Math 1553
Introduction to Linear Algebra

School of Mathematics
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Chapter 1

Overview

Linear. Algebra.

What is Linear Algebra?

Linear

- having to do with lines/planes/etc.
- For example, $x + y + 3z = 7$, not $\sin, \log, x^2$, etc.

Algebra

- solving equations involving numbers and symbols
- from al-jebr (Arabic), meaning reunion of broken parts
- 9^{th} century Abu Ja'far Muhammad ibn Muso al-Khwarizmi

Why a whole course?

But these are the easiest kind of equations! I learned how to solve them in 7th grade!

Ah, but engineers need to solve *lots* of equations in *lots* of variables.

$$\begin{array}{rcrcrcrcrcrcr}
3x_1 & + & 4x_2 & + & 10x_3 & + & 19x_4 & - & 2x_5 & - & 3x_6 & = & 141 \\
7x_1 & + & 2x_2 & - & 13x_3 & - & 7x_4 & + & 21x_5 & + & 8x_6 & = & 2567 \\
-x_1 & + & 9x_2 & + & \frac{3}{2}x_3 & + & x_4 & + & 14x_5 & + & 27x_6 & = & 26 \\
\frac{1}{2}x_1 & + & 4x_2 & + & 10x_3 & + & 11x_4 & + & 2x_5 & + & x_6 & = & -15
\end{array}$$

Often, it's enough to know some information about the set of solutions without having to solve the equations at all!

Also, what if one of the coefficients of the x_i is itself a parameter— like an unknown real number t?

In real life, the difficult part is often in recognizing that a problem can be solved using linear algebra in the first place: need *conceptual* understanding.

Linear Algebra in Engineering

Large classes of engineering problems, no matter how huge, can be reduced to linear algebra:

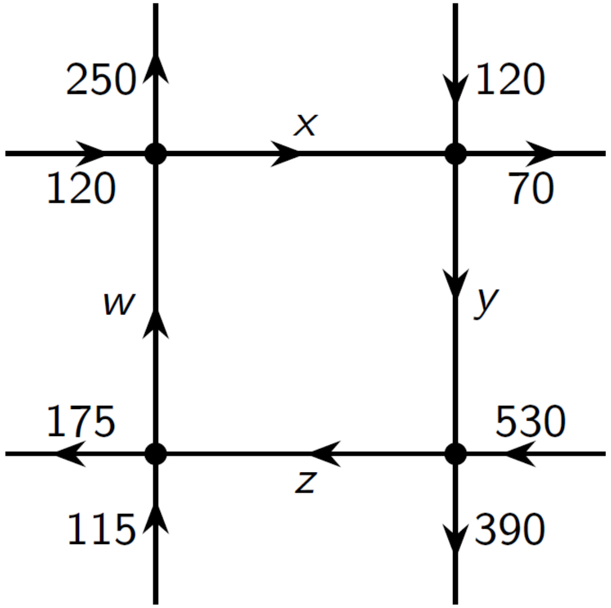
$$Ax = b \quad \text{or}$$
$$Ax = \lambda x$$

"... and now it's just linear algebra"

Applications of Linear Algebra

Civil Engineering: How much traffic flows through the four labeled segments?

Traffic flow (cars/hr)



$\rightsquigarrow$ system of linear equations:

$$
\begin{aligned}
w + 120 &= x + 250 \\
x + 120 &= y + 70 \\
y + 530 &= z + 390 \\
z + 115 &= w + 175
\end{aligned}
$$

Applications of Linear Algebra

Chemistry: Balancing reaction equations

$$\underline{\ x\ }\ C_2H_6 + \underline{\ y\ }\ O_2 \rightarrow \underline{\ z\ }\ CO_2 + \underline{\ w\ }\ H_2O$$

$\rightsquigarrow$ system of linear equations, one equation for each element.

$$\begin{aligned} 2x &= z \\ 6x &= 2w \\ 2y &= 2z + w \end{aligned}$$

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

$\rightsquigarrow$ system of linear equations:

$$\begin{aligned} 6y_{2016} + 8z_{2016} &= x_{2017} \\ \tfrac{1}{2}x_{2016} \qquad\qquad &= y_{2017} \\ \tfrac{1}{2}y_{2016} \qquad &= z_{2017} \end{aligned}$$

Question

Does the rabbit population have an asymptotic behavior? Is this even a linear algebra question? Yes, it is! [interactive]

http://textbooks.math.gatech.edu/ila/demos/rabbits/book_demo.html

Applications of Linear Algebra

Geometry and Astronomy: Find the equation of a circle passing through 3 given points, say $(1,0)$, $(0,1)$, and $(1,1)$. The general form of a circle is $a(x^2+y^2)+bx+cy+d=0$.

$\rightsquigarrow$ system of linear equations:

$$\begin{aligned} a + b \phantom{{}+c} + d &= 0 \\ a \phantom{{}+b} + c + d &= 0 \\ 2a + b + c + d &= 0 \end{aligned}$$

Very similar to: compute the orbit of a planet:

$$ax^2+by^2+cxy+dx+ey+f=0$$

Example (Astronomy). An asteroid has been observed at the following locations:

$$(0,2),\ (2,1),\ (1,-1),\ (-1,-2),\ (-3,1),\ (-1,-1).$$

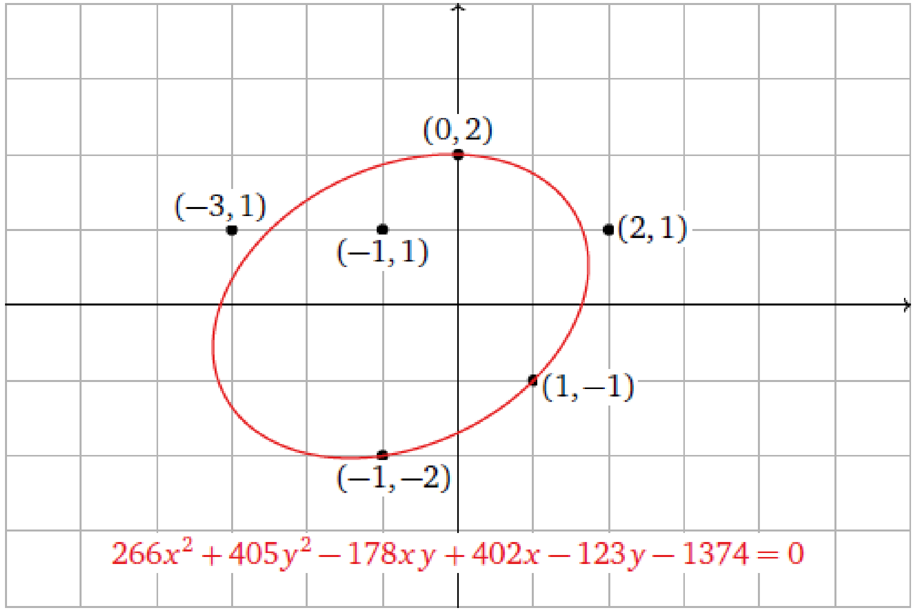
Its orbit around the sun is elliptical; it is described by an equation of the form

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

What is the most likely orbit of the asteroid, given that there was some significant error in measuring its position? Substituting the data points into the above equation yields the system

$$\begin{aligned}
(0)^2 + \quad B(2)^2 + \quad C(0)(2) + \quad D(0) + \quad E(2) + F &= 0\\
(2)^2 + \quad B(1)^2 + \quad C(2)(1) + \quad D(2) + \quad E(1) + F &= 0\\
(1)^2 + B(-1)^2 + \quad C(1)(-1) + \quad D(1) + E(-1) + F &= 0\\
(-1)^2 + B(-2)^2 + C(-1)(-2) + D(-1) + E(-2) + F &= 0\\
(-3)^2 + \quad B(1)^2 + \quad C(-3)(1) + D(-3) + \quad E(1) + F &= 0\\
(-1)^2 + B(-1)^2 + C(-1)(-1) + D(-1) + E(-1) + F &= 0.
\end{aligned}$$

There is no actual solution to this system due to measurement error, but here is the best-fitting ellipse:



Applications of Linear Algebra

Google: "The 25 billion dollar eigenvector." Each web page has some importance, which it shares via outgoing links to other pages

$\rightsquigarrow$ system of linear equations (in gazillions of variables).

Larry Page flies around in a private 747 because he paid attention in his linear algebra class!

Overview of the Course

- Solve the matrix equation $Ax = b$
  - Solve systems of linear equations using matrices, row reduction, and inverses.
  - Solve systems of linear equations with varying parameters using parametric forms for solutions, the geometry of linear transformations, the characterizations of invertible matrices, and determinants.
- Solve the matrix equation $Ax = \lambda x$
  - Solve eigenvalue problems through the use of the characteristic polynomial.
  - Understand the dynamics of a linear transformation via the computation of eigenvalues, eigenvectors, and diagonalization.
- Almost solve the equation $Ax = b$
  - Find best-fit solutions to systems of linear equations that have no actual solution using least squares approximations.

What to Expect This Semester

Your previous math courses probably focused on how to do (sometimes rather involved) computations.

- Compute the derivative of $\sin(\log x)\cos(e^x)$.
- Compute $\int_0^1 (1 - \cos(x))\,dx$.

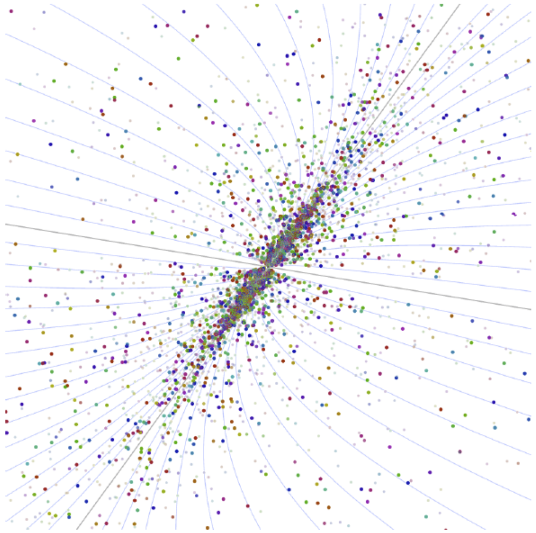
This is important, **but** ~~Matlab~~ Python can do all these problems better than any of us can. Nobody is going to hire you to do something a computer can do better.

> If a computer can do the problem better than you can, then it's just an algorithm: **this is not problem solving**.

So what are we going to do?

- About half the material focuses on how to do linear algebra computations—that is still important.
- The other half is on *conceptual* understanding of linear algebra. This is much more subtle: it's about figuring out *what question* to ask the computer, or whether you actually need to do any computations at all.

Interactive Linear Algebra



Dan Margalit and Joe Rabinoff have written a free online textbook called *Interactive Linear Algebra*, with a version specifically created for this course.

`https://textbooks.math.gatech.edu/ila/1553/`

There are about 150 interactive demonstrations in the book. They're there for a reason: you'll be expected to gain and demonstrate a **geometric** understanding of the material.

https://textbooks.math.gatech.edu/ila/1553/

How to Succeed in this Course

- Practice, practice, practice! It makes sense to most people that if you want to get good at tennis, you have to hit a million forehands and backhands. But for some reason, many people think you're either born good at math, or you're not. This is ridiculous. People who are good at math are just people who have spent a long time thinking about math. *Nobody* is born good at math.

Not good at math ⟶ 

- Do the homework carefully. Homework is practice for the quizzes. Quizzes are practice for the midterms. Remember what I said about practice?
- Study the pictures. I expect you to play around with the demos in the book until you understand them!
- Take advantage of the resources provided. Come to office hours! Read the textbook! Go to Math Lab!