Chapter 3

Systems of Linear Equations: Geometry

Section 3.2

Vector Equations and Spans

Systems of Linear Equations

Question

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Systems of Linear Equations

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This means: can we solve the equation

$$x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

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$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

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This is just a system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

Systems of Linear Equations

Continued

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \xrightarrow{\text{matrix form}} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

Systems of Linear Equations
Continued

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \xrightarrow{\text{matrix form}} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{row reduce}} \quad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{solution}} \quad \begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

[interactive] $\longleftarrow$ (this is the picture of a *consistent* linear system)

http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2,6&v2=-1,-2,-1&target=8,16,3&tlabel=b&range=20&camera=3,.5,1.5

Systems of Linear Equations

Continued

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What is the relationship between the original vectors and the matrix form of the linear equation? They have the same columns!

Shortcut: You can make the augmented matrix without writing down the system of linear equations first.

Vector Equations and Linear Equations

Summary

The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

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where the v_i's and b are the columns of the matrix.

So we now have (at least) *two* equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.

Span

It is important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$: it's exactly the collection of all b in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

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Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \;\middle|\; x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \right\}.}$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Span

Continued

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Span

Continued

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

[interactive example] $\longleftarrow$ (this is the picture of an *inconsistent* linear system)

Note: **equivalent** means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, *either* all three statements are true, *or* all three statements are false.

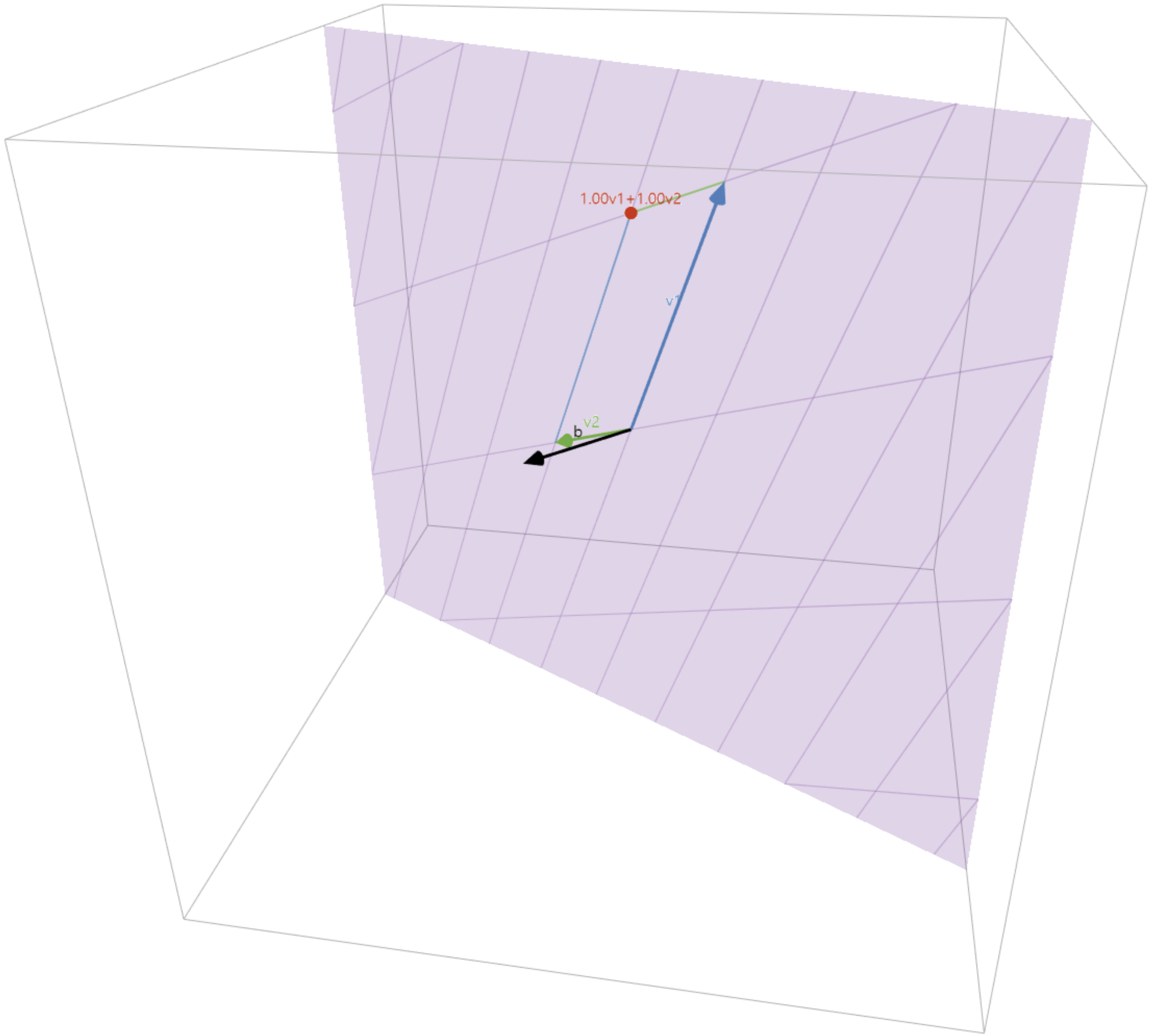
http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2,6&v2=-1,-2,-1&target=2,-2,0&tlabel=b&range=8&camera=3,.5,1.5

Solve this equation by moving the sliders:

$$\begin{bmatrix} 1.00 \\ 2.00 \\ 6.00 \end{bmatrix} + \begin{bmatrix} -1.00 \\ -2.00 \\ -1.00 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.00 \\ 5.00 \end{bmatrix} \neq \begin{bmatrix} 2.00 \\ -2.00 \\ 0.00 \end{bmatrix}.$$

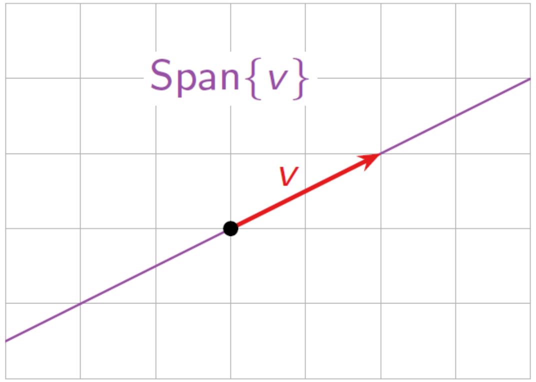
Axes	☐	
Show x.v1 + y.v2	☑	
x		1
y		1
Grid	☑	

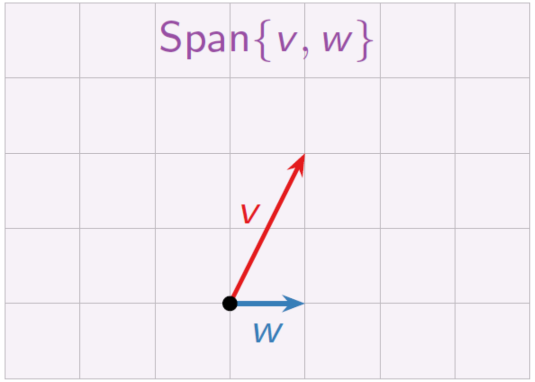
Close Controls

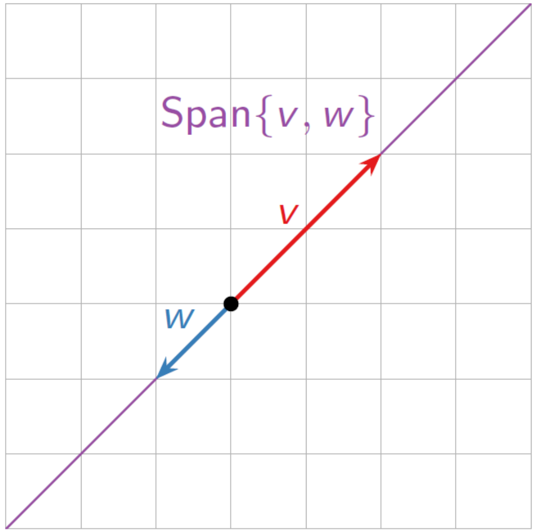


Pictures of Span

Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.



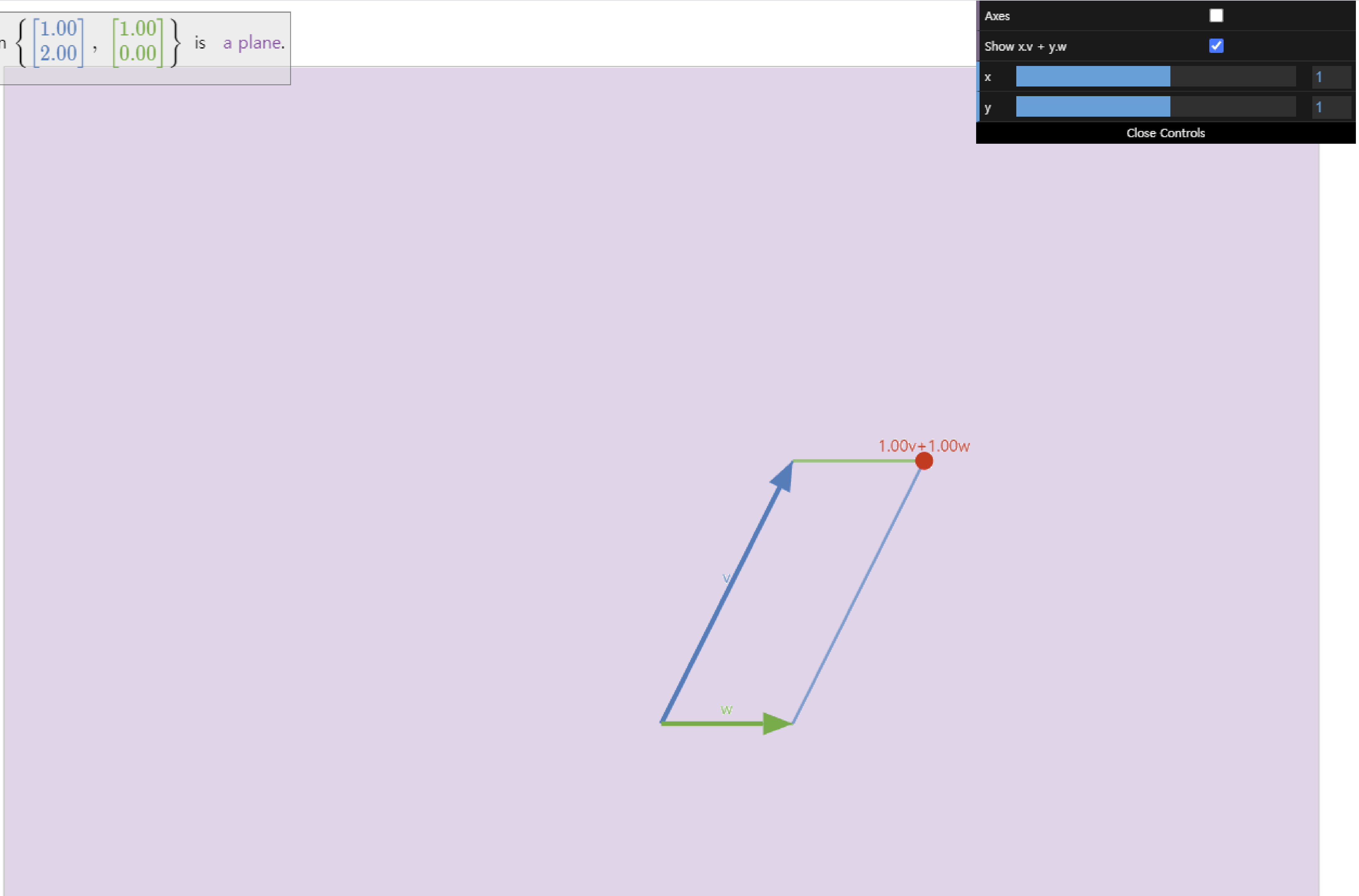




[interactive: span of two vectors in $\mathbf{R}^2$]

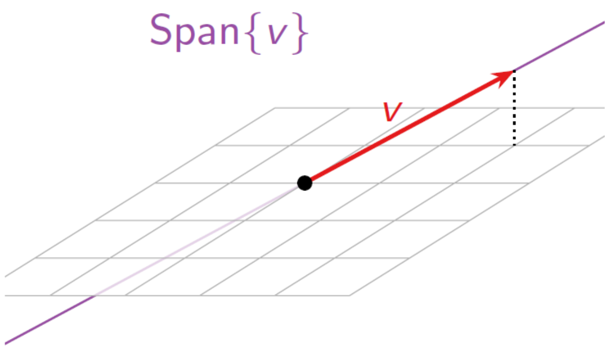
http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2&v2=1,0&range=5&labels=v,w

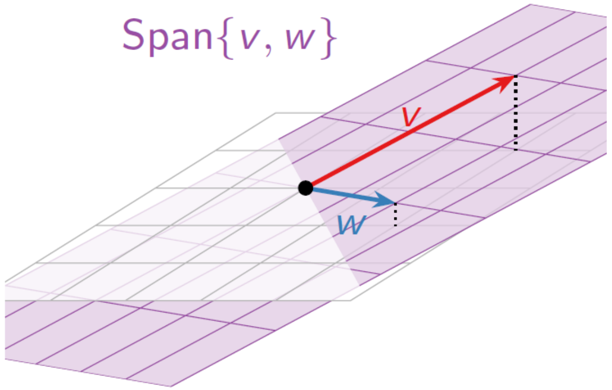
Span $\left\{ \begin{bmatrix} 1.00 \\ 2.00 \end{bmatrix}, \begin{bmatrix} 1.00 \\ 0.00 \end{bmatrix} \right\}$ is a plane.

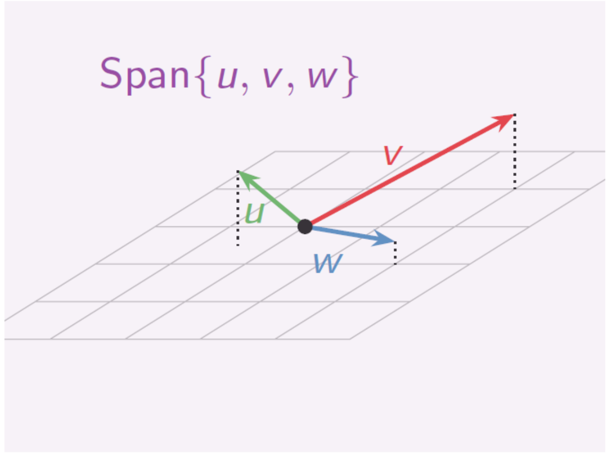


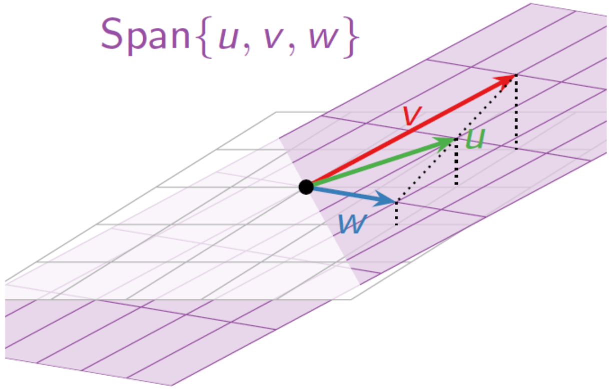
Pictures of Span

In $\mathbf{R}^3$









[interactive: span of two vectors in $\mathbf{R}^3$]   [interactive: span of three vectors in $\mathbf{R}^3$]

http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=5,3,-2&v2=3,-4,1&labels=v,w&range=8

http://textbooks.math.gatech.edu/ila/demos/spans.html?labels=u,v,w&range=8

Span $\left\{ \begin{bmatrix} 5.00 \\ 3.00 \\ -2.00 \end{bmatrix}, \begin{bmatrix} 3.00 \\ -4.00 \\ 1.00 \end{bmatrix}, \begin{bmatrix} -1.00 \\ 1.00 \\ 7.00 \end{bmatrix} \right\}$ is space.

Axes

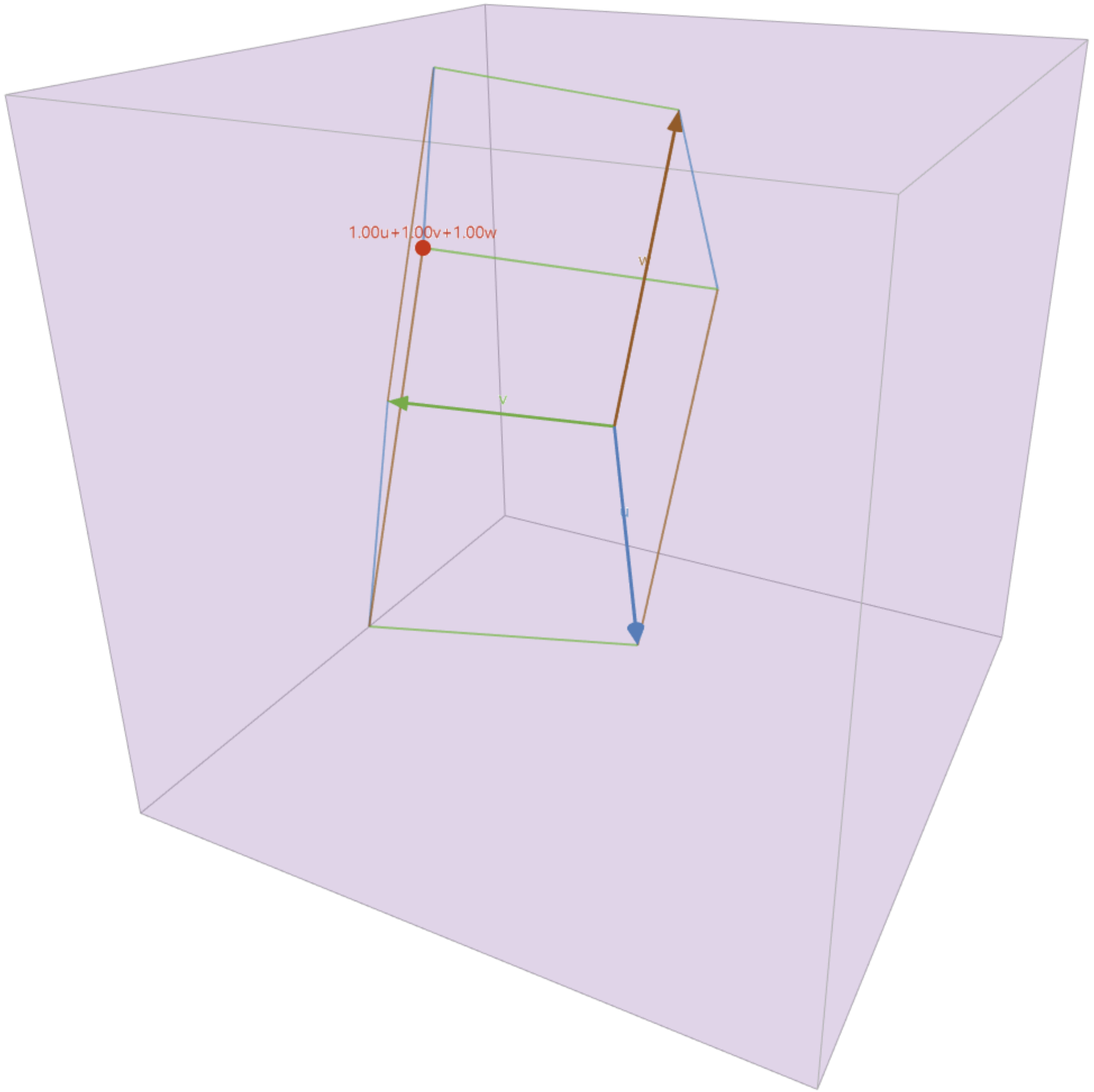
Show x.u + y.v + z.w

x 1

y 1

z 1

Close Controls



Observation about spans

How many vectors are in $\mathrm{Span}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$?

A. Zero

B. One

C. Infinity

In general, it appears that $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest "linear space" (line, plane, etc.) containing the origin and all of the vectors $v_1, v_2, \ldots, v_p$.

We will make this precise later.

Summary

The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.