Problem set 1. Question 1: Solow meets Piketty.

Thomas Piketty's book "Capital in the Twenty-First Century" was a timely contribution documenting the increase in wealth inequality across the world.

Besides presenting amazing data gathered by Piketty and coauthors, and publicly available at https://wid.world/data/, the book also introduces a conceptual framework summarized in two "fundamental laws of capitalism."

Piketty's second fundamental law of capitalism says that, as the growth rate of the economy declines due to lack of productivity (a decline in the growth rate of technology) or changing demographics (a decline in population growth), capital will become more important in the economy.

This problem set presents a formalization for his second law, and asks you to evaluate critically the assumptions behind it.

Let $\widetilde{Y}(t) = Y(t) - \delta K$ denote the output of an economy *net* of depreciation. Consider a variant of the Solow growth model where households save a constant fraction $\widetilde{s} \in (0,1)$ of their net income *plus* an amount to make up for the depreciation of capital, so that

$$\dot{K}(t) = \widetilde{s}(Y(t) - \delta K(t)).$$

Suppose that $Y(t) = F(K(t), A(t)L(t))$, and that $\dot{L}(t) = nL(t)$ and $\dot{A}(t) = gA(t)$.

1. Define an equilibrium in this economy.

2. Let $x(t) = \frac{K(t)}{A(t)L(t)}$. Derive an equation for the evolution of $x(t)$ over time, and show that $x(t) \to x^{ss} \in (0, \infty)$.

3. Show that, along an equilibrium path, $K(t)/\widetilde{Y}(t)$ converges to

$$\frac{K(t)}{\widetilde{Y}(t)} = \frac{\widetilde{s}}{g+n}.$$

4. What happens to $\frac{K}{Y}$ as $g + n \to 0$? Explain how this relates to Piketty's second law.

5. Show that the steady state of the model introduced above is equivalent to one where households save a fraction s of their gross income, where

$$s = \frac{\widetilde{s}(\delta + g + n)}{\widetilde{s}\delta + g + n}.$$

   What happens to this saving rate as $(g + n) \to 0$? Is it reasonable to expect this behavior from savings as growth or population growth slows down as in this model?

6. Use data from the Penn World Tables (available here https://www.rug.nl/ggdc/productivity/pwt/) to explore the relationship between the growth rate of aggregate GDP ($g+n$ in the model—or the growth rate of rgdpna in the dataset) and the *gross* savings rate. Although the Penn World Tables do not report savings, the data contain the investment rate (the variable csh_i).

Explain why we can use the investment rate as a proxy for savings. Then study empirically the relationship between the gross savings rate, s, and the growth rate of an economy, $g+n$. Does your evidence favor a model where s is independent of $g+n$ (as in the original Solow model) or decreasing with $g+n$ (as in the model used to derive Piketty's claim)? There are many sensible ways to explore this relationship. Please explain your procedure and why you chose it. Can you think of other tests of the second law using these data?

Problem set 1. Question 2: Self-replicating capital.

Consider an economy in discrete time where capital can be used in two ways: to produce A units of capital in the next period (replicate itself), or to produce units of a consumption good in period t (production). Independently of its use, capital depreciates at a rate $\delta \in (0,1)$.

Households own capital and they use an exogenous fraction s in the investment sector. Therefore, capital accumulates according to the equation,

$$K_{t+1} = sAK_t + (1-\delta)K_t.$$

Whereas consumption is given by $C_t = F((1-s)K_t, L)$, for some function F satisfying NC1 and NC2. Here L is a fixed level of labor that does not grow over time. Suppose also that $sA > \delta$.

This exercise is designed to explore the implications of "self-replicating" capital for growth.

Below, we will denote by $F_1(K,L)$ the partial derivative of F with respect to the first argument evaluated at (K,L) and by $F_2(K,L)$ the partial derivative of F with respect to the second argument evaluated at (K,L).

1. Define an equilibrium of this economy.

2. Show that, in any equilibrium, $K_t \to \infty$ and $C_t \to \infty$ (hint: use NC1 to show that $F((1-s)K_t, L) \geq LF_2((1-s)K_t, L)$ and then use NC2). Why is this economy capable of generating long-run growth in consumption and capital in the absence of technological change?

3. Let $R_t = F_1((1-s)K_t, L)$ be the rental rate of capital (where recall that this is the derivative of F with respect to the entire $(1-s)K$ term), and P_t the price of K_{t+1} in terms of units of the consumption good at time t. Argue that $R_t = AP_t$ (hint: capital can be used to produce or to build additional capital, and both uses should give the same return).

4. Let Y_t be GDP in this economy. Recall that output equals the *value* of consumption plus the *value* of investment. Show that we can compute output (denominated in units of the consumption good) as:

$$Y_t = C_t + sK_tAP_t.$$

5. Show that in this model, we still have $Y_t = W_tL + R_tK_t$.

6. Suppose that F is a Cobb-Douglas production function. Show that, along an equilibrium, the factor shares of capital and labor in national GDP, Y_t, are constant. Show also that the value of investment represents a fixed share of GDP.

7. Do capital, consumption, and GDP grow at the same rate? Is this a good model of growth at the frontier?