Math 1553 Worksheet §3.1, §3.2

1. Write a set of three vectors whose span is a plane in $\mathbf{R}^3$.

Define a set of vector $A = \{v_1, v_2, v_3\}$. in $\mathbb{R}^3$.

This will span a plane if a vector inside A can be derived from the scalar multiple of an another vector b.

So the answer will be set $A = \{v_1, cv_1, v_2\}$ (c: scalar, $v_1 \neq 0$, $v_2 \neq 0$, $v_2 \neq cv_1$)

if $A = \{v_1, c_1v_1, c_2v_1\}$ it will span a line.

ex. $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$v_2 = 2v_1$

2. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4. \end{aligned}$$

Question: Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

Does $\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$ reduce to a row echelon form without any contradictions?

initial matrix

b) Formulate this question as a vector equation.

Does $x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ represent a consistent system? has appropriate tuple (x, y)?

vector equation

c) What does this mean in terms of spans?

It is asking if a linear combination of given vectors can generate the target vector.

If the target vector lies within the span of the given vectors, there is a solution to the system. Otherwise, no solution exists.

d) Answer the question using the interactive demo.

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e) Answer the question using row reduction.

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -3 & -16 \\ 0 & 1 & 11 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 11 \\ 0 & -3 & -16 \end{array}\right) \rightarrow \begin{array}{c} \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & -15 \\ 0 & 1 & 11 \\ 0 & 0 & 17 \end{array}\right) \end{array} \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}$$

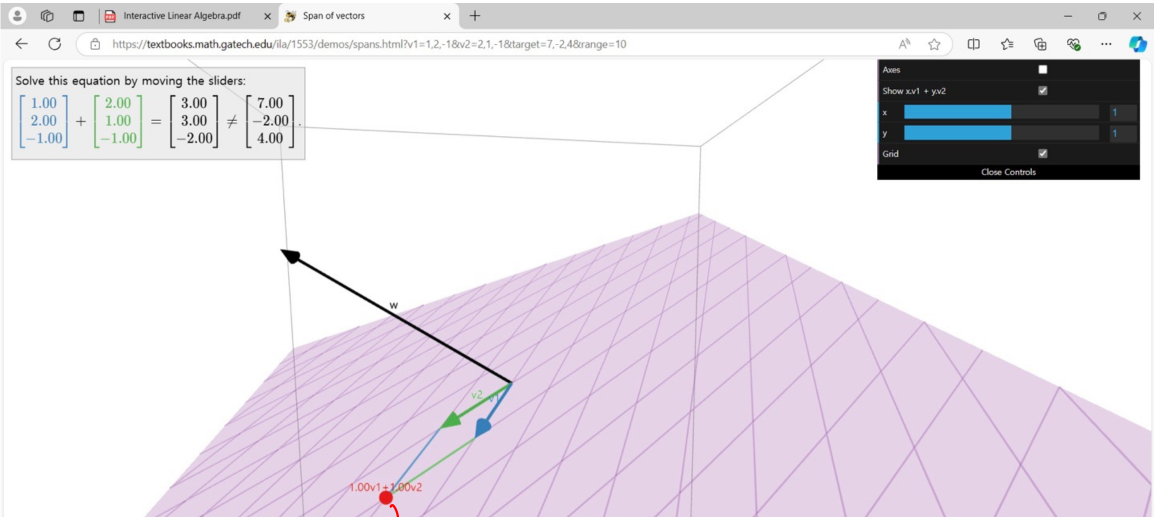
initial matrix

$R_2 \rightarrow -2R_1 + R_2$
$R_3 \rightarrow R_1 + R_3$

swap (R_2, R_3)

pivot

Since pivot is in the augmented row, this system has no solution. (= inconsistent) d)



span $\{v_1, v_2\}$ current $v_1 + v_2$

As you can see, the target vector $w = \begin{pmatrix}7\\-2\\4\end{pmatrix}$ is outside of the span $\{v_1, v_2\}$.

$\rightarrow$ No solution exists.

3. Paul Drake has challenged you to find his hidden treasure, located at some point (a, b, c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Paul's message, you have discovered that the first and second coordinates of the treasure's location are (in order) −4 and 3.

a) What is the treasure's full location?

Let this $(-4, 3, c)$.

He has honestly guaranteed you that the treasure can be found.

→ vector equation will represent a consistent system.

$$x v_1 + y v_2 + z v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

expanded form →

$$x \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + z \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$$

this can be converted to...

free variable

$$\underbrace{\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right)}_{\text{initial matrix}} \to \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & 6 & c-8 \end{array}\right) \to \begin{array}{c} \begin{array}{cccc} x & y & z & \text{aug} \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right) \end{array}$$

→ consistent 하므로 $c=5$.

$\therefore (-4, 3, 5)$

b) Give instructions for how to find the treasure by only moving in the directions given by v_1, v_2, and v_3.

We found that z is a free variable because it's pivot is in augmented row.

→ set $z = 0$ to make calculation easier.

1. go $1 \cdot v_1$ from the start $(0, 0, 0)$
   ⇒ $(1, -1, -2)$ currently.
2. go $(-1) \cdot v_1$ from the current $(1, -1, -2)$
   ⇒ $(-4, 3, 5)$ currently, which is a target.