Math 1553 Worksheet §§3.3-3.5

For problems 1 and 2 below: The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{c} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

Feel free to use a calculator to carry out arithmetic in problems 1 and 2.

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3. Write a matrix equation $Ax = b$ to relate your final grades to your scores.

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

2. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Use Problem 1 to determine which grading scheme leaves you with the highest overall course grade.

I will calculate percentage as a grade.
ex. 90% → 90

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} 90 \\ 90 \\ 85 \\ 95 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\rightsquigarrow 90 \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix} + 90 \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix} + 85 \begin{pmatrix} 0.5 \\ 0.4 \\ 0.6 \end{pmatrix} + 95 \begin{pmatrix} 0.3 \\ 0.4 \\ 0.2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left.\begin{aligned} b_1 &= 9 + 9 + 42.5 + 28.5 = 89 \\ b_2 &= 9 + 9 + 34 + 38 = 90 \\ b_3 &= 9 + 9 + 51 + 19 = 88 \end{aligned}\right\}$$

∴ Scheme 2 gives you a final grade "90", which is the highest.

3. **a)** True or false. Justify your answer:
If A is a 5×4 matrix, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^5$.

TRUE **FALSE**

ex. $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

A 5x4 matrix has 5 rows and 4 colums.
→ This means it represents a system of 5 equations and 4 variables.

Since the question asks if there are some "b" (not all) which makes a system of linear equations inconsistent, this is trivial.
... Make $Ax = b$ augmented matrix and use row operations to make it RREF.
If there is a pivot in augmented row, the system of linear equations is inconsistent!

b) Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Must it be true that $\{v_1, v_2, v_3\}$ is linearly dependent? If so, write a linear dependence relation for the vectors.

This already gave us the hint
"There is a non-zero solution" which means
linear dependence.

This equals to $-3v_1 + 2v_2 + 7v_3 = 0$.

4. Find the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ ① and $x_1 - 3x_2 + 5x_3 = 3$ ② and write them in parametric vector form. How do the solution sets compare geometrically?

① $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}} \rightarrow \text{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}\right\}$

변수가 3개, 매개변수가 2개.
→ plane in $\mathbb{R}^3$

② $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

$= x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

① equation represents the plane passes through origin and is spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

② equation represents the same plane ($= ①$) but shifted by the vector $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$
→ parallel.