Optimal Growth Model: Summary

1 Model equations

We consider a social planner who chooses the path of saving $s_t \in [0,1]$ to maximize intertemporal welfare, subject to the economy's dynamics.

Production

Output is given by a Cobb–Douglas technology

$$Y_t \;=\; A_t\,K_t^{\alpha}\,L_t^{1-\alpha},$$

where A_t is total factor productivity (TFP), K_t the capital stock, and L_t population.

Resource constraint

$$Y_t = C_t + I_t, \qquad C_t = (1-s_t)Y_t, \qquad I_t = s_t Y_t.$$

Capital accumulation

$$K_{t+1} = (1-\delta)^{\Delta} K_t + \Delta I_t,$$

with depreciation rate δ and time step Δ (in years).

Exogenous processes

TFP dynamics. The law of motion for TFP is

$$A_t \;=\; \frac{A_{t-1}}{1-g_A\,e^{-\delta_A \Delta (t-1)}},$$

where g_A is the initial productivity growth rate and δ_A controls its decline.

Population dynamics. Population follows a logistic law of motion

$$L_t = L_{t-1} \cdot \left(\frac{L_\infty}{L_{t-1}} \right)^{\ell_g},$$

with carrying capacity L_∞ and convergence parameter ℓ_g.

Preferences

The representative household has CRRA utility over per-capita consumption $c_t = C_t/L_t$:

$$U(c_t, L_t) = L_t \cdot \frac{\left(c_t\right)^{1-\gamma} - 1}{1-\gamma},$$

with risk aversion γ.

Lifetime welfare is

$$W = \sum_{t=0}^{T_{\text{planner}}} \frac{1}{(1+\rho)^{t\Delta}} U(c_t, L_t),$$

where ρ is the pure rate of time preference and T_{planner} the effective horizon determined by discounting and tolerance.

2 Calibration

3 Planner's optimization problem

The planner chooses $\{s_t\}_{t=0}^{T_{\text{planner}}}$ to solve

$$\max_{\{s_t\}} \sum_{t=0}^{T_{\text{planner}}} \frac{1}{(1+\rho)^{t\Delta}} L_t \frac{\left(c_t\right)^{1-\gamma} - 1}{1-\gamma}$$

subject to:

$$\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha}, \\
C_t &= (1-s_t)Y_t, \quad I_t = s_t Y_t, \\
K_{t+1} &= (1-\delta)^{\Delta} K_t + \Delta I_t, \\
A_t &= \frac{A_{t-1}}{1 - g_A\, e^{-\delta_A \Delta (t-1)}}, \\
L_t &= L_{t-1} \cdot \left(\frac{L_\infty}{L_{t-1}} \right)^{\ell_g}, \\
K_0 &= 3.0,\ A_0 = 1.0,\ L_0 = 1.0.
\end{aligned}$$

Parameter	Value	Description
Δ	1	Time step (years)
t_0	1	Start year
t_T	100	End year
α	0.30	Capital share
A_0	1.0	Initial productivity
g_A	0.015×0	Initial TFP growth per year (baseline set to 0)
δ_A	0.005	Decline rate of TFP growth
L_0	1.0	Initial population (normalized)
L_∞	10500	Asymptotic population (millions)
ℓ_g	0.134/5×0	Population growth parameter (baseline set to 0)
δ	0.06	Depreciation rate
K_0	3.0	Initial capital stock
ρ	0.015	Pure rate of time preference
γ	2.0	Relative risk aversion (1/IES)

Table 1: Baseline calibration of the optimal growth model.