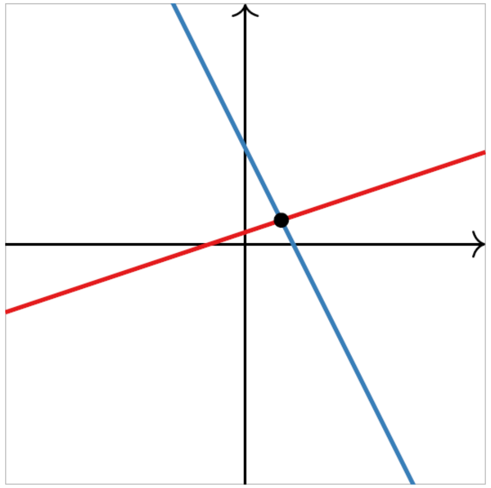
Chapter 3

Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



This will give us better insight into the properties of systems of equations and their solution sets.

Remember: I expect you to be able to draw pictures!

Section 3.1

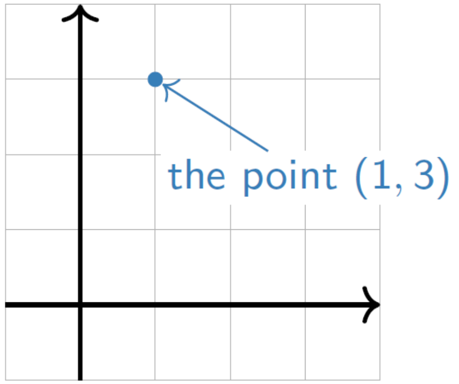
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

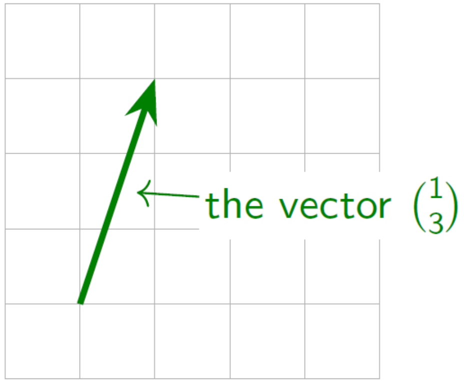
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

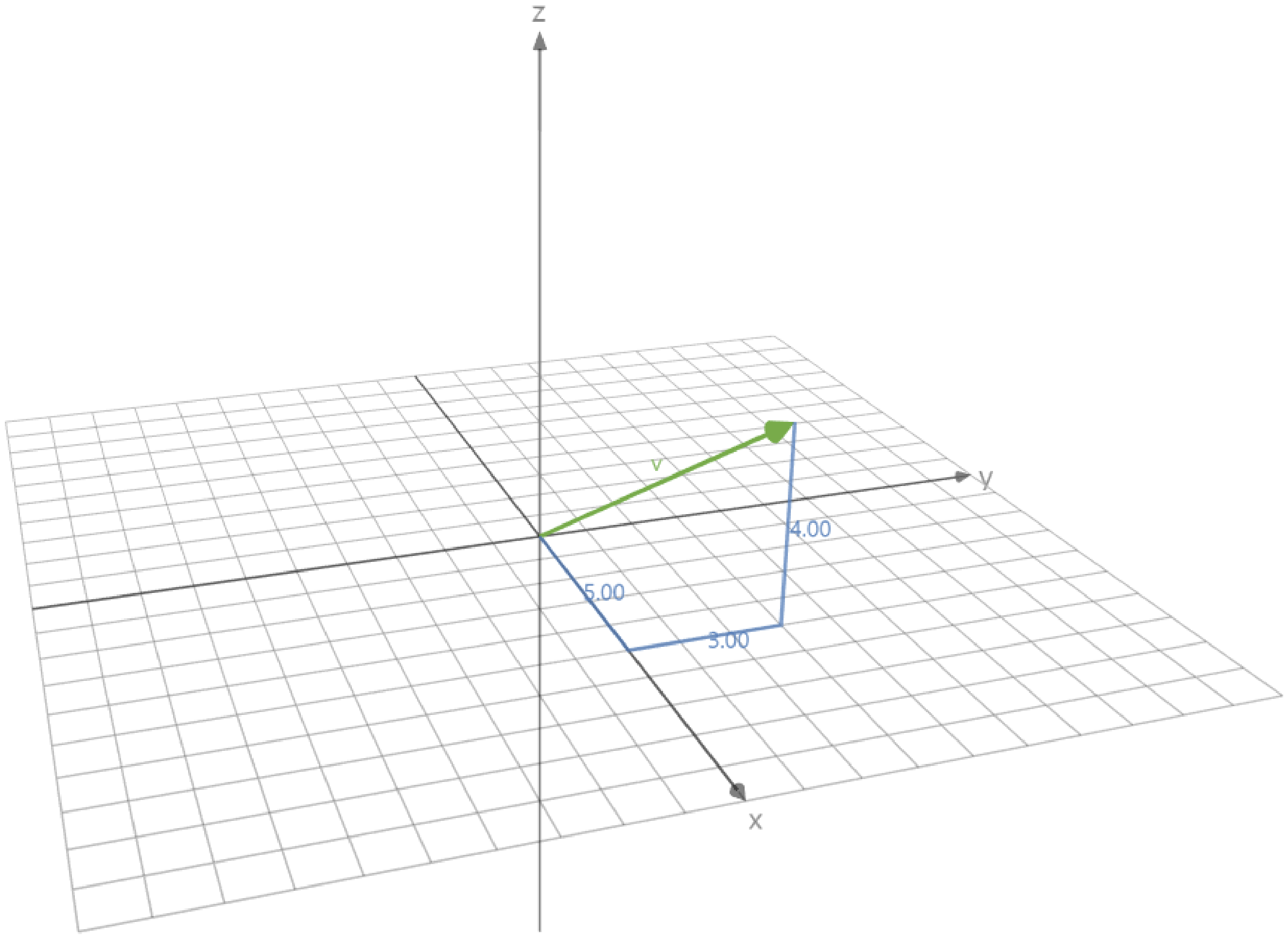
http://textbooks.math.gatech.edu/ila/demos/vector.html

$$v = \begin{bmatrix} 5.00 \\ 3.00 \\ 4.00 \end{bmatrix}$$

[click and drag the arrow head and tail]

a		5
b		3
c		4

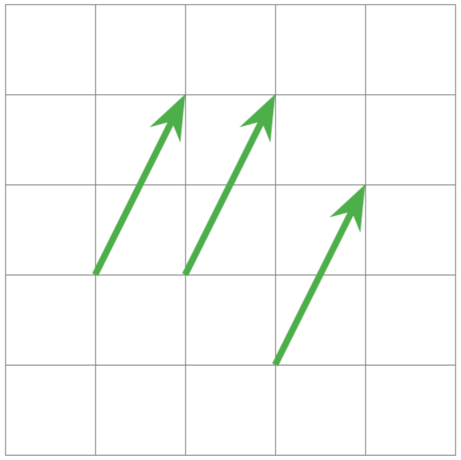
Close Controls



Points and Vectors

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.

Vector Algebra

Definition

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c:

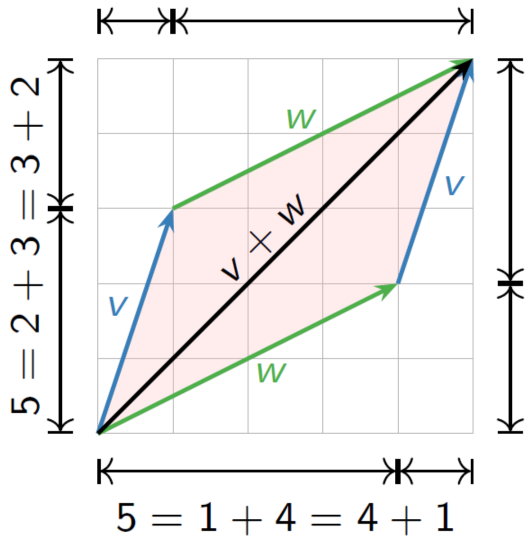
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

  We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

(And likewise for vectors of length n.) For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

Vector Addition and Subtraction: Geometry



The parallelogram law for vector addition

Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v + w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

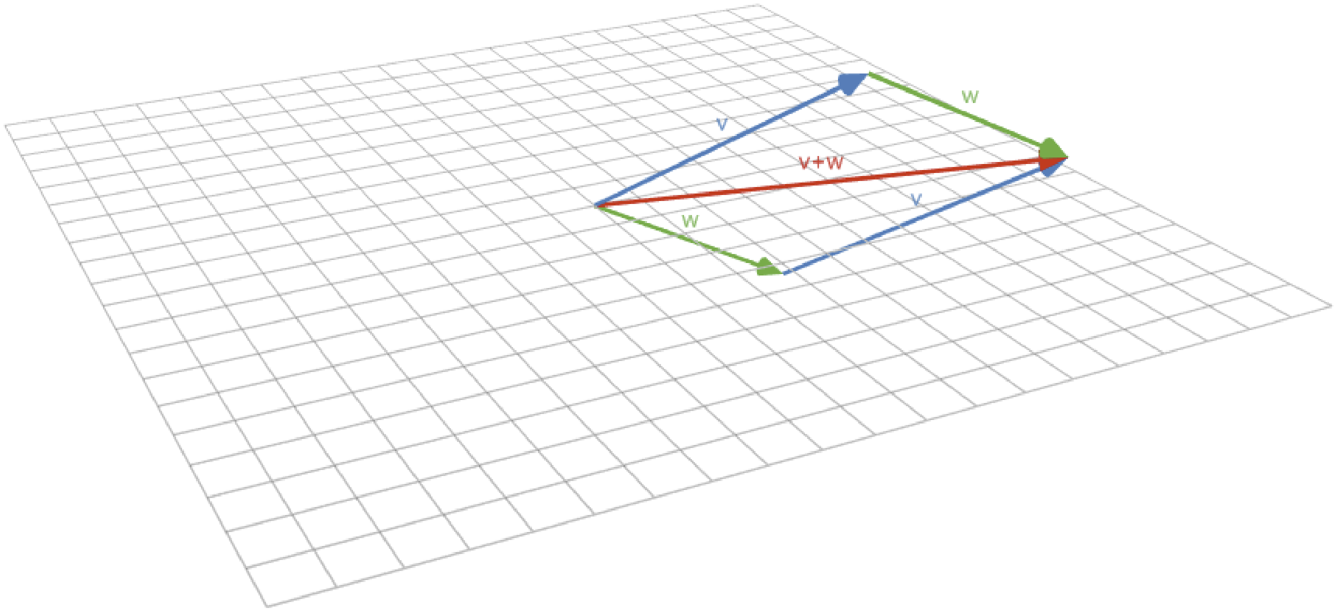
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights. [interactive]

http://textbooks.math.gatech.edu/ila/demos/vector-add.html

$$\begin{bmatrix} 3.00 \\ -5.00 \\ 4.00 \end{bmatrix} + \begin{bmatrix} 4.00 \\ -1.00 \\ -2.00 \end{bmatrix} = \begin{bmatrix} 7.00 \\ -6.00 \\ 2.00 \end{bmatrix}$$

[click and drag the heads of v and w to move them]



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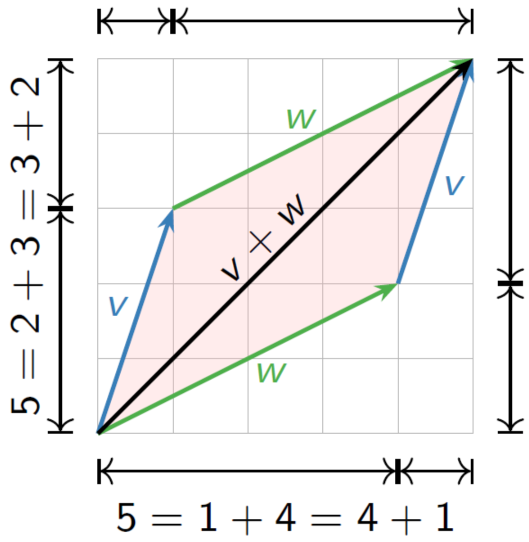

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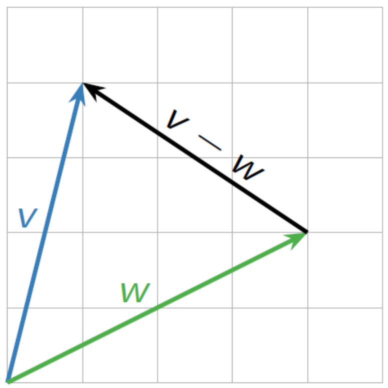


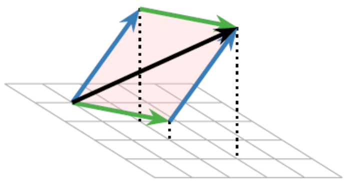
Vector subtraction

Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v. For example,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add $v - w$ to w, you get v. [interactive]

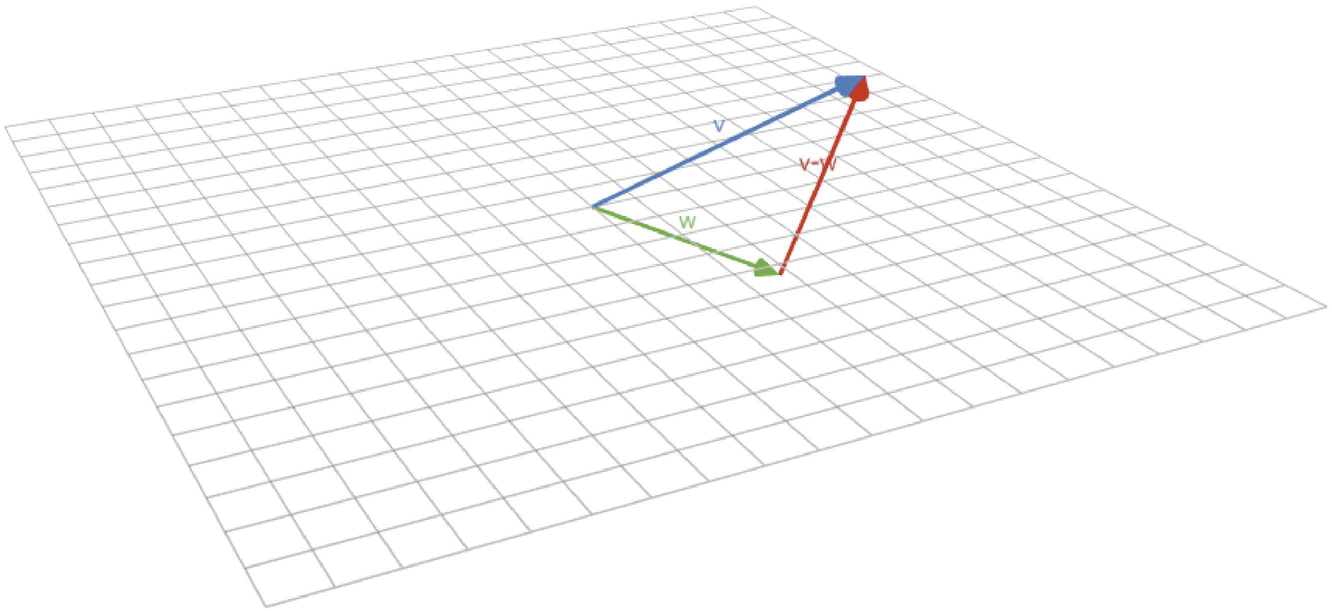
This works in higher dimensions too!



http://textbooks.math.gatech.edu/ila/demos/vector-sub.html

$$\begin{bmatrix} 3.00 \\ -5.00 \\ 4.00 \end{bmatrix} - \begin{bmatrix} 4.00 \\ -1.00 \\ -2.00 \end{bmatrix} = \begin{bmatrix} -1.00 \\ -4.00 \\ 6.00 \end{bmatrix}$$

[click and drag the heads of v and w to move them]

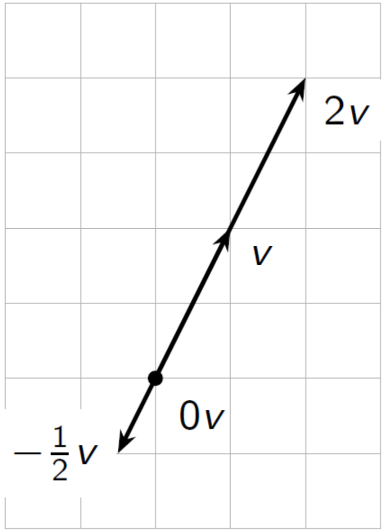


Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same *direction* but a different *length*.

Some multiples of v.



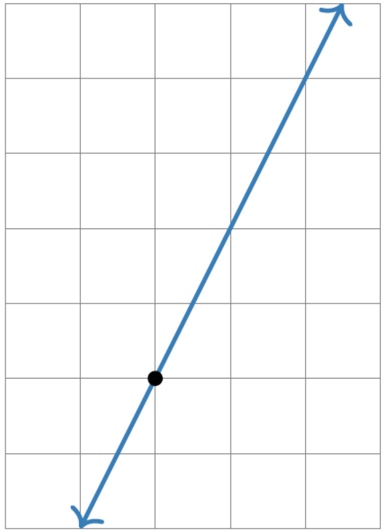
$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of v.



[interactive]

So the scalar multiples of v form a *line*.

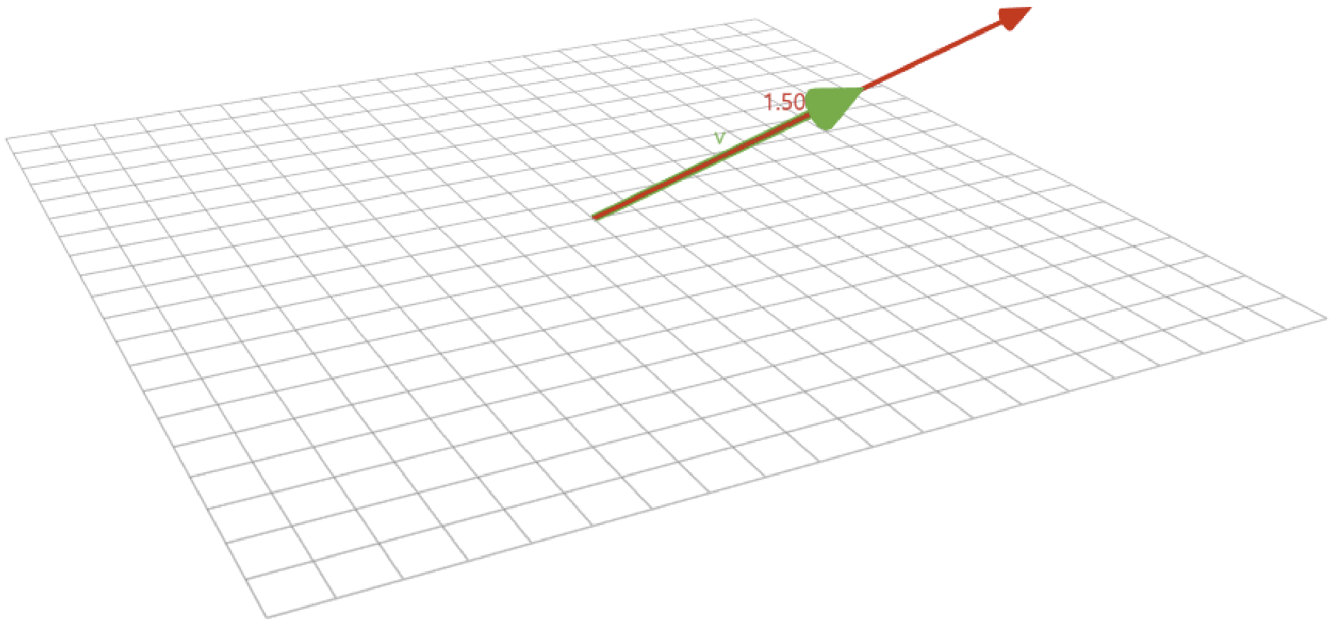
http://textbooks.math.gatech.edu/ila/demos/vector-mul.html

$$1.50 \cdot \begin{bmatrix} 3.00 \\ -5.00 \\ 4.00 \end{bmatrix} = \begin{bmatrix} 4.50 \\ -7.50 \\ 6.00 \end{bmatrix}$$

[click and drag the head of v to move it]

c 1.5

Close Controls



Linear Combinations

We can add and scalar multiply in the same equation:

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

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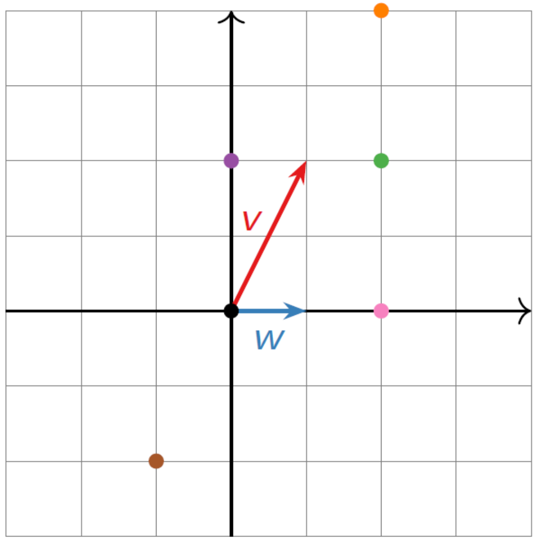
Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2&v2=1,0&range=5&captions=combo&nomove=true&labels=v,w

$$\begin{bmatrix}1.00\\2.00\end{bmatrix}+\begin{bmatrix}1.00\\0.00\end{bmatrix}=\begin{bmatrix}2.00\\2.00\end{bmatrix}$$

Axes

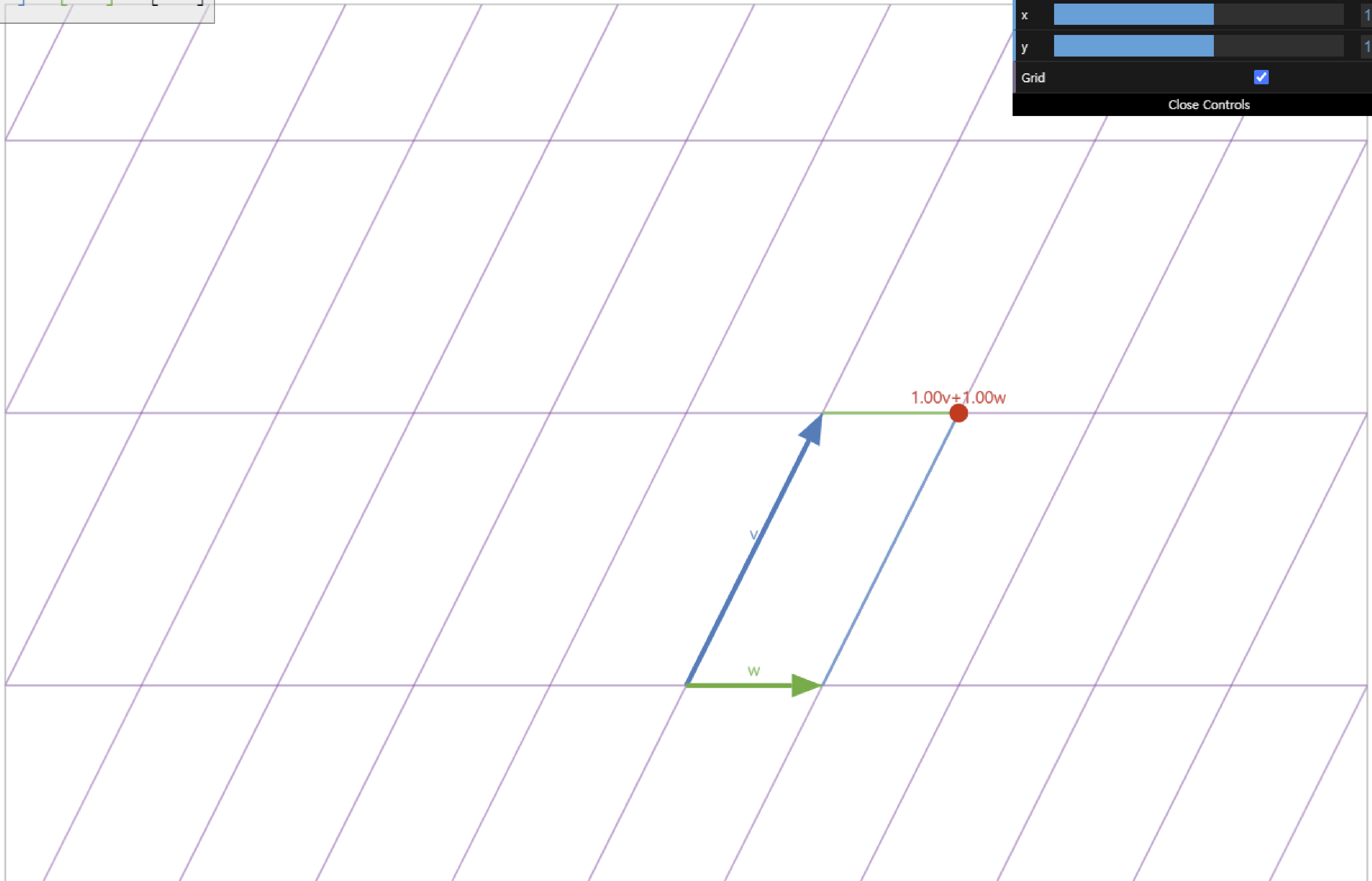
Show x.v + y.w

x 1

y 1

Grid

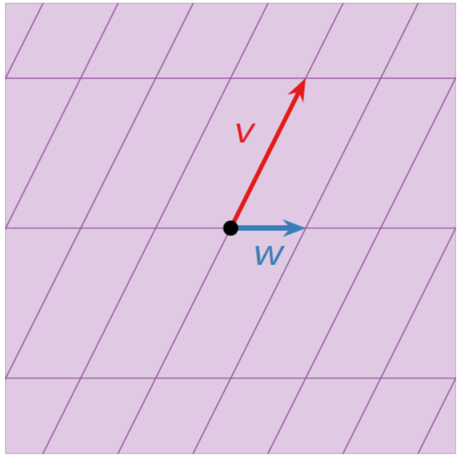
Close Controls



Poll

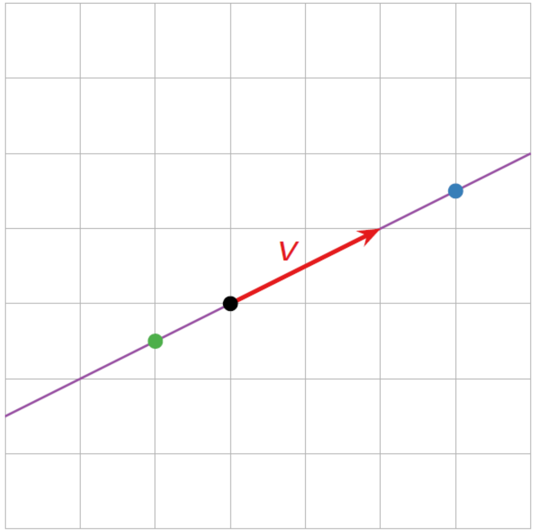
Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?



(The purple lines are to help measure *how much* of v and w you need to get to a given point.)

More Examples



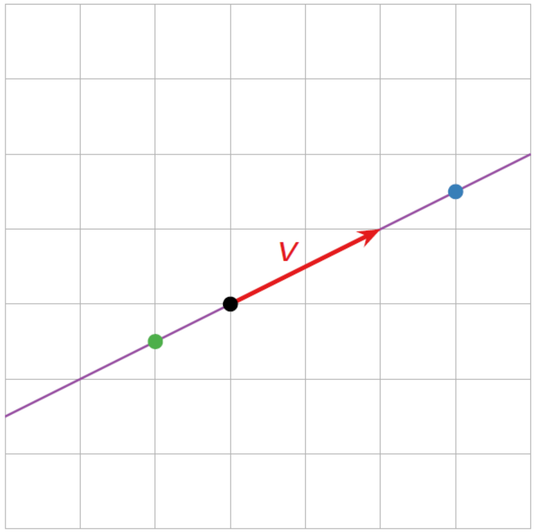
What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- $\frac{3}{2}v$
- $-\frac{1}{2}v$
- ...

What are *all* linear combinations of v?

All vectors cv for c a real number. I.e., all *scalar multiples* of v. These form a *line*.

More Examples



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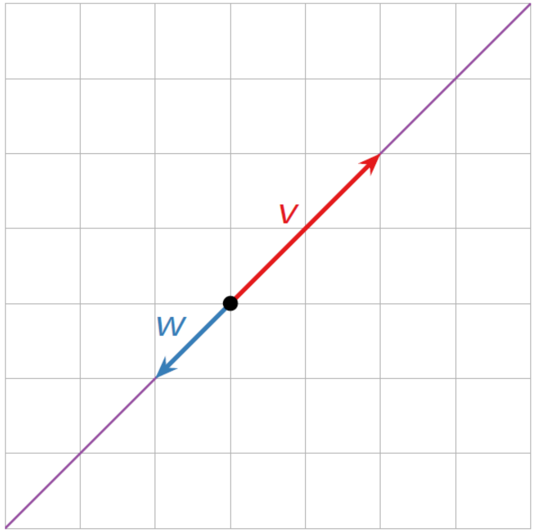
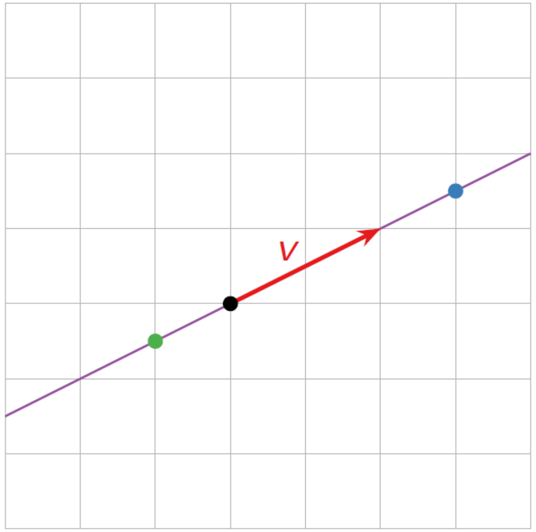


Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

More Examples



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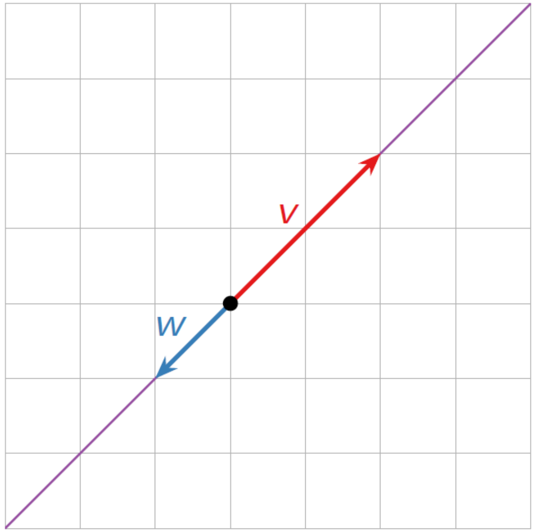

Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

Answer: The line which contains both vectors.

What's different about this example and the one on the poll? [interactive]

http://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,2&v2=-1,-1&range=5&labels=v,w&grid=disabled&captions=combo