Math 1553 Worksheet §2.2, §2.3

1. Is it possible for a linear system to have a unique solution if it has more equations than variables? If yes, give an example. If no, justify why it is impossible.

consider this one:

$$\begin{cases} x + y = 2 \\ x - y = 0 \\ 2x = 2 \end{cases} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & 2 \end{array}\right) \rightarrow \overset{\text{RREF}}{\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)}$$

Contrary to what we thought, the linear equation system has a unique solution $x=1$, $y=1$.

If we revisit the definition of "inconsistent", it refers to the case where a pivot appears in the augmented column when forming the RREF. This represents an equation of the form "$0 = n$ (where n is a constant)" which is false in general cases. However, if n is also 0, the equation is always true, meaning it has no impact on the result of the system of equations, even if such equations are added. In conclusion, having (the number of equations $\geq$ the number of variables) does not necessarily mean the system is inconsistent; the RREF form of the equations must be checked to confirm.

2. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

b) For the matrices in row echelon form, which entries are the pivots? What are the pivot columns?

1
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
1, 2, 3

2
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
1, 2, 4

3
$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$
1, 2, 4, 5

4
$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
1, 3

1. Definition of REF

① All nonzero rows are above any rows of all zeros.
② The leading coefficient (pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
③ All entries in a column below a leading coefficient are zeros.

REF: 1, 2, 3, 4

2. Definition of RREF.
① REF is needed.
② The leading entry in each nonzero is 1.
③ Each leading 1 is the only nonzero entry in its column.

RREF: 1, 2, 4

3rd matrix violates ②

3. Find the parametric form of the solutions of following system of equations in x_1, x_2, and x_3 by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables. Describe the solution set geometrically.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3. \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow[\text{Replacement}]{R_2 \to 4R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow[\text{Scale}]{R_2 \times \frac{1}{3}} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow[\text{Replacement}]{R_1 = -3R_2 + R_1,\ R_3 = 3R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \boxed{-5} & -2 \\ 0 & 1 & \boxed{2} & 1 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right)$$

RREF. → This can be a free variable. (since there is no pivot.)

This equals to...

$$\begin{cases} x_1 \quad - 5x_3 = -2 \\ \quad x_2 + 2x_3 = 1 \end{cases} \rightarrow \begin{aligned} x_1 &= 5x_3 - 2 \\ x_2 &= 1 - 2x_3 \end{aligned}$$

Solution set: $(5x_3 - 2,\ 1 - 2x_3,\ x_3)$

Let $x_3 = t$. $(t \in \mathbb{R})$

Then the solution set is $(5t - 2,\ 1 - 2t,\ t)$

→ This means a line in $\mathbb{R}^3$, parameterized with the variable "t".

The line passes through $(-2, 1, 0)$ (when $t = 0$)

and is directed along the vector $(5, -2, 1)$