# Research on matrix problems based on Quadratic Forms


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## Background

Our problem is drived from max problem of quadratic form and Bell nonlocality.

We mainly study extreme value problems,but the exat solutions were proven impossible. We break down the problem into simpler ones.The main method we use are SVD and Lagrange Multiplier .

## Methods

### Method Ⅰ: SVD

$M^tM = V\Sigma U^tU\Sigma V^t = V\Sigma\Sigma V^t$ (U and V are orthogonal matrix)

let $L = \Sigma\Sigma = \begin{pmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{pmatrix}\begin{pmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{pmatrix} = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_r\end{pmatrix}$

Then $M^tMV = VL$ and $MM^tU = UL$

To get V,we need to solve the eigenvector of $M^tM$ : $M^tM\vec{v} = \lambda\vec{v}$

$$M^tM\begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix} = \begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix}\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix} \longleftrightarrow M^tMV = VL$$

$$MM^t\begin{pmatrix}\vec{u}_1 & \vec{u}_2\end{pmatrix} = \begin{pmatrix}\vec{u}_1 & \vec{u}_2\end{pmatrix}\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix} \longleftrightarrow MM^tU = UL$$

$$L = \begin{pmatrix}\sigma_1^2 & \\ & \sigma_2^2\end{pmatrix} = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$$

### Method Ⅱ: Lagrange Multiplier Method

1) Lagrang first: $F(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$

$F'x=f'x(x,y)+\lambda\varphi'x(x,y)=0$

$F'y=f'y(x,y)+\lambda\varphi'y(x,y)=0$

$F'\lambda=\varphi(x,y)=0$

2) Lagrangian multiplier method for multi-constraints

$F(x,\lambda) = f(x) + \sum_k \lambda_k\varphi_k(x)$

$F'_x = f'(x) + \sum_k \lambda_k\varphi'(x) = 0$

$F'_{\lambda_k} = \varphi(x) = 0$

3) For matrix

$q(x) = \sum_{i,j} a_{ij}\vec{x}_i\vec{y}_j$, $\quad \sum_{i=1}^{d} x_{ki}^2 = 1$, $\quad \sum_{j=1}^{d} y_{kj}^2 = 1$

$F(\vec{x}_1,\cdots\vec{x}_n,\vec{y}_1,\cdots\vec{y}_n,\lambda_1,\cdots\lambda_n,\mu_1,\cdots\mu_n) = q(x) + \sum_{k=1}^{n}\lambda_k(\sum_{i=1}^{d}x_{ki}^2-1) + \sum_{k=1}^{n}\mu_k(\sum_{j=1}^{d}y_{kj}^2-1)$

$\frac{\partial F}{\partial x_{k\gamma}} = \sum_{j=1}^{n} a_{kj}y_{j\gamma} + 2\lambda_k x_{k\gamma} = 0$

$\frac{\partial F}{\partial y_{k\gamma}} = \sum_{i=1}^{n} a_{ik}x_{i\gamma} + 2\mu_k y_{k\gamma} = 0$

$\frac{\partial F}{\partial \lambda_k} = \sum_{i=1}^{d} x_{ki}^2 - 1 = 0$

$\frac{\partial F}{\partial \mu_k} = \sum_{j=1}^{d} y_{kj}^2 - 1 = 0$

## Special Stitutions

Given that $A^T=A$, $q(x)=x^TAx$ and $|x|=1$

$\xrightarrow{\text{Larange}}$ $F(x_1,\cdots x_n,\lambda) = q(x) + \lambda(\sum_{i=1}^{n} x_i^2 - 1)$

Last, we can get $\lambda = -q(x)$

$\xrightarrow{\text{SVD}}$ $A = U^t\Lambda U$

$q(x) = y^t\Lambda y = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$

$(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)(|y|=|x|=1)$

$q(y_1=1, y_2=0, \dots y_n=0) = \lambda_1$

$q(y_1=0, y_2=0, \dots y_n=1) = \lambda_n$

Given that A is any n-order real square matrix and $q(x)=x^TAy$ and $|x|=|y|=1$

$\xrightarrow{\text{SVD}}$ $A = U^t\Lambda U$

$x^TAy = X^T\Lambda Y = \sigma_1X_1Y_1 + \sigma_2X_2Y_2 + \dots + \sigma_rX_rY_r$

$(\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0)$

$q(X_1=1, Y_1=1, X_2=0, Y_2=0, \dots, X_r=0, Y_r=0) = \sigma_1$

$q(X_1=-1, Y_1=1, X_2=0, Y_2=0, \dots, X_r=0, Y_r=0) = -\sigma_1$

## Derivation

Given that A is any n-order real square matrix, $q(x) = \vec{x}^T A\vec{y} = \sum_{k,j=1}^{n} a_{kj}\vec{x}_k\vec{y}_j$ and $\vec{x}_i, \vec{y}_i \in R^d$ and $|\vec{x}_i| = |\vec{y}_i| = 1$,

try to find the maximum and the minimum of q(x)

Larange

$\frac{\partial q}{\partial x_{kr}} = \sum_{j=1}^{n} a_{kj}y_{jr} = 0$, $k \in \{1,2,\dots n\}$, $\gamma \in \{1,2,\dots d\}$.

$\frac{\partial q}{\partial y_{kr}} = \sum_{j=1}^{n} a_{kj}x_{jr} = 0$, $k \in \{1,2,\dots n\}$, $\gamma \in \{1,2,\dots d\}$.

boundary $\sum_{i=1}^{d} x_{ki}^2 = 1$, $\sum_{j=1}^{d} y_{kj}^2 = 1$ $k \in \{1,2,\cdots,n\}$.

so: $F(\vec{x}_1,\cdots\vec{x}_n,\vec{y}_1,\cdots,\vec{y}_n,\lambda_1,\cdots,\lambda_n,\mu_1,\cdots,\mu_n) = q(x) + \sum_{k=1}^{n}\lambda_k(\sum_{i=1}^{d}x_{ki}^2 - 1) + \sum_{k=1}^{n}u_k(\sum_{j=1}^{d}y_{kj}^2 - 1)$

then we get:

$\frac{\partial F}{\partial x_{k\gamma}} = \sum_{j=1}^{n} a_{kj}y_{j\gamma} + 2\lambda_k x_{k\gamma} = 0$ $\quad \frac{\partial F}{\partial \lambda_k} = \sum_{i=1}^{d} x_{ki}^2 - 1 = 0$

$\frac{\partial F}{\partial y_{k\gamma}} = \sum_{i=1}^{n} a_{ik}x_{i\gamma} + 2\mu_k y_{k\gamma} = 0$ $\quad \frac{\partial F}{\partial \mu_k} = \sum_{j=1}^{d} y_{kj}^2 - 1 = 0$

by moving items and substituting $-2\sum_{k=1}^{n}\lambda_k = q(x)$. $\quad -2\sum_{k=1}^{n}\mu_k = q(x)$.

We try to represent $x_{ki}, y_{ki}$ like this: $x_{k\gamma} = -\frac{1}{2\lambda_k}\sum_{j=1}^{n} a_{kj}y_{j\gamma}$

$\sum_{i=1}^{n} a_{ik}\left(-\frac{1}{2\lambda_i}\sum_{j=1}^{n} a_{ij}y_{j\gamma}\right) + 2\mu_k y_{k\gamma} = 0$

so: $\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{a_{ik}a_{ij}}{2\lambda_i}\right)y_{j\gamma} = 2\mu_k y_{k\gamma}$

$$\begin{pmatrix}a_{1k} & a_{2k} & \cdots & a_{nk}\end{pmatrix}\begin{pmatrix}\frac{1}{2\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{2\lambda_n}\end{pmatrix}A\begin{pmatrix}y_{1\gamma}\\ y_{2\gamma}\\ \vdots\\ y_{n\gamma}\end{pmatrix} = 2\mu_k y_{k\gamma}$$

Finally we get :

$$A^T\begin{pmatrix}\frac{1}{2\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{2\lambda_n}\end{pmatrix}A\begin{pmatrix}y_{1\gamma}\\ y_{2\gamma}\\ \vdots\\ y_{n\gamma}\end{pmatrix} = \begin{pmatrix}2\mu_1 & & \\ & \ddots & \\ & & 2\mu_n\end{pmatrix}\begin{pmatrix}y_{1\gamma}\\ y_{2\gamma}\\ \vdots\\ y_{n\gamma}\end{pmatrix} \cdots ①$$

Similarly :

$$A\begin{pmatrix}\frac{1}{2\mu_1} & & \\ & \ddots & \\ & & \frac{1}{2\mu_n}\end{pmatrix}A^T\begin{pmatrix}x_{1\gamma}\\ x_{2\gamma}\\ \vdots\\ x_{n\gamma}\end{pmatrix} = \begin{pmatrix}2\lambda_1 & & \\ & \ddots & \\ & & 2\lambda_n\end{pmatrix}\begin{pmatrix}x_{1\gamma}\\ x_{2\gamma}\\ \vdots\\ x_{n\gamma}\end{pmatrix} \cdots ②$$

The equations ① and ② are called generailzed singular value equation!

For the next part, We can specialize the conditions to solve the corresponding problems. we try to give verification when dimension is two and we try to assume that matrix A is a two by two matrix.

## Examples (2-dimension)

Given that $q(x) = \vec{x}^TA\vec{y} = \sum_{i,j=1}^{n} a_{ij}\vec{x}_i^T\vec{y}_j$

▸**Condition 1** where $\vec{y}_j = \vec{x}_j$, $A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, $|\vec{x}_i| = 1$, $\vec{x}_i \in R^{2\times 1}$

We can get the result like this based on that what we did before: $q(x) = -(\lambda_1 + \lambda_2)$.

$(A + A^T)\begin{pmatrix}x_{1r}\\ x_{2r}\end{pmatrix} = -\begin{pmatrix}2\lambda_1 & \\ & 2\lambda_2\end{pmatrix}\begin{pmatrix}x_{1r}\\ x_{2r}\end{pmatrix}$

We can abstract this question that: $BX = \Lambda X$, $B = \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$, $\Lambda = \begin{pmatrix}\mu_1 & \\ & \mu_2\end{pmatrix}$.

so: $(b_{11} - \mu_1)(b_{22} - \mu_2) - b_{12}b_{21} = 0$

then we get: $\mu_1\mu_2 - b_{11}\mu_2 - b_{22}\mu_1 + b_{11}b_{22} - b_{12}b_{21} = 0$

$F(\mu_1, \mu_2, \lambda) = \mu_1 + \mu_2 + \lambda(\mu_1\mu_2 - b_{11}\mu_2 - b_{22}\mu_1 + b_{11}b_{22} - b_{12}b_{21})$

then we have:
$$\begin{cases}\frac{\partial F}{\partial \mu_1} = 1 + \lambda\mu_2 - b_{22} = 0 \Longrightarrow \mu_2 = \frac{b_{22} - 1}{\lambda} \dots ①\\ \frac{\partial F}{\partial \mu_2} = 1 + \lambda\mu_1 - b_{11} = 0 \Longrightarrow \mu_1 = \frac{b_{11} - 1}{\lambda} \dots ②\\ \frac{\partial F}{\partial \lambda} = \mu_1\mu_2 - b_{11}\mu_2 - b_{22}\mu_1 + b_{11}b_{22} - b_{12}b_{21} = 0 \dots ③\end{cases}$$

If $\det(A) \neq 0$, $b_{11}b_{22} - b_{12}b_{21} \neq 0$, so we can get the solution of *lambda*,

$f(\mu_1, \mu_2) = \frac{1}{\lambda}(b_{11} + b_{22} - 1)$

So we get the extreme value: $\mu_1 + \mu_2$

▸**Condition 2** where $A^T = A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, $|\vec{x}_i| = |\vec{y}_i| = 1$, $\vec{x}_i, \vec{y}_i \in R^{2\times 1}$.

we get: $\det\left[A\begin{pmatrix}\frac{1}{2\lambda_1} & \\ & \frac{1}{2\lambda_2}\end{pmatrix}A - \begin{pmatrix}2\lambda_1 & \\ & 2\lambda_1\end{pmatrix}\right] = 0$

$F(\lambda_1, \lambda_2, \mu) = \lambda_1 + \lambda_2 + \mu(\frac{a_{12}^2a_{21}^2}{4\lambda_1\lambda_2} + 4\lambda_1\lambda_2 - \frac{\lambda_1}{\lambda_2}a_{11}^2 - \frac{\lambda_2}{\lambda_1}a_{22}^2)$

then we have: $\frac{\partial F}{\partial \lambda_1} = 0$, $\frac{\partial F}{\partial \lambda_2} = 0$, $\frac{\partial F}{\partial \mu} = 0$.

Using these 3 equations, we can solve $\lambda_1 = \frac{a_{11}}{2}$, $\lambda_2 = \frac{a_{22}}{2}$, then $f(\lambda_1 + \lambda_2) = \frac{a_{11} + a_{22}}{2}$,

▸**Condition 3** where A is a general matrix, $|\vec{x}_i| = |\vec{y}_i| = 1$, $\vec{x}_i, \vec{y}_i \in R^{2\times 1}$

This condition is more general and complicated. We can get the result based on what we did before.

We need to construct Lagrange's Multiply equation F(λ1, λ2, μ1, μ2, λ, μ) with 6 variables, then take Partial derivative for each variable to get 6 constraine quations to solve each variables.

Finally λ1 + λ2 = u1 + u2 is extreme value.

## Future Works

- In condition 1, we can get $\begin{pmatrix}x_{11}\\ x_{21}\end{pmatrix}$, $\begin{pmatrix}x_{12}\\ x_{22}\end{pmatrix}$ however we can't get $x_{11}^2 + x_{12}^2=1; x_{21}^2 + x_{22}^2=1$. More work is need .
- In condition 2, we may still try to find other $\lambda_1, \lambda_2$ to make satisfy the equation.At the same time, when we can't get $x_{11}^2 + x_{12}^2=1; x_{21}^2 + x_{22}^2=1, y_{11}^2 + y_{12}^2=1; y_{21}^2 + y_{22}^2=1$ whether we can find the max/min of $\lambda_1 + \lambda_2$.
- In condition 3, drawing graphs may be an effective way to solve the problem.But in this case, constraints seems to be lack that where we will try again.

## Reference


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