(1) $\frac{1}{2}\|Ka-y\|^2+\frac{\lambda}{2}a^TKa \to \min \quad K=\Phi\Phi^T$

$$\frac{\partial \frac{1}{2}\|Ka-y\|^2}{\partial a}=\frac{\partial \frac{1}{2}(Ka-y)^T(Ka-y)}{\partial a}=$$

$$=\frac{\partial \frac{1}{2}(Ka-y)^T}{\partial a}\cdot(Ka-y)+\frac{1}{2}(Ka-y)^T\frac{\partial(Ka-y)}{\partial a}=$$

$$=\frac{1}{2}\left(\frac{\partial(Ka-y)}{\partial a}\right)^T\cdot(Ka-y)+\frac{1}{2}(Ka-y)^T\frac{\partial(Ka-y)}{\partial a}=$$

$$=\frac{1}{2}K^T(Ka-y)+\frac{1}{2}(Ka-y)^T\cdot K=\frac{1}{2}\cdot 2K^T(Ka-y)=K^T(Ka-y)$$

$$\frac{\partial \frac{\lambda}{2}a^TKa}{\partial a}=\frac{\partial \frac{\lambda}{2}a^TK}{\partial a}\cdot a+\frac{\lambda}{2}a^TK\frac{\partial a}{\partial a}=\frac{\lambda}{2}Ka+\frac{\lambda}{2}a^TK=$$

$$=\frac{\lambda}{2}(Ka+K^Ta)=\frac{\lambda}{2}(K^Ta+K^Ta)=\lambda K^Ta$$

$$K^T(Ka-y)+\lambda K^Ta=0$$

$$Ka-y+\lambda Ia=0$$

$$(K+\lambda I)a=y$$

$$a=(K+\lambda I)^{-1}y$$

(2) $x^2+1\to\min$

$(x-2)(x-4)\le 0 \qquad x\in R$

1. $2\le x\le 4$

$(x^2+1)'=2x=0$

$x=0\notin[2;4]$

на отрезке $[2;4]$ $f(x)$ возрастает $\Rightarrow$ $x_*=2;\ f(x_*)=5$

2. $L(x,\lambda)=x^2+1+\lambda(x-2)(x-4)$

$\lambda_1=0{,}5 \quad L(x;0{,}5)=1{,}5x^2-3x+5$

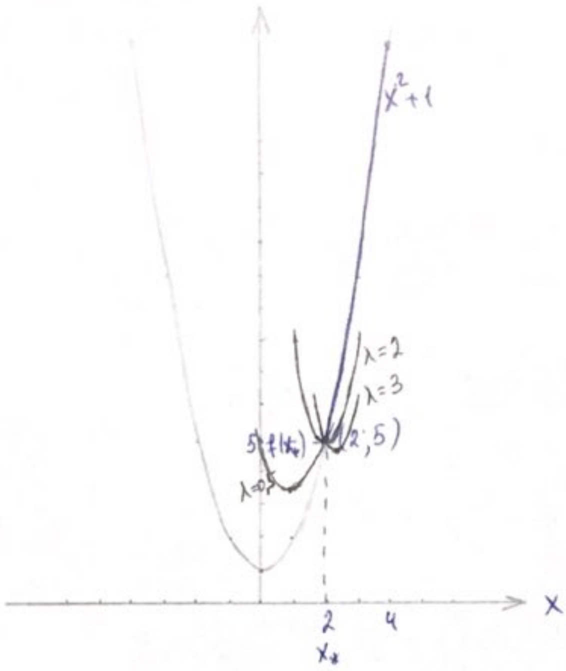
$\lambda_2=2 \quad L(x;2)=3x^2-12x+17$

$\lambda_3=3 \quad L(x;3)=4x^2-18x+25$

$f(x_*) \geq \inf_x L(x,\lambda)$

$f(x_*) = 5 \quad \inf_x L(x,\lambda) = 5 \Rightarrow$

$f(x_*) = \inf_x L(x,\lambda)$



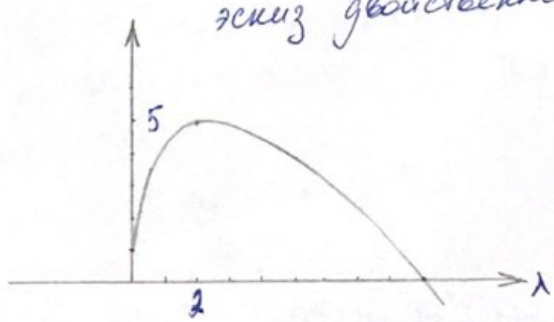
$$\frac{\partial L}{\partial x} = 2x + 2\lambda x - 6\lambda \qquad \hat{x} = \frac{6\lambda}{2+2\lambda} = \frac{3\lambda}{\lambda+1}$$

$$g(\lambda) = \left(\frac{3\lambda}{\lambda+1}\right)^2 + 1 + \lambda\left(\frac{3\lambda}{\lambda+1} - 2\right)\left(\frac{3\lambda}{\lambda+1} - 4\right) = \frac{9\lambda^2}{(\lambda+1)^2} + 1 + \frac{\lambda(\lambda-2)(-\lambda-4)}{(\lambda+1)^2} =$$

$$= \frac{9\lambda^2 + \lambda^2 + 2\lambda + 1 - \lambda^3 - 2\lambda^2 + 8\lambda}{(\lambda+1)^2} = \frac{-\lambda^3 + 8\lambda^2 + 10\lambda + 1}{(\lambda+1)^2} =$$

$$= -\lambda + 10 - \frac{9\lambda+9}{(\lambda+1)^2} = -\lambda + 10 - \frac{9}{\lambda+1}$$

эскиз двойственной ф-ии



3. $\begin{cases} g(\lambda) \to \max \\ \lambda \geq 0 \end{cases}$

$$\left(-\lambda + 10 - \frac{9}{\lambda+1}\right)' = -1 + \frac{9}{(\lambda+1)^2} = 0$$

$(\lambda+1)^2 = 9$

$\lambda + 1 = \pm 3$

$\underline{\lambda^* = 2} \qquad \lambda_2 = -4 < 0$

$g(\lambda^*) = -2 + 10 - \frac{9}{3} = \underline{5}$

$g(\lambda^*) = f(x_*)$ – строгая двойственность выполняется

4. $x^2+1 \to \min$

$(x-2)(x-4) \le u$

$x^2-6x+8-u \le 0$

$x = \dfrac{6 \pm \sqrt{4+4u}}{2} = 3 \pm \sqrt{1+u}$, $u \ge -1$

$x \in [3-\sqrt{1+u}; 3+\sqrt{1+u}]$

$f(x) = x^2+1$; $x_{min} = 0$

1) $3-\sqrt{1+u} \ge 0$ $(= x_{min})$

$\sqrt{1+u} \le 3$

$u \le 8$

$f_u(x_*) = f_u(3-\sqrt{1+u}) = (3-\sqrt{1+u})^2+1 = 9-6\sqrt{1+u}+1+u+1 =$

$= 11-6\sqrt{1+u}+u$

$$\begin{cases} f_u(x_*) = 11-6\sqrt{1+u}+u \\ u \le 8 \end{cases}$$

2) $$\begin{cases} 3-\sqrt{1+u} < x_{min} < 3+\sqrt{1+u} \\ f_u(x_*) = f_u(x_{min}) = 1 \end{cases}$$

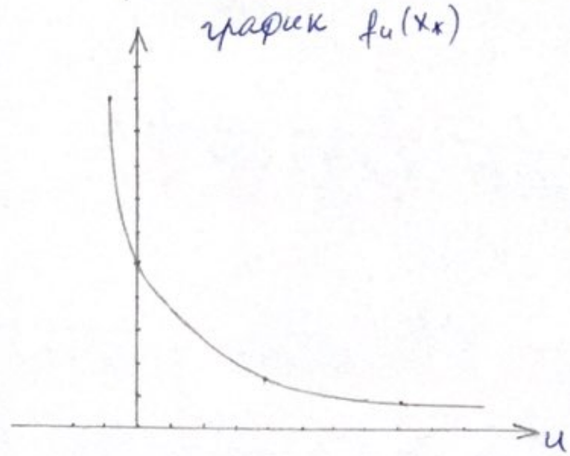
$$\begin{cases} 3-\sqrt{1+u} < 0 \\ 3+\sqrt{1+u} > 0 \end{cases} \quad \begin{cases} \sqrt{1+u} > 3 \\ \sqrt{1+u} > -3 \end{cases} \quad \begin{cases} u > 8 \\ f_u(x_*) = 1 \end{cases}$$

3) $3+\sqrt{1+u} \le 0$ $(= x_{min})$

$\sqrt{1+u} \le -3$

$\varnothing$

график $f_u(x_*)$



$$\frac{\partial f_u(x_*)}{\partial u} = \frac{\partial(11-6\sqrt{1+u}+u)}{\partial u} =$$

$$= \frac{-6}{2\sqrt{1+u}} + 1 = \frac{-3}{\sqrt{1+u}} + 1$$

$$L(x,\lambda,u) = x^2+1+\lambda(x^2-6x+8-u)$$

$$\frac{\partial L}{\partial x} = 2x+2\lambda x-6\lambda \qquad \tilde{x} = \frac{3\lambda}{\lambda+1}$$

$$g(\lambda,u) = \frac{\partial\left(-\lambda+10-\frac{9}{\lambda+1}-\lambda u\right)}{\partial\lambda} = -1+\frac{9}{(\lambda+1)^2}-u=0$$

$$\frac{9}{(\lambda+1)^2} = 1+u$$

$$(\lambda+1)^2 = \frac{9}{1+u}$$

$$\lambda+1 = \pm\frac{3}{\sqrt{1+u}}$$

$$\underline{\lambda^* = -1+\frac{3}{\sqrt{1+u}}} \qquad \lambda = -1-\frac{3}{\sqrt{1+u}} < 0$$

$$\frac{\partial f_u(x_*)}{\partial u} = 1-\frac{3}{\sqrt{1+u}} = -\lambda$$

(3) $K(x,z) = \cos(x-z) = \cos x\cos z+\sin x\cdot\sin z$

$K_1(x,z) = \cos x\cdot\cos z$ – ядро по св-ву $K_1(x,z) = f(x)f(z)$

$K_2(x,z) = \sin x\cdot\sin z$ – ядро по св-ву $K_2(x,z) = g(x)g(z)$

$K(x,z) = \underbrace{K_1(x,z)+K_2(x,z)}_{\text{ядра}}$ является ядром

(4) $K(x,z) = \dfrac{1}{1+e^{-xz}}$, $x,z\in\mathbb{R}$

Докажем, что не выполняется п.2 теоремы Мерсера для некоторых значений x, z.

Пусть $x=1$; $z=2$

$$\begin{vmatrix} \frac{1}{1+e^{-1}} & \frac{1}{1+e^{-2}} \\ \frac{1}{1+e^{-2}} & \frac{1}{1+e^{-4}} \end{vmatrix} = \frac{1}{1+e^{-1}}\cdot\frac{1}{1+e^{-4}}-\frac{1}{(1+e^{-2})^2} =$$

$$= \frac{e}{e+1}\cdot\frac{e^4}{e^4+1}-\frac{e^4}{e^4+2e^2+1} = e^4\left(\frac{e}{e^5+e^4+e+1}-\frac{1}{e^4+2e^2+1}\right) =$$

$$= e^4\left(\frac{\cancel{e^5}+2e^3+\cancel{e}-\cancel{e^5}-e^4-\cancel{e}-1}{(e+1)(e^4+1)(e^2+1)^2}\right) = \frac{e^4(2e^3-e^4-1)}{(e+1)(e^4+1)(e^2+1)^2} < 0$$ – отрицательно определена

Минор 2-го порядка <0, миноры 1-го порядка $>0 \Rightarrow$ не является ядром

⑤ $K_1(x,z) = (1+xz)^2 = 1+2xz+x^2z^2 = (1, \sqrt{2}x, x^2), (1+\sqrt{2}z, z^2)$

Спрямляющее пр-во $(1, \sqrt{2}x, x^2)$

$K_2(x,z) = 1+xz+x^2z^2 = (1, x, x^2), (1, z, z^2)$

Спрямляющее пр-во $(1, x, x^2)$

$K(x,z) = K_1(x,z) + K_2(x,z) = 2+3xz+2x^2z^2 =$

$= (\sqrt{2}, \sqrt{3}x, \sqrt{2}x^2), (\sqrt{2}, \sqrt{3}z, \sqrt{2}z^2)$

Спрямляющее пр-во $(\sqrt{2}, \sqrt{3}x, \sqrt{2}x^2)$