PROPOSITIONAL LOGIC

Discrete Structures I

COMP 1805 A

Today's Class

- **Propositional Logic**
  - A.k.a. statement logic
- Propositions
- Negations
- Conjunctions
- Disjunctions
- Solving Expressions
- Truth Tables
- Conditional Statement
- Bidirectional/Biconditional

What is a Proposition?

- **Proposition**: Declarative statement that has a value of **True (T)** or **False (F)**, but not both

- Examples (Exs):
  - I am your instructor
  - Canada is the largest country in the world

What is NOT a proposition?

- Propositions must be defined
- Not all statements are propositions
- Exs:
  - X is taller than 150 cm
  - Y + Z = 5
    - **These are not** propositions: the definition of x is not concrete, and Y and Z are completely **undefined**
  - Watch out!
  - How far is the closest washroom?
    - **Not** propositions because they are **neither true nor false**

Atomic Propositions

- Propositions that cannot be divided into smaller propositions are considered **Primitive** or **Atomic**
  - Express a single fact
  - Its truth value does not depend on other propositions

- Ex: I am a student.

- Counterex: Mercury is a metal that is liquid at room temperature.

Propositional Variables

- All variables must be defined

- We can use **propositional variables** to refer to propositions
  - Ex: let p be the proposition "…"
  - Ex: z= 'it is raining'

Logical Operators

- We can use logical operators to create *new* propositions from one or more existing propositions

- **Unary operator:** affects *one* proposition
  - **Negation**

Negation

- Let p be a proposition.
- The negation of p is a **new** proposition with the **opposite** truth value of p (i.e. **¬p, "not p"**)

- Exs:

¬m is True when m is False

¬m is False when m is True

¬(a is even) means a is odd

p is "1<3" and ¬p is "1≥3"

"Penguins cannot fly"

Logical Operators and Compound Propositions

- **Binary operator**: logical *connective* for two propositions
  - Exs: Conjunction; Disjunction

- Creates a **compound proposition:** proposition that contains one or more logical connectives

Conjunction (AND)

- Let p and q be propositions.
- The **conjunction** of p and q, denoted **$p \land q$**, is a new proposition whose truth value is true when **both p and q** are true, and false otherwise.

- In English, these are conjunctions:
  - And, but, so, also

- Ex: It is raining and I bring my umbrella

Disjunction (OR)

- The **disjunction** 'p or q' denoted **p V q**, is a new proposition whose truth value is true if **p is true**, **q is true**, or **both** p and q are true, and false otherwise.
  - ***p***V***q*** is false when both ***p*** and ***q*** are false
  - Otherwise, ***p***V***q*** has a value of true

- Inclusive 'or'

- Ex: Cats meow or mice meow

  1hr = 60 mins or 1 min= 60 sec

Exclusive Or

- The **exclusive or** of p xor q, denoted **p $\oplus$ q**, is a new proposition that is true if **exactly one** of p or q is true; otherwise false (not both)

- p $\oplus$ q is read "p xor q" or "p exclusive or q"

- Exs: Soup or salad

  I am telling the truth or I am lying

Useful Mnemonic

- ¬OT

  "¬ " is used for Negation and is similar to an "n"

- ∧ N D

  "∧" is used for Conjunction and is similar to an "A“

- OV

  "V" is used for Disjunction and is similar to an "r"

Solving Logical Expressions

- Complex expressions are formed from simpler sub-expressions

- To solve the complex expressions, the sub-expressions must be evaluated and the results are then substituted back into the expression
  - A.k.a. **reduction**

Solving an Arithmetic Expression

$$(((8 - 3) + 1) / 3) + ((7 - 4)\ 2)$$

$$=(((8 - 3) + 1) / 3) + (\mathbf{3} \cdot 2)$$

$$=((\mathbf{5 + 1}) / 3) + (3 \cdot 2)$$

$$=(\mathbf{6} / 3) + (3 \cdot 2)$$

$$=\mathbf{2} + (3 \cdot 2)$$

$$=2 + \mathbf{6}$$

$$=\mathbf{8}$$

Solving a General Expression

$(\neg(((6 + 1) > 7) \wedge (4 > 3))) \vee F$

$(\neg((\mathbf{7} > 7) \wedge (4 > 3))) \vee F$

$(\neg(\mathbf{F} \wedge (4 > 3))) \vee F$

$(\neg(F \wedge \mathbf{T})) \vee F$

$(\boldsymbol{\neg}\mathbf{F}) \vee F$

$\mathbf{T} \vee F$

$\mathbf{T}$

Solving a Logical Expression

$(((\neg F) \vee T) \wedge F) \vee (\neg(F \wedge F))$

$((\mathbf{T} \vee T) \wedge F) \vee (\neg(F \wedge F))$

$(\mathbf{T \wedge F}) \vee (\neg(F \wedge F))$

$(T \wedge F) \vee (\neg\ \mathbf{F})$

$(T \wedge F) \vee \mathbf{T}$

$\mathbf{F} \vee T$

$\mathbf{T}$

Evaluating Expressions

- Can we evaluate **x+5**?
- Without the value of x you cannot reduce…
- …but what if you knew that $x \in \mathbf{1,2,3}$?
- Since the set of possible values is finite you could enumerate the set of possible solutions

Evaluating Expressions

- In propositional logic, the set of possible values for a variable is always *finite*
  - (i.e. {True, False})

- Thus, it is *always* possible to enumerate all possible solutions to an expression

Evaluating Expressions

- What are the possible solutions to this expression?

$$x \land \text{True}$$

- x can be: True or False
  - when x is True, solution is True
  - when x is False, solution is False

Evaluating Expressions

- What are the possible solutions to this expression?

$$\boldsymbol{x} \wedge \boldsymbol{y}$$

- Possible Values for $(\boldsymbol{x}, \boldsymbol{y})$: (T, T), (T, F), (F, T), or (F, F)
  - when $(\boldsymbol{x}, \boldsymbol{y})$ is (T, T), solution is True
  - False otherwise

Binary Operation Tables

- We can use Binary Operation Tables to find truth values to *compound propositions* because:
  - Operands are finite
  - Truth values are only **T**rue or **F**alse

- Thus, create a **Truth Table** with **each distinct operator** used to construct the logical expression

- Truth tables are used to exhibit the relationship between the truth values of a compound proposition and the truth values of its component propositions

Constructing Truth Tables

- Lists all possible outcomes
- **2^n rows** (one for each possible combination of truth values for the operands required)
- Every propositional variable has its own column (on the left, alphabetized)
  - **Any helper propositions (smaller propositions) are given a column**
  - Final column specifies the result
  - Order of the T's and F's matters!

Truth Table for Negation

- If j is “My name is John”
- Then ¬j is “My name is not John”

j	¬j

- Person talking is John
- Person talking is not John

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T	
F	

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T	F
F	T

Conjunction Truth Table

- I'm Mila and I teach.
- Let "a" be "My name is Mila" and "b" be "I teach"

a	b	a ∧ b

- A teacher named Mila
- A non teacher named Mila
- A teacher named Julie
- A non teacher named Julie

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T	T	
T	F	
F	T	
F	F	

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a	b	a ∧ b
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction Truth Table

- You like computers or you need this credit.
- Let "p" be "you like computers" and "q" be "you need this credit"

- Like it and Need it
- Like it but don't need it
- Don't like it but need it
- Don't like it and don't need it

p	q	p V q

Disjunction Truth Table

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p	q	p V q
T	T	
T	F	
F	T	
F	F	

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p	q	p V q
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or Truth Table

- Soup or salad?
- Let "p" be "soup" and "q" be "salad"

p	q	$p \oplus q$
T	T	**F**
T	F	T
F	T	T
F	F	F

- You order both
- Soup
- Salad
- Neither

Evaluating Expressions

$(a \wedge b) \vee c$

- *What are the possible values for*
  - *a?*
  - *b?*
  - *c?*
  - *$a \wedge b$?*
  - *$(a \wedge b) \vee c$?*
- …Truth table!

$(a \wedge b) \vee c$

a	b	c	$(a \wedge b)$	$(a \wedge b) \vee c$

Complete Truth Table for: $(a \land b) \lor c$

- Rightmost column contains the solution for each possible combination

a	b	c	$(a \land b)$	$(a \land b) \lor c$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Complete Truth Table for: $(a \land b) \lor c$

- Rightmost column contains the solution for each possible combination

a	b	c	$(a \land b)$	$(a \land b) \lor c$
T	T	T	T	
T	T	F	T	
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	F	
F	F	T	F	
F	F	F	F	

Complete Truth Table for: ($a \wedge b$) V c

- Rightmost column contains the solution for each possible combination

a	b	c	($a \wedge b$)	($a \wedge b$) V c
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Complete Truth Table for: $(a \wedge b) \vee c$

- Rightmost column contains the solution for each possible combination

a	b	c	$(a \wedge b)$	$(a \wedge b) \vee c$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Operations Thus Far

- Lets complete the table

a	b	¬a	a∧b	a ∨ b	a ⊕ b

Operations Thus Far

- All possible truth values

a	b	¬a	a∧b	a ∨ b	a ⊕ b
T	T	F	T	**T**	**F**
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

More Connectives

- Conditional Statement/Implication (if, then)

- Bidirectional/Biconditional (iff)

Implication

- The **implication a → b** is a new proposition whose truth value is:
  - Only false when p is true and q is false
  - True otherwise
  - Read "*if* a *then* b" or "a implies b"

Implication

- $P \rightarrow Q$ can be said in many ways:
- If P, then Q
- If P, Q
- P only if Q
- Q if P
- Q when P
- Q whenever P

- P implies Q
- Q follows from P
- P is *sufficient* for Q
- A *necessary* condition for P is Q
- Q is necessary for P
- A sufficient condition for Q is P

Implication

- In a → b,
  - a is called the **antecedent** (or **hypothesis**)
  - b is called the **consequent** (or **conclusion**)
- "if a then b" means you conclude b if a is true
- **It asserts nothing if 'a' is false, so the expression must be considered true**

Implication Truth Table

- $a \rightarrow b$
  - False only when a is true and b is false
- Otherwise, $a \rightarrow b$ is True

a	b	$a \rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Implication Example

- If *you have a Canadian passport*, then *you are a Canadian citizen*
  - Let a be "*You have a Canadian Passport*"
  - Let b be "*You are a Canadian Citizen*"

- What are all possible pairs of truth values for these propositions?

Implication Example

- Case 1: $a = True$, $b = True$
  - Maybe you have Canadian Passport and you are a Canadian Citizen
- Obviously **true** for this case

a	b	$a \rightarrow b$
T	**T**	**T**
T	F	F
F	T	T
F	F	T

Implication Example

- Case 2: a = *False*, b = *False*
  - Maybe you do **not** have a Canadian Passport and you are **not** a Canadian Citizen
- **True** for this case

a	b	a → b
T	T	T
T	F	F
F	T	T
F	**F**	**T**

- True: no contradiction.

Implication Example

- Case 3: *a* = *False*, *b* = *True*
  - Maybe you do **not** have a Canadian Passport but you are a Canadian Citizen
- **True** for this case

a	b	a → b
T	T	T
T	F	F
F	**T**	**T**
F	F	T

- Maybe it expired and you didn't update

Implication Example

- Case 4: a = *True*, b = *False*
- Can someone have a passport but not be a citizen?
  - definitely not

a	b	a → b
T	T	T
T	**F**	**F**
F	T	T
F	F	T

Implication Statement Rules

- Implication: if a then b ($a \longrightarrow b$)
  - **Inverse:** if not a then not b
    - $\neg a \longrightarrow \neg b$
  - **Converse:** if b then a
    - $a \longleftarrow b$ or $b \longrightarrow a$
  - **Contrapositive:** if not b then not a
    - $\neg b \longrightarrow \neg a$

Implication Statement Rules

- If the converse is true, then inverse true too

- Contrapositive is both inverted and converted and it is the only one of the three that is equivalent to $a \rightarrow b$
  - If original statement is true, then contrapositive is true

Implication Statement Example 1

- **Statement:** If <u>two angles are congruent</u>, then <u>they have the same measures</u>
- **Inverse:** If two angles are not congruent, then they do not have the same measures
- **Converse:** If two angles have the same measures, then they are congruent
- **Contrapositive:** If two angles do not have the same measures, then they are not congruent
- Note: Here, the hypothesis and conclusion are equivalent, so all are true… Not always the case!

Implication Statement Example 2

- **Statement:** If a quadrilateral is a rectangle, then it has two pairs of parallel sides
- **Converse:** If a quadrilateral has two pairs of parallel sides, then it is a rectangle **(FALSE)**
- **Inverse:** If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides **(FALSE)**
- **Contrapositive:** If a quadrilateral does not have two pairs of parallel sides, then it is not a rectangle

Implication Truth Table

a	b	¬a	¬b	Implication $a \rightarrow b$	Inverse $\neg a \rightarrow \neg b$	Converse $b \rightarrow a$	Contrapositive $\neg b \rightarrow \neg a$
T	T						
T	F						
F	T						
F	F						

Implication Truth Table

a	b	¬a	¬b	Implication $a \to b$	Inverse $\neg a \to \neg b$	Converse $b \to a$	Contrapositive $\neg b \to \neg a$
T	T	F	F				
T	F	F	T				
F	T	T	F				
F	F	T	T				

Implication Truth Table

a	b	¬a	¬b	Implication $a \to b$	Inverse $\neg a \to \neg b$	Converse $b \to a$	Contrapositive $\neg b \to \neg a$
T	T	F	F	T			
T	F	F	T	F			
F	T	T	F	T			
F	F	T	T	T			

Implication Truth Table

a	b	¬a	¬b	Implication $a \rightarrow b$	Inverse $\neg a \rightarrow \neg b$	Converse $b \rightarrow a$	Contrapositive $\neg b \rightarrow \neg a$
T	T	F	F	T	T		
T	F	F	T	F	T		
F	T	T	F	T	F		
F	F	T	T	T	T		

Implication Truth Table

a	b	¬a	¬b	Implication $a \rightarrow b$	Inverse $\neg a \rightarrow \neg b$	Converse $b \rightarrow a$	Contrapositive $\neg b \rightarrow \neg a$
T	T	F	F	T	T	T	
T	F	F	T	F	T	T	
F	T	T	F	T	F	F	
F	F	T	T	T	T	T	

Implication Truth Table

a	b	¬a	¬b	Implication $a \rightarrow b$	Inverse $\neg a \rightarrow \neg b$	Converse $b \rightarrow a$	Contrapositive $\neg b \rightarrow \neg a$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Bidirectional/Biconditional

- We often want the conjunction of an implication and its converse: $a \rightarrow b \wedge b \rightarrow a$

If and Only If (iff): $a \leftrightarrow b$

- True when *a*'s and *b*'s truth values are the same, else false

Biconditional Truth Table

a	b	$a \leftrightarrow b$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional in English

- $p \leftrightarrow q$ can be said in a couple ways:
- *p if and only iff q*
- *p iff q*
- *if p then q and q implies p*
- *p is necessary and sufficient for q*
- *p is true whenever q is*

Biconditional Examples

- Let's assume that the pool is full of water, that it is too deep to stand, that I know how to swim, and that I am not drowning.
- I am swimming if and only if I am in a pool
  - If I am swimming then I must be in pool
  - If I am in a pool then I must be swimming
- I am breathing IFF I am alive
  - If I am breathing then I am alive
  - If I am alive then I am breathing

Questions?

- Thank you!