# Chapter 2

## Systems of Linear Equations: Algebra

# Section 2.3

## Parametric Form

# Another Example

The linear system

$$\begin{aligned} 2x + \phantom{2}y + 12z &= \phantom{-}1 \\ x + 2y + \phantom{1}9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

Let's row reduce it: [interactive row reducer]

http://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=2,1,12,1:1,2,9,-1

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$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ \color{red}{0} & -3 & -6 & 3 \end{array}\right) \quad \color{blue}{\text{(Step 1c)}}$$

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$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & \color{red}{0} & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \quad \color{blue}{\text{(Step 2c)}}$$

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The row reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \quad \text{corresponds to the linear system} \quad \begin{cases} x + 5z = \phantom{-}1 \\ y + 2z = -1 \end{cases}$$

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## Another Example

### Continued

The system

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comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

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Yes! Rewrite:

$$\begin{aligned} x &= \phantom{-}1 - 5z \\ y &= -1 - 2z \end{aligned}$$

For any value of $z$, there is exactly one value of $x$ and $y$ that makes the equations true. But $z$ can be *anything we want*!

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So we have found the solution set: it is all values $x, y, z$ where

$$\begin{aligned} x &= \phantom{-}1 - 5z \\ y &= -1 - 2z \qquad \text{for } z \text{ any real number.} \\ (z &= \phantom{-1 - 5}z) \end{aligned}$$

This is called the **parametric form** for the solution. [interactive picture]

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

http://textbooks.math.gatech.edu/ila/demos/parametric1.html

# Free Variables

## Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let $A$ be a row echelon form of the matrix for this system.

We say that $x_i$ is a **free variable** if its corresponding column in $A$ is *not* a pivot column.

**Important**

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, $z$ was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

the free variables are $x_2$ and $x_4$. (What about the last column?)

# One More Example

The reduced row echelon form of the matrix for a linear system in $x_1, x_2, x_3, x_4$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

The free variables are $x_2$ and $x_4$: they are the ones whose columns are *not* pivot columns.

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This translates into the system of equations

$$\left\{\begin{array}{rcrcl} x_1 & & + 3x_4 & = & 2 \\ & x_3 & + 4x_4 & = & -1 \end{array}\right. \quad \Longrightarrow \quad \boxed{\begin{array}{rcl} x_1 & = & 2 - 3x_4 \\ x_3 & = & -1 - 4x_4 \end{array}}$$

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$$\begin{cases} x_1 \qquad\qquad + 3x_4 = \phantom{-}2 \\ \qquad\quad x_3 + 4x_4 = -1 \end{cases} \implies \boxed{\begin{aligned} x_1 &= \phantom{-}2 - 3x_4 \\ x_3 &= -1 - 4x_4 \end{aligned}}$$

What happened to $x_2$? What is it allowed to be? Anything! The general solution is

$$(x_1,\ x_2,\ x_3,\ x_4) = (2 - 3x_4,\ x_2,\ -1 - 4x_4,\ x_4)$$

for any values of $x_2$ and $x_4$. For instance, $(2, 0, -1, 0)$ is a solution ($x_2 = x_4 = 0$), and $(5, 1, 3, -1)$ is a solution ($x_2 = 1,\ x_4 = -1$).

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The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

## Yet Another Example

The linear system

$$x + y + z = 1 \quad \text{has matrix form} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right).$$

This is in reduced row echelon form. The free variables are $y$ and $z$. The parametric form of the general solution is

$$x = 1 - y - z.$$

Rearranging:

$$(x, y, z) = (1 - y - z,\ y,\ z),$$

where $y$ and $z$ are arbitrary real numbers. This was an example in the second lecture!

[interactive]

http://textbooks.math.gatech.edu/ila/demos/plane.html

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   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

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2. Every column except the last column is a pivot column.
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3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:
   $$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

## Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables.
- A linear system has zero, one, or infinitely many solutions.