Growth and climate change: IAMs in Dynare

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Outline for today

- **Format:** 1 hour of lecture combined with 2 hours of practical work.
- **Objective:** This course is computational in nature. It introduces simple and practical methods to solve macroeconomic models, with a particular focus on the use of Dynare.
  - **Model setup in Dynare**
    - Application: a simple New Classical growth model
  - **Perturbation methods: short-run dynamics**
    - Application: a simple New Classical growth model
    - Exercise: Integrated Assessment Model (IAM) based on Heutel (2012)
  - **Perfect foresight simulations: long-run dynamics**
    - Application: a simple New Classical growth model
    - Exercise: IAM based on a toy DICE model

What is an integrated assessment model?

- IAMs bring together economics, climate science, and energy systems within a single quantitative framework.

- **Purpose:** Provide a tractable way to evaluate long-run interactions between the economy and the climate.

- **Institutional use:**
  - **IPCC** (Intergovernmental Panel on Climate Change) relies on IAMs to define emission scenarios (e.g. SSP pathways).
  - **Central banks, NGFS, OECD, IEA, World Bank** use IAM-based tools for stress testing, policy design, and energy forecasts.

- IAMs are **quantitative tools** used for scenario design, policy evaluation, and to inform international climate negotiations.
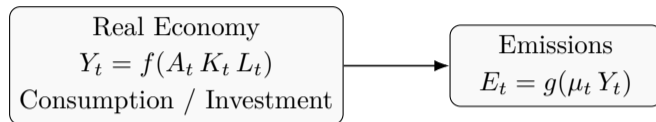
> Real Economy
> $Y_t = f(A_t\, K_t\, L_t)$
> Consumption / Investment

Intro: IAM schematic: economy-climate feedbacks

Real Economy
$Y_t = f(A_t\, K_t\, L_t)$
Consumption / Investment

$\longrightarrow$

Emissions
$E_t = g(\mu_t\, Y_t)$

Intro: IAM schematic: economy-climate feedbacks

```
┌─────────────────────────┐        ┌──────────────────┐       ┌──────────────────────────────┐
│      Real Economy       │        │    Emissions     │ GHGs  │      Carbon & Climate        │
│   $Y_t = f(A_t K_t L_t)$│ ────▶  │ $E_t = g(\mu_t Y_t)$│────▶ │   Carbon cycle $\to M_t$     │
│ Consumption / Investment│        │                  │       │ Radiative forcing $\to T_t$  │
└─────────────────────────┘        └──────────────────┘       └──────────────────────────────┘
```

Intro: IAM schematic: economy-climate feedbacks

Real Economy
$Y_t = f(A_t\,K_t\,L_t)$
Consumption / Investment

Emissions
$E_t = g(\mu_t\,Y_t)$

GHGs

Carbon & Climate
Carbon cycle $\rightarrow M_t$
Radiative forcing $\rightarrow T_t$

damages: lost output / utility

Real Economy
$Y_t = f(A_t\, K_t\, L_t)$
Consumption / Investment

Emissions
$E_t = g(\mu_t\, Y_t)$

GHGs

Carbon & Climate
Carbon cycle $\to M_t$
Radiative forcing $\to T_t$

Carbon Policy
Carbon price τ_t

damages: lost output / utility

Real Economy
$Y_t = f(A_t\, K_t\, L_t)$
Consumption / Investment

Emissions
$E_t = g(\mu_t\, Y_t)$

GHGs

Carbon & Climate
Carbon cycle $\to M_t$
Radiative forcing $\to T_t$

abatement cost

reduces intensity

Carbon Policy
Carbon price τ_t

damages: lost output / utility
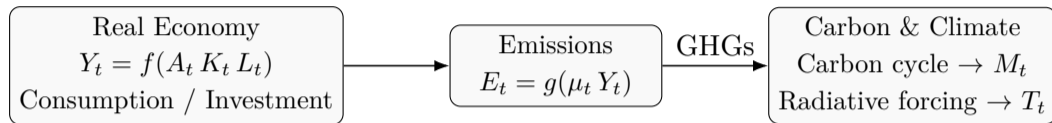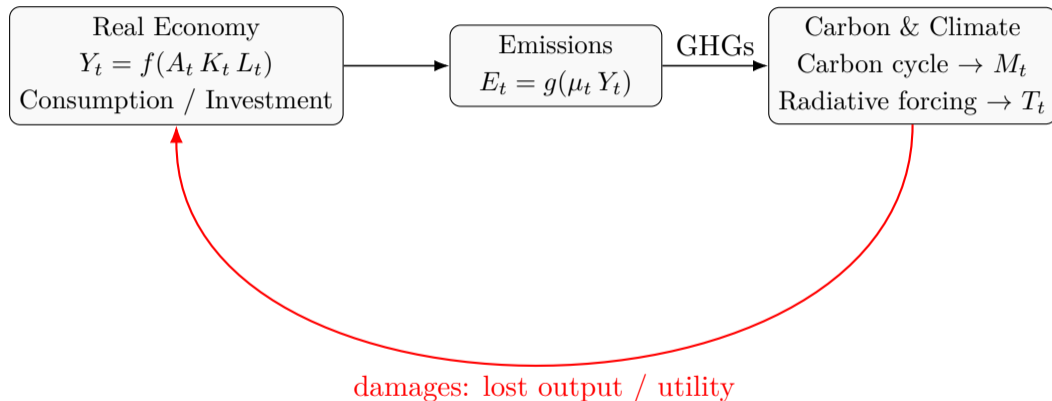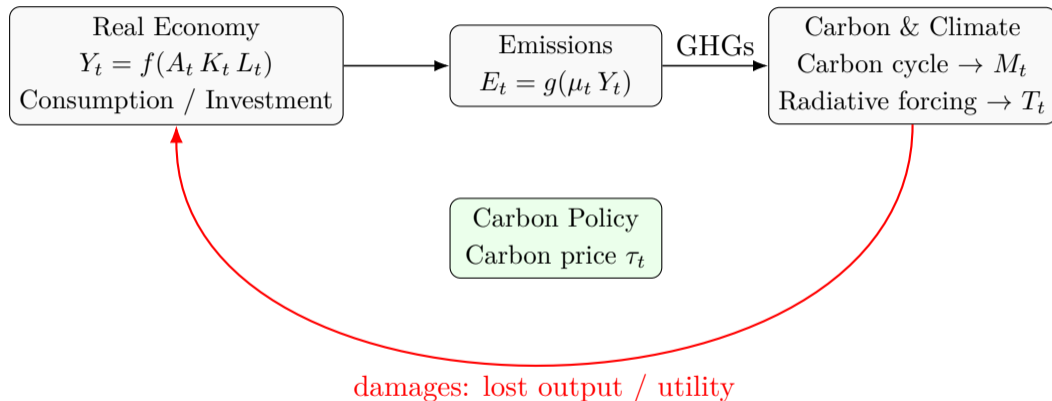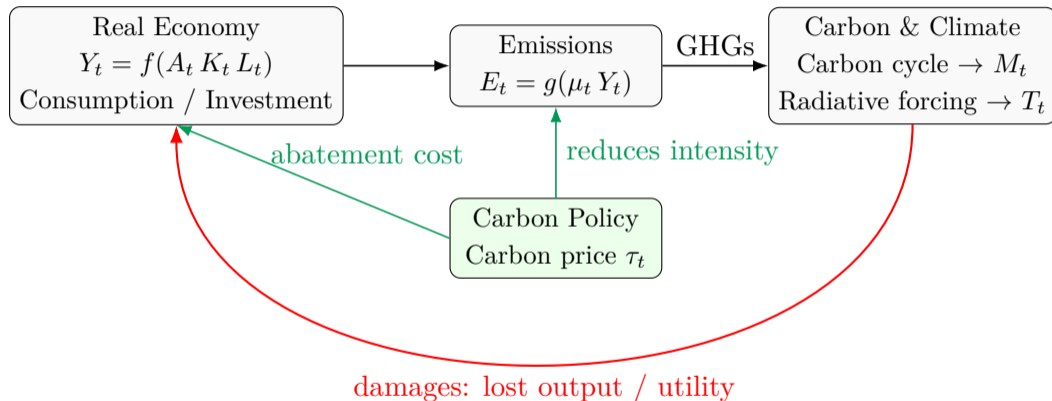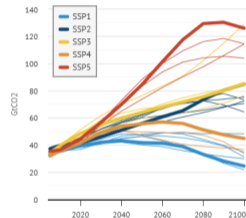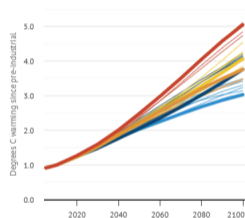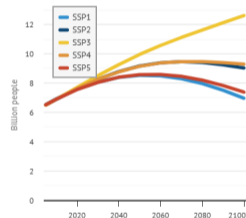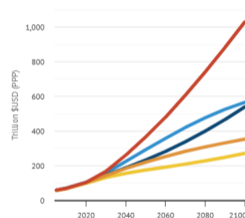
Notes: Illustrative SSP baseline trajectories across models.

Literature review I

Branch I: **Engineering-based IAMs**

- Examples: IMACLIM (CIRED), REMIND, GCAM
- Features:
  - Deterministic, large-scale, multi-sectoral
  - Frictionless (nominal rigidities, financial, etc.)
  - Backward-looking, non-optimizing
- But **not immune to Lucas critique**:
  - No expectations or uncertainty
  - No explicit optimization or welfare criterion

Branch II: **Long-run Mainstream Macro IAMs**

- Examples:
  - DICE/RICE [Nordhaus, 1991, 2008, 2017; Barrage and Nordhaus, 2023]
  - Energy-augmented growth models [Golosov et al. (2014); Hassler et al. (2021)]
  - Directed technical change [Acemoglu et al. (2012, 2016)]
- Features:
  - Forward-looking, optimizing behavior
  - Deterministic models
  - Long run analysis

Branch III: **E-DSGE models for policy analysis**

- Examples:
  - NY Fed multi-sector DSGE model with energy sectors [Del Negro et al., 2023]
  - Medium-scale DSGE with green brown energy [Coenen et al., 2024]
  - IMF energy and Phillips curve [Erceg et al., 2024]

- Features:
  - Forward-looking, optimizing behavior
  - Stochastic in neighboorhood of some steady state
  - No climate block, so not truly an IAM

Dynare: A toolbox for solving (climate) macroeconomic models

- **All these models can be solved efficiently in Dynare.**

- Dynare is a **MATLAB/Python/Julia package** that provides a high-level interface to define, solve, and simulate economic models written in *state-space form.*

- Founded by **Michel Juillard**, now hosted by **CEPREMAP** (based at PSE premise).

- Dynare serves as a **front-end** to numerical solvers:
    - *Perturbation methods* (first- or higher-order approximations)
    - *Perfect foresight* and *extended-path* deterministic solvers
    - Bayesian estimation and policy analysis

- **Dynare is the mother tongue of central banks and policy institutions.** Understanding its logic is a valuable skill for applied macro and

A toy model of the New Classical growth model

- To fix ideas, let us consider a **toy version of the New Classical growth model**:

- A representative, infinitely-lived household solves

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(C_t)$$

$$s.t. \ C_t + K_t = A_t K_{t-1}^{\alpha} + (1 - \delta) K_{t-1},$$

- The household chooses $\{C_t, K_t\}$ taking $\{A_t\}$ as given. The first-order condition (Euler equation) reads

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \left(1 - \delta + \alpha A_{t+1} K_t^{\alpha-1} \right) \right].$$

Model summary

- The model is a 3 equation model

$$-\frac{1}{C_t} + \beta \, \mathbb{E}_t\left[\frac{1}{C_{t+1}}\left(1 - \delta + \alpha A_{t+1} K_t^{\alpha-1}\right)\right] = 0$$

$$-C_t - K_t + A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} = 0$$

$$-\log A_t + \rho_A \log A_{t-1} + \varepsilon_t = 0$$

with

- 3 endogenous variables $y_t = [C_t, K_t, A_t]'$
- 1 exogenous variable $\eta_t = [\varepsilon_t]$
- 5 structural parameters $\Theta = \{\beta\,, \delta, \alpha, \rho_A, \sigma\}$

A general formulation of a structural model

- Consider a structural model written in **state-space form**:

$$\mathbb{E}_t\{f_\Theta(z_{t-1},\, z_t,\, z_{t+1},\, \eta_t)\} = 0$$

- where:
    - z_t: vector ($N_z \times 1$) of endogenous variables,
    - η_t: vector ($N_\eta \times 1$) of Gaussian innovations, $\eta_t \sim \mathcal{N}(0,\, \Sigma)$,
    - $f_\Theta(\cdot)$: system of nonlinear equilibrium conditions,
    - Θ: vector of structural parameters,
    - $\mathbb{E}_t[\cdot]$: expectation conditional on information available at t.

- *This representation nests most macroeconomic models: RBC, NK, and E-DSGE.*

Steady State Determination

- The **steady state** is a fixed point of the model where all variables are constant and all expectations are realized.
- The steady state satisfies the system of **nonlinear algebraic equations** obtained by removing time indices and shocks:

$$f_\Theta(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

- It plays two key roles in Dynare:
    - As a **fixed point** around which perturbation methods are applied (`stoch_simul`).
    - As the **terminal condition** in perfect-foresight simulations (`perfect_foresight_solver`).
- The steady state can be found:
    - *analytically* (by hand or symbolic computation), or
    - *numerically* (using nonlinear solvers such as `fsolve`).

A model declaration in Dynare

- A Dynare model is written in a single `.mod` file.
- It follows a clear and standardized structure:

1) Declaration of variables

- var [list of endogenous variables];
- varexo [list of exogenous shocks];
- parameters [list of structural parameters];

2) Model definition

- Use the model; ... end; block to list the **dynamic equilibrium equations**.
- Equations can be written in **nonlinear (levels)** or **log-linearized** form.

A model declaration in Dynare (cont.)

3) Provide steady state

- steady;   compute steady state

4) Solution and simulation (to come later)

- stoch_simul(...);   perturbation methods
- perfect_foresight_solver;   deterministic simulations

$\rightarrow$**Let's work on the exercise 1 of PC1.**

Plan

A model declaration in Dynare

- **E-DSGE** models extend standard DSGE frameworks to incorporate the **climate-economy link** (emissions, damages, and abatement decisions).

- They rely on the same **perturbation methods** used in business-cycle models (e.g. **Smets and Wouters (2007)**), which **linearize the true nonlinear model** around its deterministic steady state.

- The first stochastic E-DSGE implementation is Heutel (2012), who solves an environmental DSGE model by perturbation methods in Dynare.

- **Perturbation methods require:**
    - A **stationary economy** (no unit roots or explosive trends),
    - A well-defined **balanced growth path** (variables scaled by trend growth),
    - **Stationary shocks**, typically AR(1) processes around zero mean.

A general formulation

- Consider a function $f(x)$. A taylor expansion around a fixed point a reads as:

$$f(x) \simeq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

- Two natural questions:
    1. Which order of approximation to consider?
    2. Which value of a?

A first order taylor expansion

- Macroeconomics still dominated by linearized models. This lecture will remain on the linear version of the true nonlinear model.

- Consider a first order taylor expansion:

$$F \cdot \mathbb{E}_t\{\hat{z}_{t+1}\} + G \cdot \hat{z}_t + H \cdot \hat{z}_{t-1} + M \cdot \eta_t = 0$$

where

  - F, G, H and M are matrices stacking first order derivarives of f_Θ
  - $\hat{z}_t = z_t - a$ is typically expressed in distance from steady state $a = z$. Why? linear model combined with gaussian stochastic shocks $\eta_t \sim \mathcal{N}(0, \Sigma)$ implies that $z_t - z \sim \mathcal{N}(0, \Sigma_z)$. The smallest deviations $E((z_t - a)^2)$ are obtained when 1st order expansion computed around steady state.

A first order taylor expansion

- One looks for a recursive solution that would have the following form:

$$\hat{z}_t = P \cdot \hat{z}_{t-1} + Q \cdot \eta_t$$

where P and Q are two unknown matrices.

- To solve the problem, one must determine the expectation scheme. Lucas implements 'rational' expectations, i.e. model-based expectations: $E_t\{\hat{z}_{t+1}\} = P \cdot \hat{z}_t + Q \cdot \eta_{t+1}$, with conditional expectation $E_t\{\eta_{t+1}\} = 0$ as $\eta_t \sim \mathcal{N}(0, \Sigma)$.

- One can find P and Q by substituting $E_t\{\hat{z}_{t+1}\}$ and $\hat{z}_t$:

$$\underbrace{\left[FP^2 + GP + H\right]}_{\text{determining } P} \hat{z}_{t-1} + \underbrace{(FPQ + GQ + M)}_{\text{determining } Q} \eta_t = 0$$

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A first order taylor expansion

- Presence of quadratic matrix equation $FP^2 + GP + H = 0$ with two solutions. The usual pratice is to calculate solutions for P, pick P that have eigenvalues <1 absolute terms. If no stable solution, have a look at Blanchard and Kahn (1980).

- Puzzling aspect of linear rational expectations model: one path of the economy not stable.

- Once solved, a linear DSGE model has similar features as VAR(1) model (useful when it comes to its estimation).

→**Let's work on the exercise 2 of PC1.**

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The first E-DSGE

- Heutel (2012) develops the first fully stochastic **E-DSGE model**, built as a **stationarized version of DICE**.

- The framework introduces business-cycle dynamics into climate economics:
  - Short-run productivity shocks affect output, consumption, and emissions.
  - The model studies how optimal climate policy reacts to these fluctuations.

- A **centralized social planner** chooses consumption, investment, and abatement effort to maximize welfare, while internalizing the externality from the carbon stock.

- The model is **stationary** and solved by **perturbation method.**

The central planner's problem

$$\max_{\{z_t,\, i_t,\, c_t, y_t, \mu_t, x_t, k_t\}} E_t \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

$$\text{s.t.} \quad \begin{cases} c_t + i_t + z_t \leq y_t, \\ k_t = (1 - \delta_K)k_{t-1} + i_t, \\ x_t = (1 - \delta_X)x_{t-1} + e_t + e_t^{row}, \\ e_t = (1 - \mu_t)h(y_t), \\ z_t = g(\mu_t)y_t, \\ y_t = (1 - d(x_t))a_t f(k_{t-1}). \end{cases}$$

Key shadow values

- The planner's value for carbon reads as:

$$\lambda_t^x = \mathbf{1}_{\{\text{policy on}\}} \underbrace{\lambda_t^y \frac{d'(x_t)}{1-d(x_t)} y_t}_{\text{marginal damage}} + \underbrace{(1-\delta_X)\beta \frac{c_{t+1}^{-\phi}}{c_t^{-\phi}} \lambda_{t+1}^x}_{\text{continuation value}}$$

where $\mathbf{1}_{\{\text{policy on}\}}$ planner internalizes externality $\rightarrow$ carbon price computed.

- Firms in response engage into abatement spending:

$$\mu_t = \left[\frac{\sigma}{\theta_1 \theta_2} \lambda_t^x y_t^{-\gamma} \right]^{1/(\theta_2 - 1)}$$

$\rightarrow$**Let's work on the exercise 3 of PC1.**

Some observations

- Main contribution of Heutel (2012)
  - Introduces **business-cycle frequency** into climate-economy models.
  - Quantifies the **co-movement between output and emissions**.
  - Identifies an **optimal carbon price path** that is **procyclical**: rising in booms, falling in recessions.

- Limitations:
  - **Stationary treatment of carbon stock and emissions:** implies mean reversion, which leads to flawed medium- and long-term predictions.
  - The **carbon stock is highly abstract**: emissions are treated as a dimensionless flow, abatement $= 1$ yields 0 carbon stock!
  - As a result, the model is well suited for **short-run cyclical analysis**, but not for long-run climate dynamics.

Plan

Integrated Assessment Models (IAMs)

- **Integrated Assessment Models (IAMs)** combine economic growth, the energy system, and the climate into a single dynamic framework.

- They study the **long-run interaction** between the economy and the climate under a **deterministic** environment - no stochastic shocks, no uncertainty.

- **Key characteristics:**
  - Deterministic **transition dynamics** toward a steady state.
  - Driven by **trends** - population, technology, productivity.
  - Often feature **unbalanced growth** between capital, output, and damages.
  - Carbon price and climate change implies structural change.

- Canonical example: the **DICE model** (Nordhaus).

A sketch of the numerical solution

- Our model, with no stochastic shocks→determnistic, reads as:

$$f_\Theta\left(z_{t+1}, z_t, z_{t-1}\right) = 0$$

- The problem becomes a two-point boundary value problem:
  - An **initial condition** z_0 given by the current state of the economy
  - A **terminal condition** requiring convergence to the steady state

- The algorithm searches for the entire **transition path** that connects these two bounds: the path of endogenous variables consistent with $f_\Theta(\cdot)$.

- In Dynare, this is solved using the `perfect_foresight_solver;` command:

A sketch of the numerical solution

A sketch of the numeric problem:

- Finite horizon problem for $t = 0, 2, ...T + 1$;

- Terminal z_{T+1} and initial z_0 conditions are given $\rightarrow$ need to numerically get $z_1, z_2, ...z_T$;

- In absence of stochastic variables $\rightarrow$ deterministic problem $\rightarrow$ perfect foresight setup where any variable in z_{t+1} corresponds to the realized variable in $t + 1$;

Numeric solution

- Over the time horizon $t = 1, 2, ...T$, stacking $F(Y)$ over time yields

$$
F(Y) = \begin{bmatrix}
f(z_2, z_1, z_0) \\
f(z_3, z_2, z_1) \\
... \\
f(z_{T+1}, z_T, z_{T-1})
\end{bmatrix}
$$

  with $Y = [z_t', z_{t+1}', ..., z_T']'$ and $F : \mathbb{R}^{NT} \to \mathbb{R}^{NT}$

- Y and $F(Y)$ are two vectors of size $NT \times 1$.

Numeric solution

- The goal is to numerically solve:

$$Y^* = \arg\min_{\{Y\}} |F(Y)|$$

- How? Newton–Raphson method very efficient as shown by Laffargue (1990), Bouckekkine (1995) and Juillard et al. (1996). Basic idea:
  - Set an initial value $Y^{(0)}$.
  - n^{th} Newton iterations:

  $$Y^{(n)} = Y^{(n-1)} - J_F\left(Y^{(n-1)}\right)^{-1} F(Y^{(n-1)})$$

  where $J_F\left(Y^{(n-1)}\right)$ is Jacobian matrix of F of dimensions $NT \times NT$.
  - Stop the iterations if $|F(Y^{(n-1)})| < \varepsilon$.

Numeric solution

- Each iteration requires to solve:

$$
\begin{bmatrix}
J_{1,1} & J_{1,2} & \dots & 0_N & 0_N \\
J_{2,1} & J_{2,2} & \dots & 0_N & 0_N \\
\dots & \dots & \dots & \dots & \dots \\
0_N & 0_N & \dots & J_{T-1,T-2} & J_{T-1,T} \\
0_N & 0_N & \dots & J_{T-1,T-1} & J_{T,T}
\end{bmatrix}
\begin{bmatrix}
\Delta y_1^{(n)} \\
\Delta y_2^{(n)} \\
\dots \\
\Delta y_{T-1}^{(n)} \\
\Delta y_T^{(n)}
\end{bmatrix}
= -
\begin{bmatrix}
f_1 \\
f_2 \\
\dots \\
f_{T-1} \\
f_T
\end{bmatrix}
$$

- The (inefficient) bruteforce way:

$$
\begin{bmatrix}
\Delta y_1^{(n)} \\
\Delta y_2^{(n)} \\
\dots \\
\Delta y_{T-1}^{(n)} \\
\Delta y_T^{(n)}
\end{bmatrix}
= -J_F \left(\begin{bmatrix}
\Delta y_1^{(n-1)} \\
\Delta y_2^{(n-1)} \\
\dots \\
\Delta y_{T-1}^{(n-1)} \\
\Delta y_T^{(n-1)}
\end{bmatrix} \right)^{-1}
\begin{bmatrix}
f_1 \\
f_2 \\
\dots \\
f_{T-1} \\
f_T
\end{bmatrix}
$$

Numeric solution

- Laffargue (1990) proposes using a triangular expression of $J_F\left(Y^{(n)}\right)$ to allow backward induction.

- Linear algebra yields the following:

$$\begin{bmatrix} I_N & g_{1,2} & \ldots & g_{1,T-1} & g_{1,T} \\ 0_N & I_N & \ldots & g_{2,T-1} & g_{2,T} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0_N & 0_N & \ldots & I_N & g_{T-1,T} \\ 0_N & 0_N & \ldots & 0_N & I_N \end{bmatrix} \begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \ldots \\ \Delta y_{T-1}^{(n)} \\ \Delta y_T^{(n)} \end{bmatrix} = - \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_{T-1} \\ d_T \end{bmatrix} \quad (1)$$

- Principle: once matrices $g_{\tau,t}$ and d_t (for $\tau, t \in [1, T]$) are obtained, easy to get $\Delta y_t^{(n)}$ recursively by starting by last row of problem (1).

Numeric solution

- Thanks to this transformation of the Jacobian, each Newton step is fast and is done through a loop.
- This allows macroeconomists to solve their models in a second, without suffering much the curse of dimensionality!

→**Let's work on the exercise 4 of PC1.**

A baby DICE

- To illustrate transition dynamics with climate, let's us consider a baby DICE model:
  - Productivity grows exogenously but converges to a steady state.
  - Output is affected by climate damages.
  - Planner allocates resources between consumption and abatement.

- The presence of large nonlinear transition dynamics in productivity, damages, and abatement renders perturbation methods unreliable. Perfect-foresight simulation in Dynare provides an appropriate global solution.

TFP dynamics

$$\frac{A_t}{A_{t-1}} = \exp\left[(1 - e^{-\delta})\left(\frac{g}{\delta} - \log\left(\frac{A_{t-1}}{A_0}\right)\right)\right].$$

- A_0: initial productivity.
- g: initial growth rate, δ: rate of decline.
- $\Rightarrow$ Productivity converges to steady state $\lim_{t \to \infty} A_t = \bar{A}$.

Climate Damage and Output

Endowment output:
$$y_t = A_t(1 - \varphi(m_t)).$$

Damage function:
$$\varphi(m_t) = \xi \, (m_t - \bar{m})^2,$$

where:

- $\bar{m}$: pre-industrial carbon stock,
- ξ: elasticity of damage to excess carbon.

$\Rightarrow$ Higher m_t permanently reduces effective output.

Carbon Dynamics

Carbon accumulation:

$$m_t - \bar{m} = \gamma(m_{t-1} - \bar{m}) + e_t,$$

- γ: persistence / rate of transfer to deep oceans,
- e_t: new emissions (in Gt of carbon).

Emissions:

$$e_t = \sigma(1 - \mu_t)A_t a_t,$$

- μ_t: abatement effort ($\mu_t = 1$ means zero emissions),
- σ: carbon intensity of production,
- a_t: exogenous activity index.

Planner's Resource Constraint

Resource allocation:

$$y_t = c_t + \theta_1 \, \mu_t^{\theta_2} A_t,$$

- c_t: consumption,
- $\theta_1 \, \mu_t^{\theta_2}$: abatement (green investment) cost

Preferences:

$$
\begin{aligned}
\max_{\{c_t, m_t, \mu_t\}} \ & \mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^\tau \log(c_{t+\tau}) \\
& + \lambda_{t+\tau} \left[A_t \left(1 - \theta_1 \mu_t^{\theta_2} - \varphi\left(m_t\right) \right) - c_t \right] \\
& + \lambda_{t+\tau} q_{t+\tau} \left[m_t - \gamma m_{t-1} - \sigma(1 - \mu_t) A_t \right]
\end{aligned}
$$

Planner's Resource Constraint

- Therefore the first main equation is the social cost of carbon (ie the shadow value) that reflects the discounted sum of marginal damages of one carbon emission along its lifetime:

$$q_t = A_t \varphi'(m_t) + \gamma \beta \frac{\lambda_{t+1}}{c_{t+1}} q_{t+1} \tag{2}$$

- In addition, the optimal abatement rate reads as:

$$\theta_2 \theta_1 \mu_t^{\theta_2 - 1} = \sigma q_t \tag{3}$$

The marginal cost of reducing carbon must be equation to the expected saved output.

$\rightarrow$**Let's work on the exercise 5 of PC1.**

Thank you for your attention

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Plan

To illustrate, consider $t = 1, 2, 3$, y_0 & y_4 given, $y_{1:3}$ unknown. We are at n-step update, $\hat{y}_t = y_t^{(n-1)}$:

$$F\left(\left[\begin{array}{c} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{array}\right]\right) = \left[\begin{array}{c} f\left(\hat{y}_2, \hat{y}_1, y_0\right) \\ f\left(\hat{y}_3, \hat{y}_2, \hat{y}_1\right) \\ f\left(y_4, \hat{y}_3, \hat{y}_2\right) \end{array}\right],$$

$$J_F\left(Y^{(n-1)}\right) = \left[\begin{array}{ccc} \frac{\partial f(\hat{y}_2, \hat{y}_1, y_0)}{\partial \hat{y}_1} & \frac{\partial f(\hat{y}_2, \hat{y}_1, y_0)}{\partial \hat{y}_2} & \frac{\partial f(\hat{y}_2, \hat{y}_1, y_0)}{\partial \hat{y}_2} \\ \frac{\partial f(\hat{y}_3, \hat{y}_2, \hat{y}_1)}{\partial \hat{y}_1} & \frac{\partial f(\hat{y}_3, \hat{y}_2, \hat{y}_1)}{\partial \hat{y}_2} & \frac{\partial f(\hat{y}_3, \hat{y}_2, \hat{y}_1)}{\partial \hat{y}_3} \\ \frac{\partial f(y_4, \hat{y}_3, \hat{y}_2)}{\partial \hat{y}_1} & \frac{\partial f(y_4, \hat{y}_3, \hat{y}_2)}{\partial \hat{y}_2} & \frac{\partial f(y_4, \hat{y}_3, \hat{y}_2)}{\partial \hat{y}_3} \end{array}\right]$$

Percentage of zeros in $J_F\left(Y^{(n-1)}\right)$ grows in T. Note that $\frac{\partial f(\hat{y}_{t+1}, \hat{y}_t, y_{t-1})}{\partial \hat{y}_\tau}$ with $\tau \in t$ is a $N \times N$ matrix.

- Recall, each Newton iteration requires to solve:

$$J_F \left(Y^{(n-1)} \right)^{-1} \Delta Y^{(n)} = -F(Y^{(n-1)})$$

$$\begin{bmatrix} J_{1,1} & J_{1,2} & 0_N \\ J_{2,1} & J_{2,2} & J_{2,3} \\ 0_N & J_{3,2} & J_{3,3} \end{bmatrix} \begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \Delta y_3^{(n)} \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

where $J_{t,\tau} \; \frac{\partial f(\hat{y}_{t+1}, \hat{y}_t, y_{t-1})}{\partial \hat{y}_\tau}$ and $f_t = f\left(\hat{y}_{t+1}, \hat{y}_t, \hat{y}_{t-1} \right)$.

- Linear problem: triangular expression of J_F allows backward induction.

- 1. Solve first row to get $\Delta y_1^{(n)}$ as linear function of $\Delta y_2^{(n)}$.

$$
\begin{array}{ccccc}
I_n \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\
J_{2,1} \Delta y_1^{(n)} & +J_{2,2} \Delta y_2^{(n)} & +J_{2,3} \Delta y_3^{(n)} & = & -f_2 \\
0_N & +J_{3,2} \Delta y_2^{(n)} & J_{3,3} \Delta y_3^{(n)} & = & -f_3
\end{array}
$$

with $d_1 = J_{1,1}^{-1} f_1$ and $g_1 = J_{1,1}^{-1} J_{1,2}$.

- 2. Use first row $I_N \Delta y_1^{(n)} = -d_1 - g_1 \Delta y_2^{(n)}$ and replace to replace $\Delta y_1^{(n)}$:

$$
\begin{array}{llll}
I_N \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\
0_N & +I_N \Delta y_2^{(n)} & +g_2 \Delta y_3^{(n)} & = & -d_2 \\
0_N & +J_{3,2} \Delta y_2^{(n)} & J_{3,3} \Delta y_3^{(n)} & = & -f_3
\end{array}
$$

where $g_2 = (J_{2,2} - J_{2,1}g_1)^{-1} J_{2,3}$ and $d_2 = (J_{2,2} - J_{2,1}g_1)^{-1} (f_2 - J_{2,1}d_1)$.

- 3. Use second row $I_N \Delta y_2^{(n)} = -d_2 - g_2 \Delta y_3^{(n)}$ and replace to replace $\Delta y_2^{(n)}$:

$$\begin{array}{ccccc}
I_N \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\
0_N & +I_N \Delta y_2^{(n)} & +g_2 \Delta y_3^{(n)} & = & -d_2 \\
0_N & +0_N & +I_N \Delta y_3^{(n)} & = & -d_3
\end{array}$$

  where $d_3 = (J_{3,3} - J_{3,2} g_2)^{-1}(f_3 - J_{3,2} d_2)$.

- Going back to stacked matrix:

$$\left[\begin{array}{ccc} I_N & g_1 & 0_0 \\ 0_N & I_N & g_2 \\ 0_N & 0_N & I_N \end{array} \right] \Delta Y^{(n)} = - \left[\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array} \right]$$

  $\rightarrow$ Backward induction by solving last row of $\Delta Y^{(n)}$ recursively.

- Generalization to a T horizon yields:

$$d_1 = J_{1,1}^{-1} f_1$$
$$d_2 = (J_{2,2} - J_{2,1} g_1)^{-1} (f_2 - J_{2,1} d_1)$$
$$g_t = (J_{t,t} - J_{t,t-1} g_{t-1})^{-1} J_{t,t+1} \quad \text{for } t \in [2, T-1]$$
$$d_t = (J_{t,t} - J_{t,t-1} g_{t-1})^{-1} (f_t - J_{t,t-1} d_{t-1}) \quad \text{for } t \in [2, T-1]$$
$$d_T = (J_{T,T} - J_{T,T-1} g_{T-1})^{-1} (f_{T-1} - J_{T,T-1} d_{T-1}).$$