EC 702 Problem set 1.

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1 Question 1

Thomas Piketty's book "Capital in the Twenty-First Century" was a timely contribution documenting the increase in wealth inequality across the world. Besides presenting amazing data gathered by Piketty and coauthors, and publicly available at https://wid.world/data/, the book also introduces a conceptual framework summarized in two "fundamental laws of capitalism." Piketty's second fundamental law of capitalism says that, as the growth rate of the economy declines due to a lack of productivity (a decline in the growth rate of technology) or changing demographics (a decline in population growth), capital will become more important in the economy. This problem set presents a formalization for his second law and asks you to evaluate critically the assumptions behind it.

$$\dot{K}(t) = \tilde{s}(Y(t) - \delta K(t)) \tag{1}$$

Suppose that $Y(t) = F(K(t), A(t)L(t))$, and that $\dot{L}(t) = nL(t)$ and $\dot{A}(t) = gA(t)$.

1.1

For any given $\{K(0), L(0), A(0)\}$. The equilibrium of the economy is a sequence $\{K(t), L(t), A(t)\}_{t=0}^{\infty}$ and output and price sequence $\{Y(t), W(t), R(t)\}_{t=0}^{\infty}$.

The sequence of $\{K(t), L(t), A(t)\}_{t=0}^{\infty}$ is defined as below:

$$\begin{aligned} \dot{K}(t) = \tilde{s}(Y(t) - \delta K(t)) = \tilde{s}(F(K(t), A(t)L(t)) - \delta K(t)) \\ \dot{L}(t) = nL(t) \\ \dot{A}(t) = qA(t) \end{aligned} \tag{2}$$

The output and prices of $\{Y(t), W(t), R(t)\}_{t=0}^{\infty}$ is defined as below:

$$
\begin{aligned}
Y(t) &= F(K(t), A(t)L(t)) \\
W(t) &= A(t)F_L(K(t), A(t)L(t)) > 0 \\
R(t) &= F_K(K(t), A(t)L(t)) > 0
\end{aligned} \tag{3}
$$

1.2

We define $x(t)$, capital per efficient unit of labor as:

$$
x(t) = \frac{K(t)}{A(t)L(t)} \tag{4}
$$

Since $F(K(t), A(t)L(t))$ is homogeneous of degree one, we have constant return to scale and:

$$
\begin{aligned}
\frac{F(K(t), A(t)L(t))}{A(t)L(t)} &= F(\frac{K(t)}{A(t)L(t)}, 1) \\
&= F(x(t), 1) = f(x(t))
\end{aligned} \tag{5}
$$

From Eq. 4 we derivate the evolution of $x(t)$ as follows:

$$
\begin{aligned}
\dot{x}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{\dot{A}(t)}{A(t)}\frac{K(t)}{A(t)L(t)} - \frac{\dot{L}(t)}{L(t)}\frac{K(t)}{A(t)L(t)} \\
&= \frac{\tilde{s}(F(K(t), A(t)L(t)) - \delta K(t))}{A(t)L(t)} - (g+n)x(t) \\
&= \tilde{s}f(x(t)) - (\tilde{s}\delta + g + n)x(t)
\end{aligned} \tag{6}
$$

In the steady state, $x\dot(t) = 0$, so we have:

$$
\begin{aligned}
\tilde{s}f(x^{ss}) &= (\tilde{s}\delta + g + n)x^{ss} \\
\frac{f(x^{ss})}{x^{ss}} &= \frac{\tilde{s}\delta + g + n}{\tilde{s}}
\end{aligned} \tag{7}
$$

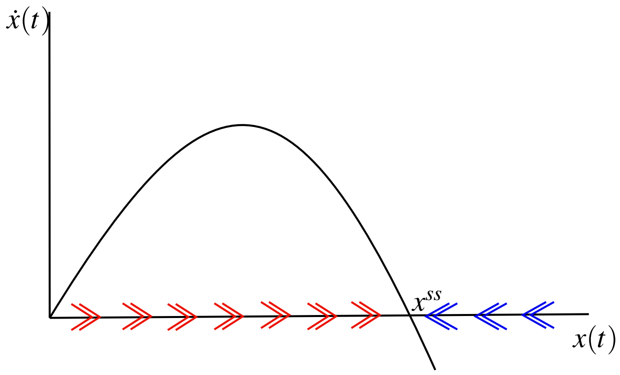
Now show that x^{ss} exists and is unique.

First, prove existence. According to NC1, $F(K(t), A(t)L(t))$ is $\mathbf{C^2}$, so $\frac{F(K(t), A(t)L(t))}{A(t)L(t)} = f(x(t))$ is continuous, and $\frac{f(x(t))}{x(t)}$ is continuous on $(0, \infty)$. According to NC2, the Inada condition, $\lim_{x(t)\to 0} \frac{f(x(t))}{x(t)} = \lim_{x(t)\to 0} f'(x(t)) = \lim_{K\to 0} F_K = \infty$, $\lim_{x(t)\to\infty} \frac{f(x(t))}{x(t)} = \lim_{x(t)\to\infty} f'(x(t)) =$

$\lim_{K\to\infty} F_K = 0$. Since $\frac{\tilde{s}\delta+g+n}{\tilde{s}} \in (0,\infty)$, by the Intermediate Value Theorem, there exist $x^{ss} \in (0,\infty)$ such that $\frac{f(x^{ss})}{x^{ss}} = \frac{\tilde{s}\delta+g+n}{\tilde{s}}$. QED.

Then prove uniqueness. $\frac{\partial \frac{f(x(t))}{x(t)}}{\partial x(t)} = \frac{f'(x(t))x(t)-f(x(t))}{x(t)^2} = -\frac{f(x(t))-f'(x(t))x(t)}{x(t)^2} = -\frac{W(t)}{x(t)^2} < 0$. So $\frac{f(x(t))}{x(t)}$ is a monotonically decreasing function. So the solution of $\frac{f(x^{ss})}{x^{ss}} = \frac{\tilde{s}\delta+g+n}{\tilde{s}}$ is unique. QED.

This can also be shown in the graph below:



The function of the curve is $x(\dot{t}) = \tilde{s}f(x(t)) - (\tilde{s}\delta + g + n)x(t)$. The shape of the curve satisfies the Inada condition. When $x(0) > x^{ss}$ the depletion is higher than the investment, therefore, by equation 6, $x(t)$ will keep decreasing on $t \in (0,T)$ until the value of $x(T)$ moves back and is equal to x^{ss} as shown by the blue arrows. Vice versa, when $x(0) < x^{ss}$, the investment is greater than the total depletion, therefore, $x(t)$ increases until x^{ss} is reached, as shown by the red arrows.

1.3

Proof:

$$\begin{aligned}\frac{K(t)}{\tilde{Y(t)}} &= \frac{K(t)}{Y(t)-\delta K(t)} \\ &= \frac{\frac{K(t)}{A(t)L(t)}}{\frac{F(K(t),A(t)L(t))-\delta K(t)}{A(t)L(t)}} \\ &= \frac{x(t)}{f(x(t))-\delta x(t)}\end{aligned} \tag{8}$$

Since $x(t) \to x^{ss}$, $\frac{x(t)}{f(x(t))-\delta x(t)} \to \frac{x^{ss}}{f(x^{ss})-\delta x^{ss}}$, therefore:

$$\frac{K(t)}{\tilde{Y(t)}} \to \frac{x^{ss}}{f(x^{ss})-\delta x^{ss}} = \frac{x^{ss}}{(\frac{\tilde{s}\delta+g+n}{\tilde{s}}-\delta)x^{ss}} = \frac{1}{(\frac{q+n}{\tilde{s}})} = \frac{\tilde{s}}{q+n} \tag{9}$$

QED.

1.4

$$
\begin{aligned}
\frac{K(t)}{\tilde{Y}(t)} = \frac{K(t)}{Y(t)-\delta K(t)} &= \frac{\tilde{s}}{g+n} \\
\iff \frac{g+n}{\tilde{s}} &= \frac{Y(t)-\delta K(t)}{K(t)} \\
\iff \frac{g+n}{\tilde{s}} + \delta &= \frac{Y(t)}{K(t)} \\
\iff \frac{Y(t)}{K(t)} &= \frac{g+n+\delta\tilde{s}}{\tilde{s}} \\
\iff \frac{K(t)}{Y(t)} &= \frac{\tilde{s}}{g+n+\delta\tilde{s}}
\end{aligned} \tag{10}
$$

Then when $n+g \to 0$:

$$
\lim_{g+n\to 0} \frac{K(t)}{Y(t)} = \frac{1}{\delta} \tag{11}
$$

The capital share of output approximates a constant value.

Piketty's second law states that if the economy keeps a constant saving rate, then capital grows more and more important with respect to output. Which is supported by the result above since $\frac{K}{Y} = \frac{k}{y}$ monotonically increases and converges to $\frac{1}{\delta}$ when $g+n$ decreases to 0.

1.5

Starting from equation 6 we have that:

$$
\dot{x}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{\dot{A}(t)}{A(t)}\frac{K(t)}{A(t)L(t)} - \frac{\dot{L}(t)}{L(t)}\frac{K(t)}{A(t)L(t)} \tag{12}
$$

In this case, the household holds a fraction s of their income therefore the evolution of capital is as follows:

$$
\dot{K}(t) = s(F(K(t),A(t)L(t)) - \delta K(t) \tag{13}
$$

Including in while the other equations in remain the same we have that in the case where the households save the evolution of $x(t)$ is as follows:

$$
\begin{aligned}
\dot{x}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{\dot{A}(t)}{A(t)}\frac{K(t)}{A(t)L(t)} - \frac{\dot{L}(t)}{L(t)}\frac{K(t)}{A(t)L(t)} \\
&= \frac{s(F(K(t),A(t)L(t)) - \delta K(t)}{A(t)L(t)} - (g+n)x(t) \\
&= \frac{s(F(K(t),A(t)L(t))}{A(t)L(t)} - \frac{\delta K(t)}{A(t)L(t)} - (g+n)x(t) \\
&= \frac{s(F(K(t),A(t)L(t))}{A(t)L(t)} - \delta x(t) - (g+n)x(t) \\
&= sf(x(t)) - (\delta+g+n)x(t)
\end{aligned}
\tag{14}
$$

Similar to subsection 1.2 the equilibrium of the model is characterised by $\dot{x}(t) = 0$ then we find that at this point:

$$
\begin{aligned}
sf(x(t)) &= (\delta+g+n)x(t) \\
s &= \frac{(\delta+g+n)x(t)}{f(x(t))}
\end{aligned}
\tag{15}
$$

With $s = \frac{\tilde{s}(\delta+g+n)}{\tilde{s}\delta+g+n}$ in equation 15 we have:

$$
\begin{aligned}
\frac{\tilde{s}(\delta+g+n)}{\tilde{s}\delta+g+n} &= \frac{(\delta+g+n)x(t)}{f(x(t))} \\
\iff \frac{\tilde{s}f(x(t))}{\tilde{s}\delta+g+n} &= \frac{(\delta+g+n)x(t)}{(\delta+g+n)} \\
\iff \tilde{s}f(x(t)) &= (\tilde{s}\delta+g+n)x(t)
\end{aligned}
\tag{16}
$$

Which is the same as equation 6 above.

When $g+n \to 0$, $s = \frac{\tilde{s}(\delta+g+n)}{\tilde{s}\delta+g+n} \to \frac{\tilde{s}\delta}{\tilde{s}\delta} = 1$. This indicates that when population growth and growth stagnate, the saving rate is 100%: households experience zero consumption and save all their income. The scenario is implausible in the real world and is not compatible with the balanced growth pattern.

1.6

In the classic Solow Model as well as in Piketty's assumption we have that households save capital at a fixed rate s and $\tilde{s}$, respectively. The setup in both versions implies that the household can either consume the final product or save a rate s for period $t+1$, then, these savings are the only investment in capital left for period $t+1$. Therefore the investment rate in the economy is a good proxy for the savings in the context of the Solow Model.

To empirically study the relationship between the savings rate, s, and the growth of an economy $g+n$, we developed a fixed effects estimation where the dependent variable was the average investment rate, and the independent variable was the average growth rate in country i during period t. We also added a country fixed-effect to address the heterogeneity across countries. In this case, since our interest is to study long-term growth, we compiled the average growth rate data on periods of 10, 15, 20, 30, and 50 years. To account for the nonlinear relationship between the savings s and growth $g+n$ given by $s = \frac{\tilde{s}(\delta+g+n)}{\tilde{s}\delta+g+n}$ we showed an additional estimation including the squared growth for each aggregated period.

Table 1 shows the results of the estimations. The first conclusion that arises from our empirical approach is that for all time aggregations we observe a positive and significant relationship between the investment and growth rates. Therefore, the evidence does not support the assumption of the original Solow Model where the savings rate s is independent of the growth $g+n$. At the same time, the evidence goes against Piketty's claim that the relationship between the variables is negative.

In addition, Columns (3), (5), (7), (9), and (11) show that the non-linear term relationship between the savings rate and the growth rate is not significant for any of the time periods, which further invalidates Piketty's assumption.

Table 1: Relationship between the investment rate and growth.

	Dependent Variable: $Investment\ rate_{i,t}$										
VARIABLES	t=1	t = 10		t = 15		t = 20		t = 30		t = 50	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
$growth\ rate_{i,t}$	0.208***	0.324***	0.323***	0.207***	0.204***	0.444***	0.497**	0.424**	0.377	0.341*	0.268**
	(0.0501)	(0.0954)	(0.123)	(0.0774)	(0.0767)	(0.140)	(0.249)	(0.177)	(0.269)	(0.175)	(0.114)
$growth\ rate^2_{i,t}$			0.00545		0.789		-0.589		0.591		2.076
			(1.213)		(0.521)		(1.857)		(2.787)		(3.856)
Constant	0.211***	0.206***	0.206***	0.209***	0.207***	0.202***	0.201***	0.205***	0.205***	0.207***	0.205***
	(0.00780)	(0.00381)	(0.00373)	(0.00286)	(0.00365)	(0.00550)	(0.00665)	(0.00694)	(0.00718)	(0.00638)	(0.00908)
Observations	10,216	1,046	1,046	817	817	637	637	523	523	362	362
R-squared	-0.148	0.613	0.613	0.626	0.627	0.653	0.653	0.676	0.676	0.726	0.726
Country FE	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES

Robust standard errors in parentheses

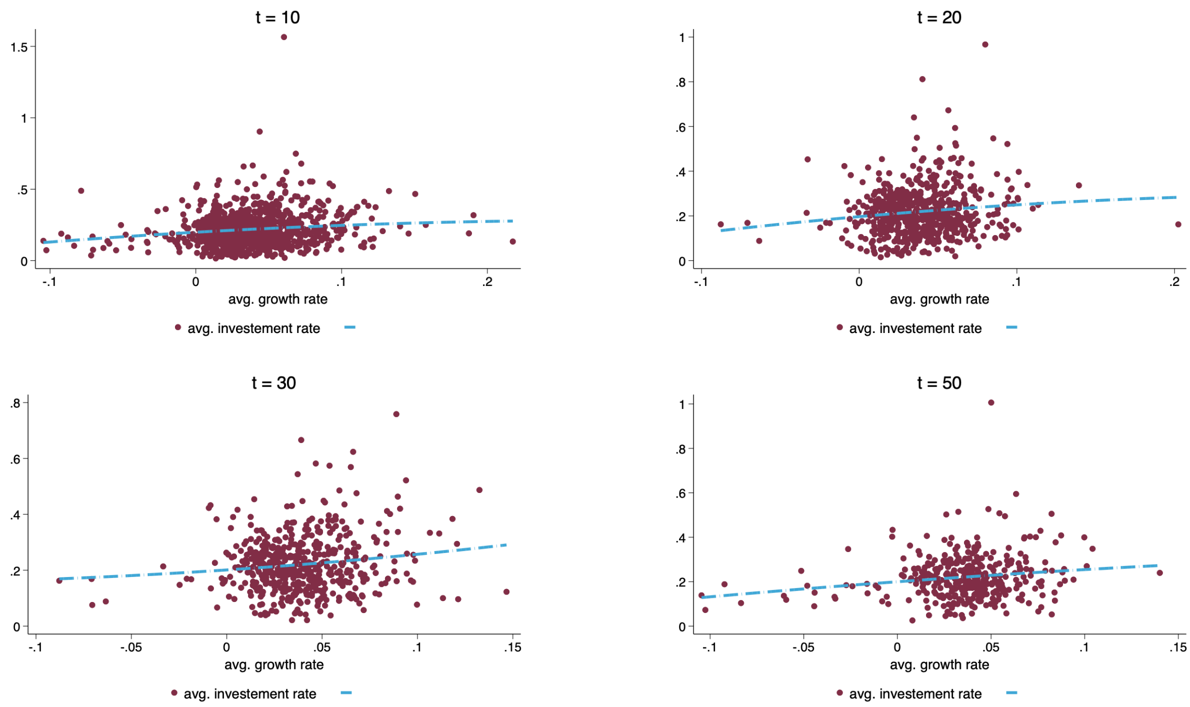
*** p<0.01, ** p<0.05, * p<0.1

However, it is important to keep in mind that this is only one approximation to empirically test Pikkety's second law. We include the squared term for growth in our estimations in table

1 to account for the non-linear relationship between growth and savings, however, this non-linearity could take other forms that could provide a different conclusion. However, as shown by figure 1 when using investment rate as a proxy for s, the correlation with growth rates appears to be linear. Moreover, in equation 16 above we can see that the savings are also affected by the depreciation rate, therefore, we could include a measure of the depreciation of capital in the construction of a proxy for savings in order to make the estimations more accurate and perhaps then we might be able to observe the non-linearity.

Lastly, if we wanted to study Piketty's second claim more accurately, we could account for differences in other factors such as the stock of capital. It might be possible that Piketty's claim holds for frontier economies that have higher stocks of capital. This would require a study similar to Acemoglu's conditional convergence.

Figure 1: Correlation between average investment rate and average growth.



Notes: Created from the information available in the Penn World Tables. Figures show average investment rates by country i and aggregated average growth of real GDP over periods t.

2 Question 2

2.1 Equilibrium

For given $\{K_0, L_0\}$, an equilibrium is given by the full-time path of stocks $\{K_t, L_t\}$ and output and factor prices $\{Y_t, W_t, R_t\}_{t=0}^{\infty}$ such that:

I Starting at $\{K_0, L_0\}$, capital and labor satisfy the following equations:

$$\begin{aligned} K_{t+1} &= sAK_t + (1-\delta)K_t \\ L_{t+1} &= L_t = L \end{aligned} \tag{17}$$

II Output and factor prices satisfy:

$$\begin{aligned} Y_t &= sAK_tP_t + F((1-s)K_t, L) \\ R_t &= F_1 \\ W_t &= F_2. \end{aligned} \tag{18}$$

2.2 Capital and Consumption Going to Infinity

Given the capital accumulation pattern, $K_{t+1} = sAK_t + (1-\delta)K_t = (1+sA-\delta)K_t$, we can derive the following expression of capital by induction:

$$K_t = (1+sA-\delta)^t K_0. \tag{19}$$

Since K_0 is fixed and $sA > \delta$, $(1+sA-\delta)^n \to \infty$, thus $K_t \to \infty$.

By NC1, the cost function, F, is homogeneous degree 1. Then, through Euler's Theorem, $(1-s)K_tF_1((1-s)K_t, L) + LF_2((1-s)K_t, L) = F((1-s)K_t, L)$.

Again by NC1, $F_1 \geq 0$. Then, $F((1-s)K_t, L) \geq LF_2((1-s)K_t, L)$.

Since F is homogeneous degree 1, F_2 then is homogeneous degree 0. Rewriting the above inequality yields:

$$\begin{aligned} \frac{F((1-s)K_t, L)}{L} &\geq F_2((1-s)K_t, L) \\ F(\frac{(1-s)K_t}{L}, 1) &\geq F_2(1, \frac{L}{(1-s)K_t}) \\ f(x_t) \geq f'(\frac{1}{x_t}), &\text{ if replacing } \frac{(1-s)K_t}{L} \text{ with } x_t. \end{aligned} \tag{20}$$

By NC2, $\lim_{x\to 0} f'(x) = \infty$, thus we have:

$$\lim_{K_t\to\infty} f'(\frac{1}{x_t}) = \lim_{\frac{1}{x_t}\to 0} f'(\frac{1}{x_t}) = \infty. \tag{21}$$

Combining this equation with $f(x_t) \geq f'(\frac{1}{x_t})$, we derive $C(t) = Lf(x_t) \to \infty$.

The economy can generate long-run growth without technological change because the capital keeps accumulating exponentially, assuring that the productivity caused by capital increase exceeds the effect of diminishing marginal return of labor.

When the capital can replicate by itself, households don't need to invest that much amount to keep a certain level of capital. Since the output is used by either consumption or investment, the consumption level must go up over time.

2.3 Rental Rate of Capital and Price of Consumption Goods

At time t, the amount of capital is K_t, which can either be entirely used for production or for replication of new capital.
If for production, the household rents out the capital to the firm, yielding revenue:

$$RV_p = K_t R_t \tag{22}$$

If for replication, then the current capital K_t will generate AK_t new capital, which can be converted to consumption goods and thus consumed during that period. Since P_t is the price of K_{t+1} in terms of the units of consumption goods at time t, yielding the revenue:

$$RV_r = (AK_t)P_t \tag{23}$$

Since the two usages should give the same return, we can let $RV_p = RV_r$, then:

$$K_t R_t = (AK_t)P_t K_t = AP_t \tag{24}$$

2.4 Computation of Output

As the household uses an exogenous fraction s, of the newly generated capital $AK(t)$, to the investment sector, the amount of capital investment is $sAK(t)$. Then the value of the investment is $I_t = sAK_tP_t$.

The output, $Y(t)$, equals the value of consumption plus the value of investment, as:

$$Y_t = C_t + I_t = C_t + sK_tAP_t \tag{25}$$

2.5 Output Equals Income

Given the fact that $R_t = AP_t$, and $F = (1-s)K_tF_1 + LF_2$, then the output can be written as:

$$\begin{aligned} Y_t &= C_t + sK_tAP_t \\ &= F + sK_tAP_t \\ &= (1-s)K_tF_1 + LF_2 + sK_tR_t \\ &= K_tF_1 + LF_2 \\ &= K_tR_t + LW_t. \end{aligned} \tag{26}$$

2.6 Cobb-Douglas F

Suppose F is in Cobb-Douglas form, $F((1-s)K_t, L) = [(1-s)K_t]^{\alpha}L^{1-\alpha}$.Then, by previous results:

$$\begin{aligned} R_t = F_1 = \alpha[(1-s)K_t]^{\alpha-1}L^{1-\alpha} &= \frac{\alpha F}{(1-s)K_t} \Rightarrow K_tR_t = \frac{\alpha}{1-s}F \\ W_t = F_2 = (1-\alpha)L^{-\alpha}[(1-s)K_t]^{\alpha} &= \frac{(1-\alpha)F}{L} \Rightarrow W_tL = (1-\alpha)F \end{aligned} \tag{27}$$

Since $Y_t = W_tL + R_tK_t$, then $Y_t = [\frac{\alpha}{1-s} + (1-\alpha)]F$. Thus the capital share and labor share are:

$$\begin{aligned} s_K = \frac{K_tR_t}{Y_t} &= \frac{\alpha}{1-s+s\alpha} \\ s_L = \frac{W_tL}{Y_t} &= \frac{(1-s)(1-\alpha)}{1-s+s\alpha} \end{aligned} \tag{28}$$

Then, the share of value of investment is $s_I = \frac{I_t}{Y_t} = \frac{sK_tR_t}{Y_t} = \frac{s\alpha}{1-s+s\alpha}$

2.7 Growth Rate for GDP, Capital and Consumption

Now we evaluate the model in Cobb-Douglas form. According to Q2.6, $\frac{C_t}{Y_t} = 1 - s_I = \frac{1-s}{1-s+s\alpha}$. Since the consumption-output ratio is constant, consumption grows proportionally with the output and has the same growth rate as the output, namely the GDP.

Since $C_t = F((1-s)K(t), L) = [(1-s)K(t)]^{\alpha} L^{1-\alpha}$, we have $\ln C_t = \alpha \ln(1-s)K_t + (1-\alpha)\ln L$. The equation indicates that $g_c = \alpha g_k, \alpha \in (0,1)$, where g_c is the growth rate of consumption and g_k is the growth rate of capital. Thus, the capital growth rate is higher than that of the consumption and GDP.

To examine the validity of a model of growth at the frontier, we can check if the results replicate the Kaldor facts.

First, from our previous arguments we can conclude that while consumption and output grow at the same rate, capital grows faster, which partially violates the first Kaldor fact. Moreover, without empirical data, we cannot examine the second fact of Kaldor that growth rates between 1% and 5%.

From equation 28, $W_t = \frac{(1-\delta)}{1-s+s\alpha}\frac{Y_t}{L}$. Therefore solving for the growth of salaries W_t we can see that the salary grows at the same rate as the output, and with L fixed, this is the same as the growth rate for the output per worker. The third Kaldor fact holds in this model.

The return of capital rate is $R_t = \alpha[(1-s)K_t]^{\alpha-1} L^{1-\alpha}$, which varies by time, since K_t goes to infinity. This violates the fourth Kaldor fact.

Lastly according to Q2.6, factor shares are constant and satisfy the fifth Kaldor fact.

We can now conclude that this model partly fails the first Kaldor fact, and completely violates the fourth. At the same time, we cannot test for the second kaldor fact. However, given the model's simple nature, these deviations could be acceptable, since it still is able to replicate 2 of the 5 facts, making this model still valid to simulate balanced growth in our view.