Gauthier Vermandel ∘ *gauthier.vermandel@polytechnique.edu*

A Crash Course on Climate Economcis

1. A Baby Decentralized Integrated Assessment Model (IAM)

This section introduces a stripped-down version of the DICE model that we adapt into a dynamic stochastic general equilibrium (DSGE) framework. Our goal is to demonstrate how climate dynamics can be integrated into a macroeconomic model that is both decentralized and subject to structural shocks, two key ingredients of standard DSGE modeling.

To this end, we construct a simple Real Business Cycle (RBC) model enriched with climate components. We refer to this as a "Baby Decentralized IAM". The model retains the core economic structure of an RBC framework while incorporating climate-related externalities in a tractable way.

The exposition proceeds in two parts:

1. First, we outline the **exogenous structural trends**, such as carbon emissions, temperature dynamics, and potential technological shifts which are crucial in shaping the long-term evolution of the economy and the environment.
2. Then, we introduce the **decentralized core** of the model, featuring households and firms making optimizing decisions in the presence of climate damages and exogenous shocks.

The model is deliberately kept minimalistic to make it easily extensible, you are encouraged to build upon this structure in your own work. Our solution strategy diverges from traditional log-linearization approaches common in tools like Dynare. Instead, we adopt a method suitable for handling structural trends and nonlinear dynamics inherent in climate-economy interactions.

For a more detailed exposition of the methodology; merging IAM components with DSGE techniques in a Dynare-compatible format, see Jondeau et al. (2022) and Sahuc et al. (2025).

1.1 The Exogenous Component of the Model

As in the original DICE framework, we begin by specifying a set of exogenous processes that drive the long-term trajectory of the economy and the climate. These exogenous variables are not influenced by the decisions of agents within the model. Their deterministic paths are particularly useful for embedding socioeconomic scenariosâ€”such as those defined by the Shared Socioeconomic Pathways (SSPs)â€”into a general equilibrium framework.

1. Productivity with declining growth. We assume that total factor productivity z_t grows over time, but at a decaying rate. Specifically, we posit:

$$z_t = (1 + g_{z,t})\, z_{t-1}$$
$$g_{z,t} = (1 - \delta_z)\, g_{z,t-1}$$

where $g_{z,t}$ is the time-varying productivity growth rate and $\delta_z \in (0, 1]$ is the decay rate. This setup implies that $g_{z,t} \to 0$ over time, and thus z_t converges to a finite level z_∞, ensuring the existence of a steady state in the long run.

2. Population dynamics. We model population l_t as converging gradually toward a long-run terminal level $l_\infty > 0$:

$$l_t = l_{t-1}^{\ell} \cdot l_\infty^{1-\ell}$$

where $\ell \in (0, 1]$ governs the speed of convergence. This simple deterministic process ensures that the population asymptotically approaches a stable level, reflecting demographic transitions commonly featured in IAM scenarios.

3. Emission intensity. Carbon emissions per unit of output are governed by an exogenous trend capturing increasing energy efficiency or decarbonization:

$$\sigma_t = (1 - \delta_\sigma)\, \sigma_{t-1}$$

with $\delta_\sigma \in (0, 1]$. As $\sigma_t \to 0$, emissions per unit of output gradually decline over time, even in the absence of explicit mitigation efforts.

4. Abatement cost. We introduce a stylized representation of the cost of emission reduction:

$$\theta_{1,t} = p_b \cdot \sigma_t \cdot \varrho_t / \theta_2$$
$$\varrho_t = (1 - \delta_\theta)\, \varrho_{t-1}$$

Here, p_b is the so-called *backstop price*â€”the marginal cost of abating the last ton of carbon to achieve net-zero emissionsâ€”and θ_2 is a scaling parameter. The process ϱ_t captures technological progress in abatement technologies, with $\delta_\theta \in (0, 1]$ being the rate at which costs fall. Over time, as $\varrho_t \to 0$, the marginal cost of abatement $\theta_{1,t}$ vanishes, representing long-run convergence to cheap clean technologies.

These exogenous processes provide the backbone of the modelâ€■s long-run structure. In what follows, we integrate them into a decentralized economy with optimizing agents, allowing us to study short-run dynamics and policy responses in a climate-aware DSGE framework.

1.2 Households

The economy is populated by a continuum of identical households indexed by $i \in [0, l_t]$, where l_t denotes the population at time t.

Each household maximizes expected lifetime utility:

$$\mathcal{W}_{it} = \sum_{s=0}^{\infty} \beta^s \left(\frac{c_{it+s}^{1-\sigma_C}}{1-\sigma_C} - \frac{\chi_{t+s}}{1+\sigma_H} h_{it+s}^{1+\sigma_H} \right)$$

where c_{it} is consumption, h_{it} is labor supply, $\sigma_C > 0$ is the inverse of the intertemporal elasticity of substitution, and $\sigma_H > 0$ is the inverse of the Frisch elasticity of labor supply. The term χ_t captures the disutility weight on labor. Since consumption grows with productivity z_t while labor remains stationary, we define $\chi_t = z_t^{1-\sigma_C}$ to ensure that utility grows in a way that supports balanced growth.

The households budget constraint is:

$$r_{t-1}b_{it-1} + w_t h_{it} + tr_{it} = c_{it} + b_{it}$$

where b_{it} is a one-period risk-free bond, r_t is the real interest rate, w_t is the real wage, and tr_{it} is a lump-sum transfer from the government.

Optimality conditions from the householdâ€■s problem are standard. The Euler equation governs intertemporal consumption choices:

$$\frac{c_{it+1}^{-\sigma_C}}{c_{it}^{-\sigma_C}} r_t = 1$$

and the intratemporal condition equates the marginal rate of substitution between consumption and labor with the real wage:

$$w_t c_{it}^{-\sigma_C} = \chi_t h_{it}^{\sigma_H}$$

In what follows, we drop the individual index i, as the representative agent structure implies symmetric behavior across households.

1.3 Firms

The economy features a continuum of perfectly competitive firms indexed by $j \in [0, l_t]$. Each firm maximizes profits by choosing labor demand, output, and abatement effort, taking all prices as given. The profit function is:

$$\Pi_{jt} = y_{jt} - w_t h_{jt} - \tau_t e_{jt} - \theta_{1,t}\mu_{jt}^{\theta_2} y_{jt} \tag{1}$$

where y_{jt} is output, h_{jt} is labor demand at wage w_t, τ_t is the carbon tax applied to emissions e_{jt}, and $\theta_{1,t}\mu_{jt}^{\theta_2}$ denotes abatement costs, which scale with output. The parameter $\theta_2 > 1$ ensures strict convexity in the abatement effort $\mu_{jt} \in [0, 1]$, meaning that marginal abatement costs increase with effort. The cost function captures no learning-by-doing: the marginal cost of abatement rises over time until cleaner technologies arrive exogenously through $\theta_{1,t} \to 0$.

Output is produced using a linear technology:

$$y_{jt} = z_t \Omega(M_{t-1}) h_{jt} \tag{2}$$

where z_t is total factor productivity and $\Omega(M_{t-1})$ is a damage function that reduces productivity as a function of the atmospheric carbon stock M_{t-1}. Following **?**, we assume:

$$\Omega(M) = \exp(-\gamma M)$$

with $\gamma > 0$. This exponential form captures the non-linear impact of climate change on productivity.

Emissions are proportional to output, net of abatement effort:

$$e_{jt} = \sigma_t (1 - \mu_{jt}) y_{jt} \tag{3}$$

Here, σ_t captures emission intensity, which falls exogenously over time due to energy efficiency improvements. In the absence of a carbon tax ($\tau_t = 0$), the firm has no private incentive to abate, since it is atomistic and does not internalize the effect of its emissions on aggregate carbon stocks or damages.

Substituting emissions and the production function into the profit expression gives:

$$\max_{\{y_{jt},\mu_{jt}\}} \left(1 - \frac{w_t}{z_t\Omega(M_{t-1})} - \tau_t\sigma_t(1-\mu_{jt}) - \theta_{1,t}\mu_{jt}^{\theta_2}\right) y_{jt}$$

The firm's first-order conditions are:

$$\text{(i) Labor demand / output condition:} \quad \left(1 - \tau_t\sigma_t(1-\mu_{jt}) - \theta_{1,t}\mu_{jt}^{\theta_2}\right) z_t\Omega(M_{t-1}) = w_t$$
$$\text{(ii) Abatement effort:} \quad \mu_{jt}^{\theta_2-1} = \frac{\tau_t\sigma_t}{\theta_2\theta_{1,t}}$$

These expressions characterize the optimal behavior of firms facing climate policy and damages. The abatement decision reflects a trade-off between paying the carbon tax and investing in emission reductions, with stronger incentives to abate when the carbon tax τ_t is high, emission intensity σ_t is high, or technology makes abatement cheap (low $\theta_{1,t}$).

1.4 Climate

The climate block of the model is governed by a system of backward-looking difference equations, which resemble a vector autoregression (VAR) process. In general, climate dynamics can be expressed as:

$$\mathbf{C}_t = \Phi_C\mathbf{C}_{t-1} + \Phi_E E_t$$

where $\mathbf{C}_t$ is a vector of climate state variables (typically including atmospheric temperature, carbon stocks in various reservoirs, or radiative forcing). The matrix Φ_C captures the persistence and cross-effects across these compartments. The term E_t are the inflow of emissions and enter the climate system via Φ_E.

To keep things tractable, we reduce the system to a single equation for the atmospheric carbon stock M_t, which evolves according to:

$$M_t = (1-\delta_m)M_{t-1} + \chi_m \int_0^{l_t} e_{jt}\,\mathrm{d}j$$

Here, $\delta_m \in [0,1]$ is the natural decay rate of atmospheric carbon, and $\chi_m = \frac{3}{11}$ is a physical conversion factor translating gigatons of CO_2 into pure carbon units. This conversion is necessary because climate dynamicsâ€”particularly radiative forcing and the greenhouse effectâ€—are governed by the carbon content, not the dioxide molecule. If $\delta_m = 0$, atmospheric carbon accumulates permanently, leading to an endogenous climate trend (an ever-worsening path, possibly requiring degrowth to stabilize). If $\delta_m > 0$, then emissions eventually decay, and the system can stabilize over time as emissions $e_{jt} \to 0$, implying $M_t \to M_\infty$. This formulation can easily be extended to multi-compartment carbon models or to include temperature dynamics, if needed for policy evaluation or scientific realism.

1.5 Government

We consider a government that imposes an exogenous carbon tax τ_t on firm-level emissions. In contrast to the DICE model, where the carbon price is derived endogenously from a social plannerâ€™s problem (e.g., via Ramsey allocations), here the tax path is taken as given. The government redistributes carbon tax revenues to households through lump-sum transfers and maintains a balanced budget. The governmentâ€™s budget constraint is:

$$r_t\int_0^{l_t} b_{it-1}\,\mathrm{d}i + \int_0^{l_t} tr_{it}\,\mathrm{d}i = \tau_t\int_0^{l_t} e_{jt}\,\mathrm{d}j + \int_0^{l_t} b_{it}\,\mathrm{d}i$$

Under the assumption of zero net asset supply:

$$\int_0^{l_t} b_{it} \, \mathrm{d}i = 0$$

the government's role reduces to collecting carbon revenues and rebating them as lump-sum transfers. This setup isolates the allocative effects of carbon taxation without introducing additional government financing frictions. We now have all the building blocks necessary to define the equilibrium and study the transition dynamics of this decentralized integrated assessment model.

1.6 General Equilibrium

The general equilibrium of this economy is defined by a sequence of prices $\{w_t, r_t\}_{t=t_0}^{\infty}$, policies $\{\tau_t\}_{t=t_0}^{\infty}$, and allocations $\{c_t, h_t, b_t, y_t, h_t, \mu_t, e_t\}_{t=t_0}^{\infty}$ such that:

- Households maximize lifetime utility subject to their budget constraint,
- Firms maximize profits given technology, prices, and climate damages,
- The government respects its intertemporal budget constraint and levies an exogenous carbon tax,
- All markets clear in aggregate.

We now define the aggregate conditions that characterize this equilibrium.

Goods Market. Output is entirely consumed in equilibrium. Aggregate output is the sum of firm-level output over the population of firms:

$$Y_t = \int_0^{l_t} y_{jt} \, \mathrm{d}j, \; C_t = \int_0^{l_t} c_{it} \, \mathrm{d}i$$

Market clearing requires:

$$Y_t = C_t + \theta_{1,t} \mu_t^{\theta_2} Y_t$$

Labor Market. Labor demand by firms equals labor supply by households:

$$\int_0^{l_t} h_{jt} \, \mathrm{d}j = \int_0^{l_t} h_{it} \, \mathrm{d}i$$

Letting h_t denote the representative labor input per agent, we write: $h_t = \int_0^{l_t} h_{jt} \, \mathrm{d}j = \int_0^{l_t} h_{it} \, \mathrm{d}i$.

Emissions. Aggregate emissions are determined by firm-level emissions across the economy:

$$E_t = \int_0^{l_t} e_{jt} \, \mathrm{d}j$$

This quantity enters the carbon dynamics equation governing the stock of atmospheric carbon.

The equilibrium is recursive: given the exogenous state variables $\{z_t, l_t, \sigma_t, \chi_t, \theta_{1,t}\}$, the endogenous state M_{t-1}, and policy path $\{\tau_t\}$, prices and allocations adjust to ensure that households and firms optimize and all markets clear.

1.7 Summary

To solve the model numerically using a perfect foresight solver, we first detrend the endogenous variables. While detrending is not strictly required for convergence, it significantly improves numerical stability and speed. The reason is simple: large or fast-growing variables (e.g., output or consumption) generate mechanically large residuals in the solution algorithm, even if the model is well-specified. Detrending brings all variables to a scale consistent with stationarity and helps the solver (which uses residual minimization) converge more quickly and accurately.

Let us define detrended variables per effective worker as:

$$\hat{c}_t = \frac{C_t}{z_t l_t}, \quad \hat{y}_t = \frac{Y_t}{z_t l_t}, \quad \hat{\tau}_t = \frac{\tau_t \sigma_t}{\theta_2 \theta_{1,t}}, \qquad \hat{w}_t = \frac{w_t}{z_t},$$

We now write the key equilibrium conditions in detrended form:

$$\begin{aligned}
\text{Euler equation:} \qquad & r_t = \left[(1+g_{z,t+1})\frac{\hat{c}_{t+1}}{\hat{c}_t}\right]^{\sigma_C} \\
\text{Labor-leisure optimality:} \qquad & \hat{w}_t = \chi h_t^{\sigma_H} \hat{c}_t^{\sigma_C} \\
\text{Production:} \qquad & \hat{y}_t = \Omega(M_{t-1}) h_t \\
\text{Firm optimality (wage condition):} \qquad & \hat{w}_t = \left(1 - \theta_{1,t}\left(\hat{\tau}_t \theta_2 (1-\mu_t) + \mu_t^{\theta_2}\right)\right)\Omega(M_{t-1}) \\
\text{Abatement effort:} \qquad & \mu_t = \hat{\tau}_t^{1/(\theta_2-1)} \\
\text{Aggregate emissions:} \qquad & E_t = \sigma_t (1-\mu_t) \hat{y}_t z_t l_t \\
\text{Carbon dynamics:} \qquad & M_t = (1-\delta_m) M_{t-1} + \chi_m E_t
\end{aligned}$$

$$\text{Resource constraint:} \quad \hat{y}_t = \hat{c}_t + \theta_{1,t}\mu_t^{\theta_2}\hat{y}_t$$

As in SSV, we control tax path as follows:

$$\hat{\tau}_t = \varphi \hat{\tau}_t^*$$

where $\hat{\tau}_t^*$ is the tax path consistent with Paris Agreement.

These equations are solved jointly with the exogenous trends:

$$\begin{aligned}
z_t &= (1+g_{z,t}) z_{t-1} \\
g_{z,t} &= (1-\delta_z) g_{z,t-1} \\
l_t &= l_{t-1}^{\ell} \cdot l_{\infty}^{1-\ell} \\
\sigma_t &= (1-\delta_\sigma)\sigma_{t-1} \\
\theta_{1,t} &= \frac{p_b \cdot \sigma_t \cdot \varrho_t}{\theta_2} \\
\varrho_t &= (1-\delta_\theta)\varrho_{t-1}
\end{aligned}$$

This completes the logics of the model. The system can now be solved as a perfect foresight transition path starting from an initial carbon stock M_{t_0-1} and policy path $\{\tau_t\}_{t \geq t_0}$. The model is readily implementable in Dynare.

2. A Modified Dynare code for Climate Dynamics

Recall that we need to think our model outside the usual steady state approach. Our dynamic problem is an extension of **?** to deal with trends:

$$\tilde{y}_t = g_\Theta\left(y_0, y_\infty, 0\right) \tag{4}$$

$$y_t = \mathbb{E}_{t,t+S}\left\{g_\Theta\left(y_{t-1}, \tilde{y}_{t+S+1}, \varepsilon_t\right)\right\} \tag{5}$$

$$\varepsilon_t \sim \mathcal{N}\left(0, \Sigma_\varepsilon\right) \tag{6}$$

where $\{\tilde{y}_t\}_{t=1}^T$ is the deterministic (transitional component) path from given state y_0 up to steady state y_∞, while $\{y_t\}_{t=1}^T$ is the stochastic path with innovations. In what follows $\mathbb{E}_{t,t+S}\{\cdot\}$ refers to expectation horizon S for updating the path when a shock emerge.

2.1 Model definition

Define the model equations, calibration as usual, including trends (see `B_RBC/model_file.mod`).

Initial vector: Next comes definition of initial state vector y_0 which are constant terms set by user:

```
endval;
Z = Z0;
L = L0;
gZ = gZ0;
M = M0;
delthet = delthet0;
THETA1 = THETA0;
y = y0;
SIG = SIG0;
end;
```

Terminal vector: we next define the terminal vector, the true steady state y_∞ of the model that the model will reach (including the effects of climate change).

```
steady_state_model;
chi = 1;
s_a = 1;
(...)
end;
```

Simulations Options. Finally, the set of options, coding the time of the simulation $T = 3,000$, the initial date $t = 0$ (here 1984Q4), the size of expectations window $S = 100$, the vector of surprises ε_t.

```
options_.initial_guess_path = 'guess_path';
options_.expectation_window = 100;
options_.forward_path                  = 3000;
options_.surprise_shocks        = logical(ones(M_.exo_nbr,1));
options_.surprise_shocks(strcmp(M_.exo_names,'e_tau'))=logical(0);
options_.dates0                         = dates('1984Q4');
```

Carbon tax path. Finally we also compute the path of the expected carbon tax:

```
idtau                                = strmatch('e_tau',M_.exo_names,'exact');
id_now                                = find(exo_init_ts.dates==dates('2023Q3')+1);
id_last                               = find(exo_init_ts.dates==dates('2100Q1'));
id_taxmax                     = find(exo_init_ts.dates==dates('2049Q4'));
exo_init                             = exo_init_ts.data;
exo_init(id_now:id_taxmax,idtau) = linspace(0,1,(id_taxmax-id_now)+1);
exo_init(id_taxmax:end,idtau)    = 1;
exo_init(:,idtau)                     = smoothdata(exo_init(:,idtau),'gaussian',10);
exo_init_ts                      = dseries(exo_init,exo_init_ts.dates(1),exo_init_ts.name);
```

2.2 Simulations

One can simulate the model easily, see in `B_RBC/run3_compare_nz0_bau_stochastic.mod`

Deterministic path. It is computed in two step. First compute the initial guess, which will reduce the computation time:

```
[y_guess,oo_,M_]      = EP_paths_init(oo_,M_,options_,exo_init_ts);
```

Use the guess to compute the deterministic path:

```
[det_bau]                    = EP_deterministic_path(y_guess,exo_init_ts,oo_,M_,options_);
```

Stochastic path. One can introduce some innovations, that we pick into `innovations_ts`.

```
[sto_bau]              = EP_stochastic_path(det_bau,innovations_ts,oo_,...
                                M_,options_,stochastic_dates);
```

References

Jondeau, E., Levieuge, G., Sahuc, J.-G., and Vermandel, G. (2022). Environmental subsidies to mitigate transition risk.

Sahuc, J.-G., Smets, F., and Vermandel, G. (2025). The new keynesian climate model.