Modélisation, estimation, simulation des risques climatiques

Introducing an Economic Block

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Technical requirements

- **Python** preferably with conda
  if not yet installed, do it now it takes ∼15min
  - `https://anaconda.org/anaconda/conda`
  - Exercises in python are done under **Jupyter notebooks**
  - All materials are available on Moodle
    `https://moodle.ip-paris.fr/course/view.php?id=10454`
  - Main python packages:

    [sys] [os] [math] [numpy] [pandas] [matplotlib] [pandas] [spicy]

Schedule of the day

Objectives

- Understanding the **basics** of macroeconomic model with saving/consumption.
- Simulating macroeconomic models .

Materials:

- OptimalGrowth_Notebook.ipynb *notebook introduction of model's options.*
- OptimalGrowth.ipynb *python package solving optimal growth models.*

Some proofs:

- Olivier Loisel's lectures on growth models [Lecture I] [Lecture II]

Introduction

Intro: what is an integrated assessment model?

- **Definition:** IAMs are a combination of science from different fields: economics, climate, and energy systems into a unified quantitative framework.
- **Purpose:** Provide a tractable way to evaluate long-run interactions between the economy and the climate.
- **Institutional use:**
  - **IPCC** (Intergovernmental Panel on Climate Change) relies on IAMs to define emission scenarios (e.g. SSP pathways).
  - **Central banks, NGFS, OECD, IEA, World Bank** use IAM-based tools for stress testing, policy design, and energy forecasts.
- IAMs are **quantitative tools** used for scenario design, policy evaluation, and to inform international climate negotiations.

Intro: IAM schematic: economy-climate feedbacks

Real Economy
$Y_t = f(A_t \, K_t \, L_t)$
Consumption / Investment

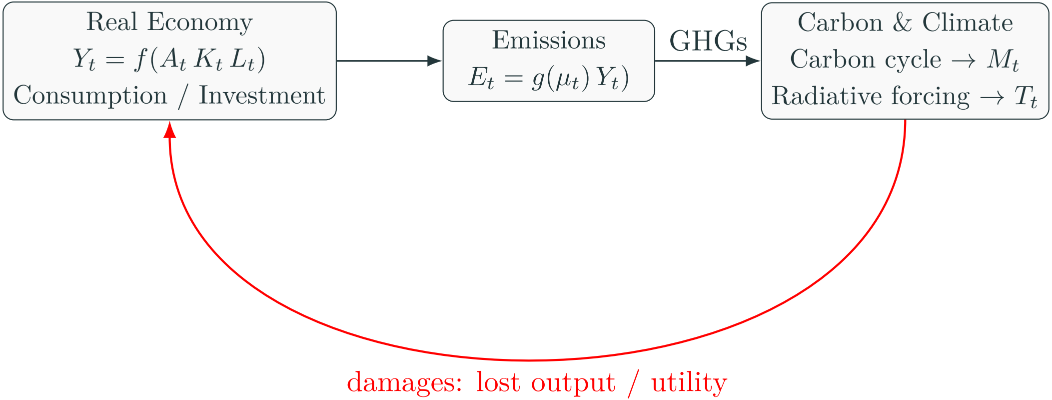
Intro: IAM schematic: economy-climate feedbacks

Real Economy
$Y_t = f(A_t\, K_t\, L_t)$
Consumption / Investment

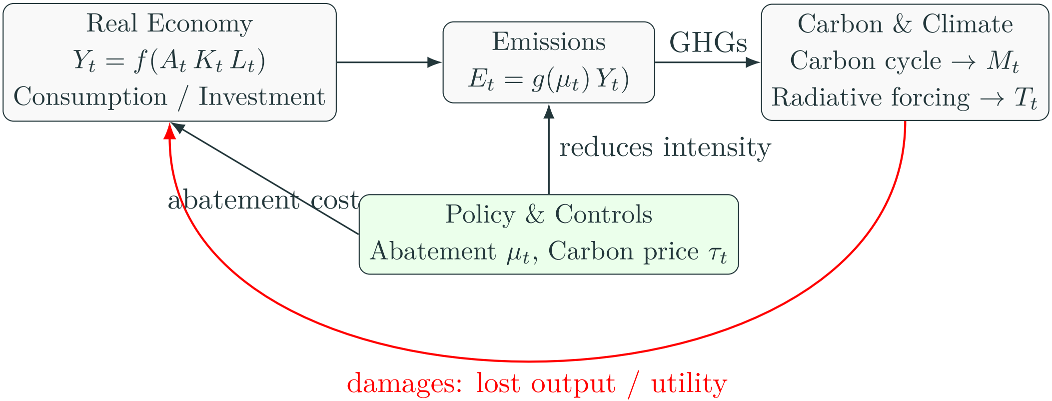
$\longrightarrow$

Emissions
$E_t = g(\mu_t)\, Y_t)$

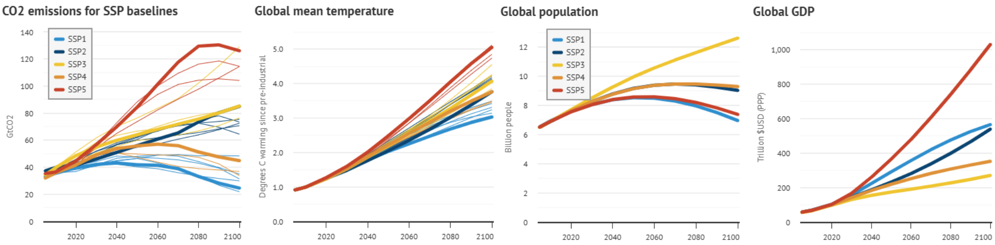
Intro: IAM schematic: economy-climate feedbacks



Intro: IAM schematic: economy-climate feedbacks



Intro: Illustration of IAM output



Notes: Illustrative SSP baseline trajectories across models.

Intro: What's inside an IAM?

- **Two core blocks** today we treat them as *separate objects*:
  - **Economic block** (2 hours today)

    Goal: optimal growth / social planner chooses saving (and later abatement). *Assumptions (standard in macro, exotic for mathematicians):* Cobb-Douglas technology, CRRA preferences, competitive pricing, simple capital law.

    Why we use them: analytical clarity, stability, (often) uniqueness of equilibrium, and transparent welfare.
  - **Climate block** (2 hours today)

    Goal: map emissions to concentrations, forcing, and temperature.

    Structure: few ODEs / linear difference equations (carbon cycle boxes, 2-box temperature), calibrated to climate science.

Intro: The big picture: from optimal growth to modern IAMs

- **Optimal growth tradition** in economics:
  - **Ramsey (1928)**: how much to save vs. consume over time?
  - **Cass (1965) - Koopmans (1965)**: rigorous optimal growth with capital accumulation, stability and uniqueness of equilibrium.

  Provides the *benchmark model* of a social planner maximizing intertemporal welfare.

- **Nordhaus (1992)**: integrated this framework with climate science.
  - Extended optimal growth to include *emissions, carbon cycle, temperature*.
  - Added *damages* and *abatement* as economic choices.
  - Created the **DICE model**, a prototype IAM (Integrated Assessment Model).

- **Big picture:** IAMs are the continuation of the optimal growth tradition, enriched with climate dynamics.

Core building blocks

Building Blocks: main ingredients

- **Discrete time** framework
  - Period length fixed (e.g. annual).
  - Easier to match with economic and climate data.

- **Notation:** X_t denotes the value of variable X in period t.

- **Centralized economy** (social planner)
  - A single agent (the planner) chooses allocations for society $\rightarrow$ deterministic optimal control problem.

- **Aggregate representation**
  - Output, capital, consumption, labor all modeled as aggregate variables.
  - Abstracts from heterogeneity to focus on long-run dynamics.

Building Blocks: Production function I

- **Gross Domestic Product (GDP)**: total value of goods and services produced in one period. $\rightarrow$ It is a **flow**, not a stock.

- GDP is calculated from a production function $F(\cdot)$ which we define as "Cobb-Douglas":

$$Y_t = F(K_t, A_t L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

  - $Y_t \geq 0$: output in period t (GDP),
  - $A_t \geq 0$: productivity (technology),
  - $K_t \geq 0$: capital,
  - $L_t \geq 0$: labor,
  - $\alpha \in [0, 1]$: capital share (~ 0.3 empirically).

Building Blocks: Production function II

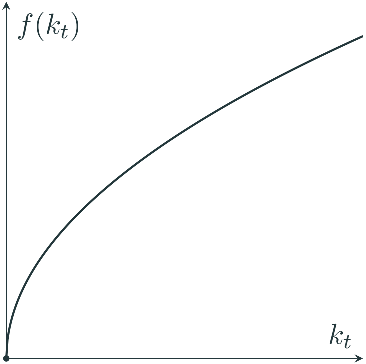
Denoting by $k_t = \frac{K_t}{A_t L_t}$ the stock of capital per effective-labor unit, we get

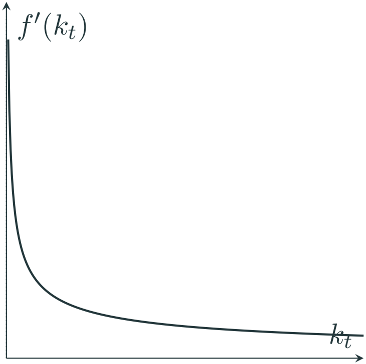
$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} F(K_t, A_t L_t) = F(k_t, 1) \equiv f(k_t)$$

where f has the following properties:

1. $f : \mathbb{R}^+ \to \mathbb{R}^+,\ k \mapsto f(k)$, with $f(0) = 0$,
2. f is **strictly increasing**: $\forall k \in \mathbb{R}^+,\ f'(k) > 0$,
3. f is **strictly concave**: $\forall k \in \mathbb{R}^+,\ f''(k) < 0$,
4. f satisfies the **Inada conditions**: $\lim_{k \to 0} f'(k) = +\infty$ and $\lim_{k \to +\infty} f'(k) = 0$:
    - $\lim_{k \to 0} f'(k)$ infinite $f'(k)$ guarantees that it is always worthwhile to accumulate at least some capital: avoid "poverty trap" at $k = 0$
    - $\lim_{k \to +\infty} f'(k)$ prevents infinite growth simply from capital accumulation alone.

Building Blocks: Production function III





Building Blocks: Why Cobb–Douglas?

- **Homogeneity of degree 1** (constant returns to scale):

$$Y(\lambda K, \lambda L) = \lambda Y(K, L)$$

  $\rightarrow$ doubling capital and labor doubles output.

- **Implications:**
  - Factor shares are constant over time.
  - Each factor has diminishing marginal productivity.
  - Guarantees stability and uniqueness of equilibrium in optimal growth models.

- Widely used in macroeconomics for simplicity and tractability.

Building Blocks: GDP allocation

- Recall: **GDP Y_t** is the total value of goods and services produced in one period. $\rightarrow$ It is a **flow**, not a stock.

- From the **demand side**, GDP is allocated to:

$$Y_t = C_t + I_t$$

  - C_t: consumption (households, public),
  - I_t: investment (new machines, buildings, infrastructure).

- Example:
  - A household buying food $\rightarrow$ consumption.
  - A firm building a new factory $\rightarrow$ investment.

Building Blocks: Capital accumulation

- **Capital dynamics:**

$$K_{t+1} = (1-\delta)K_t + I_t$$

  where:
  - K_t: stock of capital (machines, factories, infrastructure),
  - δ: depreciation rate (typically $\sim 5\%$ per year),
  - I_t: new investment.

- **Interpretation:**
  - Each period, a fraction δ of capital "wears out".
  - Investment replenishes and grows the capital stock.
  - Today's saving $\rightarrow$ tomorrow's productive capacity.

- This law of motion is the **state equation** of the optimal growth model.

Building Blocks: Preferences (setup)

- In growth models, we evaluate **per capita consumption**:

$$c_t = \frac{C_t}{L_t}$$

  where C_t is aggregate consumption, L_t is population.

- **Utility at the individual level**: captures satisfaction from consumption of goods at date t.

- Standard assumption: **CRRA utility** (constant relative risk aversion):

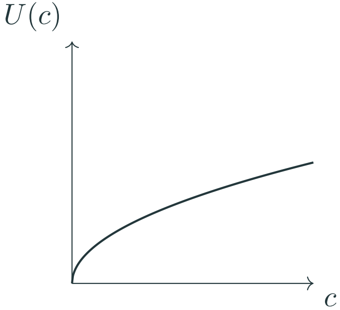
$$U(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0$$

  - γ: curvature parameter = risk aversion = 1/(intertemporal elasticity of substitution).

Building Blocks: Preferences II

- U: $\mathbb{R}^+ \to \mathbb{R}$ (U can take negative values, what matters is the relative difference)
- U is strictly increasing: $\forall c \in \mathbb{R}^+$, $u'(c) > 0$
- U is strictly concave: $\forall c \in \mathbb{R}^+$, $u''(c) < 0$
- U satisfies "Inada conditions": $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$
  - Ensures an interior optimum in dynamic optimization problems (no corner solutions).
  - Interpretation:
    $\lim_{c \to 0} u'(c)$ formalizes the idea that **starving is infinitely painful** (so agents avoid it),
    $\lim_{c \to +\infty} u'(c)$ formalizes satiation: more and **more consumption eventually adds negligible utility**.

Building Blocks: Preferences III



Building Blocks: Intertemporal welfare

- The planner values utility flows across time:

$$W = \sum_{t=0}^{\infty} \beta^t L_t U(c_t)$$

- **Discounting**:

$$\beta = \frac{1}{1+\rho}, \quad \rho > 0$$

  - ρ: pure rate of time preference.
  - Low ρ: future generations valued more (Stern).
  - High ρ: more weight on present (Nordhaus).

- Preferences combine two dimensions:
  1. *Across goods*: concavity (γ) governs curvature of $U(c)$.
  2. *Across time*: discounting (β, ρ) governs how much future is valued.

Building Blocks: Introducing the saving rate

- Recall the resource identity with $c_t = C_t/L_t$:

$$Y_t = c_t L_t + I_t$$

- Define the **saving rate** s_t:

$$I_t = s_t Y_t, \quad C_t = (1 - s_t) Y_t$$

- Interpretation:
  - s_t is the **balance-sheet view of GDP**: it tells us how much of income is devoted to *future capacity* (investment) vs *current use* (consumption).
  - This simple choice variable links today's output to tomorrow's capital.

Building Blocks: model recap

- A set of **exogenous variables** (independent of controls)

$$x_t = \{A_t, L_t\}$$

- A set of **endogenous variables**

$$y_t = \{c_t, Y_t, K_t\}$$

- A set of **control variables**

$$z_t = \{s_t\}$$

Optimal growth problem

Optimal growth problem: setup

- **Objective function:**

$$\max_{0 \leq \{s_t\}_{t=0}^{\infty} \leq 1} \quad W_t = \sum_{s=0}^{\infty} \beta^s \, L_{t+s} \, U(c_{t+s}),$$

- **Constraints:**

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \qquad \text{(production)}$$

$$Y_t = L_t c_t + I_t \qquad \text{(resource allocation)}$$

$$K_{t+1} = (1-\delta) K_t + I_t \qquad \text{(capital accumulation)}$$

- **Control:**

$$s_t \in [0,1] \quad \text{(saving rate)}$$

determines the split: $I_t = s_t Y_t$, $C_t = (1 - s_t) Y_t$.

Optimal growth problem: Ways to solve the optimal growth problem

1. **Dynamic programming (Bellman)** Value function $V(K_t)$, recursive solution. Guarantees existence/uniqueness but often costly numerically.
2. **Ramsey approach (Euler)** Derive first-order conditions (Euler eqs.), solve with Newton methods. Provides analytical intuition on intertemporal trade-offs.
3. **Brute-force / direct optimization** Parameterize $\{s_t\}$ over horizon, use a nonlinear solver. Easy to implement, but less efficient.

In class: we will use the *brute-force approach* in Python because it is very flexible, but we connect back to the Ramsey conditions to explain mechanism.

Why concavity matters

- **Concavity of utility** $U(c)$ and production $Y(K, L)$ ensures:
  - The planner's objective $W = \sum \beta^t L_t U(c_t)$ is *strictly concave* in the control sequence $\{s_t\}$.
  - The constraints (capital accumulation, resource constraint) are linear/convex.
- **Implications:**
  - The optimization problem is a **convex program**.
  - Existence of an optimal solution is guaranteed.
  - Strict concavity $\Rightarrow$ **uniqueness** of the optimal path (c_t, K_t).
- **For dynamic programming:** Concavity of the Bellman operator $\Rightarrow$ value function is concave $\Rightarrow$ optimal policy is well-defined and stable.

Optimal growth problem: the Ramsey approach I

We use the **Ramsey approach**: Optimal-control theory with setting up the Lagrangian with two constraints (capital law and resource allocation via s_t):

$$\begin{aligned}\mathcal{L} = & \textstyle\sum_{i=0}^{\infty} \beta^i \, L_{t+i} \, U(c_{t+i}) \\ & + \textstyle\sum_{i=0}^{\infty} \beta^i \lambda_{1t+i} \left[K_{t+1+i} - (1-\delta) K_{t+i} - s_{t+i} A_{t+i} K_{t+i}^{\alpha} L_{t+i}^{1-\alpha} \right] \\ & + \textstyle\sum_{i=0}^{\infty} \beta^i \lambda_{2t+i} \left[(1 - s_{t+i}) A_{t+i} K_{t+i}^{\alpha} L_{t+i}^{1-\alpha} - L_{t+i} c_{t+i} \right].\end{aligned}$$

We call $\lambda_{1t} \geq 0$ and $\lambda_{2t} \geq 0$ costate variables, while $s_t \geq 0$ is a control and $K_{t+1} \geq 0$ is a state variable

Optimal growth problem: the Ramsey approach II

First-order conditions (interior solution):

$$c_t : \quad L_t U'(c_t) - \lambda_{2t} L_t = 0$$

$$s_t : \quad -\lambda_{1t} + \lambda_{2t} = 0$$

$$K_{t+1} : \quad \lambda_{1t} = \beta\lambda_{1,t+1}\left[(1-\delta) + \alpha s_{t+1}\frac{Y_{t+1}}{K_{t+1}}\right] - \beta\lambda_{2,t+1}\left[(1-s_{t+1})\,\alpha\,\frac{Y_{t+1}}{K_{t+1}}\right].$$

Optimal growth problem: the optimal condition

Remarks

- Two multipliers enforce two constraints; at the optimum they collapse to $\lambda_t = U'(c_t)$.
- Strict concavity of U and Cobb-Douglas production ensures existence and uniqueness of the optimal path.

Ramsey rule:

$$\boxed{\mathrm{c}_t^{-\gamma} = \beta\, \mathrm{c}_{t+1}^{-\gamma} \left[(1-\delta) + \alpha\, \frac{Y_{t+1}}{K_{t+1}} \right].}$$

Interpretation:

- c_{t+1}, Y_{t+1} are forward-looking variables.
- The system links today's decisions to both future and past states: a mix of backward- and forward-looking dynamics.

Optimal growth problem: summary

Assumptions (for simplicity):

$$A_t = \bar{A} \quad \text{(constant productivity)}, \qquad L_t = \bar{L} \quad \text{(constant population)}.$$

Dynamic system:

$$\text{(Ramsey)} \quad c_t^{-\gamma} = \beta\, c_{t+1}^{-\gamma} \Big[(1-\delta) + \alpha\, K_{t+1}^{\alpha-1} \Big],$$

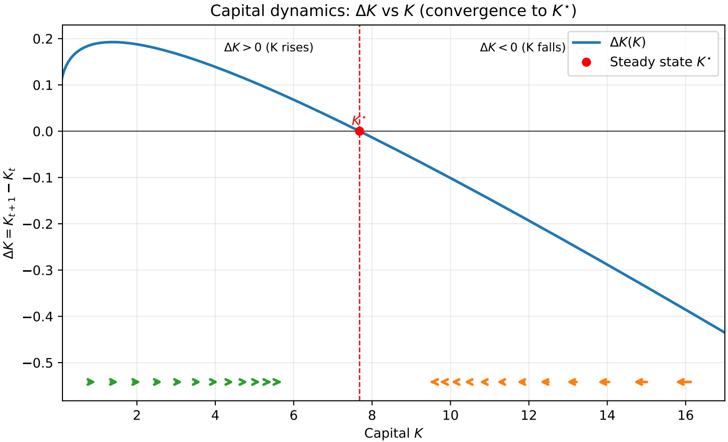
$$\text{(Capital law)} \quad K_{t+1} = (1-\delta) K_t + K_t^{\alpha} - c_t.$$

Properties:

- Given $K_0 \geq 0$, the system generates a path $\{c_t, K_t\}$ converge to a unique *steady state*:

$$\bar{K} = \left(\frac{1}{\alpha} \left[\frac{1}{\beta} - (1-\delta) \right] \right)^{\frac{1}{\alpha-1}}.$$

Optimal growth problem: Diagram



$$\alpha = 0.3,\ \beta = 1/(1 + 0.015),\ \delta = 0.10.$$

Optimal growth problem: Diagram Interpretation

Interpretation:

- Case $K_0 \leq \bar{K}$
  - Capital is too low relative to its steady-state.
  - This requires higher investment (a larger share of output devoted to I_t) and thus lower consumption in the short run and capital accumulates $\Delta K > 0$
  - Over time, capital rises toward $\bar{K}$.
- Case $K_0 \geq \bar{K}$
  - Opposite mechanism: overaccumulation of capital $\rightarrow$ more consumption.
  - Depreciation dominates $\Delta K < 0$
- Case $K_0 = \bar{K}$
  - ΔK capital stock stationary. Consumption and investment stable.

If you want the proof of convergence with saddle path see: [Lecture II]

Control Problem: a sketch of the numerical solution I

- Because the social welfare is forward looking, the system is both forward and backward looking as follows:

$$s_t = \arg \min_{\{s_t\}_{i\geq 0}} - \sum_{i=0}^{\infty} (1+\rho)^{-i} L_{t+i}\, u(c_{t+i})$$

- When a sequence $\{s_{t+i}\}_{i=0}^{\infty}$ is found, one can compute:

$$\begin{aligned} x_t &= g(x_{t-1}), & g:\ \mathbb{R}^{N_x} &\to \mathbb{R}^{N_x}, \\ y_t &= f(y_{t-1},\, x_t,\, z_t), & f:\ \mathbb{R}^{N_y} \times \mathbb{R}^{N_x} \times \mathbb{R}^2 &\to \mathbb{R}^{N_y}. \end{aligned}$$

- $x_t \in \mathbb{R}^{N_x}$: **exogenous variables** $(A_t,\ L_t)$.
- $y_t \in \mathbb{R}^{N_y}$: **endogenous states/flows** (c_t, Y_t, K_t).
- $z_t = [s_t]' \in \mathbb{R}$: **decision variables** (saving).

Control Problem: a sketch of the numerical solution II

- **Idea.** Approximate the **infinite-horizon** objective with a **finite** one. Let $\beta \equiv (1+\rho)^{-1}$. Because $\lim_{i\to\infty} \beta^i = 0$, the tail contributes little.
- **Pick the horizon $I^\star$** so that the tail weight is below a tolerance ε:

$$\beta^{I^\star} \leq \varepsilon$$

  Trade-off: smaller $I^\star \Rightarrow$ faster solve but larger truncation error.
- **Finite-horizon problem** that solves

$$\max_{\{s_{t+i},\, \mu_{t+i}\}_{i=0}^{I^\star}} \sum_{i=0}^{I^\star} \beta^{i} L_{t+i} U\left(c_{t+i}\right)$$

s.t. (resource, capital) and bounds.

Thank you!

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