF}$-result*: *result well-formed-$\mathcal{SCF}$ limit-$\mathcal{SCF}$*
⟨*proof*⟩

Results from committee functions, for the purpose of composability and

modularity given as three sets of (potentially tied) sets of alternatives or committees. [[*Not actually used yet.*]]

**global-interpretation** *committee-result*: *result*
  $\lambda\ A\ r.$ *set-equals-partition* (*Pow A*) $r \wedge$ *disjoint3 r*
  $\lambda\ A\ rs.\ \{r \cap A \mid r.\ r \in rs\}$
⟨*proof*⟩

Results from social welfare functions ($\mathcal{SWF}s$), for the purpose of composability and modularity given as three sets of (potentially tied) linear orders over the alternatives. See `Social_Welfare_Result.thy` for details.

**global-interpretation** $\mathcal{SWF}$-*result*: *result well-formed-*$\mathcal{SWF}$ *limit-*$\mathcal{SWF}$
⟨*proof*⟩

⟨*ML*⟩

**end**

## 1.9 Symmetry Properties of Functions

**theory** *Symmetry-Of-Functions*
  **imports** *HOL−Algebra.Group-Action*
    *HOL−Algebra.Generated-Groups*
**begin**

### 1.9.1 Functions

**type-synonym** $('x, 'y)$ *binary-fun* $= 'x \Rightarrow 'y \Rightarrow 'y$

**fun** *extensional-continuation* :: $('x \Rightarrow 'y) \Rightarrow 'x\ set \Rightarrow ('x \Rightarrow 'y)$ **where**
  *extensional-continuation f s* $= (\lambda\ x.$ *if* $(x \in s)$ *then* $(f\ x)$ *else undefined*)

**fun** *preimg* :: $('x \Rightarrow 'y) \Rightarrow 'x\ set \Rightarrow 'y \Rightarrow 'x\ set$ **where**
  *preimg f s x* $= \{x' \in s.\ f\ x' = x\}$

### 1.9.2 Relations for Symmetry Constructions

**fun** *restricted-rel* :: $'x\ rel \Rightarrow 'x\ set \Rightarrow 'x\ set \Rightarrow 'x\ rel$ **where**
  *restricted-rel r s s'* $= r \cap (s \times s')$

**fun** *closed-restricted-rel* :: $'x\ rel \Rightarrow 'x\ set \Rightarrow 'x\ set \Rightarrow$ *bool* **where**
  *closed-restricted-rel r s t* $=$ ((*restricted-rel r t s*) `` $t \subseteq t$)

**fun** *action-induced-rel* :: $'x\ set \Rightarrow 'y\ set \Rightarrow ('x, 'y)$ *binary-fun* $\Rightarrow 'y\ rel$ **where**
  *action-induced-rel s t* $\varphi = \{(y, y').\ y \in t \wedge (\exists\ x \in s.\ \varphi\ x\ y = y')\}$

**fun** *product* :: $'x\ rel \Rightarrow ('x * 'x)\ rel$ **where**

*product r* = {(*p*, *p'*). (*fst p*, *fst p'*) ∈ *r* ∧ (*snd p*, *snd p'*) ∈ *r*}

**fun** *equivariance* :: *'x set* ⇒ *'y set* ⇒ (*'x*, *'y*) *binary-fun* ⇒ (*'y* ∗ *'y*) *rel* **where**
*equivariance s t φ* =
{((*u*, *v*), (*x*, *y*)). (*u*, *v*) ∈ *t* × *t* ∧ (∃ *z* ∈ *s*. *x* = *φ z u* ∧ *y* = *φ z v*)}

**fun** *closed-rel* :: *'x set* ⇒ *'x rel* ⇒ *bool* **where**
*closed-rel s r* = (∀ *x y*. (*x*, *y*) ∈ *r* ⟶ *x* ∈ *s* ⟶ *y* ∈ *s*)

**fun** *singleton-set-system* :: *'x set* ⇒ *'x set set* **where**
*singleton-set-system s* = {{*x*} | *x*. *x* ∈ *s*}

**fun** *set-action* :: (*'x*, *'r*) *binary-fun* ⇒ (*'x*, *'r set*) *binary-fun* **where**
*set-action ψ x* = *image* (*ψ x*)

### 1.9.3 Invariance and Equivariance

Invariance and equivariance are symmetry properties of functions: Invariance means that related preimages have identical images and equivariance denotes consistent changes.

**datatype** (*'x*, *'y*) *symmetry* =
*Invariance 'x rel* |
*Equivariance 'x set* ((*'x* ⇒ *'x*) × (*'y* ⇒ *'y*)) *set*

**fun** *is-symmetry* :: (*'x* ⇒ *'y*) ⇒ (*'x*, *'y*) *symmetry* ⇒ *bool* **where**
*is-symmetry f* (*Invariance r*) = (∀ *x*. ∀ *y*. (*x*, *y*) ∈ *r* ⟶ *f x* = *f y*) |
*is-symmetry f* (*Equivariance s τ*) = (∀ (*φ*, *ψ*) ∈ *τ*. ∀ *x* ∈ *s*. *f* (*φ x*) = *ψ* (*f x*))

**definition** *action-induced-equivariance* :: *'z set* ⇒ *'x set* ⇒ (*'z*, *'x*) *binary-fun* ⇒
(*'z*, *'y*) *binary-fun* ⇒ (*'x*, *'y*) *symmetry* **where**
*action-induced-equivariance t s φ ψ* ≡ *Equivariance s* {(*φ z*, *ψ z*) | *z*. *z* ∈ *t*}

### 1.9.4 Auxiliary Lemmas

**lemma** *un-left-inv-singleton-set-system*: ⋃ ∘ *singleton-set-system* = *id*
⟨*proof*⟩

**lemma** *preimg-comp*:
**fixes**
*f* :: *'x* ⇒ *'y* **and**
*g* :: *'x* ⇒ *'x* **and**
*s* :: *'x set* **and**
*x* :: *'y*
**shows** *preimg f* (*g* ‘ *s*) *x* = *g* ‘ *preimg* (*f* ∘ *g*) *s x*
⟨*proof*⟩

### 1.9.5 Rewrite Rules

**theorem** *rewrite-invar-as-equivar*:

**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s :: 'x\ set$ **and**
$t :: 'z\ set$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$
**shows** $is\text{-}symmetry\ f\ (Invariance\ (action\text{-}induced\text{-}rel\ t\ s\ \varphi)) =$
$is\text{-}symmetry\ f\ (action\text{-}induced\text{-}equivariance\ t\ s\ \varphi\ (\lambda\ g.\ id))$
⟨*proof*⟩

**lemma** *rewrite-invar-ind-by-act*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s :: 'z\ set$ **and**
$t :: 'x\ set$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$
**shows** $is\text{-}symmetry\ f\ (Invariance\ (action\text{-}induced\text{-}rel\ s\ t\ \varphi)) =$
$(\forall\ x \in s.\ \forall\ y \in t.\ f\ y = f\ (\varphi\ x\ y))$
⟨*proof*⟩

**lemma** *rewrite-equivariance*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s :: 'z\ set$ **and**
$t :: 'x\ set$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$ **and**
$\psi :: ('z, 'y)\ binary\text{-}fun$
**shows** $is\text{-}symmetry\ f\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ \psi) =$
$(\forall\ x \in s.\ \forall\ y \in t.\ f\ (\varphi\ x\ y) = \psi\ x\ (f\ y))$
⟨*proof*⟩

**lemma** *rewrite-group-action-img*:
**fixes**
$m :: 'x\ monoid$ **and**
$s\ t :: 'y\ set$ **and**
$\varphi :: ('x, 'y)\ binary\text{-}fun$ **and**
$x\ y :: 'x$
**assumes**
$t \subseteq s$ **and**
$x \in carrier\ m$ **and**
$y \in carrier\ m$ **and**
$group\text{-}action\ m\ s\ \varphi$
**shows** $\varphi\ (x \otimes_m y)\ `\ t = \varphi\ x\ `\ \varphi\ y\ `\ t$
⟨*proof*⟩

**lemma** *rewrite-carrier*: $carrier\ (BijGroup\ UNIV) = \{f'.\ bij\ f'\}$
⟨*proof*⟩

**lemma** *universal-set-carrier-imp-bij-group*:
**fixes** $f :: 'a \Rightarrow 'a$

**assumes** $f \in carrier\ (BijGroup\ UNIV)$
**shows** $bij\ f$
⟨*proof*⟩

**lemma** *rewrite-sym-group*:
**fixes**
$f\ g :: 'a \Rightarrow 'a$ **and**
$s :: 'a\ set$
**assumes**
$f \in carrier\ (BijGroup\ s)$ **and**
$g \in carrier\ (BijGroup\ s)$
**shows**
*rewrite-mult*: $f \otimes_{BijGroup\ s} g = extensional\text{-}continuation\ (f \circ g)\ s$ **and**
*rewrite-mult-univ*: $s = UNIV \longrightarrow f \otimes_{BijGroup\ s} g = f \circ g$
⟨*proof*⟩

**lemma** *simp-extensional-univ*:
**fixes** $f :: 'a \Rightarrow 'b$
**shows** $extensional\text{-}continuation\ f\ UNIV = f$
⟨*proof*⟩

**lemma** *extensional-continuation-subset*:
**fixes**
$f :: 'a \Rightarrow 'b$ **and**
$s\ t :: 'a\ set$ **and**
$x :: 'a$
**assumes**
$t \subseteq s$ **and**
$x \in t$
**shows** $extensional\text{-}continuation\ f\ s\ x = extensional\text{-}continuation\ f\ t\ x$
⟨*proof*⟩

**lemma** *rel-ind-by-coinciding-action-on-subset-eq-restr*:
**fixes**
$\varphi\ \psi :: ('a, 'b)\ binary\text{-}fun$ **and**
$s :: 'a\ set$ **and**
$t\ u :: 'b\ set$
**assumes**
$u \subseteq t$ **and**
$\forall\ x \in s.\ \forall\ y \in u.\ \psi\ x\ y = \varphi\ x\ y$
**shows** $action\text{-}induced\text{-}rel\ s\ u\ \psi = restricted\text{-}rel\ (action\text{-}induced\text{-}rel\ s\ t\ \varphi)\ u\ UNIV$
⟨*proof*⟩

**lemma** *coinciding-actions-ind-equal-rel*:
**fixes**
$s :: 'x\ set$ **and**
$t :: 'y\ set$ **and**
$\varphi\ \psi :: ('x, 'y)\ binary\text{-}fun$
**assumes** $\forall\ x \in s.\ \forall\ y \in t.\ \varphi\ x\ y = \psi\ x\ y$

**shows** *action-induced-rel s t φ = action-induced-rel s t ψ*
⟨*proof*⟩

### 1.9.6 Group Actions

**lemma** *const-id-is-group-action*:
**fixes** *m* :: *'x monoid*
**assumes** *group m*
**shows** *group-action m UNIV* (λ *x*. *id*)
⟨*proof*⟩

**theorem** *group-act-induces-set-group-act*:
**fixes**
*m* :: *'x monoid* **and**
*s* :: *'y set* **and**
*φ* :: (*'x*, *'y*) *binary-fun*
**defines** *φ-img* ≡ (λ *x*. *extensional-continuation* (*image* (*φ x*)) (*Pow s*))
**assumes** *group-action m s φ*
**shows** *group-action m* (*Pow s*) *φ-img*
⟨*proof*⟩

### 1.9.7 Invariance and Equivariance

It suffices to show equivariance under the group action of a generating set of a group to show equivariance under the group action of the whole group. For example, it is enough to show invariance under transpositions to show invariance under a complete finite symmetric group.

**theorem** *equivar-generators-imp-equivar-group*:
**fixes**
*f* :: *'x* ⇒ *'y* **and**
*m* :: *'z monoid* **and**
*s* :: *'z set* **and**
*t* :: *'x set* **and**
*φ* :: (*'z*, *'x*) *binary-fun* **and**
*ψ* :: (*'z*, *'y*) *binary-fun*
**assumes**
*equivar*: *is-symmetry f* (*action-induced-equivariance s t φ ψ*) **and**
*action-φ*: *group-action m t φ* **and**
*action-ψ*: *group-action m* (*f* ‘ *t*) *ψ* **and**
*gen*: *carrier m = generate m s*
**shows** *is-symmetry f* (*action-induced-equivariance* (*carrier m*) *t φ ψ*)
⟨*proof*⟩

**lemma** *invar-parameterized-fun*:
**fixes**
*f* :: *'x* ⇒ (*'x* ⇒ *'y*) **and**
*r* :: *'x rel*
**assumes**

$\forall\ x.$ *is-symmetry* $(f\ x)$ $(Invariance\ r)$ **and**
*is-symmetry* $f$ $(Invariance\ r)$
**shows** *is-symmetry* $(\lambda\ x.\ f\ x\ x)$ $(Invariance\ r)$
⟨*proof*⟩

**lemma** *invar-under-subset-rel*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$r\ s :: 'x\ rel$
**assumes**
*subset*: $r \subseteq s$ **and**
*invar*: *is-symmetry* $f$ $(Invariance\ s)$
**shows** *is-symmetry* $f$ $(Invariance\ r)$
⟨*proof*⟩

**lemma** *equivar-ind-by-act-coincide*:
**fixes**
$s :: 'x\ set$ **and**
$t :: 'y\ set$ **and**
$f :: 'y \Rightarrow 'z$ **and**
$\varphi\ \varphi' :: ('x, 'y)\ binary\text{-}fun$ **and**
$\psi :: ('x, 'z)\ binary\text{-}fun$
**assumes** $\forall\ x \in s.\ \forall\ y \in t.\ \varphi\ x\ y = \varphi'\ x\ y$
**shows** *is-symmetry* $f$ $(action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ \psi) =$
*is-symmetry* $f$ $(action\text{-}induced\text{-}equivariance\ s\ t\ \varphi'\ \psi)$
⟨*proof*⟩

**lemma** *equivar-under-subset*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s\ t :: 'x\ set$ **and**
$\tau :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y))\ set$
**assumes**
*is-symmetry* $f$ $(Equivariance\ s\ \tau)$ **and**
$t \subseteq s$
**shows** *is-symmetry* $f$ $(Equivariance\ t\ \tau)$
⟨*proof*⟩

**lemma** *equivar-under-subset'*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s :: 'x\ set$ **and**
$\tau\ \upsilon :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y))\ set$
**assumes**
*is-symmetry* $f$ $(Equivariance\ s\ \tau)$ **and**
$\upsilon \subseteq \tau$
**shows** *is-symmetry* $f$ $(Equivariance\ s\ \upsilon)$
⟨*proof*⟩

**theorem** *group-action-equivar-f-imp-equivar-preimg*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$\mathcal{D}_f\ s :: 'x\ set$ **and**
$m :: 'z\ monoid$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$ **and**
$\psi :: ('z, 'y)\ binary\text{-}fun$ **and**
$x :: 'z$
**defines** *equivar-prop* $\equiv$ *action-induced-equivariance* $(carrier\ m)\ \mathcal{D}_f\ \varphi\ \psi$
**assumes**
*action-φ*: *group-action* $m\ s\ \varphi$ **and**
*action-res*: *group-action* $m\ UNIV\ \psi$ **and**
*dom-in-s*: $\mathcal{D}_f \subseteq s$ **and**
*closed-domain*:
*closed-restricted-rel* $(action\text{-}induced\text{-}rel\ (carrier\ m)\ s\ \varphi)\ s\ \mathcal{D}_f$ **and**
*equivar-f*: *is-symmetry f equivar-prop* **and**
*group-elem-x*: $x \in carrier\ m$
**shows** $\forall\ y.\ preimg\ f\ \mathcal{D}_f\ (\psi\ x\ y) = (\varphi\ x)\ `\ (preimg\ f\ \mathcal{D}_f\ y)$
⟨*proof*⟩

### 1.9.8 Function Composition

**lemma** *invar-comp*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$g :: 'y \Rightarrow 'z$ **and**
$r :: 'x\ rel$
**assumes** *is-symmetry f* (*Invariance r*)
**shows** *is-symmetry* $(g \circ f)$ (*Invariance r*)
⟨*proof*⟩

**lemma** *equivar-comp*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$g :: 'y \Rightarrow 'z$ **and**
$s :: 'x\ set$ **and**
$t :: 'y\ set$ **and**
$\tau :: (('x \Rightarrow 'x) \times ('y \Rightarrow 'y))\ set$ **and**
$\upsilon :: (('y \Rightarrow 'y) \times ('z \Rightarrow 'z))\ set$
**defines**
*transitive-acts* $\equiv$
$\{(\varphi, \psi).\ \exists\ \chi :: 'y \Rightarrow 'y.\ (\varphi, \chi) \in \tau \wedge (\chi, \psi) \in \upsilon \wedge \chi\ `\ f\ `\ s \subseteq t\}$
**assumes**
$f\ `\ s \subseteq t$ **and**
*is-symmetry f* (*Equivariance s τ*) **and**
*is-symmetry g* (*Equivariance t υ*)
**shows** *is-symmetry* $(g \circ f)$ (*Equivariance s transitive-acts*)
⟨*proof*⟩

**lemma** *equivar-ind-by-action-comp*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$g :: 'y \Rightarrow 'z$ **and**
$s :: 'w\ set$ **and**
$t :: 'x\ set$ **and**
$u :: 'y\ set$ **and**
$\varphi :: ('w, 'x)\ binary\text{-}fun$ **and**
$\chi :: ('w, 'y)\ binary\text{-}fun$ **and**
$\psi :: ('w, 'z)\ binary\text{-}fun$
**assumes**
$f\ `\ t \subseteq u$ **and**
$\forall\ x \in s.\ \chi\ x\ `\ f\ `\ t \subseteq u$ **and**
$is\text{-}symmetry\ f\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ \chi)$ **and**
$is\text{-}symmetry\ g\ (action\text{-}induced\text{-}equivariance\ s\ u\ \chi\ \psi)$
**shows** $is\text{-}symmetry\ (g \circ f)\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ \psi)$
⟨*proof*⟩

**lemma** *equivar-set-minus*:
**fixes**
$f\ g :: 'x \Rightarrow 'y\ set$ **and**
$s :: 'z\ set$ **and**
$t :: 'x\ set$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$ **and**
$\psi :: ('z, 'y)\ binary\text{-}fun$
**assumes**
*f-equivar*: $is\text{-}symmetry\ f\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ (set\text{-}action\ \psi))$ **and**
*g-equivar*: $is\text{-}symmetry\ g\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ (set\text{-}action\ \psi))$ **and**
*bij-a*: $\forall\ a \in s.\ bij\ (\psi\ a)$
**shows**
$is\text{-}symmetry\ (\lambda\ b.\ f\ b - g\ b)\ (action\text{-}induced\text{-}equivariance\ s\ t\ \varphi\ (set\text{-}action\ \psi))$
⟨*proof*⟩

**lemma** *equivar-union-under-image-action*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$s :: 'z\ set$ **and**
$\varphi :: ('z, 'x)\ binary\text{-}fun$
**shows** $is\text{-}symmetry\ \bigcup\ (action\text{-}induced\text{-}equivariance\ s\ UNIV$
$(set\text{-}action\ (set\text{-}action\ \varphi))\ (set\text{-}action\ \varphi))$
⟨*proof*⟩

**end**

## 1.10 Symmetry Properties of Voting Rules

**theory** *Voting-Symmetry*

**imports** *Symmetry-Of-Functions*
*Social-Choice-Result*
*Social-Welfare-Result*
*Profile*
**begin**

### 1.10.1 Definitions

**fun** (**in** *result*) *closed-elections* :: ('*a*, '*v*) *Election rel* ⇒ *bool* **where**
*closed-elections r* =
(∀ (*e*, *e'*) ∈ *r*.
*limit* (*alternatives-*$\mathcal{E}$ *e*) *UNIV* = *limit* (*alternatives-*$\mathcal{E}$ *e'*) *UNIV*)

**fun** *result-action* :: ('*x*, '*r*) *binary-fun* ⇒ ('*x*, '*r Result*) *binary-fun* **where**
*result-action* $\psi$ *x* = ($\lambda$ *r*. ($\psi$ *x* ‘ *elect-r r*, $\psi$ *x* ‘ *reject-r r*, $\psi$ *x* ‘ *defer-r r*))

#### Anonymity

**definition** *anonymity*$_{\mathcal{G}}$ :: ('*v* ⇒ '*v*) *monoid* **where**
*anonymity*$_{\mathcal{G}}$ ≡ *BijGroup* (*UNIV* :: '*v set*)

**fun** $\varphi$-*anon* :: ('*a*, '*v*) *Election set* ⇒ ('*v* ⇒ '*v*) ⇒
(('*a*, '*v*) *Election* ⇒ ('*a*, '*v*) *Election*) **where**
$\varphi$-*anon* $\mathcal{E}$ $\pi$ = *extensional-continuation* (*rename* $\pi$) $\mathcal{E}$

**fun** *anonymity*$_{\mathcal{R}}$ :: ('*a*, '*v*) *Election set* ⇒ ('*a*, '*v*) *Election rel* **where**
*anonymity*$_{\mathcal{R}}$ $\mathcal{E}$ = *action-induced-rel* (*carrier anonymity*$_{\mathcal{G}}$) $\mathcal{E}$ ($\varphi$-*anon* $\mathcal{E}$)

#### Neutrality

**fun** *rel-rename* :: ('*a* ⇒ '*a*, '*a Preference-Relation*) *binary-fun* **where**
*rel-rename* $\pi$ *r* = {($\pi$ *a*, $\pi$ *b*) | *a b*. (*a*, *b*) ∈ *r*}

**fun** *alternatives-rename* :: ('*a* ⇒ '*a*, ('*a*, '*v*) *Election*) *binary-fun* **where**
*alternatives-rename* $\pi$ $\mathcal{E}$ =
($\pi$ ‘ (*alternatives-*$\mathcal{E}$ $\mathcal{E}$), *voters-*$\mathcal{E}$ $\mathcal{E}$, (*rel-rename* $\pi$) ∘ (*profile-*$\mathcal{E}$ $\mathcal{E}$))

**definition** *neutrality*$_{\mathcal{G}}$ :: ('*a* ⇒ '*a*) *monoid* **where**
*neutrality*$_{\mathcal{G}}$ ≡ *BijGroup* (*UNIV* :: '*a set*)

**fun** $\varphi$-*neutral* :: ('*a*, '*v*) *Election set* ⇒
('*a* ⇒ '*a*, ('*a*, '*v*) *Election*) *binary-fun* **where**
$\varphi$-*neutral* $\mathcal{E}$ $\pi$ = *extensional-continuation* (*alternatives-rename* $\pi$) $\mathcal{E}$

**fun** *neutrality*$_{\mathcal{R}}$ :: ('*a*, '*v*) *Election set* ⇒ ('*a*, '*v*) *Election rel* **where**
*neutrality*$_{\mathcal{R}}$ $\mathcal{E}$ = *action-induced-rel* (*carrier neutrality*$_{\mathcal{G}}$) $\mathcal{E}$ ($\varphi$-*neutral* $\mathcal{E}$)

**fun** $\psi$-*neutral*$_{\mathrm{c}}$ :: ('*a* ⇒ '*a*, '*a*) *binary-fun* **where**
$\psi$-*neutral*$_{\mathrm{c}}$ $\pi$ *r* = $\pi$ *r*

**fun** $\psi$-*neutral*$_{\mathrm{w}}$ :: $('a \Rightarrow 'a, 'a\ rel)$ *binary-fun* **where**
$\psi$-*neutral*$_{\mathrm{w}}$ $\pi$ $r$ = *rel-rename* $\pi$ $r$

### Homogeneity

**fun** *homogeneity*$_{\mathcal{R}}$ :: $('a, 'v)$ *Election set* $\Rightarrow$ $('a, 'v)$ *Election rel* **where**
*homogeneity*$_{\mathcal{R}}$ $\mathcal{E}$ =
$\{(E, E').\ E \in \mathcal{E}$
$\wedge$ *alternatives-*$\mathcal{E}$ $E$ = *alternatives-*$\mathcal{E}$ $E'$
$\wedge$ *finite* (*voters-*$\mathcal{E}$ $E$) $\wedge$ *finite* (*voters-*$\mathcal{E}$ $E'$)
$\wedge$ $(\exists\ n > 0.\ \forall\ r :: 'a$ *Preference-Relation.*
*vote-count* $r$ $E$ = $n *$ (*vote-count* $r$ $E'$))$\}$

**fun** *copy-list* :: *nat* $\Rightarrow$ $'x$ *list* $\Rightarrow$ $'x$ *list* **where**
*copy-list* 0 $l$ = [] |
*copy-list* (*Suc* $n$) $l$ = *copy-list* $n$ $l$ @ $l$

**fun** *homogeneity*$_{\mathcal{R}}'$ :: $('a, 'v ::$ *linorder*$)$ *Election set* $\Rightarrow$ $('a, 'v)$ *Election rel* **where**
*homogeneity*$_{\mathcal{R}}'$ $\mathcal{E}$ =
$\{(E, E').\ E \in \mathcal{E}$
$\wedge$ *alternatives-*$\mathcal{E}$ $E$ = *alternatives-*$\mathcal{E}$ $E'$
$\wedge$ *finite* (*voters-*$\mathcal{E}$ $E$) $\wedge$ *finite* (*voters-*$\mathcal{E}$ $E'$)
$\wedge$ $(\exists\ n > 0.$
*to-list* (*voters-*$\mathcal{E}$ $E'$) (*profile-*$\mathcal{E}$ $E'$) =
*copy-list* $n$ (*to-list* (*voters-*$\mathcal{E}$ $E$) (*profile-*$\mathcal{E}$ $E$)))$\}$

### Reversal Symmetry

**fun** *reverse-rel* :: $'a\ rel \Rightarrow 'a\ rel$ **where**
*reverse-rel* $r$ = $\{(a, b).\ (b, a) \in r\}$

**fun** *rel-app* :: $('a\ rel \Rightarrow 'a\ rel) \Rightarrow ('a, 'v)$ *Election* $\Rightarrow ('a, 'v)$ *Election* **where**
*rel-app* $f$ $(A, V, p)$ = $(A, V, f \circ p)$

**definition** *reversal*$_{\mathcal{G}}$ :: $('a\ rel \Rightarrow 'a\ rel)$ *monoid* **where**
*reversal*$_{\mathcal{G}}$ $\equiv$ $(\!|$*carrier* = $\{$*reverse-rel*, *id*$\}$, *monoid.mult* = *comp*, *one* = *id*$|\!)$

**fun** $\varphi$-*reverse* :: $('a, 'v)$ *Election set*
$\Rightarrow$ $('a\ rel \Rightarrow 'a\ rel, ('a, 'v)$ *Election*$)$ *binary-fun* **where**
$\varphi$-*reverse* $\mathcal{E}$ $\varphi$ = *extensional-continuation* (*rel-app* $\varphi$) $\mathcal{E}$

**fun** $\psi$-*reverse* :: $('a\ rel \Rightarrow 'a\ rel, 'a\ rel)$ *binary-fun* **where**
$\psi$-*reverse* $\varphi$ $r$ = $\varphi$ $r$

**fun** *reversal*$_{\mathcal{R}}$ :: $('a, 'v)$ *Election set* $\Rightarrow$ $('a, 'v)$ *Election rel* **where**
*reversal*$_{\mathcal{R}}$ $\mathcal{E}$ = *action-induced-rel* (*carrier reversal*$_{\mathcal{G}}$) $\mathcal{E}$ ($\varphi$-*reverse* $\mathcal{E}$)

## 1.10.2 Auxiliary Lemmas

**fun** *n-app* :: *nat* $\Rightarrow$ $('x \Rightarrow 'x) \Rightarrow ('x \Rightarrow 'x)$ **where**

*n-app-id*: *n-app 0 f* = *id* |
*n-app-suc*: *n-app* (*Suc n*) *f* = *f* ∘ *n-app n f*

**lemma** *n-app-rewrite*:
**fixes**
*f* :: *'x* ⇒ *'x* **and**
*n* :: *nat* **and**
*x* :: *'x*
**shows** (*f* ∘ *n-app n f*) *x* = (*n-app n f* ∘ *f*) *x*
⟨*proof*⟩

**lemma** *n-app-leaves-set*:
**fixes**
*A B* :: *'x set* **and**
*f* :: *'x* ⇒ *'x* **and**
*x* :: *'x*
**assumes**
*fin-A*: *finite A* **and**
*fin-B*: *finite B* **and**
*x-el*: *x* ∈ *A* − *B* **and**
*bij-f*: *bij-betw f A B*
**obtains** *n* :: *nat* **where**
*n* > *0* **and**
*n-app n f x* ∈ *B* − *A* **and**
∀ *m* > *0*. *m* < *n* ⟶ *n-app m f x* ∈ *A* ∩ *B*
⟨*proof*⟩

**lemma** *n-app-rev*:
**fixes**
*A B* :: *'x set* **and**
*f* :: *'x* ⇒ *'x* **and**
*m n* :: *nat* **and**
*x y* :: *'x*
**assumes**
*x-in-A*: *x* ∈ *A* **and**
*y-in-A*: *y* ∈ *A* **and**
*n-geq-m*: *n* ≥ *m* **and**
*n-app-eq-m-n*: *n-app n f x* = *n-app m f y* **and**
*n-app-x-in-A*: ∀ *n'* < *n*. *n-app n' f x* ∈ *A* **and**
*n-app-y-in-A*: ∀ *m'* < *m*. *n-app m' f y* ∈ *A* **and**
*fin-A*: *finite A* **and**
*fin-B*: *finite B* **and**
*bij-f-A-B*: *bij-betw f A B*
**shows** *n-app* (*n* − *m*) *f x* = *y*
⟨*proof*⟩

**lemma** *n-app-inv*:
**fixes**
*A B* :: *'x set* **and**

$f :: 'x \Rightarrow 'x$ **and**
$n :: nat$ **and**
$x :: 'x$
**assumes**
$x \in B$ **and**
$\forall\ m \geq 0.\ m < n \longrightarrow n\text{-}app\ m\ (the\text{-}inv\text{-}into\ A\ f)\ x \in B$ **and**
*bij-betw f A B*
**shows** $n\text{-}app\ n\ f\ (n\text{-}app\ n\ (the\text{-}inv\text{-}into\ A\ f)\ x) = x$
⟨*proof*⟩

**lemma** *bij-betw-finite-ind-global-bij*:
**fixes**
$A\ B :: 'x\ set$ **and**
$f :: 'x \Rightarrow 'x$
**assumes**
*fin-A*: *finite A* **and**
*fin-B*: *finite B* **and**
*bij-f*: *bij-betw f A B*
**obtains** $g :: 'x \Rightarrow 'x$ **where**
*bij g* **and**
$\forall\ a \in A.\ g\ a = f\ a$ **and**
$\forall\ b \in B - A.\ g\ b \in A - B \wedge (\exists\ n > 0.\ n\text{-}app\ n\ f\ (g\ b) = b)$ **and**
$\forall\ x \in UNIV - A - B.\ g\ x = x$
⟨*proof*⟩

**lemma** *bij-betw-ext*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$X :: 'x\ set$ **and**
$Y :: 'y\ set$
**assumes** *bij-betw f X Y*
**shows** *bij-betw* (*extensional-continuation f X*) *X Y*
⟨*proof*⟩

### 1.10.3 Anonymity Lemmas

**lemma** *anon-rel-vote-count*:
**fixes**
$\mathcal{E} :: ('a, 'v)\ Election\ set$ **and**
$E\ E' :: ('a, 'v)\ Election$
**assumes**
*finite* (*voters-*$\mathcal{E}$ *E*) **and**
$(E, E') \in anonymity_{\mathcal{R}}\ \mathcal{E}$
**shows** *alternatives-*$\mathcal{E}$ $E = alternatives\text{-}\mathcal{E}\ E' \wedge E \in \mathcal{E}$
$\wedge\ (\forall\ p.\ vote\text{-}count\ p\ E = vote\text{-}count\ p\ E')$
⟨*proof*⟩

**lemma** *vote-count-anon-rel*:
**fixes**

$\mathcal{E}$ :: ('a, 'v) *Election set* **and**
$E$ $E'$ :: ('a, 'v) *Election*
**assumes**
*fin-voters-E*: *finite* (*voters-*$\mathcal{E}$ $E$) **and**
*fin-voters-E'*: *finite* (*voters-*$\mathcal{E}$ $E'$) **and**
*default-non-v*: $\forall\ v.\ v \notin$ *voters-*$\mathcal{E}$ $E \longrightarrow$ *profile-*$\mathcal{E}$ $E\ v = \{\}$ **and**
*default-non-v'*: $\forall\ v.\ v \notin$ *voters-*$\mathcal{E}$ $E' \longrightarrow$ *profile-*$\mathcal{E}$ $E'\ v = \{\}$ **and**
*eq*: *alternatives-*$\mathcal{E}$ $E$ = *alternatives-*$\mathcal{E}$ $E' \wedge (E, E') \in \mathcal{E} \times \mathcal{E}$
$\wedge\ (\forall\ p.$ *vote-count* $p\ E$ = *vote-count* $p\ E')$
**shows** $(E, E') \in$ *anonymity*$_{\mathcal{R}}$ $\mathcal{E}$
⟨*proof*⟩

**lemma** *rename-comp*:
**fixes** $\pi\ \pi'$ :: 'v ⇒ 'v
**assumes**
*bij* $\pi$ **and**
*bij* $\pi'$
**shows** *rename* $\pi \circ$ *rename* $\pi'$ = *rename* $(\pi \circ \pi')$
⟨*proof*⟩

**interpretation** *anonymous-group-action*:
*group-action anonymity*$_{\mathcal{G}}$ *well-formed-elections* $\varphi$*-anon well-formed-elections*
⟨*proof*⟩

**lemma** (**in** *result*) *anonymity*:
*is-symmetry* $(\lambda\ E.$ *limit* (*alternatives-*$\mathcal{E}$ $E$) *UNIV*)
(*Invariance* (*anonymity*$_{\mathcal{R}}$ *well-formed-elections*))
⟨*proof*⟩

### 1.10.4 Neutrality Lemmas

**lemma** *rel-rename-helper*:
**fixes**
$r$ :: 'a *rel* **and**
$\pi$ :: 'a ⇒ 'a **and**
$a\ b$ :: 'a
**assumes** *bij* $\pi$
**shows** $(\pi\ a, \pi\ b) \in \{(\pi\ x, \pi\ y) \mid x\ y.\ (x, y) \in r\}$
$\longleftrightarrow (a, b) \in \{(x, y) \mid x\ y.\ (x, y) \in r\}$
⟨*proof*⟩

**lemma** *rel-rename-comp*:
**fixes** $\pi\ \pi'$ :: 'a ⇒ 'a
**shows** *rel-rename* $(\pi \circ \pi')$ = *rel-rename* $\pi \circ$ *rel-rename* $\pi'$
⟨*proof*⟩

**lemma** *rel-rename-sound*:
**fixes**
$\pi$ :: 'a ⇒ 'a **and**

$r :: 'a\ rel$ **and**
$A :: 'a\ set$
**assumes** *inj* $\pi$
**shows**
*refl-on* $A\ r \longrightarrow$ *refl-on* $(\pi\ `\ A)\ (rel\text{-}rename\ \pi\ r)$ **and**
*antisym* $r \longrightarrow$ *antisym* $(rel\text{-}rename\ \pi\ r)$ **and**
*total-on* $A\ r \longrightarrow$ *total-on* $(\pi\ `\ A)\ (rel\text{-}rename\ \pi\ r)$ **and**
*Relation.trans* $r \longrightarrow$ *Relation.trans* $(rel\text{-}rename\ \pi\ r)$
⟨*proof*⟩

**lemma** *rename-subset*:
**fixes**
$r\ s :: 'a\ rel$ **and**
$a\ b :: 'a$ **and**
$\pi :: 'a \Rightarrow 'a$
**assumes**
*bij-π*: *bij* $\pi$ **and**
$rel\text{-}rename\ \pi\ r = rel\text{-}rename\ \pi\ s$ **and**
$(a, b) \in r$
**shows** $(a, b) \in s$
⟨*proof*⟩

**lemma** *rel-rename-bij*:
**fixes** $\pi :: 'a \Rightarrow 'a$
**assumes** *bij-π*: *bij* $\pi$
**shows** *bij* $(rel\text{-}rename\ \pi)$
⟨*proof*⟩

**lemma** *alternatives-rename-comp*:
**fixes** $\pi\ \pi' :: 'a \Rightarrow 'a$
**shows** $alternatives\text{-}rename\ \pi \circ alternatives\text{-}rename\ \pi' =$
$alternatives\text{-}rename\ (\pi \circ \pi')$
⟨*proof*⟩

**lemma** *well-formed-elects-closed*:
**fixes**
$A\ A' :: 'a\ set$ **and**
$V\ V' :: 'v\ set$ **and**
$p\ p' :: ('a, 'v)\ Profile$ **and**
$\pi :: 'a \Rightarrow 'a$
**assumes**
*bij-π*: *bij* $\pi$ **and**
*wf-elects*: $(A, V, p) \in well\text{-}formed\text{-}elections$ **and**
*renamed*: $(A', V', p') = alternatives\text{-}rename\ \pi\ (A, V, p)$
**shows** $(A', V', p') \in well\text{-}formed\text{-}elections$
⟨*proof*⟩

**lemma** *alternatives-rename-bij*:
**fixes** $\pi :: ('a \Rightarrow 'a)$

**assumes** *bij-π*: *bij π*
**shows** *bij-betw* (*alternatives-rename π*) *well-formed-elections well-formed-elections*
⟨*proof*⟩

**interpretation** *φ-neutral-action*: *group-action neutrality$_{\mathcal{G}}$ well-formed-elections*
*φ-neutral well-formed-elections*
⟨*proof*⟩

**interpretation** *ψ-neutral$_{\mathrm{c}}$-action*: *group-action neutrality$_{\mathcal{G}}$ UNIV ψ-neutral$_{\mathrm{c}}$*
⟨*proof*⟩

**interpretation** *ψ-neutral$_{\mathrm{w}}$-action*: *group-action neutrality$_{\mathcal{G}}$ UNIV ψ-neutral$_{\mathrm{w}}$*
⟨*proof*⟩

**lemma** *neutrality-SCF*: *is-symmetry* (λ *E*. *limit-SCF* (*alternatives-E E*) *UNIV*)
(*action-induced-equivariance* (*carrier neutrality$_{\mathcal{G}}$*) *well-formed-elections*
(*φ-neutral well-formed-elections*) (*set-action ψ-neutral$_{\mathrm{c}}$*))
⟨*proof*⟩

**lemma** *neutrality-SWF*: *is-symmetry* (λ *E*. *limit-SWF* (*alternatives-E E*) *UNIV*)
(*action-induced-equivariance* (*carrier neutrality$_{\mathcal{G}}$*) *well-formed-elections*
(*φ-neutral well-formed-elections*) (*set-action ψ-neutral$_{\mathrm{w}}$*))
⟨*proof*⟩

### 1.10.5 Homogeneity Lemmas

**definition** *reflp-on′* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
*reflp-on′ A r* ≡ *reflp-on A* (λ *x y*. (*x*, *y*) ∈ *r*)

**lemma** *refl-homogeneity$_{\mathcal{R}}$*:
**fixes** *E* :: (*'a*, *'v*) *Election set*
**assumes** *E* ⊆ *finite-elections-V*
**shows** *reflp-on′ E* (*homogeneity$_{\mathcal{R}}$ E*)
⟨*proof*⟩

**lemma** (**in** *result*) *homogeneity*:
*is-symmetry* (λ *E*. *limit* (*alternatives-E E*) *UNIV*)
(*Invariance* (*homogeneity$_{\mathcal{R}}$ UNIV*))
⟨*proof*⟩

**lemma** *refl-homogeneity$_{\mathcal{R}}$′*:
**fixes** *E* :: (*'a*, *'v* :: *linorder*) *Election set*
**assumes** *E* ⊆ *finite-elections-V*
**shows** *reflp-on′ E* (*homogeneity$_{\mathcal{R}}$′ E*)
⟨*proof*⟩

**lemma** (**in** *result*) *homogeneity′*:
*is-symmetry* (λ *E*. *limit* (*alternatives-E E*) *UNIV*)
(*Invariance* (*homogeneity$_{\mathcal{R}}$′ UNIV*))

⟨*proof*⟩

### 1.10.6 Reversal Symmetry Lemmas

**lemma** *reverse-reverse-id*: *reverse-rel* ∘ *reverse-rel* = *id*
⟨*proof*⟩

**lemma** *reverse-rel-limit*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a rel*
**shows** *reverse-rel* (*limit A r*) = *limit A* (*reverse-rel r*)
⟨*proof*⟩

**lemma** *reverse-rel-lin-ord*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a rel*
**assumes** *linear-order-on A r*
**shows** *linear-order-on A* (*reverse-rel r*)
⟨*proof*⟩

**interpretation** $reversal_{\mathcal{G}}$*-group*: *group* $reversal_{\mathcal{G}}$
⟨*proof*⟩

**interpretation** $\varphi$*-reverse-action*: *group-action* $reversal_{\mathcal{G}}$ *well-formed-elections*
$\varphi$*-reverse well-formed-elections*
⟨*proof*⟩

**interpretation** $\psi$*-reverse-action*: *group-action* $reversal_{\mathcal{G}}$ *UNIV* $\psi$*-reverse*
⟨*proof*⟩

**lemma** *reversal-symmetry*: *is-symmetry* (λ $\mathcal{E}$. *limit-*$\mathcal{SWF}$ (*alternatives-*$\mathcal{E}$ $\mathcal{E}$) *UNIV*)
(*action-induced-equivariance* (*carrier* $reversal_{\mathcal{G}}$) *well-formed-elections*
($\varphi$*-reverse well-formed-elections*) (*set-action* $\psi$*-reverse*))
⟨*proof*⟩

**end**

## 1.11 Result-Dependent Voting Rule Properties

**theory** *Property-Interpretations*
**imports** *Voting-Symmetry*
*Result-Interpretations*
**begin**

### 1.11.1 Property Definitions

The interpretation of equivariance properties generally depends on the result type. For example, neutrality for social choice rules means that single winners are renamed when the candidates in the votes are consistently renamed. For social welfare results, the complete result rankings must be renamed.

New result-type-dependent definitions for properties can be added here.

**locale** *result-properties* = *result* +
  **fixes** $\psi$-*neutral* :: ('*a* ⇒ '*a*, '*b*) *binary-fun* **and**
      *voter-type* :: '*v itself*
  **assumes**
    *action-neutral*: *group-action neutrality*$_{\mathcal{G}}$ *UNIV* $\psi$-*neutral* **and**
    *neutrality*:
      *is-symmetry* (λ $\mathcal{E}$ :: ('*a*, '*v*) *Election*. *limit* (*alternatives-*$\mathcal{E}$ $\mathcal{E}$) *UNIV*)
            (*action-induced-equivariance* (*carrier neutrality*$_{\mathcal{G}}$)
              *well-formed-elections*
              ($\varphi$-*neutral well-formed-elections*) (*set-action* $\psi$-*neutral*))

**sublocale** *result-properties* ⊆ *result*
  ⟨*proof*⟩

### 1.11.2 Interpretations

**global-interpretation** $\mathcal{SCF}$-*properties*: *result-properties well-formed-*$\mathcal{SCF}$
      *limit-*$\mathcal{SCF}$ $\psi$-*neutral*$_{\mathrm{c}}$
  ⟨*proof*⟩

**global-interpretation** $\mathcal{SWF}$-*properties*: *result-properties well-formed-*$\mathcal{SWF}$
      *limit-*$\mathcal{SWF}$ $\psi$-*neutral*$_{\mathrm{w}}$
  ⟨*proof*⟩

**end**

# Chapter 2

# Refined Types

## 2.1 Preference List

**theory** *Preference-List*
  **imports** *../Preference-Relation*
    *HOL−Combinatorics.Multiset-Permutations*
    *List−Index.List-Index*
**begin**

Preference lists derive from preference relations, ordered from most to least preferred alternative.

### 2.1.1 Well-Formedness

**type-synonym** *'a Preference-List* = *'a list*

**abbreviation** *well-formed-l* :: *'a Preference-List* ⇒ *bool* **where**
  *well-formed-l l* ≡ *distinct l*

### 2.1.2 Auxiliary Lemmas About Lists

**lemma** *is-arg-min-equal*:
  **fixes**
    *f g* :: *'a* ⇒ *'b* :: *ord* **and**
    *S* :: *'a set* **and**
    *x* :: *'a*
  **assumes** ∀ *x* ∈ *S*. *f x* = *g x*
  **shows** *is-arg-min f* (λ *s*. *s* ∈ *S*) *x* = *is-arg-min g* (λ *s*. *s* ∈ *S*) *x*
⟨*proof*⟩

**lemma** *list-cons-presv-finiteness*:
  **fixes**
    *A* :: *'a set* **and**
    *S* :: *'a list set*
  **assumes**

*fin-A*: *finite A* **and**
*fin-B*: *finite S*
**shows** *finite* $\{a\#l \mid a\ l.\ a \in A \wedge l \in S\}$
⟨*proof*⟩

**lemma** *listset-finiteness*:
**fixes** $l :: 'a\ set\ list$
**assumes** $\forall\ i{::}nat.\ i < length\ l \longrightarrow finite\ (l!i)$
**shows** *finite* (*listset l*)
⟨*proof*⟩

**lemma** *all-ls-elems-same-len*:
**fixes** $l :: 'a\ set\ list$
**shows** $\forall\ l' :: 'a\ list.\ l' \in listset\ l \longrightarrow length\ l' = length\ l$
⟨*proof*⟩

**lemma** *all-ls-elems-in-ls-set*:
**fixes** $l :: 'a\ set\ list$
**shows** $\forall\ l' \in listset\ l.\ \forall\ i{::}nat < length\ l'.\ l'!i \in l!i$
⟨*proof*⟩

**lemma** *all-ls-in-ls-set*:
**fixes** $l :: 'a\ set\ list$
**shows** $\forall\ l'.\ length\ l' = length\ l$
$\wedge\ (\forall\ i < length\ l'.\ l'!i \in l!i) \longrightarrow l' \in listset\ l$
⟨*proof*⟩

### 2.1.3 Ranking

Rank 1 is the top preference, rank 2 the second, and so on. Rank 0 does not exist.

**fun** *rank-l* :: $'a$ *Preference-List* ⇒ $'a$ ⇒ *nat* **where**
*rank-l l a* = (*if a* ∈ *set l then index l a* + *1 else 0*)

**fun** *rank-l-idx* :: $'a$ *Preference-List* ⇒ $'a$ ⇒ *nat* **where**
*rank-l-idx l a* =
(*let i* = *index l a in*
*if i* = *length l then 0 else i* + *1*)

**lemma** *rank-l-equiv*: *rank-l* = *rank-l-idx*
⟨*proof*⟩

**lemma** *rank-zero-imp-not-present*:
**fixes**
$p :: 'a\ Preference\text{-}List$ **and**
$a :: 'a$
**assumes** *rank-l p a* = *0*
**shows** $a \notin set\ p$
⟨*proof*⟩

**definition** *above-l* :: *'a Preference-List* ⇒ *'a* ⇒ *'a Preference-List* **where**
*above-l r a* ≡ *take* (*rank-l r a*) *r*

### 2.1.4 Definition

**fun** *is-less-preferred-than-l* :: *'a* ⇒ *'a Preference-List* ⇒ *'a* ⇒ *bool*
(- ≲- - [*50*, *1000*, *51*] *50*) **where**
*a* ≲*l b* = (*a* ∈ *set l* ∧ *b* ∈ *set l* ∧ *index l a* ≥ *index l b*)

**lemma** *rank-gt-zero*:
**fixes**
*l* :: *'a Preference-List* **and**
*a* :: *'a*
**assumes** *a* ≲*l a*
**shows** *rank-l l a* ≥ *1*
⟨*proof*⟩

**definition** *pl-α* :: *'a Preference-List* ⇒ *'a Preference-Relation* **where**
*pl-α l* ≡ {(*a*, *b*). *a* ≲*l b*}

**lemma** *rel-trans*:
**fixes** *l* :: *'a Preference-List*
**shows** *trans* (*pl-α l*)
⟨*proof*⟩

**lemma** *pl-α-lin-order*:
**fixes**
*A* :: *'a set* **and**
*r* :: *'a rel*
**assumes** *r* ∈ *pl-α* ‘ *permutations-of-set A*
**shows** *linear-order-on A r*
⟨*proof*⟩

**lemma** *lin-order-pl-α*:
**fixes**
*r* :: *'a rel* **and**
*A* :: *'a set*
**assumes**
*lin-order*: *linear-order-on A r* **and**
*fin*: *finite A*
**shows** *r* ∈ *pl-α* ‘ *permutations-of-set A*
⟨*proof*⟩

**lemma** *index-helper*:
**fixes**
*l* :: *'x list* **and**
*x* :: *'x*
**assumes**

*finite* (*set l*) **and**
*distinct l* **and**
$x \in$ *set l*
**shows** *index l x* = *card* {$y \in$ *set l*. *index l y* < *index l x*}
⟨*proof*⟩

**lemma** *pl-α-eq-imp-list-eq*:
**fixes** *l l'* :: '*x list*
**assumes**
*fin-set-l*: *finite* (*set l*) **and**
*set-eq*: *set l* = *set l'* **and**
*dist-l*: *distinct l* **and**
*dist-l'*: *distinct l'* **and**
*pl-α-eq*: *pl-α l* = *pl-α l'*
**shows** *l* = *l'*
⟨*proof*⟩

**lemma** *pl-α-bij-betw*:
**fixes** *X* :: '*x set*
**assumes** *finite X*
**shows** *bij-betw pl-α* (*permutations-of-set X*) {*r*. *linear-order-on X r*}
⟨*proof*⟩

### 2.1.5 Limited Preference

**definition** *limited* :: '*a set* ⇒ '*a Preference-List* ⇒ *bool* **where**
*limited A r* ≡ ∀ *a*. *a* ∈ *set r* ⟶ *a* ∈ *A*

**fun** *limit-l* :: '*a set* ⇒ '*a Preference-List* ⇒ '*a Preference-List* **where**
*limit-l A l* = *List.filter* (λ *a*. *a* ∈ *A*) *l*

**lemma** *limited-dest*:
**fixes**
*A* :: '*a set* **and**
*l* :: '*a Preference-List* **and**
*a b* :: '*a*
**assumes**
*a* $\lesssim_l$ *b* **and**
*limited A l*
**shows** *a* ∈ *A* ∧ *b* ∈ *A*
⟨*proof*⟩

**lemma** *limit-equiv*:
**fixes**
*A* :: '*a set* **and**
*l* :: '*a list*
**assumes** *well-formed-l l*
**shows** *pl-α* (*limit-l A l*) = *limit A* (*pl-α l*)
⟨*proof*⟩

### 2.1.6 Auxiliary Definitions

**definition** *total-on-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*total-on-l A l* ≡ ∀ *a* ∈ *A*. *a* ∈ *set l*

**definition** *refl-on-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*refl-on-l A l* ≡ (∀ *a*. *a* ∈ *set l* ⟶ *a* ∈ *A*) ∧ (∀ *a* ∈ *A*. *a* $\lesssim_l$ *a*)

**definition** *trans* :: *'a Preference-List* ⇒ *bool* **where**
*trans l* ≡ ∀ (*a*, *b*, *c*) ∈ *set l* × *set l* × *set l*. *a* $\lesssim_l$ *b* ∧ *b* $\lesssim_l$ *c* ⟶ *a* $\lesssim_l$ *c*

**definition** *preorder-on-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*preorder-on-l A l* ≡ *refl-on-l A l* ∧ *trans l*

**definition** *antisym-l* :: *'a list* ⇒ *bool* **where**
*antisym-l l* ≡ ∀ *a b*. *a* $\lesssim_l$ *b* ∧ *b* $\lesssim_l$ *a* ⟶ *a* = *b*

**definition** *partial-order-on-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*partial-order-on-l A l* ≡ *preorder-on-l A l* ∧ *antisym-l l*

**definition** *linear-order-on-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*linear-order-on-l A l* ≡ *partial-order-on-l A l* ∧ *total-on-l A l*

**definition** *connex-l* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*connex-l A l* ≡ *limited A l* ∧ (∀ *a* ∈ *A*. ∀ *b* ∈ *A*. *a* $\lesssim_l$ *b* ∨ *b* $\lesssim_l$ *a*)

**abbreviation** *ballot-on* :: *'a set* ⇒ *'a Preference-List* ⇒ *bool* **where**
*ballot-on A l* ≡ *well-formed-l l* ∧ *linear-order-on-l A l*

### 2.1.7 Auxiliary Lemmas

**lemma** *list-trans*[*simp*]:
**fixes** *l* :: *'a Preference-List*
**shows** *trans l*
⟨*proof*⟩

**lemma** *list-antisym*[*simp*]:
**fixes** *l* :: *'a Preference-List*
**shows** *antisym-l l*
⟨*proof*⟩

**lemma** *lin-order-equiv-list-of-alts*:
**fixes**
*A* :: *'a set* **and**
*l* :: *'a Preference-List*
**shows** *linear-order-on-l A l* = (*A* = *set l*)
⟨*proof*⟩

**lemma** *connex-imp-refl*:
**fixes**

*A* :: *'a set* **and**
*l* :: *'a Preference-List*
**assumes** *connex-l A l*
**shows** *refl-on-l A l*
⟨*proof*⟩

**lemma** *lin-ord-imp-connex-l*:
**fixes**
*A* :: *'a set* **and**
*l* :: *'a Preference-List*
**assumes** *linear-order-on-l A l*
**shows** *connex-l A l*
⟨*proof*⟩

**lemma** *above-trans*:
**fixes**
*l* :: *'a Preference-List* **and**
*a b* :: *'a*
**assumes**
*trans l* **and**
$a \lesssim_l b$
**shows** *set* (*above-l l b*) ⊆ *set* (*above-l l a*)
⟨*proof*⟩

**lemma** *less-preferred-l-rel-equiv*:
**fixes**
*l* :: *'a Preference-List* **and**
*a b* :: *'a*
**shows** $a \lesssim_l b =$
*Preference-Relation.is-less-preferred-than a* (*pl-α l*) *b*
⟨*proof*⟩

**theorem** *above-equiv*:
**fixes**
*l* :: *'a Preference-List* **and**
*a* :: *'a*
**shows** *set* (*above-l l a*) = *above* (*pl-α l*) *a*
⟨*proof*⟩

**theorem** *rank-equiv*:
**fixes**
*l* :: *'a Preference-List* **and**
*a* :: *'a*
**assumes** *well-formed-l l*
**shows** *rank-l l a* = *rank* (*pl-α l*) *a*
⟨*proof*⟩

**lemma** *lin-ord-equiv*:
**fixes**

$A :: 'a\ set$ **and**
$l :: 'a\ Preference\text{-}List$
**shows** *linear-order-on-l* $A\ l = linear\text{-}order\text{-}on\ A\ (pl\text{-}\alpha\ l)$
⟨*proof*⟩

### 2.1.8 First Occurrence Indices

**lemma** *pos-in-list-yields-rank*:
**fixes**
$l :: 'a\ Preference\text{-}List$ **and**
$a :: 'a$ **and**
$n :: nat$
**assumes**
$\forall\ (j::nat) \leq n.\ l!j \neq a$ **and**
$l!(n - 1) = a$
**shows** *rank-l* $l\ a = n$
⟨*proof*⟩

**lemma** *ranked-alt-not-at-pos-before*:
**fixes**
$l :: 'a\ Preference\text{-}List$ **and**
$a :: 'a$ **and**
$n :: nat$
**assumes**
$a \in set\ l$ **and**
$n < (rank\text{-}l\ l\ a) - 1$
**shows** $l!n \neq a$
⟨*proof*⟩

**lemma** *pos-in-list-yields-pos*:
**fixes**
$l :: 'a\ Preference\text{-}List$ **and**
$a :: 'a$
**assumes** $a \in set\ l$
**shows** $l!(rank\text{-}l\ l\ a - 1) = a$
⟨*proof*⟩

**lemma** *rel-of-pref-pred-for-set-eq-list-to-rel*:
**fixes** $l :: 'a\ Preference\text{-}List$
**shows** $relation\text{-}of\ (\lambda\ y\ z.\ y \lesssim_l z)\ (set\ l) = pl\text{-}\alpha\ l$
⟨*proof*⟩

**end**

## 2.2 Preference (List) Profile

**theory** *Profile-List*
  **imports** *../Profile*
    *Preference-List*
**begin**

### 2.2.1 Definition

A profile (list) contains one ballot for each voter.

**type-synonym** *'a Profile-List = 'a Preference-List list*

**type-synonym** *'a Election-List = 'a set × 'a Profile-List*

Abstraction from profile list to profile.

**fun** *pl-to-pr-α :: 'a Profile-List ⇒ ('a, nat) Profile* **where**
  *pl-to-pr-α pl = (λ n. if (n < length pl ∧ n ≥ 0)*
    *then (map (Preference-List.pl-α) pl)!n*
    *else {})*

**lemma** *prof-abstr-presv-size*:
  **fixes** *p :: 'a Profile-List*
  **shows** *length p = length (to-list {0 ..< length p} (pl-to-pr-α p))*
  ⟨*proof*⟩

### 2.2.2 Refinement Proof

A profile on a finite set of alternatives A contains only ballots that are lists of linear orders on A.

**definition** *profile-l :: 'a set ⇒ 'a Profile-List ⇒ bool* **where**
  *profile-l A p ≡ ∀ i < length p. ballot-on A (p!i)*

**lemma** *refinement*:
  **fixes**
    *A :: 'a set* **and**
    *p :: 'a Profile-List*
  **assumes** *profile-l A p*
  **shows** *profile {0 ..< length p} A (pl-to-pr-α p)*
⟨*proof*⟩

**end**

## 2.3 Ordered Relation Type

**theory** *Ordered-Relation*

**imports** *Preference-Relation*
*./Refined-Types/Preference-List*
*HOL−Combinatorics.Multiset-Permutations*
**begin**

**lemma** *fin-ordered*:
**fixes** $X :: 'x\ set$
**assumes** *finite X*
**obtains** *ord* :: $'x\ rel$ **where**
*linear-order-on X ord*
⟨*proof*⟩

**typedef** $'a$ *Ordered-Preference* =
$\{p :: 'a ::$ *finite Preference-Relation. linear-order-on* (*UNIV* :: $'a\ set$) $p\}$
**morphisms** *ord2pref pref2ord*
⟨*proof*⟩

**instance** *Ordered-Preference* :: (*finite*) *finite*
⟨*proof*⟩

**lemma** *range-ord2pref*: *range ord2pref* = $\{p.$ *linear-order* $p\}$
⟨*proof*⟩

**lemma** *card-ord-pref*: *card* (*UNIV* :: $'a ::$ *finite Ordered-Preference set*) =
*fact* (*card* (*UNIV* :: $'a\ set$))
⟨*proof*⟩

**end**

## 2.4 Alternative Election Type

**theory** *Quotient-Type-Election*
**imports** *Profile*
**begin**

**lemma** *election-equality-equiv*:
*election-equality E E* **and**
*election-equality E* $E' \longrightarrow$ *election-equality* $E'$ *E* **and**
*election-equality E* $E' \longrightarrow$ *election-equality* $E'$ *F*
$\longrightarrow$ *election-equality E F*
⟨*proof*⟩

**quotient-type** ($'a$, $'v$) $Election_{\mathcal{Q}}$ =
$'a\ set \times 'v\ set \times$ ($'a$, $'v$) *Profile* / *election-equality*
⟨*proof*⟩

**fun** $fst_{\mathcal{Q}}$ :: ($'a$, $'v$) $Election_{\mathcal{Q}} \Rightarrow 'a\ set$ **where**

$fst_{\mathcal{Q}}\ E = Product\text{-}Type.fst\ (rep\text{-}Election_{\mathcal{Q}}\ E)$

**fun** $snd_{\mathcal{Q}} :: ('a, 'v)\ Election_{\mathcal{Q}} \Rightarrow 'v\ set \times ('a, 'v)\ Profile$ **where**
$snd_{\mathcal{Q}}\ E = Product\text{-}Type.snd\ (rep\text{-}Election_{\mathcal{Q}}\ E)$

**abbreviation** $alternatives\text{-}\mathcal{E}_{\mathcal{Q}} :: ('a, 'v)\ Election_{\mathcal{Q}} \Rightarrow 'a\ set$ **where**
$alternatives\text{-}\mathcal{E}_{\mathcal{Q}}\ E \equiv fst_{\mathcal{Q}}\ E$

**abbreviation** $voters\text{-}\mathcal{E}_{\mathcal{Q}} :: ('a, 'v)\ Election_{\mathcal{Q}} \Rightarrow 'v\ set$ **where**
$voters\text{-}\mathcal{E}_{\mathcal{Q}}\ E \equiv Product\text{-}Type.fst\ (snd_{\mathcal{Q}}\ E)$

**abbreviation** $profile\text{-}\mathcal{E}_{\mathcal{Q}} :: ('a, 'v)\ Election_{\mathcal{Q}} \Rightarrow ('a, 'v)\ Profile$ **where**
$profile\text{-}\mathcal{E}_{\mathcal{Q}}\ E \equiv Product\text{-}Type.snd\ (snd_{\mathcal{Q}}\ E)$

**end**

# Chapter 3

# Quotient Rules

## 3.1 Quotients of Equivalence Relations

**theory** *Relation-Quotients*
  **imports** *../Social-Choice-Types/Symmetry-Of-Functions*
**begin**

### 3.1.1 Definitions

**fun** *singleton-set* :: *'x set* ⇒ *'x* **where**
  *singleton-set s* = (*if* (*card s* = *1*) *then* (*the-inv* (λ *x*. {*x*}) *s*) *else undefined*)
— This is undefined if *card s* ≠ *1*. Note that "*undefined* = *undefined*" is the only provable equality for *undefined*.

For a given function, we define a function on sets that maps each set to the unique image under f of its elements, if one exists. Otherwise, the result is undefined.

**fun** $\pi_{\mathcal{Q}}$ :: (*'x* ⇒ *'y*) ⇒ (*'x set* ⇒ *'y*) **where**
  $\pi_{\mathcal{Q}}$ *f s* = *singleton-set* (*f* ‘ *s*)

For a given function f on sets and a mapping from elements to sets, we define a function on the set element type that maps each element to the image of its corresponding set under f. A natural mapping is from elements to their classes under a relation.

**fun** *inv-*$\pi_{\mathcal{Q}}$ :: (*'x* ⇒ *'x set*) ⇒ (*'x set* ⇒ *'y*) ⇒ (*'x* ⇒ *'y*) **where**
  *inv-*$\pi_{\mathcal{Q}}$ *cls f x* = *f* (*cls x*)

**fun** *relation-class* :: *'x rel* ⇒ *'x* ⇒ *'x set* **where**
  *relation-class r x* = *r* ‘‘ {*x*}

### 3.1.2 Well-Definedness

**lemma** *singleton-set-undef-if-card-neq-one*:
  **fixes** *s* :: *'x set*

**assumes** *card s* $\neq$ *1*
**shows** *singleton-set s* = *undefined*
⟨*proof*⟩

**lemma** *singleton-set-def-if-card-one*:
**fixes** *s* :: *'x set*
**assumes** *card s* = *1*
**shows** $\exists!$ *x*. *x* = *singleton-set s* $\wedge$ $\{x\}$ = *s*
⟨*proof*⟩

If the given function is invariant under an equivalence relation, the induced function on sets is well-defined for all equivalence classes of that relation.

**theorem** *pass-to-quotient*:
**fixes**
*f* :: *'x* $\Rightarrow$ *'y* **and**
*r* :: *'x rel* **and**
*s* :: *'x set*
**assumes**
*f respects r* **and**
*equiv s r*
**shows** $\forall$ *t* $\in$ *s* // *r*. $\forall$ *x* $\in$ *t*. $\pi_{\mathcal{Q}}$ *f t* = *f x*
⟨*proof*⟩

A function on sets induces a function on the element type that is invariant under a given equivalence relation.

**theorem** *pass-to-quotient-inv*:
**fixes**
*f* :: *'x set* $\Rightarrow$ *'x* **and**
*r* :: *'x rel* **and**
*s* :: *'x set*
**assumes** *equiv s r*
**defines** *induced-fun* $\equiv$ (*inv-*$\pi_{\mathcal{Q}}$ (*relation-class r*) *f*)
**shows**
*induced-fun respects r* **and**
$\forall$ *A* $\in$ *s* // *r*. $\pi_{\mathcal{Q}}$ *induced-fun A* = *f A*
⟨*proof*⟩

### 3.1.3 Equivalence Relations

**lemma** *restr-equals-restricted-rel*:
**fixes**
*s t* :: *'a set* **and**
*r* :: *'a rel*
**assumes**
*closed-restricted-rel r s t* **and**
*t* $\subseteq$ *s*
**shows** *restricted-rel r t s* = *Restr r t*
⟨*proof*⟩

**lemma** *equiv-rel-restr*:
**fixes**
$s\ t :: 'x\ set$ **and**
$r :: 'x\ rel$
**assumes**
*equiv* $s\ r$ **and**
$t \subseteq s$
**shows** *equiv* $t$ (*Restr* $r\ t$)
⟨*proof*⟩

**lemma** *rel-ind-by-group-act-equiv*:
**fixes**
$m :: 'x\ monoid$ **and**
$s :: 'y\ set$ **and**
$\varphi :: ('x, 'y)\ binary\text{-}fun$
**assumes** *group-action* $m\ s\ \varphi$
**shows** *equiv* $s$ (*action-induced-rel* (*carrier* $m$) $s\ \varphi$)
⟨*proof*⟩

**end**

## 3.2 Quotients of Election Set Equivalences

**theory** *Election-Quotients*
**imports** *Relation-Quotients*
*../Social-Choice-Types/Voting-Symmetry*
*../Social-Choice-Types/Ordered-Relation*
*HOL−Analysis.Convex*
*HOL−Analysis.Cartesian-Space*
**begin**

### 3.2.1 Auxiliary Lemmas

**lemma** *obtain-partition*:
**fixes**
$A :: 'a\ set$ **and**
$N :: 'b \Rightarrow nat$ **and**
$B :: 'b\ set$
**assumes**
*finite* $A$ **and**
*finite* $B$ **and**
*sum* $N\ B = card\ A$
**shows** $\exists\ \mathcal{X}.\ A = \bigcup\ \{\mathcal{X}\ i \mid i.\ i \in B\} \wedge (\forall\ i \in B.\ card\ (\mathcal{X}\ i) = N\ i) \wedge$
$(\forall\ i\ j.\ i \neq j \longrightarrow i \in B \wedge j \in B \longrightarrow \mathcal{X}\ i \cap \mathcal{X}\ j = \{\})$
⟨*proof*⟩

### 3.2.2 Anonymity Quotient: Grid

**fun** $anonymity_{\mathcal{Q}}$ :: *'a set* ⇒ (*'a*, *'v*) *Election set set* **where**
  $anonymity_{\mathcal{Q}}$ *A* = *quotient* (*elections-*$\mathcal{A}$ *A*) ($anonymity_{\mathcal{R}}$ (*elections-*$\mathcal{A}$ *A*))

— Here, we count the occurrences of a ballot per election in a set of elections for which the occurrences of the ballot per election coincide for all elections in the set.
**fun** $vote\text{-}count_{\mathcal{Q}}$ :: *'a Preference-Relation* ⇒ (*'a*, *'v*) *Election set* ⇒ *nat* **where**
  $vote\text{-}count_{\mathcal{Q}}$ *p* = $\pi_{\mathcal{Q}}$ (*vote-count p*)

**fun** *anonymity-class* :: (*'a*::*finite*, *'v*) *Election set* ⇒
      (*nat*, *'a Ordered-Preference*) *vec* **where**
  *anonymity-class X* = (χ *p*. $vote\text{-}count_{\mathcal{Q}}$ (*ord2pref p*) *X*)

**lemma** *anon-rel-equiv*: *equiv* (*elections-*$\mathcal{A}$ *UNIV*) ($anonymity_{\mathcal{R}}$ (*elections-*$\mathcal{A}$ *UNIV*))
⟨*proof*⟩

We assume that all elections consist of a fixed finite alternative set of size $n$ and finite subsets of an infinite voter universe. Profiles are linear orders on the alternatives. Then, we can operate on the natural-number-vectors of dimension $n!$ instead of the equivalence classes of the anonymity relation: Each dimension corresponds to one possible linear order on the alternative set, i.e., the possible preferences. Each equivalence class of elections corresponds to a vector whose entries denote the amount of voters per election in that class who vote the respective corresponding preference.

**theorem** $anonymity_{\mathcal{Q}}$*-isomorphism*:
  **assumes** *infinite* (*UNIV* :: *'v set*)
  **shows** *bij-betw* (*anonymity-class* :: (*'a* :: *finite*, *'v*) *Election set*
        ⇒ *nat*^(*'a Ordered-Preference*)) ($anonymity_{\mathcal{Q}}$ (*UNIV* :: *'a set*))
      (*UNIV* :: (*nat*^(*'a Ordered-Preference*)) *set*)
⟨*proof*⟩

### 3.2.3 Homogeneity Quotient: Simplex

**fun** *vote-fraction* :: *'a Preference-Relation* ⇒ (*'a*, *'v*) *Election* ⇒ *rat* **where**
  *vote-fraction r E* =
    (*if* (*finite* (*voters-*$\mathcal{E}$ *E*) ∧ *voters-*$\mathcal{E}$ *E* ≠ {})
      *then* (*Fract* (*vote-count r E*) (*card* (*voters-*$\mathcal{E}$ *E*))) *else 0*)

**fun** $anonymity\text{-}homogeneity_{\mathcal{R}}$ :: (*'a*, *'v*) *Election set* ⇒ (*'a*, *'v*) *Election rel* **where**
  $anonymity\text{-}homogeneity_{\mathcal{R}}$ $\mathcal{E}$ =
    {(*E*, *E'*) | *E E'*. *E* ∈ $\mathcal{E}$ ∧ *E'* ∈ $\mathcal{E}$
          ∧ (*finite* (*voters-*$\mathcal{E}$ *E*) = *finite* (*voters-*$\mathcal{E}$ *E'*))
          ∧ (∀ *r*. *vote-fraction r E* = *vote-fraction r E'*)}

**fun** $anonymity\text{-}homogeneity_{\mathcal{Q}}$ :: *'a set* ⇒ (*'a*, *'v*) *Election set set* **where**
  $anonymity\text{-}homogeneity_{\mathcal{Q}}$ *A* =
    *quotient* (*elections-*$\mathcal{A}$ *A*) ($anonymity\text{-}homogeneity_{\mathcal{R}}$ (*elections-*$\mathcal{A}$ *A*))

**fun** *vote-fraction*$_{\mathcal{Q}}$ :: *'a Preference-Relation* ⇒ (*'a*, *'v*) *Election set* ⇒ *rat* **where**
*vote-fraction*$_{\mathcal{Q}}$ *p* = $\pi_{\mathcal{Q}}$ (*vote-fraction p*)

**fun** *anonymity-homogeneity-class* :: (*'a*::*finite*, *'v*) *Election set* ⇒
(*rat*, *'a Ordered-Preference*) *vec* **where**
*anonymity-homogeneity-class* $\mathcal{E}$ = (χ *p*. *vote-fraction*$_{\mathcal{Q}}$ (*ord2pref p*) $\mathcal{E}$)

Maps each rational real vector entry to the corresponding rational. If the entry is not rational, the corresponding entry will be undefined.

**fun** *rat-vector* :: *real*^'*b* ⇒ *rat*^'*b* **where**
*rat-vector v* = (χ *p*. *the-inv of-rat* (*v*$*p*))

**fun** *rat-vector-set* :: (*real*^'*b*) *set* ⇒ (*rat*^'*b*) *set* **where**
*rat-vector-set V* = *rat-vector* ' {*v* ∈ *V*. ∀ *i*. *v*$*i* ∈ ℚ}

**definition** *standard-basis* :: (*real*^'*b*) *set* **where**
*standard-basis* ≡ {*v*. ∃ *b*. *v*$*b* = *1* ∧ (∀ *c* ≠ *b*. *v*$*c* = *0*)}

The rational points in the simplex.

**definition** *vote-simplex* :: (*rat*^'*b*) *set* **where**
*vote-simplex* ≡
*insert 0* (*rat-vector-set* (*convex hull* (*standard-basis* :: (*real*^'*b*) *set*)))

## Auxiliary Lemmas

**lemma** *convex-combination-in-convex-hull*:
**fixes**
*X* :: (*real*^'*b*) *set* **and**
*x* :: *real*^'*b*
**assumes** ∃ *f* :: (*real*^'*b*) ⇒ *real*.
*sum f X* = *1* ∧ (∀ *x* ∈ *X*. *f x* ≥ *0*)
∧ *x* = *sum* (λ *x*. (*f x*) *$_R$ *x*) *X*
**shows** *x* ∈ *convex hull X*
⟨*proof*⟩

**lemma** *standard-simplex-rewrite*: *convex hull standard-basis* =
{*v* :: (*real*^'*b*). (∀ *i*. *v*$*i* ≥ *0*) ∧ *sum* (($) *v*) *UNIV* = *1*}
⟨*proof*⟩

**lemma** *fract-distr-helper*:
**fixes** *a b c* :: *int*
**assumes** *c* ≠ *0*
**shows** *Fract a c* + *Fract b c* = *Fract* (*a* + *b*) *c*
⟨*proof*⟩

**lemma** *anonymity-homogeneity-is-equivalence*:
**fixes** *X* :: (*'a*, *'v*) *Election set*
**assumes** ∀ *E* ∈ *X*. *finite* (*voters-*$\mathcal{E}$ *E*)
**shows** *equiv X* (*anonymity-homogeneity*$_{\mathcal{R}}$ *X*)

⟨*proof*⟩

**lemma** *fract-distr*:
**fixes**
*A* :: *'x set* **and**
*f* :: *'x* ⇒ *int* **and**
*b* :: *int*
**assumes**
*finite A* **and**
*b* ≠ *0*
**shows** *sum* (λ *a*. *Fract* (*f a*) *b*) *A* = *Fract* (*sum f A*) *b*
⟨*proof*⟩

### Simplex Bijection

We assume all our elections to consist of a fixed finite alternative set of size n and finite subsets of an infinite voter universe. Profiles are linear orders on the alternatives. Then we can work on the standard simplex of dimension n! instead of the equivalence classes of the equivalence relation for anonymous + homogeneous voting rules (anon hom): Each dimension corresponds to one possible linear order on the alternative set, i.e., the possible preferences. Each equivalence class of elections corresponds to a vector whose entries denote the fraction of voters per election in that class who vote the respective corresponding preference.

**theorem** *anonymity-homogeneity*$_{\mathcal{Q}}$*-isomorphism*:
**assumes** *infinite* (*UNIV* :: *'v set*)
**shows**
*bij-betw* (*anonymity-homogeneity-class* :: (*'a* :: *finite*, *'v*) *Election set* ⇒
*rat*^(*'a Ordered-Preference*)) (*anonymity-homogeneity*$_{\mathcal{Q}}$ (*UNIV* :: *'a set*))
(*vote-simplex* :: (*rat*^(*'a Ordered-Preference*)) *set*)
⟨*proof*⟩

**end**

# Chapter 4

# Component Types

## 4.1 Distance

**theory** *Distance*
  **imports** *HOL−Library.Extended-Real*
    *Social-Choice-Types/Voting-Symmetry*
**begin**

A general distance on a set X is a mapping $d$: $X \times X \mapsto R \cup \{+\infty\}$ such that for every $x$, $y$, $z$ in X, the following four conditions are satisfied:

- $d(x, y) \geq 0$ (non-negativity);
- $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
- $d(x, y) = d(y, x)$ (symmetry);
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

  Moreover, a mapping that satisfies all but the second conditions is called a pseudo-distance, whereas a quasi-distance needs to satisfy the first three conditions (and not necessarily the last one).

### 4.1.1 Definition

**type-synonym** *'a Distance* = *'a* ⇒ *'a* ⇒ *ereal*

The un-curried version of a distance is defined on tuples.

**fun** *tup* :: *'a Distance* ⇒ (*'a* ∗ *'a* ⇒ *ereal*) **where**
  *tup d* = (λ *pair*. *d* (*fst pair*) (*snd pair*))

**definition** *distance* :: *'a set* ⇒ *'a Distance* ⇒ *bool* **where**
  *distance S d* ≡ ∀ *x y*. *x* ∈ *S* ∧ *y* ∈ *S* ⟶ *d x x* = 0 ∧ 0 ≤ *d x y*

### 4.1.2 Conditions

**definition** *symmetric* :: *'a set* ⇒ *'a Distance* ⇒ *bool* **where**
*symmetric S d* ≡ ∀ *x y*. *x* ∈ *S* ∧ *y* ∈ *S* ⟶ *d x y* = *d y x*

**definition** *triangle-ineq* :: *'a set* ⇒ *'a Distance* ⇒ *bool* **where**
*triangle-ineq S d* ≡ ∀ *x y z*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *z* ∈ *S* ⟶ *d x z* ≤ *d x y* + *d y z*

**definition** *eq-if-zero* :: *'a set* ⇒ *'a Distance* ⇒ *bool* **where**
*eq-if-zero S d* ≡ ∀ *x y*. *x* ∈ *S* ∧ *y* ∈ *S* ⟶ *d x y* = *0* ⟶ *x* = *y*

**definition** *vote-distance* :: (*'a Vote set* ⇒ *'a Vote Distance* ⇒ *bool*) ⇒
*'a Vote Distance* ⇒ *bool* **where**
*vote-distance π d* ≡ *π* {(*A*, *p*). *linear-order-on A p* ∧ *finite A*} *d*

**definition** *election-distance* :: ((*'a*, *'v*) *Election set* ⇒
(*'a*, *'v*) *Election Distance* ⇒ *bool*) ⇒
(*'a*, *'v*) *Election Distance* ⇒ *bool* **where**
*election-distance π d* ≡ *π* {(*A*, *V*, *p*). *finite-profile V A p*} *d*

### 4.1.3 Standard-Distance Property

**definition** *standard* :: (*'a*, *'v*) *Election Distance* ⇒ *bool* **where**
*standard d* ≡
∀ *A A' V V' p p'*. *A* ≠ *A'* ∨ *V* ≠ *V'* ⟶ *d* (*A*, *V*, *p*) (*A'*, *V'*, *p'*) = ∞

### 4.1.4 Auxiliary Lemmas

**fun** *arg-min-set* :: (*'b* ⇒ *'a* :: *ord*) ⇒ *'b set* ⇒ *'b set* **where**
*arg-min-set f A* = *Collect* (*is-arg-min f* (λ *a*. *a* ∈ *A*))

**lemma** *arg-min-subset*:
**fixes**
*B* :: *'b set* **and**
*f* :: *'b* ⇒ *'a* :: *ord*
**shows** *arg-min-set f B* ⊆ *B*
⟨*proof*⟩

**lemma** *sum-monotone*:
**fixes**
*A* :: *'a set* **and**
*f g* :: *'a* ⇒ *int*
**assumes** ∀ *a* ∈ *A*. *f a* ≤ *g a*
**shows** (∑ *a* ∈ *A*. *f a*) ≤ (∑ *a* ∈ *A*. *g a*)
⟨*proof*⟩

**lemma** *distrib*:
**fixes**
*A* :: *'a set* **and**
*f g* :: *'a* ⇒ *int*

**shows** $(\sum\ a \in A.\ f\ a) + (\sum\ a \in A.\ g\ a) = (\sum\ a \in A.\ f\ a + g\ a)$
⟨*proof*⟩

**lemma** *distrib-ereal*:
**fixes**
$A :: 'a\ set$ **and**
$f\ g :: 'a \Rightarrow int$
**shows** *ereal* (*real-of-int* $((\sum\ a \in A.\ (f :: 'a \Rightarrow int)\ a) + (\sum\ a \in A.\ g\ a))) =$
*ereal* (*real-of-int* $((\sum\ a \in A.\ (f\ a) + (g\ a))))$
⟨*proof*⟩

**lemma** *uneq-ereal*:
**fixes** $x\ y :: int$
**assumes** $x \leq y$
**shows** *ereal* (*real-of-int* $x$) $\leq$ *ereal* (*real-of-int* $y$)
⟨*proof*⟩

### 4.1.5 Swap Distance

**fun** *neq-ord* :: $'a$ *Preference-Relation* $\Rightarrow$ $'a$ *Preference-Relation* $\Rightarrow$
$'a \Rightarrow 'a \Rightarrow bool$ **where**
*neq-ord* $r\ s\ a\ b = ((a \preceq_r b \wedge b \preceq_s a) \vee (b \preceq_r a \wedge a \preceq_s b))$

**fun** *pairwise-disagreements* :: $'a\ set \Rightarrow$ $'a$ *Preference-Relation* $\Rightarrow$
$'a$ *Preference-Relation* $\Rightarrow$ $('a \times 'a)\ set$ **where**
*pairwise-disagreements* $A\ r\ s = \{(a, b) \in A \times A.\ a \neq b \wedge$ *neq-ord* $r\ s\ a\ b\}$

**fun** *pairwise-disagreements'* :: $'a\ set \Rightarrow$ $'a$ *Preference-Relation* $\Rightarrow$
$'a$ *Preference-Relation* $\Rightarrow$ $('a \times 'a)\ set$ **where**
*pairwise-disagreements'* $A\ r\ s =$
*Set.filter* $(\lambda\ (a, b).\ a \neq b \wedge$ *neq-ord* $r\ s\ a\ b)\ (A \times A)$

**lemma** *set-eq-filter*:
**fixes**
$X :: 'a\ set$ **and**
$P :: 'a \Rightarrow bool$
**shows** $\{x \in X.\ P\ x\} =$ *Set.filter* $P\ X$
⟨*proof*⟩

**lemma** *pairwise-disagreements-eq*[*code*]: *pairwise-disagreements* = *pairwise-disagreements'*
⟨*proof*⟩

**fun** *swap* :: $'a$ *Vote Distance* **where**
*swap* $(A, r)\ (A', r') =$
(*if* $A = A'$
*then card* (*pairwise-disagreements* $A\ r\ r'$)
*else* $\infty$)

**lemma** *swap-case-infinity*:

**fixes** $x\ y :: 'a\ Vote$
**assumes** $alts\text{-}\mathcal{V}\ x \neq alts\text{-}\mathcal{V}\ y$
**shows** $swap\ x\ y = \infty$
⟨*proof*⟩

**lemma** *swap-case-fin*:
**fixes** $x\ y :: 'a\ Vote$
**assumes** $alts\text{-}\mathcal{V}\ x = alts\text{-}\mathcal{V}\ y$
**shows** $swap\ x\ y = card\ (pairwise\text{-}disagreements\ (alts\text{-}\mathcal{V}\ x)\ (pref\text{-}\mathcal{V}\ x)\ (pref\text{-}\mathcal{V}\ y))$
⟨*proof*⟩

### 4.1.6 Spearman Distance

**fun** *spearman* :: $'a\ Vote\ Distance$ **where**
$spearman\ (A,\ x)\ (A',\ y) =$
$(if\ A = A'$
$then\ \sum\ a \in A.\ abs\ (int\ (rank\ x\ a) - int\ (rank\ y\ a))$
$else\ \infty)$

**lemma** *spearman-case-inf*:
**fixes** $x\ y :: 'a\ Vote$
**assumes** $alts\text{-}\mathcal{V}\ x \neq alts\text{-}\mathcal{V}\ y$
**shows** $spearman\ x\ y = \infty$
⟨*proof*⟩

**lemma** *spearman-case-fin*:
**fixes** $x\ y :: 'a\ Vote$
**assumes** $alts\text{-}\mathcal{V}\ x = alts\text{-}\mathcal{V}\ y$
**shows** $spearman\ x\ y =$
$(\sum\ a \in alts\text{-}\mathcal{V}\ x.\ abs\ (int\ (rank\ (pref\text{-}\mathcal{V}\ x)\ a) - int\ (rank\ (pref\text{-}\mathcal{V}\ y)\ a)))$
⟨*proof*⟩

### 4.1.7 Properties

Distances that are invariant under specific relations induce symmetry properties in distance rationalized voting rules.

#### Definitions

**fun** $total\text{-}invariance_{\mathcal{D}} :: 'x\ Distance \Rightarrow 'x\ rel \Rightarrow bool$ **where**
$total\text{-}invariance_{\mathcal{D}}\ d\ rel = is\text{-}symmetry\ (tup\ d)\ (Invariance\ (product\ rel))$

**fun** $invariance_{\mathcal{D}} :: 'y\ Distance \Rightarrow 'x\ set \Rightarrow 'y\ set \Rightarrow$
$('x,\ 'y)\ binary\text{-}fun \Rightarrow bool$ **where**
$invariance_{\mathcal{D}}\ d\ X\ Y\ \varphi = is\text{-}symmetry\ (tup\ d)\ (Invariance\ (equivariance\ X\ Y\ \varphi))$

**definition** *distance-anonymity* :: $('a,\ 'v)\ Election\ Distance \Rightarrow bool$ **where**
$distance\text{-}anonymity\ d \equiv$
$\forall\ A\ A'\ V\ V'\ p\ p'\ \pi :: ('v \Rightarrow 'v).$

$(bij\ \pi \longrightarrow$
$(d\ (A,\ V,\ p)\ (A',\ V',\ p')) =$
$(d\ (rename\ \pi\ (A,\ V,\ p)))\ (rename\ \pi\ (A',\ V',\ p')))$

**fun** *distance-anonymity'* :: $('a,\ 'v)$ *Election set* ⇒
$('a,\ 'v)$ *Election Distance* ⇒ *bool* **where**
*distance-anonymity'* $X\ d = invariance_{\mathcal{D}}\ d\ (carrier\ anonymity_{\mathcal{G}})\ X\ (\varphi\text{-}anon\ X)$

**fun** *distance-neutrality* :: $('a,\ 'v)$ *Election set* ⇒
$('a,\ 'v)$ *Election Distance* ⇒ *bool* **where**
*distance-neutrality* $X\ d = invariance_{\mathcal{D}}\ d\ (carrier\ neutrality_{\mathcal{G}})\ X\ (\varphi\text{-}neutral\ X)$

**fun** *distance-reversal-symmetry* :: $('a,\ 'v)$ *Election set* ⇒
$('a,\ 'v)$ *Election Distance* ⇒ *bool* **where**
*distance-reversal-symmetry* $X\ d =$
$invariance_{\mathcal{D}}\ d\ (carrier\ reversal_{\mathcal{G}})\ X\ (\varphi\text{-}reverse\ X)$

**definition** *distance-homogeneity'* :: $('a,\ 'v :: linorder)$ *Election set* ⇒
$('a,\ 'v)$ *Election Distance* ⇒ *bool* **where**
*distance-homogeneity'* $X\ d \equiv total\text{-}invariance_{\mathcal{D}}\ d\ (homogeneity_{\mathcal{R}}'\ X)$

**definition** *distance-homogeneity* :: $('a,\ 'v)$ *Election set* ⇒
$('a,\ 'v)$ *Election Distance* ⇒ *bool* **where**
*distance-homogeneity* $X\ d \equiv total\text{-}invariance_{\mathcal{D}}\ d\ (homogeneity_{\mathcal{R}}\ X)$

## Auxiliary Lemmas

**lemma** *rewrite-total-invariance*$_{\mathcal{D}}$:
**fixes**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$
**shows** $total\text{-}invariance_{\mathcal{D}}\ d\ r = (\forall\ (x,\ y) \in r.\ \forall\ (a,\ b) \in r.\ d\ a\ x = d\ b\ y)$
⟨*proof*⟩

**lemma** *rewrite-invariance*$_{\mathcal{D}}$:
**fixes**
$d :: 'y\ Distance$ **and**
$X :: 'x\ set$ **and**
$Y :: 'y\ set$ **and**
$\varphi :: ('x,\ 'y)\ binary\text{-}fun$
**shows** $invariance_{\mathcal{D}}\ d\ X\ Y\ \varphi =$
$(\forall\ x \in X.\ \forall\ y \in Y.\ \forall\ z \in Y.\ d\ y\ z = d\ (\varphi\ x\ y)\ (\varphi\ x\ z))$
⟨*proof*⟩

**lemma** *invar-dist-image*:
**fixes**
$d :: 'y\ Distance$ **and**
$G :: 'x\ monoid$ **and**
$Y\ Y' :: 'y\ set$ **and**

$\varphi :: ('x, 'y)$ *binary-fun* **and**
$y :: 'y$ **and**
$g :: 'x$
**assumes**
*invar-d*: $invariance_{\mathcal{D}}\ d\ (carrier\ G)\ Y\ \varphi$ **and**
*Y'-in-Y*: $Y' \subseteq Y$ **and**
*action-φ*: *group-action* $G\ Y\ \varphi$ **and**
*g-carrier*: $g \in carrier\ G$ **and**
*y-in-Y*: $y \in Y$
**shows** $d\ (\varphi\ g\ y)\ `\ (\varphi\ g)\ `\ Y' = d\ y\ `\ Y'$
⟨*proof*⟩

**lemma** *swap-neutral*: $invariance_{\mathcal{D}}\ swap\ (carrier\ neutrality_{\mathcal{G}})$
$UNIV\ (\lambda\ \pi\ (A, q).\ (\pi\ `\ A, rel\text{-}rename\ \pi\ q))$
⟨*proof*⟩

**end**

## 4.2 Votewise Distance

**theory** *Votewise-Distance*
**imports** *Social-Choice-Types/Norm*
*Distance*
**begin**

Votewise distances are a natural class of distances on elections which depend on the submitted votes in a simple and transparent manner. They are formed by using any distance d on individual orders and combining the components with a norm on $\mathbb{R}^n$.

### 4.2.1 Definition

**fun** *votewise-distance* :: *'a Vote Distance* ⇒ *Norm* ⇒
$('a, 'v :: linorder)$ *Election Distance* **where**
*votewise-distance d n* $(A, V, p)\ (A', V', p') =$
(*if* (*finite V*) ∧ $V = V' \wedge (V \neq \{\} \vee A = A')$
*then n* (*map2* $(\lambda\ q\ q'.\ d\ (A, q)\ (A', q'))$ (*to-list V p*) (*to-list V' p'*))
*else* ∞)

### 4.2.2 Inference Rules

**lemma** *symmetric-norm-inv-under-map-permute*:
**fixes**
*d* :: *'a Vote Distance* **and**
*n* :: *Norm* **and**
*A A'* :: *'a set* **and**

$\varphi :: nat \Rightarrow nat$ **and**
$p\ p' :: ('a\ Preference\text{-}Relation)\ list$
**assumes**
*perm*: $\varphi\ permutes\ \{0\ ..<\ length\ p\}$ **and**
*len-eq*: $length\ p = length\ p'$ **and**
*sym-n*: $symmetry\ n$
**shows** $n\ (map2\ (\lambda\ q\ q'.\ d\ (A,\ q)\ (A',\ q'))\ p\ p') =$
$n\ (map2\ (\lambda\ q\ q'.\ d\ (A,\ q)\ (A',\ q'))\ (permute\text{-}list\ \varphi\ p)\ (permute\text{-}list\ \varphi\ p'))$
⟨*proof*⟩

**lemma** *permute-invariant-under-map*:
**fixes** $l\ l' :: 'a\ list$
**assumes** $l <^\sim\!\sim> l'$
**shows** $map\ f\ l <^\sim\!\sim> map\ f\ l'$
⟨*proof*⟩

**lemma** *linorder-rank-injective*:
**fixes**
$V :: 'v :: linorder\ set$ **and**
$v\ v' :: 'v$
**assumes**
*v-in-V*: $v \in V$ **and**
*v'-in-V*: $v' \in V$ **and**
*v'-neq-v*: $v' \neq v$ **and**
*fin-V*: $finite\ V$
**shows** $card\ \{x \in V.\ x < v\} \neq card\ \{x \in V.\ x < v'\}$
⟨*proof*⟩

**lemma** *permute-invariant-under-coinciding-funs*:
**fixes**
$l :: 'v\ list$ **and**
$\pi_1\ \pi_2 :: nat \Rightarrow nat$
**assumes** $\forall\ i < length\ l.\ \pi_1\ i = \pi_2\ i$
**shows** $permute\text{-}list\ \pi_1\ l = permute\text{-}list\ \pi_2\ l$
⟨*proof*⟩

**lemma** *symmetric-norm-imp-distance-anonymous*:
**fixes**
$d :: 'a\ Vote\ Distance$ **and**
$n :: Norm$
**assumes** $symmetry\ n$
**shows** $distance\text{-}anonymity\ (votewise\text{-}distance\ d\ n)$
⟨*proof*⟩

**lemma** *neutral-dist-imp-neutral-votewise-dist*:
**fixes**
$d :: 'a\ Vote\ Distance$ **and**
$n :: Norm$
**defines** $vote\text{-}action \equiv (\lambda\ \pi\ (A,\ q).\ (\pi\ `\ A,\ rel\text{-}rename\ \pi\ q))$

**assumes** *invar*: $invariance_{\mathcal{D}}\ d\ (carrier\ neutrality_{\mathcal{G}})\ UNIV\ vote\text{-}action$
**shows** *distance-neutrality well-formed-elections* (*votewise-distance d n*)
⟨*proof*⟩

**end**

## 4.3 Consensus

**theory** *Consensus*
**imports** *Social-Choice-Types/Voting-Symmetry*
**begin**

An election consisting of a set of alternatives and preferential votes for each voter (a profile) is a consensus if it has an undisputed winner reflecting a certain concept of fairness in the society.

### 4.3.1 Definition

**type-synonym** $('a, 'v)\ Consensus = ('a, 'v)\ Election \Rightarrow bool$

### 4.3.2 Consensus Conditions

Nonempty alternative set.

**fun** $nonempty\text{-}set_{\mathcal{C}} :: ('a, 'v)\ Consensus$ **where**
$nonempty\text{-}set_{\mathcal{C}}\ (A,\ V,\ p) = (A \neq \{\})$

Nonempty profile, i.e., nonempty voter set. Note that this is also true if p(v) = holds for all voters v in V.

**fun** $nonempty\text{-}profile_{\mathcal{C}} :: ('a, 'v)\ Consensus$ **where**
$nonempty\text{-}profile_{\mathcal{C}}\ (A,\ V,\ p) = (V \neq \{\})$

Equal top ranked alternatives.

**fun** $equal\text{-}top_{\mathcal{C}}' :: 'a \Rightarrow ('a, 'v)\ Consensus$ **where**
$equal\text{-}top_{\mathcal{C}}'\ a\ (A,\ V,\ p) = (a \in A \wedge (\forall\ v \in V.\ above\ (p\ v)\ a = \{a\}))$

**fun** $equal\text{-}top_{\mathcal{C}} :: ('a, 'v)\ Consensus$ **where**
$equal\text{-}top_{\mathcal{C}}\ c = (\exists\ a.\ equal\text{-}top_{\mathcal{C}}'\ a\ c)$

Equal votes.

**fun** $equal\text{-}vote_{\mathcal{C}}' :: 'a\ Preference\text{-}Relation \Rightarrow ('a, 'v)\ Consensus$ **where**
$equal\text{-}vote_{\mathcal{C}}'\ r\ (A,\ V,\ p) = (\forall\ v \in V.\ (p\ v) = r)$

**fun** $equal\text{-}vote_{\mathcal{C}} :: ('a, 'v)\ Consensus$ **where**
$equal\text{-}vote_{\mathcal{C}}\ c = (\exists\ r.\ equal\text{-}vote_{\mathcal{C}}'\ r\ c)$

Unanimity condition.

**fun** *unanimity*$_{\mathcal{C}}$ :: $('a, 'v)$ *Consensus* **where**
*unanimity*$_{\mathcal{C}}$ $c$ = (*nonempty-set*$_{\mathcal{C}}$ $c \wedge$ *nonempty-profile*$_{\mathcal{C}}$ $c \wedge$ *equal-top*$_{\mathcal{C}}$ $c$)

Strong unanimity condition.

**fun** *strong-unanimity*$_{\mathcal{C}}$ :: $('a, 'v)$ *Consensus* **where**
*strong-unanimity*$_{\mathcal{C}}$ $c$ = (*nonempty-set*$_{\mathcal{C}}$ $c \wedge$ *nonempty-profile*$_{\mathcal{C}}$ $c \wedge$ *equal-vote*$_{\mathcal{C}}$ $c$)

### 4.3.3 Properties

**definition** *consensus-anonymity* :: $('a, 'v)$ *Consensus* $\Rightarrow$ *bool* **where**
*consensus-anonymity* $c \equiv$
($\forall$ $A$ $V$ $p$ $\pi$ :: $('v \Rightarrow 'v)$.
*bij* $\pi \longrightarrow$
(*let* $(A', V', q)$ = (*rename* $\pi$ $(A, V, p)$) *in*
*profile* $V$ $A$ $p \longrightarrow$ *profile* $V'$ $A'$ $q$
$\longrightarrow$ $c$ $(A, V, p) \longrightarrow$ $c$ $(A', V', q)$)))

**fun** *consensus-neutrality* :: $('a, 'v)$ *Election set* $\Rightarrow$ $('a, 'v)$ *Consensus* $\Rightarrow$ *bool* **where**
*consensus-neutrality* $X$ $c$ = *is-symmetry* $c$ (*Invariance* (*neutrality*$_{\mathcal{R}}$ $X$))

### 4.3.4 Auxiliary Lemmas

**lemma** *cons-anon-conj*:
**fixes** $c$ $c'$ :: $('a, 'v)$ *Consensus*
**assumes**
*consensus-anonymity* $c$ **and**
*consensus-anonymity* $c'$
**shows** *consensus-anonymity* $(\lambda$ $e$. $c$ $e \wedge$ $c'$ $e)$
⟨*proof*⟩

**theorem** *cons-conjunction-invariant*:
**fixes**
$\mathfrak{C}$ :: $('a, 'v)$ *Consensus set* **and**
*rel* :: $('a, 'v)$ *Election rel*
**defines** $C \equiv (\lambda$ $E$. ($\forall$ $C' \in \mathfrak{C}$. $C'$ $E))$
**assumes** $\forall$ $C'$. $C' \in \mathfrak{C} \longrightarrow$ *is-symmetry* $C'$ (*Invariance rel*)
**shows** *is-symmetry* $C$ (*Invariance rel*)
⟨*proof*⟩

**lemma** *cons-anon-invariant*:
**fixes**
$c$ :: $('a, 'v)$ *Consensus* **and**
$A$ $A'$ :: $'a$ *set* **and**
$V$ $V'$ :: $'v$ *set* **and**
$p$ $q$ :: $('a, 'v)$ *Profile* **and**
$\pi$ :: $'v \Rightarrow 'v$
**assumes**
*anon*: *consensus-anonymity* $c$ **and**

*bij-π*: *bij π* **and**
*prof-p*: *profile V A p* **and**
*renamed*: *rename π* $(A,\ V,\ p) = (A',\ V',\ q)$ **and**
*cond-c*: *c* $(A,\ V,\ p)$
**shows** *c* $(A',\ V',\ q)$
⟨*proof*⟩

**lemma** *ex-anon-cons-imp-cons-anonymous*:
**fixes**
$b :: ('a,\ 'v)$ *Consensus* **and**
$b' :: 'b \Rightarrow ('a,\ 'v)$ *Consensus*
**assumes**
*general-cond-b*: $b = (\lambda\ E.\ \exists\ x.\ b'\ x\ E)$ **and**
*all-cond-anon*: $\forall\ x.$ *consensus-anonymity* $(b'\ x)$
**shows** *consensus-anonymity b*
⟨*proof*⟩

### 4.3.5 Theorems

#### Anonymity

**lemma** *nonempty-set-cons-anonymous*: *consensus-anonymity nonempty-set*$_{\mathcal{C}}$
⟨*proof*⟩

**lemma** *nonempty-profile-cons-anonymous*: *consensus-anonymity nonempty-profile*$_{\mathcal{C}}$
⟨*proof*⟩

**lemma** *equal-top-cons′-anonymous*:
**fixes** $a :: 'a$
**shows** *consensus-anonymity* (*equal-top*$_{\mathcal{C}}$′ *a*)
⟨*proof*⟩

**lemma** *eq-top-cons-anon*: *consensus-anonymity equal-top*$_{\mathcal{C}}$
⟨*proof*⟩

**lemma** *eq-vote-cons′-anonymous*:
**fixes** $r :: 'a$ *Preference-Relation*
**shows** *consensus-anonymity* (*equal-vote*$_{\mathcal{C}}$′ *r*)
⟨*proof*⟩

**lemma** *eq-vote-cons-anonymous*: *consensus-anonymity equal-vote*$_{\mathcal{C}}$
⟨*proof*⟩

#### Neutrality

**lemma** *nonempty-set*$_{\mathcal{C}}$*-neutral*: *consensus-neutrality well-formed-elections nonempty-set*$_{\mathcal{C}}$
⟨*proof*⟩

**lemma** *nonempty-profile*$_{\mathcal{C}}$*-neutral*: *consensus-neutrality well-formed-elections nonempty-profile*$_{\mathcal{C}}$
⟨*proof*⟩

**lemma** *equal-vote$_{\mathcal{C}}$-neutral*: *consensus-neutrality well-formed-elections equal-vote$_{\mathcal{C}}$*
⟨*proof*⟩

**lemma** *strong-unanimity$_{\mathcal{C}}$-neutral*: *consensus-neutrality*
*well-formed-elections strong-unanimity$_{\mathcal{C}}$*
⟨*proof*⟩

**end**

## 4.4 Electoral Module

**theory** *Electoral-Module*
**imports** *Social-Choice-Types/Property-Interpretations*
**begin**

Electoral modules are the principal component type of the composable modules voting framework, as they are a generalization of voting rules in the sense of social choice functions. These are only the types used for electoral modules. Further restrictions are encompassed by the electoral-module predicate.

An electoral module does not need to make final decisions for all alternatives, but can instead defer the decision for some or all of them to other modules. Hence, electoral modules partition the received (possibly empty) set of alternatives into elected, rejected and deferred alternatives. In particular, any of those sets, e.g., the set of winning (elected) alternatives, may also be left empty, as long as they collectively still hold all the received alternatives. Just like a voting rule, an electoral module also receives a profile which holds the voters preferences, which, unlike a voting rule, consider only the (sub-)set of alternatives that the module receives.

### 4.4.1 Definition

An electoral module maps an election to a result. To enable currying, the Election type is not used here because that would require tuples.

**type-synonym** ('*a*, '*v*, '*r*) *Electoral-Module* = '*v set* ⇒ '*a set* ⇒
('*a*, '*v*) *Profile* ⇒ '*r*

**fun** *fun$_{\mathcal{E}}$* :: ('*v set* ⇒ '*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ '*r*) ⇒
(('*a*, '*v*) *Election* ⇒ '*r*) **where**
*fun$_{\mathcal{E}}$ m* = (λ *E*. *m* (*voters-$\mathcal{E}$ E*) (*alternatives-$\mathcal{E}$ E*) (*profile-$\mathcal{E}$ E*))

The next three functions take an electoral module and turn it into a function only outputting the elect, reject, or defer set respectively.

**abbreviation** *elect* :: ('*a*, '*v*, '*r Result*) *Electoral-Module* ⇒ '*v set* ⇒ '*a set* ⇒
('*a*, '*v*) *Profile* ⇒ '*r set* **where**
*elect m V A p* ≡ *elect-r* (*m V A p*)

**abbreviation** *reject* :: ('*a*, '*v*, '*r Result*) *Electoral-Module* ⇒ '*v set* ⇒ '*a set* ⇒
('*a*, '*v*) *Profile* ⇒ '*r set* **where**
*reject m V A p* ≡ *reject-r* (*m V A p*)

**abbreviation** *defer* :: ('*a*, '*v*, '*r Result*) *Electoral-Module* ⇒ '*v set* ⇒ '*a set* ⇒
('*a*, '*v*) *Profile* ⇒ '*r set* **where**
*defer m V A p* ≡ *defer-r* (*m V A p*)

### 4.4.2 Auxiliary Definitions

Electoral modules partition a given set of alternatives A into a set of elected alternatives e, a set of rejected alternatives r, and a set of deferred alternatives d, using a profile. e, r, and d partition A. Electoral modules can be used as voting rules. They can also be composed in multiple structures to create more complex electoral modules.

**fun** (**in** *result*) *electoral-module* :: ('*a*, '*v*, ('*r Result*)) *Electoral-Module* ⇒
*bool* **where**
*electoral-module m* = (∀ *A V p*. *profile V A p* ⟶ *well-formed A* (*m V A p*))

**fun** *voters-determine-election* :: ('*a*, '*v*, ('*r Result*)) *Electoral-Module* ⇒ *bool* **where**
*voters-determine-election m* =
(∀ *A V p p'*. (∀ *v* ∈ *V*. *p v* = *p' v*) ⟶ *m V A p* = *m V A p'*)

**lemma** (**in** *result*) *electoral-modI*:
**fixes** *m* :: ('*a*, '*v*, ('*r Result*)) *Electoral-Module*
**assumes** ∀ *A V p*. *profile V A p* ⟶ *well-formed A* (*m V A p*)
**shows** *electoral-module m*
⟨*proof*⟩

### 4.4.3 Properties

We only require voting rules to behave a specific way on admissible elections, i.e., elections that are valid profiles (= votes are linear orders on the alternatives). Note that we do not assume finiteness of voter or alternative sets by default.

#### Anonymity

An electoral module is anonymous iff the result is invariant under renamings of voters, i.e., any permutation of the voter set that does not change the preferences leads to an identical result.

**definition** (**in** *result*) *anonymity* :: ('*a*, '*v*, ('*r Result*)) *Electoral-Module* ⇒
    *bool* **where**
  *anonymity m* ≡
    *electoral-module m* ∧
      (∀ *A V p π* :: ('*v* ⇒ '*v*).
        *bij π* ⟶ (*let* (*A'*, *V'*, *q*) = (*rename π* (*A*, *V*, *p*)) *in*
          *profile V A p* ∧ *profile V' A' q* ⟶ *m V A p* = *m V' A' q*))

Anonymity can alternatively be described as invariance under the voter permutation group acting on elections via the rename function.

**fun** *anonymity'* :: ('*a*, '*v*) *Election set* ⇒ ('*a*, '*v*, '*r*) *Electoral-Module* ⇒ *bool* **where**
  *anonymity' X m* = *is-symmetry* (*fun*$_{\mathcal{E}}$ *m*) (*Invariance* (*anonymity*$_{\mathcal{R}}$ *X*))

### Homogeneity

A voting rule is homogeneous if copying an election does not change the result. For ordered voter types and finite elections, we use the notion of copying ballot lists to define copying an election. The more general definition of homogeneity for unordered voter types already implies anonymity.

**fun** (**in** *result*) *homogeneity* :: ('*a*, '*v*) *Election set* ⇒
    ('*a*, '*v*, ('*r Result*)) *Electoral-Module* ⇒ *bool* **where**
  *homogeneity X m* = *is-symmetry* (*fun*$_{\mathcal{E}}$ *m*) (*Invariance* (*homogeneity*$_{\mathcal{R}}$ *X*))
— This does not require any specific behaviour on infinite voter sets ... It might make sense to extend the definition to that case somehow.

**fun** *homogeneity'* :: ('*a*, '*v* :: *linorder*) *Election set* ⇒
    ('*a*, '*v*, '*b Result*) *Electoral-Module* ⇒ *bool* **where**
  *homogeneity' X m* = *is-symmetry* (*fun*$_{\mathcal{E}}$ *m*) (*Invariance* (*homogeneity*$_{\mathcal{R}}$' *X*))

**lemma** (**in** *result*) *hom-imp-anon*:
  **fixes**
    *X* :: ('*a*, '*v*) *Election set* **and**
    *m* :: ('*a*, '*v*, ('*r Result*)) *Electoral-Module*
  **assumes**
    *homogeneity X m* **and**
    ∀ *E* ∈ *X*. *finite* (*voters-*$\mathcal{E}$ *E*)
  **shows** *anonymity' X m*
⟨*proof*⟩

### Neutrality

Neutrality is equivariance under consistent renaming of candidates in the candidate set and election results.

**fun** (**in** *result-properties*) *neutrality* :: ('*a*, '*v*) *Election set* ⇒
    ('*a*, '*v*, '*b Result*) *Electoral-Module* ⇒ *bool* **where**
  *neutrality X m* =
    *is-symmetry* (*fun*$_{\mathcal{E}}$ *m*) (*action-induced-equivariance* (*carrier neutrality*$_{\mathcal{G}}$) *X*
      (*φ-neutral X*) (*result-action ψ-neutral*))

### 4.4.4 Social-Welfare Properties

#### Reversal Symmetry

A social welfare rule is reversal symmetric if reversing all voters' preferences reverses the result rankings as well.

**definition** *reversal-symmetry* :: ('*a*, '*v*) *Election set* ⇒
('*a*, '*v*, '*a rel Result*) *Electoral-Module* ⇒ *bool* **where**
*reversal-symmetry X m* ≡
*is-symmetry* (*fun*$_{\mathcal{E}}$ *m*) (*action-induced-equivariance* (*carrier reversal*$_{\mathcal{G}}$) *X*
(*φ-reverse X*) (*result-action ψ-reverse*))

### 4.4.5 Social-Choice Modules

The following results require electoral modules to return social choice results, i.e., sets of elected, rejected and deferred alternatives. In order to export code, we use the hack provided by Locale-Code.

"defers n" is true for all electoral modules that defer exactly n alternatives, whenever there are n or more alternatives.

**definition** *defers* :: *nat* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*defers n m* ≡
*SCF-result.electoral-module m* ∧
(∀ *A V p*. (*card A* ≥ *n* ∧ *finite A* ∧ *profile V A p*)
⟶ *card* (*defer m V A p*) = *n*)

"rejects n" is true for all electoral modules that reject exactly n alternatives, whenever there are n or more alternatives.

**definition** *rejects* :: *nat* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*rejects n m* ≡
*SCF-result.electoral-module m* ∧
(∀ *A V p*. (*card A* ≥ *n* ∧ *finite A* ∧ *profile V A p*)
⟶ *card* (*reject m V A p*) = *n*)

As opposed to "rejects", "eliminates" allows to stop rejecting if no alternatives were to remain.

**definition** *eliminates* :: *nat* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*eliminates n m* ≡
*SCF-result.electoral-module m* ∧
(∀ *A V p*. (*card A* > *n* ∧ *profile V A p*) ⟶ *card* (*reject m V A p*) = *n*)

"elects n" is true for all electoral modules that elect exactly n alternatives, whenever there are n or more alternatives.

**definition** *elects* :: *nat* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*elects n m* ≡
*SCF-result.electoral-module m* ∧
(∀ *A V p*. (*card A* ≥ *n* ∧ *profile V A p*) ⟶ *card* (*elect m V A p*) = *n*)

An electoral module is independent of an alternative a iff a's ranking does not influence the outcome.

**definition** *indep-of-alt* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*v set* ⇒
'*a set* ⇒ '*a* ⇒ *bool* **where**
*indep-of-alt m V A a* ≡
$\mathcal{SCF}$*-result.electoral-module m*
∧ (∀ *p q*. *equiv-prof-except-a V A p q a* ⟶ *m V A p* = *m V A q*)

**definition** *unique-winner-if-profile-non-empty* :: ('*a*, '*v*, '*a Result*)
*Electoral-Module* ⇒ *bool* **where**
*unique-winner-if-profile-non-empty m* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p*. (*A* ≠ {} ∧ *V* ≠ {} ∧ *profile V A p*) ⟶
(∃ *a* ∈ *A*. *m V A p* = ({*a*}, *A* − {*a*}, {})))

### 4.4.6 Equivalence Definitions

**definition** *prof-contains-result* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*v set* ⇒
'*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*prof-contains-result m V A p q a* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
*profile V A p* ∧ *profile V A q* ∧ *a* ∈ *A* ∧
(*a* ∈ *elect m V A p* ⟶ *a* ∈ *elect m V A q*) ∧
(*a* ∈ *reject m V A p* ⟶ *a* ∈ *reject m V A q*) ∧
(*a* ∈ *defer m V A p* ⟶ *a* ∈ *defer m V A q*)

**definition** *prof-leq-result* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*v set* ⇒
'*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*prof-leq-result m V A p q a* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
*profile V A p* ∧ *profile V A q* ∧ *a* ∈ *A* ∧
(*a* ∈ *reject m V A p* ⟶ *a* ∈ *reject m V A q*) ∧
(*a* ∈ *defer m V A p* ⟶ *a* ∉ *elect m V A q*)

**definition** *prof-geq-result* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*v set* ⇒
'*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*prof-geq-result m V A p q a* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
*profile V A p* ∧ *profile V A q* ∧ *a* ∈ *A* ∧
(*a* ∈ *elect m V A p* ⟶ *a* ∈ *elect m V A q*) ∧
(*a* ∈ *defer m V A p* ⟶ *a* ∉ *reject m V A q*)

**definition** *mod-contains-result* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*v set* ⇒ '*a set* ⇒
('*a*, '*v*) *Profile* ⇒ '*a* ⇒ *bool* **where**
*mod-contains-result m n V A p a* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
$\mathcal{SCF}$*-result.electoral-module n* ∧
*profile V A p* ∧ *a* ∈ *A* ∧

$(a \in elect\ m\ V\ A\ p \longrightarrow a \in elect\ n\ V\ A\ p) \wedge$
$(a \in reject\ m\ V\ A\ p \longrightarrow a \in reject\ n\ V\ A\ p) \wedge$
$(a \in defer\ m\ V\ A\ p \longrightarrow a \in defer\ n\ V\ A\ p)$

**definition** *mod-contains-result-sym* :: $('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow$
$('a, 'v, 'a\ Result)\ Electoral\text{-}Module \Rightarrow 'v\ set \Rightarrow 'a\ set \Rightarrow$
$('a, 'v)\ Profile \Rightarrow 'a \Rightarrow bool$ **where**
*mod-contains-result-sym m n V A p a* $\equiv$
$\mathcal{SCF}$*-result.electoral-module m* $\wedge$
$\mathcal{SCF}$*-result.electoral-module n* $\wedge$
*profile V A p* $\wedge$ $a \in A \wedge$
$(a \in elect\ m\ V\ A\ p \longleftrightarrow a \in elect\ n\ V\ A\ p) \wedge$
$(a \in reject\ m\ V\ A\ p \longleftrightarrow a \in reject\ n\ V\ A\ p) \wedge$
$(a \in defer\ m\ V\ A\ p \longleftrightarrow a \in defer\ n\ V\ A\ p)$

### 4.4.7 Auxiliary Lemmas

**lemma** *elect-rej-def-combination*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$V :: 'v\ set$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$e\ r\ d :: 'a\ set$
**assumes**
$elect\ m\ V\ A\ p = e$ **and**
$reject\ m\ V\ A\ p = r$ **and**
$defer\ m\ V\ A\ p = d$
**shows** $m\ V\ A\ p = (e, r, d)$
⟨*proof*⟩

**lemma** *par-comp-result-sound*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$\mathcal{SCF}$*-result.electoral-module m* **and**
*profile V A p*
**shows** *well-formed-*$\mathcal{SCF}$ *A* $(m\ V\ A\ p)$
⟨*proof*⟩

**lemma** *result-presv-alts*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**

*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** $(elect\ m\ V\ A\ p) \cup (reject\ m\ V\ A\ p) \cup (defer\ m\ V\ A\ p) = A$
⟨*proof*⟩

**lemma** *result-disj*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$V :: 'v\ set$
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows**
$(elect\ m\ V\ A\ p) \cap (reject\ m\ V\ A\ p) = \{\} \wedge$
$(elect\ m\ V\ A\ p) \cap (defer\ m\ V\ A\ p) = \{\} \wedge$
$(reject\ m\ V\ A\ p) \cap (defer\ m\ V\ A\ p) = \{\}$
⟨*proof*⟩

**lemma** *elect-in-alts*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** $elect\ m\ V\ A\ p \subseteq A$
⟨*proof*⟩

**lemma** *reject-in-alts*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** $reject\ m\ V\ A\ p \subseteq A$
⟨*proof*⟩

**lemma** *defer-in-alts*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$

**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** *defer m V A p* $\subseteq$ *A*
⟨*proof*⟩

**lemma** *def-presv-prof*:
**fixes**
*m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** *let new-A* = *defer m V A p in profile V new-A* (*limit-profile new-A p*)
⟨*proof*⟩

An electoral module can never reject, defer or elect more than |A| alternatives.

**lemma** *upper-card-bounds-for-result*:
**fixes**
*m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p* **and**
*finite A*
**shows**
*upper-card-bound-for-elect*: *card* (*elect m V A p*) $\leq$ *card A* **and**
*upper-card-bound-for-reject*: *card* (*reject m V A p*) $\leq$ *card A* **and**
*upper-card-bound-for-defer*: *card* (*defer m V A p*) $\leq$ *card A*
⟨*proof*⟩

**lemma** *reject-not-elected-or-deferred*:
**fixes**
*m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*SCF-result.electoral-module m* **and**
*profile V A p*
**shows** *reject m V A p* = *A* − (*elect m V A p*) − (*defer m V A p*)
⟨*proof*⟩

**lemma** *elec-and-def-not-rej*:
**fixes**

$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
$profile\ V\ A\ p$
**shows** $elect\ m\ V\ A\ p \cup defer\ m\ V\ A\ p = A - (reject\ m\ V\ A\ p)$
⟨*proof*⟩

**lemma** *defer-not-elec-or-rej*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
$profile\ V\ A\ p$
**shows** $defer\ m\ V\ A\ p = A - (elect\ m\ V\ A\ p) - (reject\ m\ V\ A\ p)$
⟨*proof*⟩

**lemma** *electoral-mod-defer-elem*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
$profile\ V\ A\ p$ **and**
$a \in A$ **and**
$a \notin elect\ m\ V\ A\ p$ **and**
$a \notin reject\ m\ V\ A\ p$
**shows** $a \in defer\ m\ V\ A\ p$
⟨*proof*⟩

**lemma** *mod-contains-result-comm*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes** $mod\text{-}contains\text{-}result\ m\ n\ V\ A\ p\ a$
**shows** $mod\text{-}contains\text{-}result\ n\ m\ V\ A\ p\ a$
⟨*proof*⟩

**lemma** *not-rej-imp-elec-or-defer*:

**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
$profile\ V\ A\ p$ **and**
$a \in A$ **and**
$a \notin reject\ m\ V\ A\ p$
**shows** $a \in elect\ m\ V\ A\ p \vee a \in defer\ m\ V\ A\ p$
⟨*proof*⟩

**lemma** *single-elim-imp-red-def-set*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$eliminates\ 1\ m$ **and**
$card\ A > 1$ **and**
$profile\ V\ A\ p$
**shows** $defer\ m\ V\ A\ p \subset A$
⟨*proof*⟩

**lemma** *eq-alts-in-profs-imp-eq-results*:
**fixes**
$m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p\ q :: ('a, 'v)\ Profile$
**assumes**
*eq*: $\forall\ a \in A.\ prof\text{-}contains\text{-}result\ m\ V\ A\ p\ q\ a$ **and**
*mod-m*: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**
*prof-p*: $profile\ V\ A\ p$ **and**
*prof-q*: $profile\ V\ A\ q$
**shows** $m\ V\ A\ p = m\ V\ A\ q$
⟨*proof*⟩

**lemma** *eq-def-and-elect-imp-eq*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p\ q :: ('a, 'v)\ Profile$
**assumes**
*mod-m*: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ m$ **and**

*mod-n*: $\mathcal{SCF}$*-result.electoral-module n* **and**
*fin-p*: *profile V A p* **and**
*fin-q*: *profile V A q* **and**
*elec-eq*: *elect m V A p* = *elect n V A q* **and**
*def-eq*: *defer m V A p* = *defer n V A q*
**shows** *m V A p* = *n V A q*
⟨*proof*⟩

### 4.4.8 Non-Blocking

An electoral module is non-blocking iff this module never rejects all alternatives.

**definition** *non-blocking* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*non-blocking m* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p*. ((*A* ≠ {} ∧ *finite A* ∧ *profile V A p*) ⟶ *reject m V A p* ≠ *A*))

### 4.4.9 Electing

An electoral module is electing iff it always elects at least one alternative.

**definition** *electing* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*electing m* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p*. (*A* ≠ {} ∧ *finite A* ∧ *profile V A p*) ⟶ *elect m V A p* ≠ {})

**lemma** *electing-for-only-alt*:
**fixes**
*m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**assumes**
*one-alt*: *card A* = *1* **and**
*electing*: *electing m* **and**
*prof*: *profile V A p*
**shows** *elect m V A p* = *A*
⟨*proof*⟩

**theorem** *electing-imp-non-blocking*:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes** *electing m*
**shows** *non-blocking m*
⟨*proof*⟩

### 4.4.10 Properties

An electoral module is non-electing iff it never elects an alternative.

**definition** *non-electing* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*non-electing m* ≡
*SCF-result.electoral-module m*
∧ (∀ *A V p*. *profile V A p* ⟶ *elect m V A p* = {})

**lemma** *single-rej-decr-def-card*:
**fixes**
*m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**assumes**
*rejecting*: *rejects 1 m* **and**
*non-electing*: *non-electing m* **and**
*f-prof*: *finite-profile V A p*
**shows** *card* (*defer m V A p*) = *card A* − *1*
⟨*proof*⟩

**lemma** *single-elim-decr-def-card'*:
**fixes**
*m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**assumes**
*eliminating*: *eliminates 1 m* **and**
*non-electing*: *non-electing m* **and**
*not-empty*: *card A* > *1* **and**
*prof-p*: *profile V A p*
**shows** *card* (*defer m V A p*) = *card A* − *1*
⟨*proof*⟩

An electoral module is defer-deciding iff this module chooses exactly 1 alternative to defer and rejects any other alternative. Note that 'rejects n-1 m' can be omitted due to the well-formedness property.

**definition** *defer-deciding* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*defer-deciding m* ≡
*SCF-result.electoral-module m* ∧ *non-electing m* ∧ *defers 1 m*

An electoral module decrements iff this module rejects at least one alternative whenever possible (|A| > 1).

**definition** *decrementing* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**
*decrementing m* ≡
*SCF-result.electoral-module m* ∧
(∀ *A V p*. *profile V A p* ∧ *card A* > *1* ⟶ *card* (*reject m V A p*) ≥ *1*)

**definition** *defer-condorcet-consistency* :: ('*a*, '*v*, '*a Result*)
*Electoral-Module* ⇒ *bool* **where**
*defer-condorcet-consistency m* ≡

$\mathcal{SCF}$-result.electoral-module $m$ ∧
(∀ $A$ $V$ $p$ $a$. condorcet-winner $V$ $A$ $p$ $a$ ⟶
($m$ $V$ $A$ $p$ = ({}, $A$ − (defer $m$ $V$ $A$ $p$), {$d$ ∈ $A$. condorcet-winner $V$ $A$ $p$ $d$}))))

**definition** *condorcet-compatibility* :: ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ bool **where**
*condorcet-compatibility* $m$ ≡
$\mathcal{SCF}$-result.electoral-module $m$ ∧
(∀ $A$ $V$ $p$ $a$. condorcet-winner $V$ $A$ $p$ $a$ ⟶
($a$ ∉ reject $m$ $V$ $A$ $p$ ∧
(∀ $b$. ¬ condorcet-winner $V$ $A$ $p$ $b$ ⟶ $b$ ∉ elect $m$ $V$ $A$ $p$) ∧
($a$ ∈ elect $m$ $V$ $A$ $p$ ⟶
(∀ $b$ ∈ $A$. ¬ condorcet-winner $V$ $A$ $p$ $b$ ⟶ $b$ ∈ reject $m$ $V$ $A$ $p$))))

An electoral module is defer-monotone iff, when a deferred alternative is lifted, this alternative remains deferred.

**definition** *defer-monotonicity* :: ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ bool **where**
*defer-monotonicity* $m$ ≡
$\mathcal{SCF}$-result.electoral-module $m$ ∧
(∀ $A$ $V$ $p$ $q$ $a$.
($a$ ∈ defer $m$ $V$ $A$ $p$ ∧ lifted $V$ $A$ $p$ $q$ $a$) ⟶ $a$ ∈ defer $m$ $V$ $A$ $q$)

An electoral module is defer-lift-invariant iff lifting a deferred alternative does not affect the outcome.

**definition** *defer-lift-invariance* :: ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ bool **where**
*defer-lift-invariance* $m$ ≡
$\mathcal{SCF}$-result.electoral-module $m$ ∧
(∀ $A$ $V$ $p$ $q$ $a$. ($a$ ∈ (defer $m$ $V$ $A$ $p$) ∧ lifted $V$ $A$ $p$ $q$ $a$)
⟶ $m$ $V$ $A$ $p$ = $m$ $V$ $A$ $q$)

**fun** *dli-rel* :: ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ ('$a$, '$v$) Election rel **where**
*dli-rel* $m$ = {(($A$, $V$, $p$), ($A$, $V$, $q$)) |$A$ $V$ $p$ $q$. (∃ $a$ ∈ defer $m$ $V$ $A$ $p$. lifted $V$ $A$ $p$ $q$ $a$)}

**lemma** *rewrite-dli-as-invariance*:
**fixes** $m$ :: ('$a$, '$v$, '$a$ Result) Electoral-Module
**shows**
*defer-lift-invariance* $m$ =
($\mathcal{SCF}$-result.electoral-module $m$
∧ (*is-symmetry* (*fun*$_{\mathcal{E}}$ $m$) (*Invariance* (*dli-rel* $m$))))
⟨*proof*⟩

Two electoral modules are disjoint-compatible if they only make decisions over disjoint sets of alternatives. Electoral modules reject alternatives for which they make no decision.

**definition** *disjoint-compatibility* :: ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ ('$a$, '$v$, '$a$ Result) Electoral-Module ⇒ bool **where**
*disjoint-compatibility* $m$ $n$ ≡

$\mathcal{SCF}$-result.electoral-module $m \wedge$ $\mathcal{SCF}$-result.electoral-module $n \wedge$
(∀ $V$.
(∀ $A$.
(∃ $B \subseteq A$.
(∀ $a \in B$. indep-of-alt $m$ $V$ $A$ $a \wedge$
(∀ $p$. profile $V$ $A$ $p \longrightarrow a \in$ reject $m$ $V$ $A$ $p$)) $\wedge$
(∀ $a \in A - B$. indep-of-alt $n$ $V$ $A$ $a \wedge$
(∀ $p$. profile $V$ $A$ $p \longrightarrow a \in$ reject $n$ $V$ $A$ $p$)))))

Lifting an elected alternative a from an invariant-monotone electoral module either does not change the elect set, or makes a the only elected alternative.

**definition** *invariant-monotonicity* :: ('$a$, '$v$, '$a$ *Result*)
*Electoral-Module* ⇒ *bool* **where**
*invariant-monotonicity* $m \equiv$
$\mathcal{SCF}$-result.electoral-module $m \wedge$
(∀ $A$ $V$ $p$ $q$ $a$. ($a \in$ elect $m$ $V$ $A$ $p \wedge$ lifted $V$ $A$ $p$ $q$ $a$) $\longrightarrow$
(elect $m$ $V$ $A$ $q$ = elect $m$ $V$ $A$ $p \vee$ elect $m$ $V$ $A$ $q = \{a\}$))

Lifting a deferred alternative a from a defer-invariant-monotone electoral module either does not change the defer set, or makes a the only deferred alternative.

**definition** *defer-invariant-monotonicity* :: ('$a$, '$v$, '$a$ *Result*)
*Electoral-Module* ⇒ *bool* **where**
*defer-invariant-monotonicity* $m \equiv$
$\mathcal{SCF}$-result.electoral-module $m \wedge$ non-electing $m \wedge$
(∀ $A$ $V$ $p$ $q$ $a$. ($a \in$ defer $m$ $V$ $A$ $p \wedge$ lifted $V$ $A$ $p$ $q$ $a$) $\longrightarrow$
(defer $m$ $V$ $A$ $q$ = defer $m$ $V$ $A$ $p \vee$ defer $m$ $V$ $A$ $q = \{a\}$))

### 4.4.11 Inference Rules

**lemma** *ccomp-and-dd-imp-def-only-winner*:
**fixes**
$m$ :: ('$a$, '$v$, '$a$ *Result*) *Electoral-Module* **and**
$A$ :: '$a$ *set* **and**
$V$ :: '$v$ *set* **and**
$p$ :: ('$a$, '$v$) *Profile* **and**
$a$ :: '$a$
**assumes**
*ccomp*: *condorcet-compatibility* $m$ **and**
*dd*: *defer-deciding* $m$ **and**
*winner*: *condorcet-winner* $V$ $A$ $p$ $a$
**shows** *defer* $m$ $V$ $A$ $p = \{a\}$
⟨*proof*⟩

**theorem** *ccomp-and-dd-imp-dcc*[*simp*]:
**fixes** $m$ :: ('$a$, '$v$, '$a$ *Result*) *Electoral-Module*
**assumes**
*ccomp*: *condorcet-compatibility* $m$ **and**
*dd*: *defer-deciding* $m$

**shows** *defer-condorcet-consistency m*
⟨*proof*⟩

If m and n are disjoint compatible, so are n and m.

**theorem** *disj-compat-comm*[*simp*]:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes** *disjoint-compatibility m n*
**shows** *disjoint-compatibility n m*
⟨*proof*⟩

Every electoral module which is defer-lift-invariant is also defer-monotone.

**theorem** *dl-inv-imp-def-mono*[*simp*]:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes** *defer-lift-invariance m*
**shows** *defer-monotonicity m*
⟨*proof*⟩

### 4.4.12 Social-Choice Properties

#### Condorcet Consistency

**definition** *condorcet-consistency* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
*bool* **where**
*condorcet-consistency m* ≡
$\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p a. condorcet-winner V A p a* ⟶
(*m V A p* = ({*e* ∈ *A. condorcet-winner V A p e*}, *A* − (*elect m V A p*), {})))

**lemma** *condorcet-consistency'*:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**shows** *condorcet-consistency m* =
($\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p a. condorcet-winner V A p a* ⟶
(*m V A p* = ({*a*}, *A* − (*elect m V A p*), {}))))
⟨*proof*⟩

**lemma** *condorcet-consistency''*:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**shows** *condorcet-consistency m* =
($\mathcal{SCF}$*-result.electoral-module m* ∧
(∀ *A V p a.*
*condorcet-winner V A p a* ⟶ *m V A p* = ({*a*}, *A* − {*a*}, {})))
⟨*proof*⟩

#### (Weak) Monotonicity

An electoral module is monotone iff when an elected alternative is lifted, this alternative remains elected.

**definition** *monotonicity* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ *bool* **where**

*monotonicity* $m \equiv$
$\mathcal{SCF}$*-result.electoral-module* $m \wedge$
$(\forall\ A\ V\ p\ q\ a.\ a \in \mathit{elect}\ m\ V\ A\ p \wedge \mathit{lifted}\ V\ A\ p\ q\ a \longrightarrow a \in \mathit{elect}\ m\ V\ A\ q)$

**end**

## 4.5 Electoral Module on Election Quotients

**theory** *Quotient-Module*
**imports** *Quotients/Relation-Quotients*
*Electoral-Module*
**begin**

**lemma** *invariance-is-congruence*:
**fixes**
$m :: ('a, 'v, 'r)$ *Electoral-Module* **and**
$r :: ('a, 'v)$ *Election rel*
**shows** $(\mathit{is\text{-}symmetry}\ (\mathit{fun}_{\mathcal{E}}\ m)\ (\mathit{Invariance}\ r)) = (\mathit{fun}_{\mathcal{E}}\ m\ \mathit{respects}\ r)$
⟨*proof*⟩

**lemma** *invariance-is-congruence′*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$r :: 'x\ \mathit{rel}$
**shows** $(\mathit{is\text{-}symmetry}\ f\ (\mathit{Invariance}\ r)) = (f\ \mathit{respects}\ r)$
⟨*proof*⟩

**theorem** *pass-to-election-quotient*:
**fixes**
$m :: ('a, 'v, 'r)$ *Electoral-Module* **and**
$r :: ('a, 'v)$ *Election rel* **and**
$X :: ('a, 'v)$ *Election set*
**assumes**
*equiv* $X\ r$ **and**
*is-symmetry* $(\mathit{fun}_{\mathcal{E}}\ m)\ (\mathit{Invariance}\ r)$
**shows** $\forall\ A \in X\ //\ r.\ \forall\ E \in A.\ \pi_{\mathcal{Q}}\ (\mathit{fun}_{\mathcal{E}}\ m)\ A = \mathit{fun}_{\mathcal{E}}\ m\ E$
⟨*proof*⟩

**end**

## 4.6 Evaluation Function

**theory** *Evaluation-Function*

**imports** *Social-Choice-Types/Profile*
**begin**

This is the evaluation function. From a set of currently eligible alternatives, the evaluation function computes a numerical value that is then to be used for further (s)election, e.g., by the elimination module.

### 4.6.1 Definition

**type-synonym** $('a, 'v)$ *Evaluation-Function* =
$'v$ *set* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set* $\Rightarrow$ $('a, 'v)$ *Profile* $\Rightarrow$ *enat*

### 4.6.2 Property

An Evaluation function is a Condorcet-rating iff the following holds: If a Condorcet Winner w exists, w and only w has the highest value.

**definition** *condorcet-rating* :: $('a, 'v)$ *Evaluation-Function* $\Rightarrow$ *bool* **where**
*condorcet-rating* $f \equiv$
$\forall\ A\ V\ p\ w\ .$ *condorcet-winner* $V\ A\ p\ w \longrightarrow$
$(\forall\ l \in A\ .\ l \neq w \longrightarrow f\ V\ l\ A\ p < f\ V\ w\ A\ p)$

An Evaluation function is dependent only on the participating voters iff it is invariant under profile changes that only impact non-voters.

**fun** *voters-determine-evaluation* :: $('a, 'v)$ *Evaluation-Function* $\Rightarrow$ *bool* **where**
*voters-determine-evaluation* $f$ =
$(\forall\ A\ V\ p\ p'.\ (\forall\ v \in V.\ p\ v = p'\ v) \longrightarrow (\forall\ a \in A.\ f\ V\ a\ A\ p = f\ V\ a\ A\ p'))$

### 4.6.3 Theorems

If e is Condorcet-rating, the following holds: If a Condorcet winner w exists, w has the maximum evaluation value.

**theorem** *cond-winner-imp-max-eval-val*:
**fixes**
$e$ :: $('a, 'v)$ *Evaluation-Function* **and**
$A$ :: $'a$ *set* **and**
$V$ :: $'v$ *set* **and**
$p$ :: $('a, 'v)$ *Profile* **and**
$a$ :: $'a$
**assumes**
*rating*: *condorcet-rating* $e$ **and**
*f-prof*: *finite-profile* $V\ A\ p$ **and**
*winner*: *condorcet-winner* $V\ A\ p\ a$
**shows** $e\ V\ a\ A\ p = Max\ \{e\ V\ b\ A\ p \mid b.\ b \in A\}$
⟨*proof*⟩

If e is Condorcet-rating, the following holds: If a Condorcet Winner w exists, a non-Condorcet winner has a value lower than the maximum evaluation value.

**theorem** *non-cond-winner-not-max-eval*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a b* :: '*a*
**assumes**
*rating*: *condorcet-rating e* **and**
*f-prof*: *finite-profile V A p* **and**
*winner*: *condorcet-winner V A p a* **and**
*lin-A*: $b \in A$ **and**
*loser*: $a \neq b$
**shows** *e V b A p* < *Max* {*e V c A p* | *c*. $c \in A$}
⟨*proof*⟩

**end**

## 4.7 Elimination Module

**theory** *Elimination-Module*
**imports** *Evaluation-Function*
*Electoral-Module*
**begin**

This is the elimination module. It rejects a set of alternatives only if these are not all alternatives. The alternatives potentially to be rejected are put in a so-called elimination set. These are all alternatives that score below a preset threshold value that depends on the specific voting rule.

### 4.7.1 General Definitions

**type-synonym** *Threshold-Value* = *enat*

**type-synonym** *Threshold-Relation* = *enat* ⇒ *enat* ⇒ *bool*

**type-synonym** ('*a*, '*v*) *Electoral-Set* = '*v set* ⇒ '*a set* ⇒ ('*a*, '*v*) *Profile* ⇒ '*a set*

**fun** *elimination-set* :: ('*a*, '*v*) *Evaluation-Function* ⇒ *Threshold-Value* ⇒
*Threshold-Relation* ⇒ ('*a*, '*v*) *Electoral-Set* **where**
*elimination-set e t r V A p* = (*if finite A then* {*a* ∈ *A* . *r* (*e V a A p*) *t*} *else* {})

**fun** *average* :: ('*a*, '*v*) *Evaluation-Function* ⇒ '*v set* ⇒ '*a set* ⇒ ('*a*, '*v*) *Profile* ⇒
*Threshold-Value* **where**
*average e V A p* = (*let sum* = (∑ *x* ∈ *A*. *e V x A p*) *in*
(*if sum* = ∞ *then* ∞ *else* (*the-enat sum div* (*card A*))))

### 4.7.2 Social-Choice Definitions

**fun** *elimination-module* :: ('*a*, '*v*) *Evaluation-Function* ⇒ *Threshold-Value* ⇒
*Threshold-Relation* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*elimination-module e t r V A p* =
(*if* (*elimination-set e t r V A p*) ≠ *A*
*then* ({}, (*elimination-set e t r V A p*), *A* − (*elimination-set e t r V A p*))
*else* ({}, {}, *A*))

### 4.7.3 Social-Choice Eliminators

**fun** *less-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒ *Threshold-Value* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*less-eliminator e t V A p* = *elimination-module e t* (<) *V A p*

**fun** *max-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*max-eliminator e V A p* =
*less-eliminator e* (*Max* {*e V x A p* | *x*. *x* ∈ *A*}) *V A p*

**fun** *leq-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒ *Threshold-Value* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*leq-eliminator e t V A p* = *elimination-module e t* (≤) *V A p*

**fun** *min-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*min-eliminator e V A p* =
*leq-eliminator e* (*Min* {*e V x A p* | *x*. *x* ∈ *A*}) *V A p*

**fun** *less-average-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*less-average-eliminator e V A p* = *less-eliminator e* (*average e V A p*) *V A p*

**fun** *leq-average-eliminator* :: ('*a*, '*v*) *Evaluation-Function* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*leq-average-eliminator e V A p* = *leq-eliminator e* (*average e V A p*) *V A p*

### 4.7.4 Soundness

**lemma** *elim-mod-sound*[*simp*]:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value* **and**
*r* :: *Threshold-Relation*
**shows** *SCF-result.electoral-module* (*elimination-module e t r*)
⟨*proof*⟩

**lemma** *less-elim-sound*[*simp*]:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**

$t$ :: *Threshold-Value*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*less-eliminator e t*)
⟨*proof*⟩

**lemma** *leq-elim-sound*[*simp*]:
**fixes**
$e$ :: ('*a*, '*v*) *Evaluation-Function* **and**
$t$ :: *Threshold-Value*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*leq-eliminator e t*)
⟨*proof*⟩

**lemma** *max-elim-sound*[*simp*]:
**fixes** $e$ :: ('*a*, '*v*) *Evaluation-Function*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*max-eliminator e*)
⟨*proof*⟩

**lemma** *min-elim-sound*[*simp*]:
**fixes** $e$ :: ('*a*, '*v*) *Evaluation-Function*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*min-eliminator e*)
⟨*proof*⟩

**lemma** *less-avg-elim-sound*[*simp*]:
**fixes** $e$ :: ('*a*, '*v*) *Evaluation-Function*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*less-average-eliminator e*)
⟨*proof*⟩

**lemma** *leq-avg-elim-sound*[*simp*]:
**fixes** $e$ :: ('*a*, '*v*) *Evaluation-Function*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*leq-average-eliminator e*)
⟨*proof*⟩

### 4.7.5 Independence of Non-Voters

**lemma** *voters-determine-elim-mod*[*simp*]:
**fixes**
$e$ :: ('*a*, '*v*) *Evaluation-Function* **and**
$t$ :: *Threshold-Value* **and**
$r$ :: *Threshold-Relation*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*elimination-module e t r*)
⟨*proof*⟩

**lemma** *voters-determine-less-elim*[*simp*]:
**fixes**
$e$ :: ('*a*, '*v*) *Evaluation-Function* **and**
$t$ :: *Threshold-Value*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*less-eliminator e t*)
⟨*proof*⟩

**lemma** *voters-determine-leq-elim*[*simp*]:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*leq-eliminator e t*)
⟨*proof*⟩

**lemma** *voters-determine-max-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*max-eliminator e*)
⟨*proof*⟩

**lemma** *voters-determine-min-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*min-eliminator e*)
⟨*proof*⟩

**lemma** *voters-determine-less-avg-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*less-average-eliminator e*)
⟨*proof*⟩

**lemma** *voters-determine-leq-avg-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *voters-determine-evaluation e*
**shows** *voters-determine-election* (*leq-average-eliminator e*)
⟨*proof*⟩

### 4.7.6 Non-Blocking

**lemma** *elim-mod-non-blocking*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value* **and**
*r* :: *Threshold-Relation*
**shows** *non-blocking* (*elimination-module e t r*)
⟨*proof*⟩

**lemma** *less-elim-non-blocking*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value*
**shows** *non-blocking* (*less-eliminator e t*)
⟨*proof*⟩

**lemma** *leq-elim-non-blocking*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value*
**shows** *non-blocking* (*leq-eliminator e t*)
⟨*proof*⟩

**lemma** *max-elim-non-blocking*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-blocking* (*max-eliminator e*)
⟨*proof*⟩

**lemma** *min-elim-non-blocking*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-blocking* (*min-eliminator e*)
⟨*proof*⟩

**lemma** *less-avg-elim-non-blocking*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-blocking* (*less-average-eliminator e*)
⟨*proof*⟩

**lemma** *leq-avg-elim-non-blocking*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-blocking* (*leq-average-eliminator e*)
⟨*proof*⟩

### 4.7.7 Non-Electing

**lemma** *elim-mod-non-electing*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value* **and**
*r* :: *Threshold-Relation*
**shows** *non-electing* (*elimination-module e t r*)
⟨*proof*⟩

**lemma** *less-elim-non-electing*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value*
**shows** *non-electing* (*less-eliminator e t*)
⟨*proof*⟩

**lemma** *leq-elim-non-electing*:
**fixes**
*e* :: ('*a*, '*v*) *Evaluation-Function* **and**
*t* :: *Threshold-Value*

**shows** *non-electing* (*leq-eliminator e t*)
⟨*proof*⟩

**lemma** *max-elim-non-electing*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-electing* (*max-eliminator e*)
⟨*proof*⟩

**lemma** *min-elim-non-electing*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-electing* (*min-eliminator e*)
⟨*proof*⟩

**lemma** *less-avg-elim-non-electing*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-electing* (*less-average-eliminator e*)
⟨*proof*⟩

**lemma** *leq-avg-elim-non-electing*:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**shows** *non-electing* (*leq-average-eliminator e*)
⟨*proof*⟩

### 4.7.8 Inference Rules

If the used evaluation function is Condorcet rating, max-eliminator is Condorcet compatible.

**theorem** *cr-eval-imp-ccomp-max-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *condorcet-rating e*
**shows** *condorcet-compatibility* (*max-eliminator e*)
⟨*proof*⟩

If the used evaluation function is Condorcet rating, max-eliminator is defer-Condorcet-consistent.

**theorem** *cr-eval-imp-dcc-max-elim*[*simp*]:
**fixes** *e* :: ('*a*, '*v*) *Evaluation-Function*
**assumes** *condorcet-rating e*
**shows** *defer-condorcet-consistency* (*max-eliminator e*)
⟨*proof*⟩

**end**

## 4.8 Aggregator

**theory** *Aggregator*
  **imports** *Social-Choice-Types/Social-Choice-Result*
**begin**

An aggregator gets two partitions (results of electoral modules) as input and output another partition. They are used to aggregate results of parallel composed electoral modules. They are commutative, i.e., the order of the aggregated modules does not affect the resulting aggregation. Moreover, they are conservative in the sense that the resulting decisions are subsets of the two given partitions' decisions.

### 4.8.1 Definition

**type-synonym** *'a Aggregator* = *'a set* ⇒ *'a Result* ⇒ *'a Result* ⇒ *'a Result*

**definition** *aggregator* :: *'a Aggregator* ⇒ *bool* **where**
  *aggregator agg* ≡
    ∀ *A e e' d d' r r'*.
      (*well-formed-SCF A* (*e*, *r*, *d*) ∧ *well-formed-SCF A* (*e'*, *r'*, *d'*)) ⟶
      *well-formed-SCF A* (*agg A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*))

### 4.8.2 Properties

**definition** *agg-commutative* :: *'a Aggregator* ⇒ *bool* **where**
  *agg-commutative agg* ≡
    *aggregator agg* ∧ (∀ *A e e' d d' r r'*.
      *agg A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*) = *agg A* (*e'*, *r'*, *d'*) (*e*, *r*, *d*))

**definition** *agg-conservative* :: *'a Aggregator* ⇒ *bool* **where**
  *agg-conservative agg* ≡
    *aggregator agg* ∧
    (∀ *A e e' d d' r r'*.
      ((*well-formed-SCF A* (*e*, *r*, *d*) ∧ *well-formed-SCF A* (*e'*, *r'*, *d'*)) ⟶
        *elect-r* (*agg A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*)) ⊆ (*e* ∪ *e'*) ∧
        *reject-r* (*agg A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*)) ⊆ (*r* ∪ *r'*) ∧
        *defer-r* (*agg A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*)) ⊆ (*d* ∪ *d'*)))

**end**

## 4.9 Maximum Aggregator

**theory** *Maximum-Aggregator*

**imports** *Aggregator*
**begin**

The max(imum) aggregator takes two partitions of an alternative set A as input. It returns a partition where every alternative receives the maximum result of the two input partitions.

### 4.9.1 Definition

**fun** *max-aggregator* :: *'a Aggregator* **where**
*max-aggregator A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*) =
(*e* ∪ *e'*,
*A* − (*e* ∪ *e'* ∪ *d* ∪ *d'*),
(*d* ∪ *d'*) − (*e* ∪ *e'*))

### 4.9.2 Auxiliary Lemma

**lemma** *max-agg-rej-set*:
**fixes**
*A e e' d d' r r'* :: *'a set* **and**
*a* :: *'a*
**assumes**
*wf-first-mod*: *well-formed-SCF A* (*e*, *r*, *d*) **and**
*wf-second-mod*: *well-formed-SCF A* (*e'*, *r'*, *d'*)
**shows** *reject-r* (*max-aggregator A* (*e*, *r*, *d*) (*e'*, *r'*, *d'*)) = *r* ∩ *r'*
⟨*proof*⟩

### 4.9.3 Soundness

**theorem** *max-agg-sound*[*simp*]: *aggregator max-aggregator*
⟨*proof*⟩

### 4.9.4 Properties

The max-aggregator is conservative.

**theorem** *max-agg-consv*[*simp*]: *agg-conservative max-aggregator*
⟨*proof*⟩

The max-aggregator is commutative.

**theorem** *max-agg-comm*[*simp*]: *agg-commutative max-aggregator*
⟨*proof*⟩

**end**

## 4.10 Termination Condition

**theory** *Termination-Condition*
  **imports** *Social-Choice-Types/Result*
**begin**

The termination condition is used in loops. It decides whether or not to terminate the loop after each iteration, depending on the current state of the loop.

**type-synonym** *'r Termination-Condition* = *'r Result* ⇒ *bool*

**end**

## 4.11 Defer Equal Condition

**theory** *Defer-Equal-Condition*
  **imports** *Termination-Condition*
**begin**

This is a family of termination conditions. For a natural number n, the according defer-equal condition is true if and only if the given result's defer-set contains exactly n elements.

**fun** *defer-equal-condition* :: *nat* ⇒ *'a Termination-Condition* **where**
  *defer-equal-condition n* (*e*, *r*, *d*) = (*card d* = *n*)

**end**

# Chapter 5

# Basic Modules

## 5.1 Defer Module

**theory** *Defer-Module*
**imports** *Component-Types/Electoral-Module*
**begin**

The defer module is not concerned about the voter's ballots, and simply defers all alternatives. It is primarily used for defining an empty loop.

### 5.1.1 Definition

**fun** *defer-module* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*defer-module V A p* = ({}, {}, *A*)

### 5.1.2 Soundness

**theorem** *def-mod-sound*[*simp*]: $\mathcal{SCF}$*-result.electoral-module defer-module*
⟨*proof*⟩

### 5.1.3 Properties

**theorem** *def-mod-non-electing*: *non-electing defer-module*
⟨*proof*⟩

**theorem** *def-mod-def-lift-inv*: *defer-lift-invariance defer-module*
⟨*proof*⟩

**end**

## 5.2 Elect-First Module

**theory** *Elect-First-Module*

**imports** *Component-Types/Electoral-Module*
**begin**

The elect first module elects the alternative that is most preferred on the first ballot and rejects all other alternatives.

### 5.2.1 Definition

**fun** *least* :: $'v$ :: *wellorder set* ⇒ $'v$ **where**
*least* $V$ = (*Least* (λ $v$. $v \in V$))

**fun** *elect-first-module* :: ($'a$, $'v$ :: *wellorder*, $'a$ *Result*) *Electoral-Module* **where**
*elect-first-module* $V$ $A$ $p$ =
({$a \in A$. *above* ($p$ (*least* $V$)) $a$ = {$a$}},
{$a \in A$. *above* ($p$ (*least* $V$)) $a \neq$ {$a$}},
{})

### 5.2.2 Soundness

**theorem** *elect-first-mod-sound*: $\mathcal{SCF}$*-result.electoral-module elect-first-module*
⟨*proof*⟩

**end**

## 5.3 Consensus Class

**theory** *Consensus-Class*
**imports** *Consensus*
*../Defer-Module*
*../Elect-First-Module*
**begin**

A consensus class is a pair of a set of elections and a mapping that assigns a unique alternative to each election in that set (of elections). This alternative is then called the consensus alternative (winner). Here, we model the mapping by an electoral module that defers alternatives which are not in the consensus.

### 5.3.1 Definition

**type-synonym** ($'a$, $'v$, $'r$) *Consensus-Class* =
($'a$, $'v$) *Consensus* × ($'a$, $'v$, $'r$) *Electoral-Module*

**fun** *consensus-*$\mathcal{K}$ :: ($'a$, $'v$, $'r$) *Consensus-Class* ⇒ ($'a$, $'v$) *Consensus* **where**
*consensus-*$\mathcal{K}$ $K$ = *fst* $K$

**fun** *rule-$\mathcal{K}$* :: ('*a*, '*v*, '*r*) *Consensus-Class* ⇒ ('*a*, '*v*, '*r*) *Electoral-Module* **where**
*rule-$\mathcal{K}$ K* = *snd K*

### 5.3.2 Consensus Choice

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

**fun** $\mathcal{K}_{\mathcal{E}}$ :: ('*a*, '*v*, '*r Result*) *Consensus-Class* ⇒ '*r* ⇒ ('*a*, '*v*) *Election set* **where**
$\mathcal{K}_{\mathcal{E}}$ *K w* =
{(*A*, *V*, *p*) | *A V p*. (*consensus-$\mathcal{K}$ K*) (*A*, *V*, *p*) ∧ *finite-profile V A p*
∧ *elect* (*rule-$\mathcal{K}$ K*) *V A p* = {*w*}}

**fun** *elections-$\mathcal{K}$* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* ⇒ ('*a*, '*v*) *Election set* **where**
*elections-$\mathcal{K}$ K* = ⋃ (($\mathcal{K}_{\mathcal{E}}$ *K*) ' *UNIV*)

A consensus class is deemed well-formed if the result of its mapping is completely determined by its consensus, the elected set of the electoral module's result.

**definition** *well-formed* :: ('*a*, '*v*) *Consensus* ⇒ ('*a*, '*v*, '*r*) *Electoral-Module* ⇒
*bool* **where**
*well-formed c m* ≡
∀ *A V V' p p'*.
*profile V A p* ∧ *profile V' A p'* ∧ *c* (*A*, *V*, *p*) ∧ *c* (*A*, *V'*, *p'*)
⟶ *m V A p* = *m V' A p'*

A sensible social choice rule for a given arbitrary consensus and social choice rule r is the one that chooses the result of r for all consensus elections and defers all candidates otherwise.

**fun** *consensus-choice* :: ('*a*, '*v*) *Consensus* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Consensus-Class* **where**
*consensus-choice c m* =
(*let*
*w* = (λ *V A p*. *if c* (*A*, *V*, *p*) *then m V A p else defer-module V A p*)
*in* (*c*, *w*))

### 5.3.3 Auxiliary Lemmas

**lemma** *unanimity'-consensus-imp-elect-fst-mod-well-formed*:
**fixes** *a* :: '*a*
**shows** *well-formed*
(λ *c*. *nonempty-set$_{\mathcal{C}}$ c* ∧ *nonempty-profile$_{\mathcal{C}}$ c*
∧ *equal-top$_{\mathcal{C}}$' a c*) *elect-first-module*
⟨*proof*⟩

**lemma** *strong-unanimity'consensus-imp-elect-fst-mod-completely-determined*:
**fixes** *r* :: '*a Preference-Relation*

**shows** *well-formed*
(λ *c*. *nonempty-set*$_{\mathcal{C}}$ *c* ∧ *nonempty-profile*$_{\mathcal{C}}$ *c* ∧ *equal-vote*$_{\mathcal{C}}$′ *r c*) *elect-first-module*
⟨*proof*⟩

**lemma** *strong-unanimity′consensus-imp-elect-fst-mod-well-formed*:
**fixes** *r* :: ′*a Preference-Relation*
**shows** *well-formed*
(λ *c*. *nonempty-set*$_{\mathcal{C}}$ *c* ∧ *nonempty-profile*$_{\mathcal{C}}$ *c*
∧ *equal-vote*$_{\mathcal{C}}$′ *r c*) *elect-first-module*
⟨*proof*⟩

**lemma** *cons-domain-valid*:
**fixes** *C* :: (′*a*, ′*v*, ′*r Result*) *Consensus-Class*
**shows** *elections-*$\mathcal{K}$ *C* ⊆ *well-formed-elections*
⟨*proof*⟩

**lemma** *cons-domain-finite*:
**fixes** *C* :: (′*a*, ′*v*, ′*r Result*) *Consensus-Class*
**shows**
*finite*: *elections-*$\mathcal{K}$ *C* ⊆ *finite-elections* **and**
*finite-voters*: *elections-*$\mathcal{K}$ *C* ⊆ *finite-elections-*$\mathcal{V}$
⟨*proof*⟩

### 5.3.4 Consensus Rules

**definition** *non-empty-set* :: (′*a*, ′*v*, ′*r*) *Consensus-Class* ⇒ *bool* **where**
*non-empty-set c* ≡ ∃ *K*. *consensus-*$\mathcal{K}$ *c K*

Unanimity condition.

**definition** *unanimity* :: (′*a*, ′*v* :: *wellorder*, ′*a Result*) *Consensus-Class* **where**
*unanimity* ≡ *consensus-choice unanimity*$_{\mathcal{C}}$ *elect-first-module*

Strong unanimity condition.

**definition** *strong-unanimity* :: (′*a*, ′*v* :: *wellorder*, ′*a Result*) *Consensus-Class* **where**
*strong-unanimity* ≡ *consensus-choice strong-unanimity*$_{\mathcal{C}}$ *elect-first-module*

### 5.3.5 Properties

**definition** *consensus-rule-anonymity* :: (′*a*, ′*v*, ′*r*) *Consensus-Class* ⇒ *bool* **where**
*consensus-rule-anonymity c* ≡
(∀ *A V p* π :: (′*v* ⇒ ′*v*).
*bij* π ⟶
(*let* (*A*′, *V*′, *q*) = (*rename* π (*A*, *V*, *p*)) *in*
*profile V A p* ⟶ *profile V*′ *A*′ *q*
⟶ *consensus-*$\mathcal{K}$ *c* (*A*, *V*, *p*)
⟶ (*consensus-*$\mathcal{K}$ *c* (*A*′, *V*′, *q*) ∧ (*rule-*$\mathcal{K}$ *c V A p* = *rule-*$\mathcal{K}$ *c V*′ *A*′ *q*))))

**fun** *consensus-rule-anonymity′* :: (′*a*, ′*v*) *Election set* ⇒

('a, 'v, 'r *Result*) *Consensus-Class* ⇒ *bool* **where**
*consensus-rule-anonymity'* *X C* =
*is-symmetry* (*elect-r* ∘ $fun_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*)) (*Invariance* ($anonymity_{\mathcal{R}}$ *X*))

**fun** (**in** *result-properties*) *consensus-rule-neutrality* :: ('a, 'v) *Election set* ⇒
('a, 'v, 'b *Result*) *Consensus-Class* ⇒ *bool* **where**
*consensus-rule-neutrality X C* =
*is-symmetry* (*elect-r* ∘ $fun_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*))
(*action-induced-equivariance*
(*carrier neutrality*$_{\mathcal{G}}$) *X* (*φ-neutral X*) (*set-action ψ-neutral*))

**fun** *consensus-rule-reversal-symmetry* :: ('a, 'v) *Election set* ⇒
('a, 'v, 'a *rel Result*) *Consensus-Class* ⇒ *bool* **where**
*consensus-rule-reversal-symmetry X C* = *is-symmetry* (*elect-r* ∘ $fun_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*))
(*action-induced-equivariance* (*carrier reversal*$_{\mathcal{G}}$) *X* (*φ-reverse X*) (*set-action ψ-reverse*))

### 5.3.6 Inference Rules

**lemma** *if-else-cons-equivar*:
**fixes**
*m n* :: ('a, 'v, 'a *Result*) *Electoral-Module* **and**
*c* :: ('a, 'v) *Consensus* **and**
*G* :: 'b *set* **and**
*X* :: ('a, 'v) *Election set* **and**
*φ* :: ('b, ('a, 'v) *Election*) *binary-fun* **and**
*ψ* :: ('b, 'a) *binary-fun* **and**
*f* :: 'a *Result* ⇒ 'a *set*
**defines**
*equivar* ≡ *action-induced-equivariance G X φ* (*set-action ψ*) **and**
*if-else-cons* ≡ (*c*, (λ *V A p*. *if c* (*A*, *V*, *p*) *then m V A p else n V A p*))
**assumes**
*equivar-m*: *is-symmetry* (*f* ∘ $fun_{\mathcal{E}}$ *m*) *equivar* **and**
*equivar-n*: *is-symmetry* (*f* ∘ $fun_{\mathcal{E}}$ *n*) *equivar* **and**
*invar-cons*: *is-symmetry c* (*Invariance* (*action-induced-rel G X φ*))
**shows** *is-symmetry* (*f* ∘ $fun_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *if-else-cons*))
(*action-induced-equivariance G X φ* (*set-action ψ*))
⟨*proof*⟩

**lemma** *consensus-choice-anonymous*:
**fixes**
*α β* :: ('a, 'v) *Consensus* **and**
*m* :: ('a, 'v, 'a *Result*) *Electoral-Module* **and**
*β'* :: 'b ⇒ ('a, 'v) *Consensus*
**assumes**
*beta-sat*: *β* = (λ *E*. ∃ *a*. *β' a E*) **and**
*beta'-anon*: ∀ *x*. *consensus-anonymity* (*β' x*) **and**
*anon-cons-cond*: *consensus-anonymity α* **and**
*conditions-univ*: ∀ *x*. *well-formed* (λ *E*. *α E* ∧ *β' x E*) *m*

**shows** *consensus-rule-anonymity* (*consensus-choice* ($\lambda$ *E*. $\alpha$ *E* $\wedge$ $\beta$ *E*) *m*)
⟨*proof*⟩

### 5.3.7 Theorems

#### Anonymity

**lemma** *unanimity-anonymous*: *consensus-rule-anonymity unanimity*
⟨*proof*⟩

**lemma** *strong-unanimity-anonymous*: *consensus-rule-anonymity strong-unanimity*
⟨*proof*⟩

#### Neutrality

**lemma** *defer-winners-equivariant*:
**fixes**
*G* :: *'b set* **and**
*E* :: (*'a*, *'v*) *Election set* **and**
$\varphi$ :: (*'b*, (*'a*, *'v*) *Election*) *binary-fun* **and**
$\psi$ :: (*'b*, *'a*) *binary-fun*
**shows** *is-symmetry* (*elect-r* $\circ$ $fun_{\mathcal{E}}$ *defer-module*)
(*action-induced-equivariance G E* $\varphi$ (*set-action* $\psi$))
⟨*proof*⟩

**lemma** *elect-first-winners-neutral*: *is-symmetry* (*elect-r* $\circ$ $fun_{\mathcal{E}}$ *elect-first-module*)
(*action-induced-equivariance* (*carrier* $neutrality_{\mathcal{G}}$)
*well-formed-elections* ($\varphi$-*neutral well-formed-elections*)
(*set-action* $\psi$-$neutral_{\mathrm{c}}$))
⟨*proof*⟩

**lemma** *strong-unanimity-neutral*:
**defines** *domain* $\equiv$ *well-formed-elections* $\cap$ *Collect* $strong\text{-}unanimity_{\mathcal{C}}$
— We want to show neutrality on a set as general as possible, as this implies subset neutrality.
**shows** $\mathcal{SCF}$-*properties.consensus-rule-neutrality domain strong-unanimity*
⟨*proof*⟩

**lemma** *strong-unanimity-neutral'*: $\mathcal{SCF}$-*properties.consensus-rule-neutrality*
(*elections-*$\mathcal{K}$ *strong-unanimity*) *strong-unanimity*
⟨*proof*⟩

**lemma** *strong-unanimity-closed-under-neutrality*: *closed-restricted-rel*
($neutrality_{\mathcal{R}}$ *well-formed-elections*) *well-formed-elections*
(*elections-*$\mathcal{K}$ *strong-unanimity*)
⟨*proof*⟩

**end**

## 5.4 Distance Rationalization

**theory** *Distance-Rationalization*
  **imports** *Social-Choice-Types/Refined-Types/Preference-List*
    *Consensus-Class*
    *Distance*
**begin**

A distance rationalization of a voting rule is its interpretation as a procedure that elects an uncontroversial winner if there is one, and otherwise elects the alternatives that are as close to becoming an uncontroversial winner as possible. Within general distance rationalization, a voting rule is characterized by a distance on profiles and a consensus class.

### 5.4.1 Definitions

Returns the distance of an election to the preimage of a unique winner under the given consensus elections and consensus rule.

**fun** *score* :: ('*a*, '*v*) *Election Distance* ⇒ ('*a*, '*v*, '*r Result*) *Consensus-Class* ⇒
    ('*a*, '*v*) *Election* ⇒ '*r* ⇒ *ereal* **where**
  *score d K E w* = *Inf* (*d E* ‘ (*K*$_{\mathcal{E}}$ *K w*))

**fun** (**in** *result*) $\mathcal{R}_{\mathcal{W}}$ :: ('*a*, '*v*) *Election Distance* ⇒
    ('*a*, '*v*, '*r Result*) *Consensus-Class* ⇒ '*v set* ⇒ '*a set* ⇒ ('*a*, '*v*) *Profile* ⇒
    '*r set* **where**
  $\mathcal{R}_{\mathcal{W}}$ *d K V A p* = *arg-min-set* (*score d K* (*A*, *V*, *p*)) (*limit A UNIV*)

**fun** (**in** *result*) *distance-*$\mathcal{R}$ :: ('*a*, '*v*) *Election Distance* ⇒
    ('*a*, '*v*, '*r Result*) *Consensus-Class* ⇒
    ('*a*, '*v*, '*r Result*) *Electoral-Module* **where**
  *distance-*$\mathcal{R}$ *d K V A p* =
    ($\mathcal{R}_{\mathcal{W}}$ *d K V A p*, (*limit A UNIV*) − $\mathcal{R}_{\mathcal{W}}$ *d K V A p*, {})

### 5.4.2 Standard Definitions

**definition** *standard* :: ('*a*, '*v*) *Election Distance* ⇒ *bool* **where**
 *standard d* ≡
   ∀ *A A*′ *V V*′ *p p*′. (*V* ≠ *V*′ ∨ *A* ≠ *A*′) ⟶ *d* (*A*, *V*, *p*) (*A*′, *V*′, *p*′) = ∞

**definition** *voters-determine-distance* :: ('*a*, '*v*) *Election Distance* ⇒ *bool* **where**
  *voters-determine-distance d* ≡
    ∀ *A A*′ *V V*′ *p q p*′.
      (∀ *v* ∈ *V*. *p v* = *q v*)
        ⟶ (*d* (*A*, *V*, *p*) (*A*′, *V*′, *p*′) = *d* (*A*, *V*, *q*) (*A*′, *V*′, *p*′)
          ∧ (*d* (*A*′, *V*′, *p*′) (*A*, *V*, *p*) = *d* (*A*′, *V*′, *p*′) (*A*, *V*, *q*)))

Creates a set of all possible profiles on a finite alternative set that are empty everywhere outside of a given finite voter set.

**fun** *profiles* :: *'v set* ⇒ *'a set* ⇒ ((*'a*, *'v*) *Profile*) *set* **where**
  *profiles V A* =
    (*if* (*infinite A* ∨ *infinite V*)
      *then* {} *else* {*p*. *p* ‘ *V* ⊆ (*pl-α* ‘ *permutations-of-set A*)})

**fun** $\mathcal{K}_{\mathcal{E}}$*-std* :: (*'a*, *'v*, *'r Result*) *Consensus-Class* ⇒ *'r* ⇒ *'a set* ⇒ *'v set* ⇒
      (*'a*, *'v*) *Election set* **where**
  $\mathcal{K}_{\mathcal{E}}$*-std K w A V* =
    (λ *p*. (*A*, *V*, *p*)) ‘ (*Set.filter*
      (λ *p*. (*consensus-*$\mathcal{K}$ *K*) (*A*, *V*, *p*) ∧ *elect* (*rule-*$\mathcal{K}$ *K*) *V A p* = {*w*})
      (*profiles V A*))

Returns those consensus elections on a given alternative and voter set from a given consensus that are mapped to the given unique winner by a given consensus rule.

**fun** *score-std* :: (*'a*, *'v*) *Election Distance* ⇒ (*'a*, *'v*, *'r Result*) *Consensus-Class* ⇒
      (*'a*, *'v*) *Election* ⇒ *'r* ⇒ *ereal* **where**
  *score-std d K E w* =
      (*if* $\mathcal{K}_{\mathcal{E}}$*-std K w* (*alternatives-*$\mathcal{E}$ *E*) (*voters-*$\mathcal{E}$ *E*) = {}
        *then* ∞ *else Min* (*d E* ‘ ($\mathcal{K}_{\mathcal{E}}$*-std K w* (*alternatives-*$\mathcal{E}$ *E*) (*voters-*$\mathcal{E}$ *E*))))

**fun** (**in** *result*) $\mathcal{R}_{\mathcal{W}}$*-std* :: (*'a*, *'v*) *Election Distance* ⇒
      (*'a*, *'v*, *'r Result*) *Consensus-Class* ⇒ *'v set* ⇒ *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒
      *'r set* **where**
  $\mathcal{R}_{\mathcal{W}}$*-std d K V A p* = *arg-min-set* (*score-std d K* (*A*, *V*, *p*)) (*limit A UNIV*)

**fun** (**in** *result*) *distance-*$\mathcal{R}$*-std* :: (*'a*, *'v*) *Election Distance* ⇒
      (*'a*, *'v*, *'r Result*) *Consensus-Class* ⇒
      (*'a*, *'v*, *'r Result*) *Electoral-Module* **where**
  *distance-*$\mathcal{R}$*-std d K V A p* =
    ($\mathcal{R}_{\mathcal{W}}$*-std d K V A p*, (*limit A UNIV*) − $\mathcal{R}_{\mathcal{W}}$*-std d K V A p*, {})

### 5.4.3 Auxiliary Lemmas

**lemma** *fin-*$\mathcal{K}_{\mathcal{E}}$:
  **fixes** *C* :: (*'a*, *'v*, *'r Result*) *Consensus-Class*
  **shows** *elections-*$\mathcal{K}$ *C* ⊆ *finite-elections*
⟨*proof*⟩

**lemma** *univ-*$\mathcal{K}_{\mathcal{E}}$:
  **fixes** *C* :: (*'a*, *'v*, *'r Result*) *Consensus-Class*
  **shows** *elections-*$\mathcal{K}$ *C* ⊆ *UNIV*
  ⟨*proof*⟩

**lemma** *list-cons-presv-finiteness*:
  **fixes**
    *A* :: *'a set* **and**
    *S* :: *'a list set*
  **assumes**

*fin-A*: *finite A* **and**
*fin-B*: *finite S*
**shows** *finite* $\{a\#l \mid a\ l.\ a \in A \wedge l \in S\}$
⟨*proof*⟩

**lemma** *listset-finiteness*:
**fixes** $l :: 'a\ set\ list$
**assumes** $\forall\ i{::}nat.\ i < length\ l \longrightarrow finite\ (l!i)$
**shows** *finite* (*listset l*)
⟨*proof*⟩

**lemma** *ls-entries-empty-imp-ls-set-empty*:
**fixes** $l :: 'a\ set\ list$
**assumes**
$0 < length\ l$ **and**
$\forall\ i{::}nat.\ i < length\ l \longrightarrow l!i = \{\}$
**shows** $listset\ l = \{\}$
⟨*proof*⟩

**lemma** *all-ls-elems-same-len*:
**fixes** $l :: 'a\ set\ list$
**shows** $\forall\ l' :: 'a\ list.\ l' \in listset\ l \longrightarrow length\ l' = length\ l$
⟨*proof*⟩

**lemma** *fin-all-profs*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$x :: 'a\ Preference\text{-}Relation$
**assumes**
*fin-A*: *finite A* **and**
*fin-V*: *finite V*
**shows** *finite* $(profiles\ V\ A \cap \{p.\ \forall\ v.\ v \notin V \longrightarrow p\ v = x\})$
⟨*proof*⟩

**lemma** *profile-permutation-set*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$
**shows** $profiles\ V\ A = \{p :: ('a, 'v)\ Profile.\ finite\text{-}profile\ V\ A\ p\}$
⟨*proof*⟩

### 5.4.4 Soundness

**lemma** (**in** *result*) $\mathcal{R}$*-sound*:
**fixes**
$K :: ('a, 'v, 'r\ Result)\ Consensus\text{-}Class$ **and**
$d :: ('a, 'v)\ Election\ Distance$
**shows** *electoral-module* $(distance\text{-}\mathcal{R}\ d\ K)$

⟨*proof*⟩

### 5.4.5 Inference Rules

**lemma** (**in** *result*) *standard-distance-imp-equal-score*:
  **fixes**
    *d* :: ('*a*, '*v*) *Election Distance* **and**
    *K* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
    *A* :: '*a set* **and**
    *V* :: '*v set* **and**
    *p* :: ('*a*, '*v*) *Profile* **and**
    *w* :: '*r*
  **assumes**
    *irr-non-V*: *voters-determine-distance d* **and**
    *std*: *standard d*
  **shows** *score d K* (*A*, *V*, *p*) *w* = *score-std d K* (*A*, *V*, *p*) *w*
⟨*proof*⟩

**lemma** (**in** *result*) *anonymous-distance-and-consensus-imp-rule-anonymity*:
  **fixes**
    *d* :: ('*a*, '*v*) *Election Distance* **and**
    *K* :: ('*a*, '*v*, '*r Result*) *Consensus-Class*
  **assumes**
    *d-anon*: *distance-anonymity d* **and**
    *K-anon*: *consensus-rule-anonymity K*
  **shows** *anonymity* (*distance-*$\mathcal{R}$ *d K*)
⟨*proof*⟩

**end**

## 5.5 Votewise Distance Rationalization

**theory** *Votewise-Distance-Rationalization*
  **imports** *Distance-Rationalization*
      *Votewise-Distance*
**begin**

A votewise distance rationalization of a voting rule is its distance rationalization with a distance function that depends on the submitted votes in a simple and a transparent manner by using a distance on individual orders and combining the components with a norm on R to n.

### 5.5.1 Common Rationalizations

**fun** *swap-*$\mathcal{R}$ :: ('*a*, '*v* :: *linorder*, '*a Result*) *Consensus-Class* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *swap-*$\mathcal{R}$ *K* = $\mathcal{SCF}$*-result.distance-*$\mathcal{R}$ (*votewise-distance swap l-one*) *K*

### 5.5.2 Theorems

**lemma** *votewise-non-voters-irrelevant*:
  **fixes**
    *d* :: *'a Vote Distance* **and**
    *N* :: *Norm*
  **shows** *voters-determine-distance* (*votewise-distance d N*)
⟨*proof*⟩

**lemma** *swap-standard*: *standard* (*votewise-distance swap l-one*)
⟨*proof*⟩

### 5.5.3 Equivalence Lemmas

**type-synonym** (*'a*, *'v*) *score-type* = (*'a*, *'v*) *Election Distance* ⇒
  (*'a*, *'v*, *'a Result*) *Consensus-Class* ⇒ (*'a*, *'v*) *Election* ⇒ *'a* ⇒ *ereal*

**type-synonym** (*'a*, *'v*) *dist-rat-type* = (*'a*, *'v*) *Election Distance* ⇒
  (*'a*, *'v*, *'a Result*) *Consensus-Class* ⇒ *'v set* ⇒ *'a set* ⇒ (*'a*, *'v*) *Profile* ⇒ *'a set*

**type-synonym** (*'a*, *'v*) *dist-rat-std-type* = (*'a*, *'v*) *Election Distance* ⇒
  (*'a*, *'v*, *'a Result*) *Consensus-Class* ⇒ (*'a*, *'v*, *'a Result*) *Electoral-Module*

**type-synonym** (*'a*, *'v*) *dist-type* = (*'a*, *'v*) *Election Distance* ⇒
  (*'a*, *'v*, *'a Result*) *Consensus-Class* ⇒ (*'a*, *'v*, *'a Result*) *Electoral-Module*

**lemma** *equal-score-swap*: (*score* :: (*'a*, *'v* :: *linorder*) *score-type*)
      (*votewise-distance swap l-one*) = *score-std* (*votewise-distance swap l-one*)
  ⟨*proof*⟩

**lemma** *swap-$\mathcal{R}$-code*[*code*]: *swap-$\mathcal{R}$* =
      (*$\mathcal{SCF}$-result.distance-$\mathcal{R}$-std* :: (*'a*, *'v* :: *linorder*) *dist-rat-std-type*)
          (*votewise-distance swap l-one*)
⟨*proof*⟩

**end**

## 5.6 Symmetry in Distance-Rationalizable Rules

**theory** *Distance-Rationalization-Symmetry*
  **imports** *Distance-Rationalization*
**begin**

### 5.6.1 Minimizer Function

**fun** *distance-infimum* :: *'a Distance* ⇒ *'a set* ⇒ *'a* ⇒ *ereal* **where**
  *distance-infimum d A a* = *Inf* (*d a* ‘ *A*)

**fun** *closest-preimg-distance* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a\ Distance \Rightarrow$
$'a \Rightarrow 'b \Rightarrow ereal$ **where**
*closest-preimg-distance f* $domain_f$ *d a b* =
*distance-infimum d* (*preimg f* $domain_f$ *b*) *a*

**fun** *minimizer* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a\ Distance \Rightarrow 'b\ set \Rightarrow 'a \Rightarrow 'b\ set$ **where**
*minimizer f* $domain_f$ *d A a* =
*arg-min-set* (*closest-preimg-distance f* $domain_f$ *d a*) *A*

## Auxiliary Lemmas

**lemma** *rewrite-arg-min-set*:
**fixes**
$f :: 'a \Rightarrow 'b :: linorder$ **and**
$A :: 'a\ set$
**shows** *arg-min-set f A* = $\bigcup$ (*preimg f A* ‘ $\{y \in (f\ `\ A).\ \forall\ z \in f\ `\ A.\ y \leq z\}$)
⟨*proof*⟩

## Equivariance

**abbreviation** *Restrp* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ rel$ **where**
*Restrp r A* ≡ *r Int* ($A \times UNIV$)

**lemma** *restr-induced-rel*:
**fixes**
$A :: 'a\ set$ **and**
$B\ B' :: 'b\ set$ **and**
$\varphi :: ('a, 'b)\ binary\text{-}fun$
**assumes** $B' \subseteq B$
**shows** *Restrp* (*action-induced-rel A B* $\varphi$) $B'$ = *action-induced-rel A* $B'$ $\varphi$
⟨*proof*⟩

**theorem** *group-action-invar-dist-and-equivar-f-imp-equivar-minimizer*:
**fixes**
$f :: 'a \Rightarrow 'b$ **and**
$domain_f\ X :: 'a\ set$ **and**
*d* :: *'a Distance* **and**
*well-formed-img* :: $'a \Rightarrow 'b\ set$ **and**
*G* :: *'c monoid* **and**
$\varphi :: ('c, 'a)\ binary\text{-}fun$ **and**
$\psi :: ('c, 'b)\ binary\text{-}fun$
**defines** *equivar-prop-set-valued* ≡
*action-induced-equivariance* (*carrier G*) *X* $\varphi$ (*set-action* $\psi$)
**assumes**
*action-*$\varphi$: *group-action G X* $\varphi$ **and**
*group-action-res*: *group-action G UNIV* $\psi$ **and**
*dom-in-X*: $domain_f \subseteq X$ **and**
*closed-domain*:
*closed-restricted-rel* (*action-induced-rel* (*carrier G*) *X* $\varphi$) *X* $domain_f$ **and**
*equivar-img*: *is-symmetry well-formed-img equivar-prop-set-valued* **and**

*invar-d*: $invariance_{\mathcal{D}}\ d\ (carrier\ G)\ X\ \varphi$ **and**
*equivar-f*:
*is-symmetry f* $(action\text{-}induced\text{-}equivariance\ (carrier\ G)\ domain_f\ \varphi\ \psi)$
**shows** *is-symmetry* $(\lambda\ x.\ minimizer\ f\ domain_f\ d\ (well\text{-}formed\text{-}img\ x)\ x)$ *equivar-prop-set-valued*
⟨*proof*⟩

## Invariance

**lemma** *closest-dist-invar-under-refl-rel-and-tot-invar-dist*:
**fixes**
$f :: 'a \Rightarrow 'b$ **and**
$domain_f :: 'a\ set$ **and**
$d :: 'a\ Distance$ **and**
$rel :: 'a\ rel$
**assumes**
*r-refl*: *reflp-on'* $domain_f$ $(Restrp\ rel\ domain_f)$ **and**
*tot-invar-d*: $total\text{-}invariance_{\mathcal{D}}\ d\ rel$
**shows** *is-symmetry* $(closest\text{-}preimg\text{-}distance\ f\ domain_f\ d)$ $(Invariance\ rel)$
⟨*proof*⟩

**lemma** *refl-rel-and-tot-invar-dist-imp-invar-minimizer*:
**fixes**
$f :: 'a \Rightarrow 'b$ **and**
$domain_f :: 'a\ set$ **and**
$d :: 'a\ Distance$ **and**
$rel :: 'a\ rel$ **and**
$img :: 'b\ set$
**assumes**
*r-refl*: *reflp-on'* $domain_f$ $(Restrp\ rel\ domain_f)$ **and**
*tot-invar-d*: $total\text{-}invariance_{\mathcal{D}}\ d\ rel$
**shows** *is-symmetry* $(minimizer\ f\ domain_f\ d\ img)$ $(Invariance\ rel)$
⟨*proof*⟩

**theorem** *group-act-invar-dist-and-invar-f-imp-invar-minimizer*:
**fixes**
$f :: 'a \Rightarrow 'b$ **and**
$domain_f\ A :: 'a\ set$ **and**
$d :: 'a\ Distance$ **and**
$img :: 'b\ set$ **and**
$G :: 'c\ monoid$ **and**
$\varphi :: ('c, 'a)\ binary\text{-}fun$
**defines**
$rel \equiv action\text{-}induced\text{-}rel\ (carrier\ G)\ A\ \varphi$ **and**
$rel' \equiv action\text{-}induced\text{-}rel\ (carrier\ G)\ domain_f\ \varphi$
**assumes**
*action-φ*: *group-action* $G\ A\ \varphi$ **and**
$domain_f \subseteq A$ **and**
*closed-domain*: $closed\text{-}restricted\text{-}rel\ rel\ A\ domain_f$ **and**
*invar-d*: $invariance_{\mathcal{D}}\ d\ (carrier\ G)\ A\ \varphi$ **and**

*invar-f*: *is-symmetry f* (*Invariance rel'*)
**shows** *is-symmetry* (*minimizer f domain$_f$ d img*) (*Invariance rel*)
⟨*proof*⟩

### 5.6.2 Minimizer Translation

**lemma** $\mathcal{K}_\mathcal{E}$*-is-preimg*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*E* :: ('*a*, '*v*) *Election* **and**
*w* :: '*r*
**shows** *preimg* (*elect-r* ∘ *fun*$_\mathcal{E}$ (*rule-*$\mathcal{K}$ *C*)) (*elections-*$\mathcal{K}$ *C*) {*w*} = $\mathcal{K}_\mathcal{E}$ *C w*
⟨*proof*⟩

**lemma** *score-is-closest-preimg-dist*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*E* :: ('*a*, '*v*) *Election* **and**
*w* :: '*r*
**shows** *score d C E w* =
*closest-preimg-distance* (*elect-r* ∘ *fun*$_\mathcal{E}$ (*rule-*$\mathcal{K}$ *C*)) (*elections-*$\mathcal{K}$ *C*) *d E* {*w*}
⟨*proof*⟩

**lemma** (**in** *result*) $\mathcal{R}_\mathcal{W}$*-is-minimizer*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class*
**shows** *fun*$_\mathcal{E}$ ($\mathcal{R}_\mathcal{W}$ *d C*) =
(λ *E*. ⋃ (*minimizer* (*elect-r* ∘ *fun*$_\mathcal{E}$ (*rule-*$\mathcal{K}$ *C*)) (*elections-*$\mathcal{K}$ *C*) *d*
(*singleton-set-system* (*limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*)) *E*))
⟨*proof*⟩

#### Invariance

**theorem** (**in** *result*) *tot-invar-dist-imp-invar-dr-rule*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*rel* :: ('*a*, '*v*) *Election rel*
**assumes**
*r-refl*: *reflp-on'* (*elections-*$\mathcal{K}$ *C*) (*Restrp rel* (*elections-*$\mathcal{K}$ *C*)) **and**
*tot-invar-d*: *total-invariance*$_\mathcal{D}$ *d rel* **and**
*invar-res*:
*is-symmetry* (λ *E*. *limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*) (*Invariance rel*)
**shows** *is-symmetry* (*fun*$_\mathcal{E}$ (*distance-*$\mathcal{R}$ *d C*)) (*Invariance rel*)
⟨*proof*⟩

**theorem** (**in** *result*) *invar-dist-cons-imp-invar-dr-rule*:

**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*G* :: '*b monoid* **and**
$\varphi$ :: ('*b*, ('*a*, '*v*) *Election*) *binary-fun* **and**
*B* :: ('*a*, '*v*) *Election set*
**defines**
*rel* ≡ *action-induced-rel* (*carrier G*) *B* $\varphi$ **and**
*rel'* ≡ *action-induced-rel* (*carrier G*) (*elections-*$\mathcal{K}$ *C*) $\varphi$
**assumes**
*action-*$\varphi$: *group-action G B* $\varphi$ **and**
*consensus-C-in-B*: *elections-*$\mathcal{K}$ *C* ⊆ *B* **and**
*closed-domain*:
*closed-restricted-rel rel B* (*elections-*$\mathcal{K}$ *C*) **and**
*invar-res*:
*is-symmetry* (λ *E*. *limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*) (*Invariance rel*) **and**
*invar-d*: *invariance*$_{\mathcal{D}}$ *d* (*carrier G*) *B* $\varphi$ **and**
*invar-C-winners*: *is-symmetry* (*elect-r* ∘ *fun*$_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*)) (*Invariance rel'*)
**shows** *is-symmetry* (*fun*$_{\mathcal{E}}$ (*distance-*$\mathcal{R}$ *d C*)) (*Invariance rel*)
⟨*proof*⟩

## Equivariance

**theorem** (**in** *result*) *invar-dist-equivar-cons-imp-equivar-dr-rule*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*G* :: '*b monoid* **and**
$\varphi$ :: ('*b*, ('*a*, '*v*) *Election*) *binary-fun* **and**
$\psi$ :: ('*b*, '*r*) *binary-fun* **and**
*B* :: ('*a*, '*v*) *Election set*
**defines**
*rel* ≡ *action-induced-rel* (*carrier G*) *B* $\varphi$ **and**
*rel'* ≡ *action-induced-rel* (*carrier G*) (*elections-*$\mathcal{K}$ *C*) $\varphi$ **and**
*equivar-prop* ≡
*action-induced-equivariance* (*carrier G*) (*elections-*$\mathcal{K}$ *C*)
$\varphi$ (*set-action* $\psi$) **and**
*equivar-prop-global-set-valued* ≡
*action-induced-equivariance* (*carrier G*) *B* $\varphi$ (*set-action* $\psi$) **and**
*equivar-prop-global-result-valued* ≡
*action-induced-equivariance* (*carrier G*) *B* $\varphi$ (*result-action* $\psi$)
**assumes**
*action-*$\varphi$: *group-action G B* $\varphi$ **and**
*group-act-res*: *group-action G UNIV* $\psi$ **and**
*cons-elect-set*: *elections-*$\mathcal{K}$ *C* ⊆ *B* **and**
*closed-domain*: *closed-restricted-rel rel B* (*elections-*$\mathcal{K}$ *C*) **and**
*equivar-res*:
*is-symmetry* (λ *E*. *limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*)
*equivar-prop-global-set-valued* **and**

*invar-d*: $invariance_{\mathcal{D}}$ *d* (*carrier G*) *B* $\varphi$ **and**
*equivar-C-winners*: *is-symmetry* (*elect-r* $\circ$ $fun_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*)) *equivar-prop*
**shows** *is-symmetry* ($fun_{\mathcal{E}}$ (*distance-*$\mathcal{R}$ *d C*)) *equivar-prop-global-result-valued*
⟨*proof*⟩

### 5.6.3 Inference Rules

**theorem** (**in** *result*) *anon-dist-and-cons-imp-anon-dr*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class*
**assumes**
*anon-d*: *distance-anonymity'* *well-formed-elections d* **and**
*anon-C*: *consensus-rule-anonymity'* (*elections-*$\mathcal{K}$ *C*) *C* **and**
*closed-C*: *closed-restricted-rel* ($anonymity_{\mathcal{R}}$ *well-formed-elections*)
*well-formed-elections* (*elections-*$\mathcal{K}$ *C*)
**shows** *anonymity' well-formed-elections* (*distance-*$\mathcal{R}$ *d C*)
⟨*proof*⟩

**theorem** (**in** *result-properties*) *neutr-dist-and-cons-imp-neutr-dr*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*b Result*) *Consensus-Class*
**assumes**
*neutral-d*: *distance-neutrality well-formed-elections d* **and**
*neutral-C*: *consensus-rule-neutrality* (*elections-*$\mathcal{K}$ *C*) *C* **and**
*closed-C*: *closed-restricted-rel* ($neutrality_{\mathcal{R}}$ *well-formed-elections*)
*well-formed-elections* (*elections-*$\mathcal{K}$ *C*)
**shows** *neutrality well-formed-elections* (*distance-*$\mathcal{R}$ *d C*)
⟨*proof*⟩

**theorem** *reversal-sym-dist-and-cons-imp-reversal-sym-dr*:
**fixes**
*d* :: ('*a*, '*c*) *Election Distance* **and**
*C* :: ('*a*, '*c*, '*a rel Result*) *Consensus-Class*
**assumes**
*reverse-sym-d*: *distance-reversal-symmetry well-formed-elections d* **and**
*reverse-sym-C*: *consensus-rule-reversal-symmetry* (*elections-*$\mathcal{K}$ *C*) *C* **and**
*closed-C*: *closed-restricted-rel* ($reversal_{\mathcal{R}}$ *well-formed-elections*)
*well-formed-elections* (*elections-*$\mathcal{K}$ *C*)
**shows** *reversal-symmetry well-formed-elections* ($\mathcal{SWF}$*-result.distance-*$\mathcal{R}$ *d C*)
⟨*proof*⟩

**theorem** (**in** *result*) *tot-hom-dist-imp-hom-dr*:
**fixes**
*d* :: ('*a*, *nat*) *Election Distance* **and**
*C* :: ('*a*, *nat*, '*r Result*) *Consensus-Class*
**assumes** *distance-homogeneity finite-elections-*$\mathcal{V}$ *d*
**shows** *homogeneity finite-elections-*$\mathcal{V}$ (*distance-*$\mathcal{R}$ *d C*)

⟨*proof*⟩

**theorem** (**in** *result*) *tot-hom-dist-imp-hom-dr'*:
  **fixes**
    $d$ :: ('*a*, '*v* :: *linorder*) *Election Distance* **and**
    $C$ :: ('*a*, '*v*, '*r Result*) *Consensus-Class*
  **assumes** *distance-homogeneity' finite-elections-*$\mathcal{V}$ $d$
  **shows** *homogeneity' finite-elections-*$\mathcal{V}$ (*distance-*$\mathcal{R}$ $d$ $C$)
⟨*proof*⟩

### 5.6.4 Properties

**fun** *decisiveness* :: ('*a*, '*v*) *Election set* ⇒ ('*a*, '*v*) *Election Distance* ⇒
      ('*a*, '*v*, '*r Result*) *Electoral-Module* ⇒ *bool* **where**
  *decisiveness X d m* =
    (∄ *E*. *E* ∈ *X*
    ∧ (∃ δ > *0*. ∀ *E'* ∈ *X*. *d E E'* < δ ⟶ *card* (*elect-r* (*fun*$_{\mathcal{E}}$ *m E'*)) > *1*))

**end**

## 5.7 Distance Rationalization on Election Quotients

**theory** *Quotient-Distance-Rationalization*
  **imports** *Quotient-Module*
      *Distance-Rationalization-Symmetry*
**begin**

### 5.7.1 Distances

**fun** *distance*$_{\mathcal{Q}}$ :: '*x Distance* ⇒ '*x set Distance* **where**
  *distance*$_{\mathcal{Q}}$ *d A B* = (*if* (*A* = {} ∧ *B* = {}) *then 0 else*
          (*if* (*A* = {} ∨ *B* = {}) *then* ∞ *else*
            $\pi_{\mathcal{Q}}$ (*tup d*) (*A* × *B*)))

**fun** *relation-paths* :: '*x rel* ⇒ '*x list set* **where**
  *relation-paths r* =
    {*p*. ∃ *k*. (*length p* = *2* ∗ *k* ∧ (∀ *i* < *k*. (*p*!(*2* ∗ *i*), *p*!(*2* ∗ *i* + *1*)) ∈ *r*))}

**fun** *admissible-paths* :: '*x rel* ⇒ '*x set* ⇒ '*x set* ⇒ '*x list set* **where**
  *admissible-paths r X Y* =
    {*x*#*p*@[*y*] | *x y p*. *x* ∈ *X* ∧ *y* ∈ *Y* ∧ *p* ∈ *relation-paths r*}

**fun** *path-length* :: '*x list* ⇒ '*x Distance* ⇒ *ereal* **where**
  *path-length* [] *d* = *0* |
  *path-length* [*x*] *d* = *0* |
  *path-length* (*x*#*y*#*xs*) *d* = *d x y* + *path-length xs d*

**fun** *quotient-dist* :: *'x rel* ⇒ *'x Distance* ⇒ *'x set Distance* **where**
*quotient-dist r d A B* =
*Inf* (⋃ {{*path-length p d* | *p*. *p* ∈ *admissible-paths r A B*}})

**fun** *distance-infimum*$_{\mathcal{Q}}$ :: *'x Distance* ⇒ *'x set Distance* **where**
*distance-infimum*$_{\mathcal{Q}}$ *d A B* = *Inf* {*d a b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}

**fun** *simple* :: *'x rel* ⇒ *'x set* ⇒ *'x Distance* ⇒ *bool* **where**
*simple r X d* =
(∀ *A* ∈ *X* // *r*.
(∃ *a* ∈ *A*. ∀ *B* ∈ *X* // *r*.
*distance-infimum*$_{\mathcal{Q}}$ *d A B* = *Inf* {*d a b* | *b*. *b* ∈ *B*}))
— We call a distance simple with respect to a relation if for all relation classes, there is an $a$ in $A$ that minimizes the infimum distance between $A$ and all $B$ such that the infimum distance between these sets coincides with the infimum distance over all $b$ in $B$ for a fixed $a$.

**fun** *product'* :: *'x rel* ⇒ (*'x* ∗ *'x*) *rel* **where**
*product' r* = {(*p*$_1$, *p*$_2$). ((*fst p*$_1$, *fst p*$_2$) ∈ *r* ∧ *snd p*$_1$ = *snd p*$_2$)
∨ ((*snd p*$_1$, *snd p*$_2$) ∈ *r* ∧ *fst p*$_1$ = *fst p*$_2$)}

## Auxiliary Lemmas

**lemma** *tot-dist-invariance-is-congruence*:
**fixes**
*d* :: *'x Distance* **and**
*r* :: *'x rel*
**shows** (*total-invariance*$_{\mathcal{D}}$ *d r*) = (*tup d respects* (*product r*))
⟨*proof*⟩

**lemma** *product-helper*:
**fixes**
*r* :: *'x rel* **and**
*X* :: *'x set*
**shows**
*trans-imp*: *Relation.trans r* ⟶ *Relation.trans* (*product r*) **and**
*refl-imp*: *refl-on X r* ⟶ *refl-on* (*X* × *X*) (*product r*) **and**
*sym*: *sym-on X r* ⟶ *sym-on* (*X* × *X*) (*product r*)
⟨*proof*⟩

**theorem** *dist-pass-to-quotient*:
**fixes**
*d* :: *'x Distance* **and**
*r* :: *'x rel* **and**
*X* :: *'x set*
**assumes**
*equiv-X-r*: *equiv X r* **and**
*tot-inv-dist-d-r*: *total-invariance*$_{\mathcal{D}}$ *d r*
**shows** ∀ *A B*. *A* ∈ *X* // *r* ∧ *B* ∈ *X* // *r*

$\longrightarrow (\forall\ a\ b.\ a \in A \wedge b \in B \longrightarrow distance_{\mathcal{Q}}\ d\ A\ B = d\ a\ b)$
⟨*proof*⟩

**lemma** *relation-paths-subset*:
**fixes**
$n :: nat$ **and**
$p :: 'x\ list$ **and**
$r :: 'x\ rel$ **and**
$X :: 'x\ set$
**assumes** $r \subseteq X \times X$
**shows** $\forall\ p.\ p \in relation\text{-}paths\ r \longrightarrow (\forall\ i < length\ p.\ p!i \in X)$
⟨*proof*⟩

**lemma** *admissible-path-len*:
**fixes**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$ **and**
$X :: 'x\ set$ **and**
$a\ b :: 'x$ **and**
$p :: 'x\ list$
**assumes** *refl-on* $X\ r$
**shows** $triangle\text{-}ineq\ X\ d \wedge p \in relation\text{-}paths\ r \wedge total\text{-}invariance_{\mathcal{D}}\ d\ r$
$\wedge\ a \in X \wedge b \in X \longrightarrow path\text{-}length\ (a\#p@[b])\ d \geq d\ a\ b$
⟨*proof*⟩

**lemma** *quotient-dist-coincides-with-dist*$_{\mathcal{Q}}$:
**fixes**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$ **and**
$X :: 'x\ set$
**assumes**
*equiv*: $equiv\ X\ r$ **and**
*tri*: $triangle\text{-}ineq\ X\ d$ **and**
*invar*: $total\text{-}invariance_{\mathcal{D}}\ d\ r$
**shows** $\forall\ A \in X\ //\ r.\ \forall\ B \in X\ //\ r.\ quotient\text{-}dist\ r\ d\ A\ B = distance_{\mathcal{Q}}\ d\ A\ B$
⟨*proof*⟩

**lemma** *inf-dist-coincides-with-dist*$_{\mathcal{Q}}$:
**fixes**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$ **and**
$X :: 'x\ set$
**assumes**
*equiv-X-r*: $equiv\ X\ r$ **and**
*tot-inv-d-r*: $total\text{-}invariance_{\mathcal{D}}\ d\ r$
**shows** $\forall\ A \in X\ //\ r.\ \forall\ B \in X\ //\ r.$
$distance\text{-}infimum_{\mathcal{Q}}\ d\ A\ B = distance_{\mathcal{Q}}\ d\ A\ B$
⟨*proof*⟩

**lemma** *inf-helper*:
**fixes**
$A\ B :: 'x\ set$ **and**
$d :: 'x\ Distance$
**shows** $Inf\ \{d\ a\ b \mid a\ b.\ a \in A \wedge b \in B\} =$
$Inf\ \{Inf\ \{d\ a\ b \mid b.\ b \in B\} \mid a.\ a \in A\}$
⟨*proof*⟩

**lemma** *invar-dist-simple*:
**fixes**
$d :: 'y\ Distance$ **and**
$G :: 'x\ monoid$ **and**
$Y :: 'y\ set$ **and**
$\varphi :: ('x, 'y)\ binary\text{-}fun$
**assumes**
*action-φ*: *group-action* $G\ Y\ \varphi$ **and**
*invar*: $invariance_{\mathcal{D}}\ d\ (carrier\ G)\ Y\ \varphi$
**shows** $simple\ (action\text{-}induced\text{-}rel\ (carrier\ G)\ Y\ \varphi)\ Y\ d$
⟨*proof*⟩

**lemma** *tot-invar-dist-simple*:
**fixes**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$ **and**
$X :: 'x\ set$
**assumes**
*equiv-on-X*: $equiv\ X\ r$ **and**
*invar*: $total\text{-}invariance_{\mathcal{D}}\ d\ r$
**shows** $simple\ r\ X\ d$
⟨*proof*⟩

### 5.7.2 Consensus and Results

**fun** $elections\text{-}\mathcal{K}_{\mathcal{Q}} :: ('a, 'v)\ Election\ rel \Rightarrow ('a, 'v, 'r\ Result)\ Consensus\text{-}Class \Rightarrow$
$('a, 'v)\ Election\ set\ set$ **where**
$elections\text{-}\mathcal{K}_{\mathcal{Q}}\ r\ C = (elections\text{-}\mathcal{K}\ C)\ //\ r$

**fun** (**in** *result*) $limit_{\mathcal{Q}} :: ('a, 'v)\ Election\ set \Rightarrow 'r\ set \Rightarrow 'r\ set$ **where**
$limit_{\mathcal{Q}}\ X\ res = \bigcap\ \{limit\ (alternatives\text{-}\mathcal{E}\ E)\ res \mid E.\ E \in X\}$

#### Auxiliary Lemmas

**lemma** *closed-under-equiv-rel-subset*:
**fixes**
$X\ Y\ Z :: 'x\ set$ **and**
$r :: 'x\ rel$
**assumes**
$equiv\ X\ r$ **and**
$Y \subseteq X$ **and**
$Z \subseteq X$ **and**

$Z \in Y \mathbin{//} r$ **and**
*closed-restricted-rel* $r\ X\ Y$
**shows** $Z \subseteq Y$
⟨*proof*⟩

**lemma** (**in** *result*) *limit-invar*:
**fixes**
$d :: ('a, 'v)$ *Election Distance* **and**
$r :: ('a, 'v)$ *Election rel* **and**
$C :: ('a, 'v, 'r\ Result)$ *Consensus-Class* **and**
$X\ A :: ('a, 'v)$ *Election set*
**assumes**
*quot-class*: $A \in X \mathbin{//} r$ **and**
*equiv-rel*: *equiv* $X\ r$ **and**
*cons-subset*: *elections-*$\mathcal{K}\ C \subseteq X$ **and**
*invar-res*: *is-symmetry* $(\lambda\ E.\ limit\ (alternatives\text{-}\mathcal{E}\ E)\ UNIV)$ *(Invariance* $r)$
**shows** $\forall\ a \in A.\ limit\ (alternatives\text{-}\mathcal{E}\ a)\ UNIV = limit_{\mathcal{Q}}\ A\ UNIV$
⟨*proof*⟩

**lemma** (**in** *result*) *preimg-invar*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$domain_f\ X :: 'x\ set$ **and**
$d :: 'x\ Distance$ **and**
$r :: 'x\ rel$
**assumes**
*equiv-rel*: *equiv* $X\ r$ **and**
*cons-subset*: $domain_f \subseteq X$ **and**
*closed-domain*: *closed-restricted-rel* $r\ X\ domain_f$ **and**
*invar-f*: *is-symmetry* $f$ *(Invariance (Restr* $r\ domain_f))$
**shows** $\forall\ y.\ (preimg\ f\ domain_f\ y) \mathbin{//} r = preimg\ (\pi_{\mathcal{Q}}\ f)\ (domain_f \mathbin{//} r)\ y$
⟨*proof*⟩

**lemma** *minimizer-helper*:
**fixes**
$f :: 'x \Rightarrow 'y$ **and**
$domain_f :: 'x\ set$ **and**
$d :: 'x\ Distance$ **and**
$Y :: 'y\ set$ **and**
$x :: 'x$ **and**
$y :: 'y$
**shows** $y \in minimizer\ f\ domain_f\ d\ Y\ x =$
$(y \in Y \wedge (\forall\ y' \in Y.$
$Inf\ (d\ x\ `\ (preimg\ f\ domain_f\ y)) \leq Inf\ (d\ x\ `\ (preimg\ f\ domain_f\ y'))))$
⟨*proof*⟩

**lemma** *rewr-singleton-set-system-union*:
**fixes**
$Y :: 'x\ set\ set$ **and**

$X :: 'x\ set$
**assumes** $Y \subseteq$ *singleton-set-system* $X$
**shows**
*singleton-set-union*: $x \in \bigcup\ Y \longleftrightarrow \{x\} \in Y$ **and**
*obtain-singleton*: $A \in$ *singleton-set-system* $X \longleftrightarrow (\exists\ x \in X.\ A = \{x\})$
⟨*proof*⟩

**lemma** *union-inf*:
**fixes** $X ::$ *ereal set set*
**shows** *Inf* $\{Inf\ A \mid A.\ A \in X\} = Inf\ (\bigcup\ X)$
⟨*proof*⟩

### 5.7.3 Distance Rationalization

**fun** (**in** *result*) $\mathcal{R}_\mathcal{Q} ::$ $('a, 'v)$ *Election rel* ⇒ $('a, 'v)$ *Election Distance* ⇒
$('a, 'v, 'r\ Result)$ *Consensus-Class* ⇒ $('a, 'v)$ *Election set* ⇒ $'r\ set$ **where**
$\mathcal{R}_\mathcal{Q}\ r\ d\ C\ A =$
$\bigcup$ (*minimizer* ($\pi_\mathcal{Q}$ (*elect-r* ∘ *fun*$_\mathcal{E}$ (*rule-*$\mathcal{K}$ $C$))) (*elections-*$\mathcal{K}_\mathcal{Q}$ $r$ $C$)
(*distance-infimum*$_\mathcal{Q}$ $d$) (*singleton-set-system* (*limit*$_\mathcal{Q}$ $A$ *UNIV*)) $A$)

**fun** (**in** *result*) *distance-*$\mathcal{R}_\mathcal{Q} ::$ $('a, 'v)$ *Election rel* ⇒ $('a, 'v)$ *Election Distance* ⇒
$('a, 'v, 'r\ Result)$ *Consensus-Class* ⇒ $('a, 'v)$ *Election set* ⇒ $'r\ Result$ **where**
*distance-*$\mathcal{R}_\mathcal{Q}$ $r\ d\ C\ A =$
($\mathcal{R}_\mathcal{Q}\ r\ d\ C\ A$,
$\pi_\mathcal{Q}$ ($\lambda\ E.$ *limit* (*alternatives-*$\mathcal{E}$ $E$) *UNIV*) $A - \mathcal{R}_\mathcal{Q}\ r\ d\ C\ A$,
{})

Proposition 4.17 by Hadjibeyli and Wilson [3].

**theorem** (**in** *result*) *invar-dr-simple-dist-imp-quotient-dr-winners*:
**fixes**
$d ::$ $('a, 'v)$ *Election Distance* **and**
$C ::$ $('a, 'v, 'r\ Result)$ *Consensus-Class* **and**
$r ::$ $('a, 'v)$ *Election rel* **and**
$X\ A ::$ $('a, 'v)$ *Election set*
**assumes**
*simple*: *simple* $r\ X\ d$ **and**
*closed-domain*: *closed-restricted-rel* $r\ X$ (*elections-*$\mathcal{K}$ $C$) **and**
*invar-res*:
*is-symmetry* ($\lambda\ E.$ *limit* (*alternatives-*$\mathcal{E}$ $E$) *UNIV*) (*Invariance* $r$) **and**
*invar-C*: *is-symmetry* (*elect-r* ∘ *fun*$_\mathcal{E}$ (*rule-*$\mathcal{K}$ $C$))
(*Invariance* (*Restr* $r$ (*elections-*$\mathcal{K}$ $C$))) **and**
*invar-dr*: *is-symmetry* (*fun*$_\mathcal{E}$ ($\mathcal{R}_\mathcal{W}$ $d$ $C$)) (*Invariance* $r$) **and**
*quot-class*: $A \in X\ //\ r$ **and**
*equiv-rel*: *equiv* $X\ r$ **and**
*cons-subset*: *elections-*$\mathcal{K}$ $C \subseteq X$
**shows** $\pi_\mathcal{Q}$ (*fun*$_\mathcal{E}$ ($\mathcal{R}_\mathcal{W}$ $d$ $C$)) $A = \mathcal{R}_\mathcal{Q}\ r\ d\ C\ A$
⟨*proof*⟩

**theorem** (**in** *result*) *invar-dr-simple-dist-imp-quotient-dr*:

**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*C* :: ('*a*, '*v*, '*r Result*) *Consensus-Class* **and**
*r* :: ('*a*, '*v*) *Election rel* **and**
*X A* :: ('*a*, '*v*) *Election set*
**assumes**
*simple*: *simple r X d* **and**
*closed-domain*: *closed-restricted-rel r X* (*elections-*$\mathcal{K}$ *C*) **and**
*invar-res*:
*is-symmetry* ($\lambda$ *E*. *limit* (*alternatives-*$\mathcal{E}$ *E*) *UNIV*)
(*Invariance r*) **and**
*invar-C*: *is-symmetry* (*elect-r* $\circ$ *fun*$_{\mathcal{E}}$ (*rule-*$\mathcal{K}$ *C*))
(*Invariance* (*Restr r* (*elections-*$\mathcal{K}$ *C*))) **and**
*invar-dr*: *is-symmetry* (*fun*$_{\mathcal{E}}$ ($\mathcal{R}_{\mathcal{W}}$ *d C*)) (*Invariance r*) **and**
*quot-class*: *A* $\in$ *X* // *r* **and**
*equiv-rel*: *equiv X r* **and**
*cons-subset*: *elections-*$\mathcal{K}$ *C* $\subseteq$ *X*
**shows** $\pi_{\mathcal{Q}}$ (*fun*$_{\mathcal{E}}$ (*distance-*$\mathcal{R}$ *d C*)) *A* = *distance-*$\mathcal{R}_{\mathcal{Q}}$ *r d C A*
⟨*proof*⟩

**end**

## 5.8 Code Generation Interpretations for Results and Properties

**theory** *Interpretation-Code*
**imports** *Electoral-Module*
*Distance-Rationalization*
**begin**
⟨*ML*⟩

### 5.8.1 Code Lemmas

Lemmas stating the explicit instantiations of interpreted abstract functions from locales.

**lemma** *electoral-module-*$\mathcal{SCF}$*-code-lemma*:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**shows** $\mathcal{SCF}$*-result.electoral-module m* =
(∀ *A V p*. *profile V A p* ⟶ *well-formed-*$\mathcal{SCF}$ *A* (*m V A p*))
⟨*proof*⟩

**lemma** $\mathcal{R}_{\mathcal{W}}$*-*$\mathcal{SCF}$*-code-lemma*:
**fixes**
*d* :: ('*a*, '*v*) *Election Distance* **and**
*K* :: ('*a*, '*v*, '*a Result*) *Consensus-Class* **and**
*V* :: '*v set* **and**

$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\ d\ K\ V\ A\ p =$
$arg\text{-}min\text{-}set\ (score\ d\ K\ (A, V, p))\ (limit\text{-}\mathcal{SCF}\ A\ UNIV)$
⟨*proof*⟩

**lemma** *distance-*$\mathcal{R}$*-*$\mathcal{SCF}$*-code-lemma*:
**fixes**
$d :: ('a, 'v)\ Election\ Distance$ **and**
$K :: ('a, 'v, 'a\ Result)\ Consensus\text{-}Class$ **and**
$V :: 'v\ set$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $\mathcal{SCF}\text{-}result.distance\text{-}\mathcal{R}\ d\ K\ V\ A\ p =$
$(\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\ d\ K\ V\ A\ p,$
$(limit\text{-}\mathcal{SCF}\ A\ UNIV) - \mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\ d\ K\ V\ A\ p,$
$\{\})$
⟨*proof*⟩

**lemma** $\mathcal{R}_{\mathcal{W}}$*-std-*$\mathcal{SCF}$*-code-lemma*:
**fixes**
$d :: ('a, 'v)\ Election\ Distance$ **and**
$K :: ('a, 'v, 'a\ Result)\ Consensus\text{-}Class$ **and**
$V :: 'v\ set$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std\ d\ K\ V\ A\ p =$
$arg\text{-}min\text{-}set\ (score\text{-}std\ d\ K\ (A, V, p))\ (limit\text{-}\mathcal{SCF}\ A\ UNIV)$
⟨*proof*⟩

**lemma** *distance-*$\mathcal{R}$*-std-*$\mathcal{SCF}$*-code-lemma*:
**fixes**
$d :: ('a, 'v)\ Election\ Distance$ **and**
$K :: ('a, 'v, 'a\ Result)\ Consensus\text{-}Class$ **and**
$V :: 'v\ set$ **and**
$A :: 'a\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $\mathcal{SCF}\text{-}result.distance\text{-}\mathcal{R}\text{-}std\ d\ K\ V\ A\ p =$
$(\mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std\ d\ K\ V\ A\ p,$
$(limit\text{-}\mathcal{SCF}\ A\ UNIV) - \mathcal{SCF}\text{-}result.\mathcal{R}_{\mathcal{W}}\text{-}std\ d\ K\ V\ A\ p,$
$\{\})$
⟨*proof*⟩

**lemma** *anonymity-*$\mathcal{SCF}$*-code-lemma*: $\mathcal{SCF}\text{-}result.anonymity =$
$(\lambda\ m :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module.$
$\mathcal{SCF}\text{-}result.electoral\text{-}module\ m \wedge$
$(\forall\ A\ V\ p\ \pi :: ('v \Rightarrow 'v).$
$bij\ \pi \longrightarrow (let\ (A', V', q) = (rename\ \pi\ (A, V, p))\ in$
$profile\ V\ A\ p \wedge profile\ V'\ A'\ q \longrightarrow m\ V\ A\ p = m\ V'\ A'\ q)))$

⟨*proof*⟩

### 5.8.2 Interpretation Declarations and Constants

Declarations for replacing interpreted abstract functions from locales by their explicit instantiations.

**declare** [[*lc-add* $\mathcal{SCF}$*-result.electoral-module electoral-module-*$\mathcal{SCF}$*-code-lemma*]]
**declare** [[*lc-add* $\mathcal{SCF}$*-result.*$\mathcal{R}_\mathcal{W}$ $\mathcal{R}_\mathcal{W}$*-*$\mathcal{SCF}$*-code-lemma*]]
**declare** [[*lc-add* $\mathcal{SCF}$*-result.*$\mathcal{R}_\mathcal{W}$*-std* $\mathcal{R}_\mathcal{W}$*-std-*$\mathcal{SCF}$*-code-lemma*]]
**declare** [[*lc-add* $\mathcal{SCF}$*-result.distance-*$\mathcal{R}$ *distance-*$\mathcal{R}$*-*$\mathcal{SCF}$*-code-lemma*]]
**declare** [[*lc-add* $\mathcal{SCF}$*-result.distance-*$\mathcal{R}$*-std distance-*$\mathcal{R}$*-std-*$\mathcal{SCF}$*-code-lemma*]]
**declare** [[*lc-add* $\mathcal{SCF}$*-result.anonymity anonymity-*$\mathcal{SCF}$*-code-lemma*]]

Constant aliases to use instead of the interpreted functions.

**definition** $\mathcal{R}_\mathcal{W}$*-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.*$\mathcal{R}_\mathcal{W}$
**definition** $\mathcal{R}_\mathcal{W}$*-std-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.*$\mathcal{R}_\mathcal{W}$*-std*
**definition** *distance-*$\mathcal{R}$*-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.distance-*$\mathcal{R}$
**definition** *distance-*$\mathcal{R}$*-std-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.distance-*$\mathcal{R}$*-std*
**definition** *electoral-module-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.electoral-module*
**definition** *anonymity-*$\mathcal{SCF}$*-code* ≡ $\mathcal{SCF}$*-result.anonymity*

⟨*ML*⟩

**end**

## 5.9 Drop Module

**theory** *Drop-Module*
  **imports** *Component-Types/Electoral-Module*
    *Component-Types/Social-Choice-Types/Result*
**begin**

This is a family of electoral modules. For a natural number n and a lexicon (linear order) r of all alternatives, the according drop module rejects the lexicographically first n alternatives (from A) and defers the rest. It is primarily used as counterpart to the pass module in a parallel composition, in order to segment the alternatives into two groups.

### 5.9.1 Definition

**fun** *drop-module* :: *nat* ⇒ *'a Preference-Relation* ⇒
    (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
  *drop-module n r V A p* =

$(\{\},$
$\{a \in A.\ rank\ (limit\ A\ r)\ a \leq n\},$
$\{a \in A.\ rank\ (limit\ A\ r)\ a > n\})$

### 5.9.2 Soundness

**theorem** *drop-mod-sound*[*simp*]:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$n :: nat$
**shows** $\mathcal{SCF}$*-result.electoral-module* (*drop-module n r*)
⟨*proof*⟩

**lemma** *voters-determine-drop-mod*:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$n :: nat$
**shows** *voters-determine-election* (*drop-module n r*)
⟨*proof*⟩

### 5.9.3 Non-Electing

The drop module is non-electing.

**theorem** *drop-mod-non-electing*[*simp*]:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$n :: nat$
**shows** *non-electing* (*drop-module n r*)
⟨*proof*⟩

### 5.9.4 Properties

The drop module is strictly defer-monotone.

**theorem** *drop-mod-def-lift-inv*[*simp*]:
**fixes**
$r :: 'a$ *Preference-Relation* **and**
$n :: nat$
**shows** *defer-lift-invariance* (*drop-module n r*)
⟨*proof*⟩

**end**

## 5.10 Pass Module

**theory** *Pass-Module*
  **imports** *Component-Types/Electoral-Module*
**begin**

This is a family of electoral modules. For a natural number n and a lexicon (linear order) r of all alternatives, the according pass module defers the lexicographically first n alternatives (from A) and rejects the rest. It is primarily used as counterpart to the drop module in a parallel composition in order to segment the alternatives into two groups.

### 5.10.1 Definition

**fun** *pass-module* :: *nat* ⇒ *'a Preference-Relation* ⇒
      (*'a*, *'v*, *'a Result*) *Electoral-Module* **where**
  *pass-module n r V A p* =
    ({},
    {*a* ∈ *A*. *rank* (*limit A r*) *a* > *n*},
    {*a* ∈ *A*. *rank* (*limit A r*) *a* ≤ *n*})

### 5.10.2 Soundness

**theorem** *pass-mod-sound*[*simp*]:
  **fixes**
    *r* :: *'a Preference-Relation* **and**
    *n* :: *nat*
  **shows** *SCF-result.electoral-module* (*pass-module n r*)
⟨*proof*⟩

**lemma** *voters-determine-pass-mod*:
  **fixes**
    *r* :: *'a Preference-Relation* **and**
    *n* :: *nat*
  **shows** *voters-determine-election* (*pass-module n r*)
  ⟨*proof*⟩

### 5.10.3 Non-Blocking

The pass module is non-blocking.

**theorem** *pass-mod-non-blocking*[*simp*]:
  **fixes**
    *r* :: *'a Preference-Relation* **and**
    *n* :: *nat*
  **assumes**
    *order*: *linear-order r* **and**
    *greater-zero*: *n* > *0*
  **shows** *non-blocking* (*pass-module n r*)
⟨*proof*⟩

### 5.10.4 Non-Electing

The pass module is non-electing.

**theorem** *pass-mod-non-electing*[*simp*]:
**fixes**
*r* :: *'a Preference-Relation* **and**
*n* :: *nat*
**assumes** *linear-order r*
**shows** *non-electing* (*pass-module n r*)
⟨*proof*⟩

### 5.10.5 Properties

The pass module is strictly defer-monotone.

**theorem** *pass-mod-dl-inv*[*simp*]:
**fixes**
*r* :: *'a Preference-Relation* **and**
*n* :: *nat*
**assumes** *linear-order r*
**shows** *defer-lift-invariance* (*pass-module n r*)
⟨*proof*⟩

**theorem** *pass-zero-mod-def-zero*[*simp*]:
**fixes** *r* :: *'a Preference-Relation*
**assumes** *linear-order r*
**shows** *defers 0* (*pass-module 0 r*)
⟨*proof*⟩

For any natural number n and any linear order, the according pass module defers n alternatives (if there are n alternatives). NOTE: The induction proof is still missing. The following are the proofs for n=1 and n=2.

**theorem** *pass-one-mod-def-one*[*simp*]:
**fixes** *r* :: *'a Preference-Relation*
**assumes** *linear-order r*
**shows** *defers 1* (*pass-module 1 r*)
⟨*proof*⟩

**theorem** *pass-two-mod-def-two*:
**fixes** *r* :: *'a Preference-Relation*
**assumes** *linear-order r*
**shows** *defers 2* (*pass-module 2 r*)
⟨*proof*⟩

**end**

## 5.11 Elect Module

**theory** *Elect-Module*
  **imports** *Component-Types/Electoral-Module*
**begin**

The elect module is not concerned about the voter's ballots, and just elects all alternatives. It is primarily used in sequence after an electoral module that only defers alternatives to finalize the decision, thereby inducing a proper voting rule in the social choice sense.

### 5.11.1 Definition

**fun** *elect-module* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *elect-module V A p* = (*A*, {}, {})

### 5.11.2 Soundness

**theorem** *elect-mod-sound*[*simp*]: $\mathcal{SCF}$*-result.electoral-module elect-module*
  ⟨*proof*⟩

**lemma** *elect-mod-only-voters*: *voters-determine-election elect-module*
  ⟨*proof*⟩

### 5.11.3 Electing

**theorem** *elect-mod-electing*[*simp*]: *electing elect-module*
  ⟨*proof*⟩

**end**

## 5.12 Plurality Module

**theory** *Plurality-Module*
  **imports** *Component-Types/Elimination-Module*
**begin**

The plurality module implements the plurality voting rule. The plurality rule elects all modules with the maximum amount of top preferences among all alternatives, and rejects all the other alternatives. It is electing and induces the classical plurality (voting) rule from social-choice theory.

### 5.12.1 Definition

**fun** *plurality-score* :: ('*a*, '*v*) *Evaluation-Function* **where**
*plurality-score V x A p* = *win-count V p x*

**fun** *plurality* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*plurality V A p* = *max-eliminator plurality-score V A p*

**fun** *plurality'* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*plurality' V A p* =
(*if finite A*
*then* ({},
{*a* ∈ *A*. ∃ *x* ∈ *A*. *win-count V p x* > *win-count V p a*},
{*a* ∈ *A*. ∀ *x* ∈ *A*. *win-count V p x* ≤ *win-count V p a*})
*else* ({}, {}, *A*))

**lemma** *enat-leq-enat-set-max*:
**fixes**
*x* :: *enat* **and**
*X* :: *enat set*
**assumes**
*x* ∈ *X* **and**
*finite X*
**shows** *x* ≤ *Max X*
⟨*proof*⟩

**lemma** *plurality-mod-equiv*:
**fixes**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**shows** *plurality V A p* = *plurality' V A p*
⟨*proof*⟩

### 5.12.2 Soundness

**theorem** *plurality-sound*[*simp*]: *SCF-result.electoral-module plurality*
⟨*proof*⟩

**theorem** *plurality'-sound*[*simp*]: *SCF-result.electoral-module plurality'*
⟨*proof*⟩

**lemma** *voters-determine-plurality-score*: *voters-determine-evaluation plurality-score*
⟨*proof*⟩

**lemma** *voters-determine-plurality*: *voters-determine-election plurality*
⟨*proof*⟩

**lemma** *voters-determine-plurality'*: *voters-determine-election plurality'*
⟨*proof*⟩

### 5.12.3 Non-Blocking

The plurality module is non-blocking.

**theorem** *plurality-mod-non-blocking*[*simp*]: *non-blocking plurality*
⟨*proof*⟩

**theorem** *plurality'-mod-non-blocking*[*simp*]: *non-blocking plurality'*
⟨*proof*⟩

### 5.12.4 Non-Electing

The plurality module is non-electing.

**theorem** *plurality-non-electing*[*simp*]: *non-electing plurality*
⟨*proof*⟩

**theorem** *plurality'-non-electing*[*simp*]: *non-electing plurality'*
⟨*proof*⟩

### 5.12.5 Property

**lemma** *plurality-def-inv-mono-alts*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes**
*defer-a*: *a* ∈ *defer plurality V A p* **and**
*lift-a*: *lifted V A p q a*
**shows** *defer plurality V A q* = *defer plurality V A p*
∨ *defer plurality V A q* = {*a*}
⟨*proof*⟩

**lemma** *plurality'-def-inv-mono-alts*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes**
*a* ∈ *defer plurality' V A p* **and**
*lifted V A p q a*
**shows** *defer plurality' V A q* = *defer plurality' V A p*
∨ *defer plurality' V A q* = {*a*}
⟨*proof*⟩

The plurality rule is invariant-monotone.

**theorem** *plurality-mod-def-inv-mono*[*simp*]: *defer-invariant-monotonicity plurality*

⟨*proof*⟩

**theorem** *plurality'-mod-def-inv-mono*[*simp*]: *defer-invariant-monotonicity plurality'*
⟨*proof*⟩

**end**

## 5.13 Borda Module

**theory** *Borda-Module*
**imports** *Component-Types/Elimination-Module*
**begin**

This is the Borda module used by the Borda rule. The Borda rule is a voting rule, where on each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

### 5.13.1 Definition

**fun** *borda-score* :: ('*a*, '*v*) *Evaluation-Function* **where**
*borda-score V x A p* = (∑ *y* ∈ *A*. (*prefer-count V p x y*))

**fun** *borda* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*borda V A p* = *max-eliminator borda-score V A p*

### 5.13.2 Soundness

**theorem** *borda-sound*: *SCF-result.electoral-module borda*
⟨*proof*⟩

### 5.13.3 Non-Blocking

The Borda module is non-blocking.

**theorem** *borda-mod-non-blocking*[*simp*]: *non-blocking borda*
⟨*proof*⟩

### 5.13.4 Non-Electing

The Borda module is non-electing.

**theorem** *borda-mod-non-electing*[*simp*]: *non-electing borda*
⟨*proof*⟩

**end**

## 5.14 Condorcet Module

**theory** *Condorcet-Module*
**imports** *Component-Types/Elimination-Module*
**begin**

This is the Condorcet module used by the Condorcet (voting) rule. The Condorcet rule is a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

### 5.14.1 Definition

**fun** *condorcet-score* :: ('*a*, '*v*) *Evaluation-Function* **where**
*condorcet-score V x A p* =
(*if* (*condorcet-winner V A p x*) *then 1 else 0*)

**fun** *condorcet* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*condorcet V A p* = (*max-eliminator condorcet-score*) *V A p*

### 5.14.2 Soundness

**theorem** *condorcet-sound*: $\mathcal{SCF}$*-result.electoral-module condorcet*
⟨*proof*⟩

### 5.14.3 Property

**theorem** *condorcet-score-is-condorcet-rating*: *condorcet-rating condorcet-score*
⟨*proof*⟩

**theorem** *condorcet-is-dcc*: *defer-condorcet-consistency condorcet*
⟨*proof*⟩

**end**

## 5.15 Copeland Module

**theory** *Copeland-Module*
  **imports** *Component-Types/Elimination-Module*
**begin**

This is the Copeland module used by the Copeland voting rule. The Copeland rule elects the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

### 5.15.1 Definition

**fun** *copeland-score* :: ('*a*, '*v*) *Evaluation-Function* **where**
  *copeland-score V x A p* =
    *card* {*y* ∈ *A* . *wins V x p y*} − *card* {*y* ∈ *A* . *wins V y p x*}

**fun** *copeland* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *copeland V A p* = *max-eliminator copeland-score V A p*

### 5.15.2 Soundness

**theorem** *copeland-sound*: $\mathcal{SCF}$*-result.electoral-module copeland*
  ⟨*proof*⟩

### 5.15.3 Lemmas

**lemma** *voters-determine-copeland-score*: *voters-determine-evaluation copeland-score*
⟨*proof*⟩

**theorem** *voters-determine-copeland*: *voters-determine-election copeland*
  ⟨*proof*⟩

For a Condorcet winner w, we have: "|{*y* ∈ *A* . *wins V w p y*}| = |*A*| − *1*".

**lemma** *cond-winner-imp-win-count*:
  **fixes**
    *A* :: '*a set* **and**
    *V* :: '*v set* **and**
    *p* :: ('*a*, '*v*) *Profile* **and**
    *w* :: '*a*
  **assumes** *condorcet-winner V A p w*
  **shows** *card* {*a* ∈ *A*. *wins V w p a*} = *card A* − *1*
⟨*proof*⟩

For a Condorcet winner w, we have: "|{*y* ∈ *A* . *wins V y p w*}| = *0*".

**lemma** *cond-winner-imp-loss-count*:

**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*w* :: *'a*
**assumes** *condorcet-winner V A p w*
**shows** *card* {*a* ∈ *A*. *wins V a p w*} = *0*
⟨*proof*⟩

Copeland score of a Condorcet winner.

**lemma** *cond-winner-imp-copeland-score*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*w* :: *'a*
**assumes** *condorcet-winner V A p w*
**shows** *copeland-score V w A p* = *card A* − *1*
⟨*proof*⟩

For a non-Condorcet winner l, we have: "|{*y* ∈ *A* . *wins V l p y*}| = |*A*| − *2*".

**lemma** *non-cond-winner-imp-win-count*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*w l* :: *'a*
**assumes**
*winner*: *condorcet-winner V A p w* **and**
*loser*: *l* ≠ *w* **and**
*l-in-A*: *l* ∈ *A*
**shows** *card* {*a* ∈ *A* . *wins V l p a*} ≤ *card A* − *2*
⟨*proof*⟩

### 5.15.4 Property

The Copeland score is Condorcet rating.

**theorem** *copeland-score-is-cr*: *condorcet-rating copeland-score*
⟨*proof*⟩

**theorem** *copeland-is-dcc*: *defer-condorcet-consistency copeland*
⟨*proof*⟩

**end**

## 5.16 Minimax Module

**theory** *Minimax-Module*
  **imports** *Component-Types/Elimination-Module*
**begin**

This is the Minimax module used by the Minimax voting rule. The Minimax rule elects the alternatives with the highest Minimax score. The module implemented herein only rejects the alternatives not elected by the voting rule, and defers the alternatives that would be elected by the full voting rule.

### 5.16.1 Definition

**fun** *minimax-score* :: ('*a*, '*v*) *Evaluation-Function* **where**
  *minimax-score V x A p* =
    *Min* {*prefer-count V p x y* | *y* . *y* ∈ *A* − {*x*}}

**fun** *minimax* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *minimax A p* = *max-eliminator minimax-score A p*

### 5.16.2 Soundness

**theorem** *minimax-sound*: $\mathcal{SCF}$*-result.electoral-module minimax*
  ⟨*proof*⟩

### 5.16.3 Lemma

**lemma** *non-cond-winner-minimax-score*:
  **fixes**
    *A* :: '*a set* **and**
    *V* :: '*v set* **and**
    *p* :: ('*a*, '*v*) *Profile* **and**
    *w l* :: '*a*
  **assumes**
    *prof*: *profile V A p* **and**
    *winner*: *condorcet-winner V A p w* **and**
    *l-in-A*: *l* ∈ *A* **and**
    *l-neq-w*: *l* ≠ *w*
  **shows** *minimax-score V l A p* ≤ *prefer-count V p l w*
⟨*proof*⟩

### 5.16.4 Property

**theorem** *minimax-score-cond-rating*: *condorcet-rating minimax-score*
⟨*proof*⟩

**theorem** *minimax-is-dcc*: *defer-condorcet-consistency minimax*
⟨*proof*⟩

**end**

# Chapter 6

# Compositional Structures

## 6.1 Drop- and Pass-Compatibility

**theory** *Drop-And-Pass-Compatibility*
  **imports** *Basic-Modules/Drop-Module*
    *Basic-Modules/Pass-Module*
**begin**

This is a collection of properties about the interplay and compatibility of both the drop module and the pass module.

**theorem** *drop-zero-mod-rej-zero*[*simp*]:
  **fixes** *r* :: *'a Preference-Relation*
  **assumes** *linear-order r*
  **shows** *rejects 0* (*drop-module 0 r*)
⟨*proof*⟩

The drop module rejects n alternatives (if there are at least n alternatives).

**theorem** *drop-two-mod-rej-n*[*simp*]:
  **fixes** *r* :: *'a Preference-Relation*
  **assumes** *linear-order r*
  **shows** *rejects n* (*drop-module n r*)
⟨*proof*⟩

The pass and drop module are (disjoint-)compatible.

**theorem** *drop-pass-disj-compat*[*simp*]:
  **fixes**
    *r* :: *'a Preference-Relation* **and**
    *n* :: *nat*
  **assumes** *linear-order r*
  **shows** *disjoint-compatibility* (*drop-module n r*) (*pass-module n r*)
⟨*proof*⟩

**end**

## 6.2 Revision Composition

**theory** *Revision-Composition*
  **imports** *Basic-Modules/Component-Types/Electoral-Module*
**begin**

A revised electoral module rejects all originally rejected or deferred alternatives, and defers the originally elected alternatives. It does not elect any alternatives.

### 6.2.1 Definition

**fun** *revision-composition* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *revision-composition m V A p* = ({}, *A* − *elect m V A p*, *elect m V A p*)

**abbreviation** *rev* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* (-↓ *50*) **where**
  *m*↓ ≡ *revision-composition m*

### 6.2.2 Soundness

**theorem** *rev-comp-sound*[*simp*]:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *SCF-result.electoral-module m*
  **shows** *SCF-result.electoral-module* (*revision-composition m*)
⟨*proof*⟩

**lemma** *voters-determine-rev-comp*:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *voters-determine-election m*
  **shows** *voters-determine-election* (*revision-composition m*)
  ⟨*proof*⟩

### 6.2.3 Composition Rules

An electoral module received by revision is never electing.

**theorem** *rev-comp-non-electing*[*simp*]:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *SCF-result.electoral-module m*
  **shows** *non-electing* (*m*↓)
  ⟨*proof*⟩

Revising an electing electoral module results in a non-blocking electoral module.

**theorem** *rev-comp-non-blocking*[*simp*]:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *electing m*
  **shows** *non-blocking* (*m*↓)
⟨*proof*⟩

Revising an invariant monotone electoral module results in a defer-invariant-monotone electoral module.

**theorem** *rev-comp-def-inv-mono*[*simp*]:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *invariant-monotonicity m*
  **shows** *defer-invariant-monotonicity* (*m*↓)
⟨*proof*⟩

**end**

## 6.3 Sequential Composition

**theory** *Sequential-Composition*
  **imports** *Basic-Modules/Component-Types/Electoral-Module*
**begin**

The sequential composition creates a new electoral module from two electoral modules. In a sequential composition, the second electoral module makes decisions over alternatives deferred by the first electoral module.

### 6.3.1 Definition

**fun** *sequential-composition* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *sequential-composition m n V A p* =
    (*let new-A* = *defer m V A p*;
      *new-p* = *limit-profile new-A p in* (
          (*elect m V A p*) ∪ (*elect n V new-A new-p*),
          (*reject m V A p*) ∪ (*reject n V new-A new-p*),
          *defer n V new-A new-p*))

**abbreviation** *sequence* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module*
      (**infix** ▷ *50*) **where**
  *m* ▷ *n* ≡ *sequential-composition m n*

**fun** *sequential-composition′* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒

```
    ('a, 'v, 'a Result) Electoral-Module ⇒
    ('a, 'v, 'a Result) Electoral-Module where
  sequential-composition' m n V A p =
    (let (m-e, m-r, m-d) = m V A p; new-A = m-d;
        new-p = limit-profile new-A p;
        (n-e, n-r, n-d) = n V new-A new-p in
          (m-e ∪ n-e, m-r ∪ n-r, n-d))

lemma voters-determine-seq-comp:
  fixes m n :: ('a, 'v, 'a Result) Electoral-Module
  assumes voters-determine-election m ∧ voters-determine-election n
  shows voters-determine-election (m ▷ n)
⟨proof⟩

lemma seq-comp-presv-disj:
  fixes
    m n :: ('a, 'v, 'a Result) Electoral-Module and
    A :: 'a set and
    V :: 'v set and
    p :: ('a, 'v) Profile
  assumes
    module-m: SCF-result.electoral-module m and
    module-n: SCF-result.electoral-module n and
    prof: profile V A p
  shows disjoint3 ((m ▷ n) V A p)
⟨proof⟩

lemma seq-comp-presv-alts:
  fixes
    m n :: ('a, 'v, 'a Result) Electoral-Module and
    A :: 'a set and
    V :: 'v set and
    p :: ('a, 'v) Profile
  assumes
    module-m: SCF-result.electoral-module m and
    module-n: SCF-result.electoral-module n and
    prof: profile V A p
  shows set-equals-partition A ((m ▷ n) V A p)
⟨proof⟩

lemma seq-comp-alt-eq[fundef-cong, code]: sequential-composition = sequential-composition'
⟨proof⟩
```

### 6.3.2 Soundness

```
theorem seq-comp-sound[simp]:
  fixes m n :: ('a, 'v, 'a Result) Electoral-Module
  assumes
    SCF-result.electoral-module m and
```

*SCF-result.electoral-module n*
**shows** *SCF-result.electoral-module* $(m \triangleright n)$
⟨*proof*⟩

### 6.3.3 Lemmas

**lemma** *seq-comp-decrease-only-defer*:
**fixes**
$m\ n$ :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
$A$ :: $'a$ *set* **and**
$V$ :: $'v$ *set* **and**
$p$ :: $('a, 'v)$ *Profile*
**assumes**
*module-m*: *SCF-result.electoral-module* $m$ **and**
*module-n*: *SCF-result.electoral-module* $n$ **and**
*prof*: *profile* $V\ A\ p$ **and**
*empty-defer*: *defer* $m\ V\ A\ p = \{\}$
**shows** $(m \triangleright n)\ V\ A\ p = m\ V\ A\ p$
⟨*proof*⟩

**lemma** *seq-comp-def-then-elect*:
**fixes**
$m\ n$ :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
$A$ :: $'a$ *set* **and**
$V$ :: $'v$ *set* **and**
$p$ :: $('a, 'v)$ *Profile*
**assumes**
*n-electing-m*: *non-electing* $m$ **and**
*def-one-m*: *defers 1* $m$ **and**
*electing-n*: *electing* $n$ **and**
*f-prof*: *finite-profile* $V\ A\ p$
**shows** *elect* $(m \triangleright n)\ V\ A\ p = defer\ m\ V\ A\ p$
⟨*proof*⟩

**lemma** *seq-comp-def-card-bounded*:
**fixes**
$m\ n$ :: $('a, 'v, 'a\ Result)$ *Electoral-Module* **and**
$A$ :: $'a$ *set* **and**
$V$ :: $'v$ *set* **and**
$p$ :: $('a, 'v)$ *Profile*
**assumes**
*SCF-result.electoral-module* $m$ **and**
*SCF-result.electoral-module* $n$ **and**
*finite-profile* $V\ A\ p$
**shows** *card* $(defer\ (m \triangleright n)\ V\ A\ p) \leq card\ (defer\ m\ V\ A\ p)$
⟨*proof*⟩

**lemma** *seq-comp-def-set-bounded*:
**fixes**

*m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*SCF-result.electoral-module m* **and**
*SCF-result.electoral-module n* **and**
*profile V A p*
**shows** *defer* (*m* ▷ *n*) *V A p* ⊆ *defer m V A p*
⟨*proof*⟩

**lemma** *seq-comp-defers-def-set*:
**fixes**
*m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**shows** *defer* (*m* ▷ *n*) *V A p* =
*defer n V* (*defer m V A p*) (*limit-profile* (*defer m V A p*) *p*)
⟨*proof*⟩

**lemma** *seq-comp-def-then-elect-elec-set*:
**fixes**
*m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**shows** *elect* (*m* ▷ *n*) *V A p* =
*elect n V* (*defer m V A p*)
(*limit-profile* (*defer m V A p*) *p*) ∪ (*elect m V A p*)
⟨*proof*⟩

**lemma** *seq-comp-elim-one-red-def-set*:
**fixes**
*m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile*
**assumes**
*SCF-result.electoral-module m* **and**
*eliminates 1 n* **and**
*profile V A p* **and**
*card* (*defer m V A p*) > *1*
**shows** *defer* (*m* ▷ *n*) *V A p* ⊂ *defer m V A p*
⟨*proof*⟩

**lemma** *seq-comp-def-set-trans*:
**fixes**
*m n* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**

$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
$a \in (defer\ (m \triangleright n)\ V\ A\ p)$ **and**
$\mathcal{SCF}$*-result.electoral-module* $m \wedge \mathcal{SCF}$*-result.electoral-module* $n$ **and**
*profile* $V\ A\ p$
**shows** $a \in defer\ n\ V\ (defer\ m\ V\ A\ p)\ (limit\text{-}profile\ (defer\ m\ V\ A\ p)\ p) \wedge$
$a \in defer\ m\ V\ A\ p$
⟨*proof*⟩

### 6.3.4 Composition Rules

The sequential composition preserves the non-blocking property.

**theorem** *seq-comp-presv-non-blocking*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*non-blocking-m*: *non-blocking* $m$ **and**
*non-blocking-n*: *non-blocking* $n$
**shows** *non-blocking* $(m \triangleright n)$
⟨*proof*⟩

Sequential composition preserves the non-electing property.

**theorem** *seq-comp-presv-non-electing*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*non-electing* $m$ **and**
*non-electing* $n$
**shows** *non-electing* $(m \triangleright n)$
⟨*proof*⟩

Composing an electoral module that defers exactly 1 alternative in sequence after an electoral module that is electing results (still) in an electing electoral module.

**theorem** *seq-comp-electing*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*def-one-m*: *defers 1* $m$ **and**
*electing-n*: *electing* $n$
**shows** *electing* $(m \triangleright n)$
⟨*proof*⟩

**lemma** *def-lift-inv-seq-comp-help*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**

$p\ q :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*monotone-m*: *defer-lift-invariance* $m$ **and**
*monotone-n*: *defer-lift-invariance* $n$ **and**
*voters-determine-n*: *voters-determine-election* $n$ **and**
*def-and-lifted*: $a \in (defer\ (m \triangleright n)\ V\ A\ p) \wedge lifted\ V\ A\ p\ q\ a$
**shows** $(m \triangleright n)\ V\ A\ p = (m \triangleright n)\ V\ A\ q$
⟨*proof*⟩

Sequential composition preserves the property defer-lift-invariance.

**theorem** *seq-comp-presv-def-lift-inv*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*defer-lift-invariance* $m$ **and**
*defer-lift-invariance* $n$ **and**
*voters-determine-election* $n$
**shows** *defer-lift-invariance* $(m \triangleright n)$
⟨*proof*⟩

Composing a non-blocking, non-electing electoral module in sequence with an electoral module that defers exactly one alternative results in an electoral module that defers exactly one alternative.

**theorem** *seq-comp-def-one*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*non-blocking-m*: *non-blocking* $m$ **and**
*non-electing-m*: *non-electing* $m$ **and**
*def-one-n*: *defers 1* $n$
**shows** *defers 1* $(m \triangleright n)$
⟨*proof*⟩

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

**theorem** *disj-compat-seq*[*simp*]:
**fixes** $m\ m'\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*compatible*: *disjoint-compatibility* $m\ n$ **and**
*module-m'*: $\mathcal{SCF}$*-result.electoral-module* $m'$ **and**
*voters-determine-m'*: *voters-determine-election* $m'$
**shows** *disjoint-compatibility* $(m \triangleright m')\ n$
⟨*proof*⟩

**theorem** *seq-comp-cond-compat*[*simp*]:
**fixes** $m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$
**assumes**
*dcc-m*: *defer-condorcet-consistency* $m$ **and**

*nb-n*: *non-blocking n* **and**
*ne-n*: *non-electing n*
**shows** *condorcet-compatibility* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-condorcet-consistent electoral module in sequence with a non-blocking and non-electing electoral module results in a defer-condorcet-consistent module.

**theorem** *seq-comp-dcc*[*simp*]:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes**
*dcc-m*: *defer-condorcet-consistency m* **and**
*nb-n*: *non-blocking n* **and**
*ne-n*: *non-electing n*
**shows** *defer-condorcet-consistency* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-lift invariant and a non-electing electoral module that defers exactly one alternative in sequence with an electing electoral module results in a monotone electoral module.

**theorem** *seq-comp-mono*[*simp*]:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes**
*def-monotone-m*: *defer-lift-invariance m* **and**
*non-ele-m*: *non-electing m* **and**
*def-one-m*: *defers 1 m* **and**
*electing-n*: *electing n*
**shows** *monotonicity* (*m* ▷ *n*)
⟨*proof*⟩

Composing a defer-invariant-monotone electoral module in sequence before a non-electing, defer-monotone electoral module that defers exactly 1 alternative results in a defer-lift-invariant electoral module.

**theorem** *def-inv-mono-imp-def-lift-inv*[*simp*]:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes**
*strong-def-mon-m*: *defer-invariant-monotonicity m* **and**
*non-electing-n*: *non-electing n* **and**
*defers-one*: *defers 1 n* **and**
*defer-monotone-n*: *defer-monotonicity n* **and**
*voters-determine-n*: *voters-determine-election n*
**shows** *defer-lift-invariance* (*m* ▷ *n*)
⟨*proof*⟩

**end**

## 6.4 Parallel Composition

**theory** *Parallel-Composition*
  **imports** *Basic-Modules/Component-Types/Aggregator*
    *Basic-Modules/Component-Types/Electoral-Module*
**begin**

The parallel composition composes a new electoral module from two electoral modules combined with an aggregator. Therein, the two modules each make a decision and the aggregator combines them to a single (aggregated) result.

### 6.4.1 Definition

**fun** *parallel-composition* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
    ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*a Aggregator* ⇒
    ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *parallel-composition m n agg V A p* = *agg A* (*m V A p*) (*n V A p*)

**abbreviation** *parallel* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*a Aggregator* ⇒
    ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module*
    (- $\|_-$ - [*50*, *1000*, *51*] *50*) **where**
  *m* $\|_a$ *n* ≡ *parallel-composition m n a*

### 6.4.2 Soundness

**theorem** *par-comp-sound*[*simp*]:
  **fixes**
    *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
    *a* :: '*a Aggregator*
  **assumes**
    $\mathcal{SCF}$*-result.electoral-module m* **and**
    $\mathcal{SCF}$*-result.electoral-module n* **and**
    *aggregator a*
  **shows** $\mathcal{SCF}$*-result.electoral-module* (*m* $\|_a$ *n*)
⟨*proof*⟩

### 6.4.3 Composition Rule

Using a conservative aggregator, the parallel composition preserves the property non-electing.

**theorem** *conserv-agg-presv-non-electing*[*simp*]:
  **fixes**
    *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
    *a* :: '*a Aggregator*
  **assumes**
    *non-electing-m*: *non-electing m* **and**

*non-electing-n*: *non-electing n* **and**
*conservative*: *agg-conservative a*
**shows** *non-electing* (*m* $\|_a$ *n*)
⟨*proof*⟩

**end**

## 6.5 Loop Composition

**theory** *Loop-Composition*
**imports** *Basic-Modules/Component-Types/Termination-Condition*
*Basic-Modules/Defer-Module*
*Sequential-Composition*
**begin**

The loop composition uses the same module in sequence, combined with a termination condition, until either

- the termination condition is met or
- no new decisions are made (i.e., a fixed point is reached).

### 6.5.1 Definition

**lemma** *loop-termination-helper*:
**fixes**
*m acc* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*t* :: '*a Termination-Condition* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**assumes**
¬ *t* (*acc V A p*) **and**
*defer* (*acc* ▷ *m*) *V A p* ⊂ *defer acc V A p* **and**
*finite* (*defer acc V A p*)
**shows** ((*acc* ▷ *m*, *m*, *t*, *V*, *A*, *p*), (*acc*, *m*, *t*, *V*, *A*, *p*)) ∈
*measure* (λ (*acc*, *m*, *t*, *V*, *A*, *p*). *card* (*defer acc V A p*))
⟨*proof*⟩

This function handles the accumulator for the following loop composition function.

**function** *loop-comp-helper* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒ '*a Termination-Condition* ⇒

```
    ('a, 'v, 'a Result) Electoral-Module where
  loop-comp-helper-finite:
  finite (defer acc V A p) ∧ (defer (acc ▷ m) V A p) ⊂ (defer acc V A p)
    ⟶ t (acc V A p) ⟹
  loop-comp-helper acc m t V A p = acc V A p |
  loop-comp-helper-infinite:
  ¬ (finite (defer acc V A p) ∧ (defer (acc ▷ m) V A p) ⊂ (defer acc V A p)
    ⟶ t (acc V A p)) ⟹
  loop-comp-helper acc m t V A p = loop-comp-helper (acc ▷ m) m t V A p
⟨proof⟩
termination
⟨proof⟩

lemma loop-comp-code-helper[code]:
  fixes
    m acc :: ('a, 'v, 'a Result) Electoral-Module and
    t :: 'a Termination-Condition and
    A :: 'a set and
    V :: 'v set and
    p :: ('a, 'v) Profile
  shows
    loop-comp-helper acc m t V A p =
      (if (t (acc V A p) ∨ ¬ ((defer (acc ▷ m) V A p) ⊂ (defer acc V A p))
      ∨ infinite (defer acc V A p))
      then (acc V A p) else (loop-comp-helper (acc ▷ m) m t V A p))
  ⟨proof⟩

function loop-composition :: ('a, 'v, 'a Result) Electoral-Module ⇒
      'a Termination-Condition ⇒ ('a, 'v, 'a Result) Electoral-Module where
  t ({}, {}, A)
    ⟹ loop-composition m t V A p = defer-module V A p |
  ¬(t ({}, {}, A))
    ⟹ loop-composition m t V A p = (loop-comp-helper m m t) V A p
  ⟨proof⟩
termination
  ⟨proof⟩

abbreviation loop :: ('a, 'v, 'a Result) Electoral-Module ⇒
      'a Termination-Condition ⇒ ('a, 'v, 'a Result) Electoral-Module
      (- ↻- 50) where
  m ↻t ≡ loop-composition m t

lemma loop-comp-code[code]:
  fixes
    m :: ('a, 'v, 'a Result) Electoral-Module and
    t :: 'a Termination-Condition and
    A :: 'a set and
    V :: 'v set and
    p :: ('a, 'v) Profile
```

**shows** *loop-composition m t V A p* =
(*if* (*t* ({}, {}, *A*))
*then* (*defer-module V A p*) *else* (*loop-comp-helper m m t*) *V A p*)
⟨*proof*⟩

**lemma** *loop-comp-helper-imp-partit*:
**fixes**
*m acc* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*t* :: '*a Termination-Condition* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*n* :: *nat*
**assumes**
*module-m*: $\mathcal{SCF}$*-result.electoral-module m* **and**
*profile*: *profile V A p* **and**
*module-acc*: $\mathcal{SCF}$*-result.electoral-module acc* **and**
*defer-card-n*: *n* = *card* (*defer acc V A p*)
**shows** *well-formed-*$\mathcal{SCF}$ *A* (*loop-comp-helper acc m t V A p*)
⟨*proof*⟩

### 6.5.2 Soundness

**theorem** *loop-comp-sound*:
**fixes**
*m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*t* :: '*a Termination-Condition*
**assumes** $\mathcal{SCF}$*-result.electoral-module m*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*m* $\circlearrowleft_t$)
⟨*proof*⟩

**lemma** *loop-comp-helper-imp-no-def-incr*:
**fixes**
*m acc* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*t* :: '*a Termination-Condition* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*n* :: *nat*
**assumes**
*module-m*: $\mathcal{SCF}$*-result.electoral-module m* **and**
*profile*: *profile V A p* **and**
*mod-acc*: $\mathcal{SCF}$*-result.electoral-module acc* **and**
*card-n-defer-acc*: *n* = *card* (*defer acc V A p*)
**shows** *defer* (*loop-comp-helper acc m t*) *V A p* ⊆ *defer acc V A p*
⟨*proof*⟩

### 6.5.3 Lemmas

**lemma** *loop-comp-helper-def-lift-inv-helper*:

**fixes**
*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*n* :: *nat*
**assumes**
*monotone-m*: *defer-lift-invariance m* **and**
*prof*: *profile V A p* **and**
*dli-acc*: *defer-lift-invariance acc* **and**
*card-n-defer*: *n* = *card* (*defer acc V A p*) **and**
*defer-finite*: *finite* (*defer acc V A p*) **and**
*voters-determine-m*: *voters-determine-election m*
**shows**
∀ *q a*. *a* ∈ (*defer* (*loop-comp-helper acc m t*) *V A p*) ∧ *lifted V A p q a* ⟶
(*loop-comp-helper acc m t*) *V A p* = (*loop-comp-helper acc m t*) *V A q*
⟨*proof*⟩

**lemma** *loop-comp-helper-def-lift-inv*:
**fixes**
*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes**
*defer-lift-invariance m* **and**
*voters-determine-election m* **and**
*defer-lift-invariance acc* **and**
*profile V A p* **and**
*lifted V A p q a* **and**
*a* ∈ *defer* (*loop-comp-helper acc m t*) *V A p*
**shows** (*loop-comp-helper acc m t*) *V A p* = (*loop-comp-helper acc m t*) *V A q*
⟨*proof*⟩

**lemma** *lifted-imp-fin-prof*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes** *lifted V A p q a*
**shows** *finite-profile V A p*
⟨*proof*⟩

**lemma** *loop-comp-helper-presv-def-lift-inv*:
**fixes**

*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition*
**assumes**
*defer-lift-invariance m* **and**
*voters-determine-election m* **and**
*defer-lift-invariance acc*
**shows** *defer-lift-invariance* (*loop-comp-helper acc m t*)
⟨*proof*⟩

**lemma** *loop-comp-presv-non-electing-helper*:
**fixes**
*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*n* :: *nat*
**assumes**
*non-electing-m*: *non-electing m* **and**
*non-electing-acc*: *non-electing acc* **and**
*prof*: *profile V A p* **and**
*acc-defer-card*: *n* = *card* (*defer acc V A p*)
**shows** *elect* (*loop-comp-helper acc m t*) *V A p* = {}
⟨*proof*⟩

**lemma** *loop-comp-helper-iter-elim-def-n-helper*:
**fixes**
*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*n x* :: *nat*
**assumes**
*non-electing-m*: *non-electing m* **and**
*single-elimination*: *eliminates 1 m* **and**
*terminate-if-n-left*: ∀ *r*. *t r* = (*card* (*defer-r r*) = *x*) **and**
*x-greater-zero*: *x* > *0* **and**
*prof*: *profile V A p* **and**
*n-acc-defer-card*: *n* = *card* (*defer acc V A p*) **and**
*n-ge-x*: *n* ≥ *x* **and**
*def-card-gt-one*: *card* (*defer acc V A p*) > *1* **and**
*acc-nonelect*: *non-electing acc*
**shows** *card* (*defer* (*loop-comp-helper acc m t*) *V A p*) = *x*
⟨*proof*⟩

**lemma** *loop-comp-helper-iter-elim-def-n*:
**fixes**

*m acc* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*x* :: *nat*
**assumes**
*non-electing m* **and**
*eliminates 1 m* **and**
∀ *r*. (*t r*) = (*card* (*defer-r r*) = *x*) **and**
*x* > *0* **and**
*profile V A p* **and**
*card* (*defer acc V A p*) ≥ *x* **and**
*non-electing acc*
**shows** *card* (*defer* (*loop-comp-helper acc m t*) *V A p*) = *x*
⟨*proof*⟩

**lemma** *iter-elim-def-n-helper*:
**fixes**
*m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition* **and**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p* :: (*'a*, *'v*) *Profile* **and**
*x* :: *nat*
**assumes**
*non-electing-m*: *non-electing m* **and**
*single-elimination*: *eliminates 1 m* **and**
*terminate-if-n-left*: ∀ *r*. (*t r*) = (*card* (*defer-r r*) = *x*) **and**
*x-greater-zero*: *x* > *0* **and**
*prof*: *profile V A p* **and**
*enough-alternatives*: *card A* ≥ *x*
**shows** *card* (*defer* (*m* ↻$_t$) *V A p*) = *x*
⟨*proof*⟩

### 6.5.4 Composition Rules

The loop composition preserves defer-lift-invariance.

**theorem** *loop-comp-presv-def-lift-inv*[*simp*]:
**fixes**
*m* :: (*'a*, *'v*, *'a Result*) *Electoral-Module* **and**
*t* :: *'a Termination-Condition*
**assumes**
*defer-lift-invariance m* **and**
*voters-determine-election m*
**shows** *defer-lift-invariance* (*m* ↻$_t$)
⟨*proof*⟩

The loop composition preserves the property non-electing.

**theorem** *loop-comp-presv-non-electing*[*simp*]:
  **fixes**
    $m$ :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
    $t$ :: '*a Termination-Condition*
  **assumes** *non-electing m*
  **shows** *non-electing* ($m$ $\circlearrowleft_t$)
⟨*proof*⟩

**theorem** *iter-elim-def-n*[*simp*]:
  **fixes**
    $m$ :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
    $t$ :: '*a Termination-Condition* **and**
    $n$ :: *nat*
  **assumes**
    *non-electing-m*: *non-electing m* **and**
    *single-elimination*: *eliminates 1 m* **and**
    *terminate-if-n-left*: $\forall\ r.\ t\ r = (card\ (defer\text{-}r\ r) = n)$ **and**
    *x-greater-zero*: $n > 0$
  **shows** *defers n* ($m$ $\circlearrowleft_t$)
⟨*proof*⟩

**end**

## 6.6 Maximum Parallel Composition

**theory** *Maximum-Parallel-Composition*
  **imports** *Basic-Modules/Component-Types/Maximum-Aggregator*
      *Parallel-Composition*
**begin**

This is a family of parallel compositions. It composes a new electoral module from two electoral modules combined with the maximum aggregator. Therein, the two modules each make a decision and then a partition is returned where every alternative receives the maximum result of the two input partitions. This means that, if any alternative is elected by at least one of the modules, then it gets elected, if any non-elected alternative is deferred by at least one of the modules, then it gets deferred, only alternatives rejected by both modules get rejected.

### 6.6.1 Definition

**fun** *maximum-parallel-composition* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**

*maximum-parallel-composition m n* =
(*let a* = *max-aggregator in* (*m* $\|_a$ *n*))

**abbreviation** *max-parallel* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* (**infix** $\|_\uparrow$ *50*) **where**
*m* $\|_\uparrow$ *n* ≡ *maximum-parallel-composition m n*

### 6.6.2 Soundness

**theorem** *max-par-comp-sound*:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes**
$\mathcal{SCF}$*-result.electoral-module m* **and**
$\mathcal{SCF}$*-result.electoral-module n*
**shows** $\mathcal{SCF}$*-result.electoral-module* (*m* $\|_\uparrow$ *n*)
⟨*proof*⟩

**lemma** *voters-determine-max-par-comp*:
**fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes**
*voters-determine-election m* **and**
*voters-determine-election n*
**shows** *voters-determine-election* (*m* $\|_\uparrow$ *n*)
⟨*proof*⟩

### 6.6.3 Lemmas

**lemma** *max-agg-eq-result*:
**fixes**
*m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a* :: '*a*
**assumes**
*module-m*: $\mathcal{SCF}$*-result.electoral-module m* **and**
*module-n*: $\mathcal{SCF}$*-result.electoral-module n* **and**
*prof-p*: *profile V A p* **and**
*a-in-A*: *a* ∈ *A*
**shows** *mod-contains-result* (*m* $\|_\uparrow$ *n*) *m V A p a* ∨
*mod-contains-result* (*m* $\|_\uparrow$ *n*) *n V A p a*
⟨*proof*⟩

**lemma** *max-agg-rej-iff-both-reject*:
**fixes**
*m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**

$a :: 'a$
**assumes**
*finite-profile* $V\ A\ p$ **and**
$\mathcal{SCF}$*-result.electoral-module* $m$ **and**
$\mathcal{SCF}$*-result.electoral-module* $n$
**shows** $(a \in reject\ (m \parallel_{\uparrow} n)\ V\ A\ p) =$
$(a \in reject\ m\ V\ A\ p \wedge a \in reject\ n\ V\ A\ p)$
⟨*proof*⟩

**lemma** *max-agg-rej-fst-imp-seq-contained*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*f-prof*: *finite-profile* $V\ A\ p$ **and**
*module-m*: $\mathcal{SCF}$*-result.electoral-module* $m$ **and**
*module-n*: $\mathcal{SCF}$*-result.electoral-module* $n$ **and**
*rejected*: $a \in reject\ n\ V\ A\ p$
**shows** *mod-contains-result* $m\ (m \parallel_{\uparrow} n)\ V\ A\ p\ a$
⟨*proof*⟩

**lemma** *max-agg-rej-fst-equiv-seq-contained*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*finite-profile* $V\ A\ p$ **and**
$\mathcal{SCF}$*-result.electoral-module* $m$ **and**
$\mathcal{SCF}$*-result.electoral-module* $n$ **and**
$a \in reject\ n\ V\ A\ p$
**shows** *mod-contains-result-sym* $(m \parallel_{\uparrow} n)\ m\ V\ A\ p\ a$
⟨*proof*⟩

**lemma** *max-agg-rej-snd-imp-seq-contained*:
**fixes**
$m\ n :: ('a, 'v, 'a\ Result)\ Electoral\text{-}Module$ **and**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$ **and**
$a :: 'a$
**assumes**
*f-prof*: *finite-profile* $V\ A\ p$ **and**
*module-m*: $\mathcal{SCF}$*-result.electoral-module* $m$ **and**

*module-n*: $\mathcal{SCF}$*-result.electoral-module n* **and**
*rejected*: *a* ∈ *reject m V A p*
**shows** *mod-contains-result n* (*m* ‖↑ *n*) *V A p a*
⟨*proof*⟩

**lemma** *max-agg-rej-snd-equiv-seq-contained*:
**fixes**
*m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile* **and**
*a* :: '*a*
**assumes**
*finite-profile V A p* **and**
$\mathcal{SCF}$*-result.electoral-module m* **and**
$\mathcal{SCF}$*-result.electoral-module n* **and**
*a* ∈ *reject m V A p*
**shows** *mod-contains-result-sym* (*m* ‖↑ *n*) *n V A p a*
⟨*proof*⟩

**lemma** *max-agg-rej-intersect*:
**fixes**
*m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*p* :: ('*a*, '*v*) *Profile*
**assumes**
$\mathcal{SCF}$*-result.electoral-module m* **and**
$\mathcal{SCF}$*-result.electoral-module n* **and**
*profile V A p* **and**
*finite A*
**shows** *reject* (*m* ‖↑ *n*) *V A p* = (*reject m V A p*) ∩ (*reject n V A p*)
⟨*proof*⟩

**lemma** *dcompat-dec-by-one-mod*:
**fixes**
*m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
*A* :: '*a set* **and**
*V* :: '*v set* **and**
*a* :: '*a*
**assumes**
*disjoint-compatibility m n* **and**
*a* ∈ *A*
**shows**
(∀ *p*. *finite-profile V A p* ⟶ *mod-contains-result m* (*m* ‖↑ *n*) *V A p a*)
∨ (∀ *p*. *finite-profile V A p* ⟶ *mod-contains-result n* (*m* ‖↑ *n*) *V A p a*)
⟨*proof*⟩

### 6.6.4 Composition Rules

Using a conservative aggregator, the parallel composition preserves the property non-electing.

**theorem** *conserv-max-agg-presv-non-electing*[*simp*]:
  **fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes**
    *non-electing m* **and**
    *non-electing n*
  **shows** *non-electing* (*m* $\|_{\uparrow}$ *n*)
  ⟨*proof*⟩

Using the max aggregator, composing two compatible electoral modules in parallel preserves defer-lift-invariance.

**theorem** *par-comp-def-lift-inv*[*simp*]:
  **fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes**
    *compatible*: *disjoint-compatibility m n* **and**
    *monotone-m*: *defer-lift-invariance m* **and**
    *monotone-n*: *defer-lift-invariance n*
  **shows** *defer-lift-invariance* (*m* $\|_{\uparrow}$ *n*)
⟨*proof*⟩

**lemma** *par-comp-rej-card*:
  **fixes**
    *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **and**
    *A* :: '*a set* **and**
    *V* :: '*v set* **and**
    *p* :: ('*a*, '*v*) *Profile* **and**
    *c* :: *nat*
  **assumes**
    *compatible*: *disjoint-compatibility m n* **and**
    *prof*: *profile V A p* **and**
    *fin-A*: *finite A* **and**
    *reject-sum*: *card* (*reject m V A p*) + *card* (*reject n V A p*) = *card A* + *c*
  **shows** *card* (*reject* (*m* $\|_{\uparrow}$ *n*) *V A p*) = *c*
⟨*proof*⟩

Using the max-aggregator for composing two compatible modules in parallel, whereof the first one is non-electing and defers exactly one alternative, and the second one rejects exactly two alternatives, the composition results in an electoral module that eliminates exactly one alternative.

**theorem** *par-comp-elim-one*[*simp*]:
  **fixes** *m n* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes**
    *defers-m-one*: *defers 1 m* **and**
    *non-elec-m*: *non-electing m* **and**
    *rejec-n-two*: *rejects 2 n* **and**

*disj-comp*: *disjoint-compatibility m n*
**shows** *eliminates 1* (*m* $\|_{\uparrow}$ *n*)
⟨*proof*⟩

**end**

## 6.7 Elect Composition

**theory** *Elect-Composition*
**imports** *Basic-Modules/Elect-Module*
*Sequential-Composition*
**begin**

The elect composition sequences an electoral module and the elect module. It finalizes the module's decision as it simply elects all their non-rejected alternatives. Thereby, any such elect-composed module induces a proper voting rule in the social choice sense, as all alternatives are either rejected or elected.

### 6.7.1 Definition

**fun** *elector* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*elector m* = (*m* ▷ *elect-module*)

### 6.7.2 Auxiliary Lemmas

**lemma** *elector-seqcomp-assoc*:
**fixes** *a b* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**shows** (*a* ▷ (*elector b*)) = (*elector* (*a* ▷ *b*))
⟨*proof*⟩

### 6.7.3 Soundness

**theorem** *elector-sound*[*simp*]:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes** $\mathcal{SCF}$-*result.electoral-module m*
**shows** $\mathcal{SCF}$-*result.electoral-module* (*elector m*)
⟨*proof*⟩

**lemma** *voters-determine-elector*:
**fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
**assumes** *voters-determine-election m*
**shows** *voters-determine-election* (*elector m*)
⟨*proof*⟩

### 6.7.4 Electing

**theorem** *elector-electing*[*simp*]:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes**
    *module-m*: $\mathcal{SCF}$*-result.electoral-module m* **and**
    *non-block-m*: *non-blocking m*
  **shows** *electing* (*elector m*)
⟨*proof*⟩

### 6.7.5 Composition Rule

If m is defer-Condorcet-consistent, then elector(m) is Condorcet consistent.

**lemma** *dcc-imp-cc-elector*:
  **fixes** *m* :: ('*a*, '*v*, '*a Result*) *Electoral-Module*
  **assumes** *defer-condorcet-consistency m*
  **shows** *condorcet-consistency* (*elector m*)
⟨*proof*⟩

**end**

## 6.8 Defer-One Loop Composition

**theory** *Defer-One-Loop-Composition*
  **imports** *Basic-Modules*/*Component-Types*/*Defer-Equal-Condition*
        *Loop-Composition*
        *Elect-Composition*
**begin**

This is a family of loop compositions. It uses the same module in sequence until either no new decisions are made or only one alternative is remaining in the defer-set. The second family herein uses the above family and subsequently elects the remaining alternative.

**fun** *iter* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *iter m* =
    (*let t* = *defer-equal-condition 1 in*
      (*m* ↻$_t$))

**abbreviation** *defer-one-loop* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒
      ('*a*, '*v*, '*a Result*) *Electoral-Module* (-↻$_{\exists\,!d}$ *50*) **where**
  *m* ↻$_{\exists\,!d}$ ≡ *iter m*

**fun** *iter-elect* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* ⇒

('a, 'v, 'a Result) Electoral-Module **where**
*iter-elect* $m$ = *elector* $(m\ \circlearrowleft_{\exists\,!d})$

### 6.8.1 Soundness

**theorem** *defer-one-loop-comp-sound*:
**fixes**
$m$ :: ('a, 'v, 'a Result) Electoral-Module **and**
$t$ :: 'a Termination-Condition
**assumes** $\mathcal{SCF}$-result.electoral-module $m$
**shows** $\mathcal{SCF}$-result.electoral-module $(m\ \circlearrowleft_{\exists\,!d})$
⟨*proof*⟩

**end**

# Chapter 7

# Voting Rules

## 7.1 Plurality Rule

**theory** *Plurality-Rule*
  **imports** *Compositional-Structures/Basic-Modules/Plurality-Module*
    *Compositional-Structures/Revision-Composition*
    *Compositional-Structures/Elect-Composition*
**begin**

This is a definition of the plurality voting rule as elimination module as well as directly. In the former one, the max operator of the set of the scores of all alternatives is evaluated and is used as the threshold value.

### 7.1.1 Definition

**fun** *plurality-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *plurality-rule V A p* = *elector plurality V A p*

**fun** *plurality-rule'* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *plurality-rule' V A p* =
    (*if finite A*
      *then* ({*a* ∈ *A*. ∀ *x* ∈ *A*. *win-count V p x* ≤ *win-count V p a*},
          {*a* ∈ *A*. ∃ *x* ∈ *A*. *win-count V p x* > *win-count V p a*},
          {})
      *else* (*A*, {}, {}))

**lemma** *plurality-revision-equiv*:
  **fixes**
    *A* :: '*a set* **and**
    *V* :: '*v set* **and**
    *p* :: ('*a*, '*v*) *Profile*
  **shows** *plurality V A p* = (*plurality-rule*↓) *V A p*
  ⟨*proof*⟩

**lemma** *plurality'-revision-equiv*:

**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $plurality'\ V\ A\ p = (plurality\text{-}rule'{\downarrow})\ V\ A\ p$
⟨*proof*⟩

**lemma** *plurality-rule-equiv*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**shows** $plurality\text{-}rule\ V\ A\ p = plurality\text{-}rule'\ V\ A\ p$
⟨*proof*⟩

### 7.1.2 Soundness

**theorem** *plurality-rule-sound*[*simp*]: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ plurality\text{-}rule$
⟨*proof*⟩

**theorem** *plurality-rule′-sound*[*simp*]: $\mathcal{SCF}\text{-}result.electoral\text{-}module\ plurality\text{-}rule'$
⟨*proof*⟩

**lemma** *voters-determine-plurality-rule*: *voters-determine-election plurality-rule*
⟨*proof*⟩

**lemma** *voters-determine-plurality-rule′*: *voters-determine-election plurality-rule′*
⟨*proof*⟩

### 7.1.3 Electing

**lemma** *plurality-rule-elect-non-empty*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$A \neq \{\}$ **and**
*finite A*
**shows** *elect plurality-rule V A p* $\neq \{\}$
⟨*proof*⟩

**lemma** *plurality-rule′-elect-non-empty*:
**fixes**
$A :: 'a\ set$ **and**
$V :: 'v\ set$ **and**
$p :: ('a, 'v)\ Profile$
**assumes**
$A \neq \{\}$ **and**
*profile V A p* **and**

*finite A*
**shows** *elect plurality-rule' V A p* ≠ {}
⟨*proof*⟩

The plurality module is electing.

**theorem** *plurality-rule-electing*[*simp*]: *electing plurality-rule*
⟨*proof*⟩

**theorem** *plurality-rule'-electing*[*simp*]: *electing plurality-rule'*
⟨*proof*⟩

### 7.1.4 Properties

**lemma** *plurality-rule-inv-mono-eq*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes**
*elect-a*: *a* ∈ *elect plurality-rule V A p* **and**
*lift-a*: *lifted V A p q a*
**shows** *elect plurality-rule V A q* = *elect plurality-rule V A p*
∨ *elect plurality-rule V A q* = {*a*}
⟨*proof*⟩

**lemma** *plurality-rule'-inv-mono-eq*:
**fixes**
*A* :: *'a set* **and**
*V* :: *'v set* **and**
*p q* :: (*'a*, *'v*) *Profile* **and**
*a* :: *'a*
**assumes**
*a* ∈ *elect plurality-rule' V A p* **and**
*lifted V A p q a*
**shows** *elect plurality-rule' V A q* = *elect plurality-rule' V A p*
∨ *elect plurality-rule' V A q* = {*a*}
⟨*proof*⟩

The plurality rule is invariant-monotone.

**theorem** *plurality-rule-inv-mono*[*simp*]: *invariant-monotonicity plurality-rule*
⟨*proof*⟩

**theorem** *plurality-rule'-inv-mono*[*simp*]: *invariant-monotonicity plurality-rule'*
⟨*proof*⟩

**end**

## 7.2 Borda Rule

**theory** *Borda-Rule*
**imports** *Compositional-Structures/Basic-Modules/Borda-Module*
*Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization*
*Compositional-Structures/Elect-Composition*
**begin**

This is the Borda rule. On each ballot, each alternative is assigned a score that depends on how many alternatives are ranked below. The sum of all such scores for an alternative is hence called their Borda score. The alternative with the highest Borda score is elected.

### 7.2.1 Definition

**fun** *borda-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*borda-rule V A p* = *elector borda V A p*

**fun** *borda-rule*$_{\mathcal{R}}$ :: ('*a*, '*v* :: *wellorder*, '*a Result*) *Electoral-Module* **where**
*borda-rule*$_{\mathcal{R}}$ *V A p* = *swap-*$\mathcal{R}$ *unanimity V A p*

### 7.2.2 Soundness

**theorem** *borda-rule-sound*: $\mathcal{SCF}$*-result.electoral-module borda-rule*
⟨*proof*⟩

**theorem** *borda-rule*$_{\mathcal{R}}$*-sound*: $\mathcal{SCF}$*-result.electoral-module borda-rule*$_{\mathcal{R}}$
⟨*proof*⟩

### 7.2.3 Anonymity

**theorem** *borda-rule*$_{\mathcal{R}}$*-anonymous*: $\mathcal{SCF}$*-result.anonymity borda-rule*$_{\mathcal{R}}$
⟨*proof*⟩

**end**

## 7.3 Pairwise Majority Rule

**theory** *Pairwise-Majority-Rule*
**imports** *Compositional-Structures/Basic-Modules/Condorcet-Module*
*Compositional-Structures/Defer-One-Loop-Composition*
**begin**

This is the pairwise majority rule, a voting rule that implements the Condorcet criterion, i.e., it elects the Condorcet winner if it exists, otherwise a tie remains between all alternatives.

### 7.3.1 Definition

**fun** *pairwise-majority-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*pairwise-majority-rule V A p* = *elector condorcet V A p*

**fun** *condorcet'* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*condorcet' V A p* = ((*min-eliminator condorcet-score*) $\circlearrowleft_{\exists !d}$) *V A p*

**fun** *pairwise-majority-rule'* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*pairwise-majority-rule' V A p* = *iter-elect condorcet' V A p*

### 7.3.2 Soundness

**theorem** *pairwise-majority-rule-sound*: $\mathcal{SCF}$*-result.electoral-module pairwise-majority-rule*
⟨*proof*⟩

**theorem** *condorcet'-sound*: $\mathcal{SCF}$*-result.electoral-module condorcet'*
⟨*proof*⟩

**theorem** *pairwise-majority-rule'-sound*: $\mathcal{SCF}$*-result.electoral-module pairwise-majority-rule'*
⟨*proof*⟩

### 7.3.3 Condorcet Consistency

**theorem** *condorcet-condorcet*: *condorcet-consistency pairwise-majority-rule*
⟨*proof*⟩

**end**

## 7.4 Copeland Rule

**theory** *Copeland-Rule*
**imports** *Compositional-Structures/Basic-Modules/Copeland-Module*
*Compositional-Structures/Elect-Composition*
**begin**

This is the Copeland voting rule. The idea is to elect the alternatives with the highest difference between the amount of simple-majority wins and the amount of simple-majority losses.

### 7.4.1 Definition

**fun** *copeland-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*copeland-rule V A p* = *elector copeland V A p*

### 7.4.2 Soundness

**theorem** *copeland-rule-sound*: $\mathcal{SCF}$*-result.electoral-module copeland-rule*
⟨*proof*⟩

### 7.4.3 Condorcet Consistency

**theorem** *copeland-condorcet*: *condorcet-consistency copeland-rule*
⟨*proof*⟩

**end**

## 7.5 Minimax Rule

**theory** *Minimax-Rule*
**imports** *Compositional-Structures/Basic-Modules/Minimax-Module*
*Compositional-Structures/Elect-Composition*
**begin**

This is the Minimax voting rule. It elects the alternatives with the highest Minimax score.

### 7.5.1 Definition

**fun** *minimax-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
*minimax-rule V A p* = *elector minimax V A p*

### 7.5.2 Soundness

**theorem** *minimax-rule-sound*: $\mathcal{SCF}$*-result.electoral-module minimax-rule*
⟨*proof*⟩

### 7.5.3 Condorcet Consistency

**theorem** *minimax-condorcet*: *condorcet-consistency minimax-rule*
⟨*proof*⟩

**end**

## 7.6 Black's Rule

**theory** *Blacks-Rule*
  **imports** *Pairwise-Majority-Rule*
    *Borda-Rule*
**begin**

This is Black's voting rule. It is composed of a function that determines the Condorcet winner, i.e., the Pairwise Majority rule, and the Borda rule. Whenever there exists no Condorcet winner, it elects the choice made by the Borda rule, otherwise the Condorcet winner is elected.

### 7.6.1 Definition

**fun** *black* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *black A p* = (*condorcet* ▷ *borda*) *A p*

**fun** *blacks-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *blacks-rule A p* = *elector black A p*

### 7.6.2 Soundness

**theorem** *blacks-sound*: $\mathcal{SCF}$*-result.electoral-module black*
  ⟨*proof*⟩

**theorem** *blacks-rule-sound*: $\mathcal{SCF}$*-result.electoral-module blacks-rule*
  ⟨*proof*⟩

### 7.6.3 Condorcet Consistency

**theorem** *black-is-dcc*: *defer-condorcet-consistency black*
  ⟨*proof*⟩

**theorem** *black-condorcet*: *condorcet-consistency blacks-rule*
  ⟨*proof*⟩

**end**

## 7.7 Nanson-Baldwin Rule

**theory** *Nanson-Baldwin-Rule*
  **imports** *Compositional-Structures/Basic-Modules/Borda-Module*
    *Compositional-Structures/Defer-One-Loop-Composition*
**begin**

This is the Nanson-Baldwin voting rule. It excludes alternatives with the lowest Borda score from the set of possible winners and then adjusts the Borda score to the new (remaining) set of still eligible alternatives.

### 7.7.1 Definition

**fun** *nanson-baldwin-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *nanson-baldwin-rule A p* =
    ((*min-eliminator borda-score*) $\circlearrowleft_{\exists!d}$) *A p*

### 7.7.2 Soundness

**theorem** *nanson-baldwin-rule-sound*: $\mathcal{SCF}$*-result.electoral-module nanson-baldwin-rule*
  ⟨*proof*⟩

**end**

## 7.8 Classic Nanson Rule

**theory** *Classic-Nanson-Rule*
  **imports** *Compositional-Structures/Basic-Modules/Borda-Module*
      *Compositional-Structures/Defer-One-Loop-Composition*
**begin**

This is the classic Nanson's voting rule, i.e., the rule that was originally invented by Nanson, but not the Nanson-Baldwin rule. The idea is similar, however, as alternatives with a Borda score less or equal than the average Borda score are excluded. The Borda scores of the remaining alternatives are hence adjusted to the new set of (still) eligible alternatives.

### 7.8.1 Definition

**fun** *classic-nanson-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *classic-nanson-rule V A p* =
    ((*leq-average-eliminator borda-score*) $\circlearrowleft_{\exists!d}$) *V A p*

### 7.8.2 Soundness

**theorem** *classic-nanson-rule-sound*: $\mathcal{SCF}$*-result.electoral-module classic-nanson-rule*
  ⟨*proof*⟩

**end**

## 7.9 Schwartz Rule

**theory** *Schwartz-Rule*
  **imports** *Compositional-Structures/Basic-Modules/Borda-Module*
    *Compositional-Structures/Defer-One-Loop-Composition*
**begin**

This is the Schwartz voting rule. Confusingly, it is sometimes also referred as Nanson's rule. The Schwartz rule proceeds as in the classic Nanson's rule, but excludes alternatives with a Borda score that is strictly less than the average Borda score.

### 7.9.1 Definition

**fun** *schwartz-rule* :: ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *schwartz-rule V A p* =
    ((*less-average-eliminator borda-score*) $\circlearrowleft_{\exists\,!d}$) *V A p*

### 7.9.2 Soundness

**theorem** *schwartz-rule-sound*: $\mathcal{SCF}$*-result.electoral-module schwartz-rule*
  ⟨*proof*⟩

**end**

## 7.10 Sequential Majority Comparison

**theory** *Sequential-Majority-Comparison*
  **imports** *Plurality-Rule*
    *Compositional-Structures/Drop-And-Pass-Compatibility*
    *Compositional-Structures/Revision-Composition*
    *Compositional-Structures/Maximum-Parallel-Composition*
    *Compositional-Structures/Defer-One-Loop-Composition*
**begin**

Sequential majority comparison compares two alternatives by plurality voting. The loser gets rejected, and the winner is compared to the next alternative. This process is repeated until only a single alternative is left, which is then elected.

### 7.10.1 Definition

**fun** *smc* :: '*a Preference-Relation* ⇒ ('*a*, '*v*, '*a Result*) *Electoral-Module* **where**
  *smc x V A p* =

((*elector* ((((*pass-module 2 x*) ▷ ((*plurality-rule*↓) ▷ (*pass-module 1 x*))) ‖↑ (*drop-module 2 x*)) ⟳∃ !*d*)) *V A p*)

### 7.10.2 Soundness

As all base components are electoral modules (, aggregators, or termination conditions), and all used compositional structures create electoral modules, sequential majority comparison unsurprisingly is an electoral module.

**theorem** *smc-sound*:
  **fixes** *x* :: *'a Preference-Relation*
  **shows** *SCF-result.electoral-module* (*smc x*)
⟨*proof*⟩

### 7.10.3 Electing

The sequential majority comparison electoral module is electing. This property is needed to convert electoral modules to a social choice function. Apart from the very last proof step, it is a part of the monotonicity proof below.

**theorem** *smc-electing*:
  **fixes** *x* :: *'a Preference-Relation*
  **assumes** *linear-order x*
  **shows** *electing* (*smc x*)
⟨*proof*⟩

### 7.10.4 (Weak) Monotonicity

The following proof is a fully modular proof for weak monotonicity of sequential majority comparison. It is composed of many small steps.

**theorem** *smc-monotone*:
  **fixes** *x* :: *'a Preference-Relation*
  **assumes** *linear-order x*
  **shows** *monotonicity* (*smc x*)
⟨*proof*⟩

**end**

## 7.11 Kemeny Rule

**theory** *Kemeny-Rule*
  **imports**
    *Compositional-Structures/Basic-Modules/Component-Types/Votewise-Distance-Rationalization*
    *Compositional-Structures/Basic-Modules/Component-Types/Distance-Rationalization-Symmetry*
**begin**

This is the Kemeny rule. It creates a complete ordering of alternatives and evaluates each ordering of the alternatives in terms of the sum of preference reversals on each ballot that would have to be performed in order to produce that transitive ordering. The complete ordering which requires the fewest preference reversals is the final result of the method.

### 7.11.1 Definition

**fun** *kemeny-rule* :: ('*a*, '*v* :: *wellorder*, '*a Result*) *Electoral-Module* **where**
*kemeny-rule V A p* = *swap-*$\mathcal{R}$ *strong-unanimity V A p*

### 7.11.2 Soundness

**theorem** *kemeny-rule-sound*: $\mathcal{SCF}$*-result.electoral-module kemeny-rule*
⟨*proof*⟩

### 7.11.3 Anonymity

**theorem** *kemeny-rule-anonymous*: $\mathcal{SCF}$*-result.anonymity kemeny-rule*
⟨*proof*⟩

### 7.11.4 Neutrality

**lemma** *swap-dist-neutral*: *distance-neutrality well-formed-elections*
(*votewise-distance swap l-one*)
⟨*proof*⟩

**theorem** *kemeny-rule-neutral*: $\mathcal{SCF}$*-properties.neutrality*
*well-formed-elections kemeny-rule*
⟨*proof*⟩

**end**

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