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Bachelor's Thesis

Gödel's Incompleteness Theorems

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Ich versichere, dass ich diese Bachelor-Arbeit selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Abstract

In this thesis we are going to elaborate on Gödel's Incompleteneness Theorems. Gödel's First Incompleteness Theorem states that every theory with enough arithmetic is incomplete. This implies that there does not exist a deductive system which proves all of the true sentences in the arithmetic $\mathcal{N} = (\mathbb{N}; +, \cdot, S; 0)$. Gödel's Second Incompleteness Theorem depicts that any theory with enough arithmetic which is consistent cannot prove its own consistency.

In the first part of this thesis we prove Gödels First Incompleteness Theorem. Hereby, we construct a sentence in a theory with enough arithmetic σ which says about itself that it is unprovable in this theory. In the second part we utilize the Formalized First Theorem in order to show the Second Incompleteness Theorem. Lastly, we examine a similar incompleteness result from algorithmic complexity involving the Kolmogorov Complexity. The Kolmogorov Complexity of a string can be defined as the minimum length of a program that outputs that string and stops. Chaitin's Incompleteness Theorem states that there exists a number c such that any consistent theory with enough arithmetic cannot prove that a string has Kolmogorov complexity larger than c.

Zusammenfassung

In dieser Bachelorarbeit werden wir die Gödelschen Unvollständigkeitssätze behandeln. Gödels Erster Unvollständigkeitssatz besagt, dass jede Theorie mit genügend Arithmetik unvollständig ist. Dies impliziert, dass es keinen Kalkül gibt, in dem alle in $\mathcal{N} = (\mathbb{N}; +, \cdot, S; 0)$ geltenden Aussagen beweisbar sind. Gödels Zweiter Unvollständigkeitssatz zeigt, dass jede konsistente Theorie mit genügend Arithmetik nicht beweisen kann, dass sie konsistent ist.

Im ersten Teil der Arbeit beweisen wir Gödels Ersten Gödelschen Unvollständigkeitssatz. In einer Theorie mit genügend Arithmetik konstruieren wir einen Satz σ, welcher über sich selbst sagt, dass er nicht beweisbar in dieser Theorie ist. Im zweiten Teil verwenden wir den Formalen Ersten Satz um den Zweiten Unvollständigkeitssatz zu beweisen. Zuletzt schauen wir uns ein ähnliches Unvollständigkeitsresultat aus der algorithmischen Komplexität an, welches die Kolmogorov Komplexität involviert. Die Kolmogorov Komplexität von einem Wort ist definiert als die minimale Länge eines Programms, welches dieses Wort ausgibt und dabei stoppt. Chaitins Unvollständigkeitssatz besagt, dass eine Zahl c existiert, sodass eine beliebige konsistente Theorie mit genügend Arithmetik nicht beweisen kann, dass ein Wort eine höhere Kolmogorov Komplexität hat als c.

Contents

1 Introduction

1.1 Preliminaries

In this chapter we deal with the preliminaries in order to prove Gödels Incompleteness Theorems. In the first part we summarize the basic concepts from first-order logic and in the second part we emphasize important facts from computability theory. Lastly, in the third part we discuss the historical background of Gödel's Incompleteness Theorems.

1.1.1 Basic Concepts of First-Order Logic

In the following we introduce the basic concepts of first-order logic.

Structures and Signatures A *structure* is a quadruple

$$\mathcal{M} = (M; (R_i^{\mathcal{M}}|i \in I); (f_j^{\mathcal{M}}|j \in J); (c_k^{\mathcal{M}}|k \in K))$$

where I, J, K are arbitrary (possibly empty or infinite) sets and the following holds:

- M is a nonempty set (the *universe of* $\mathcal{M}$; the elements of M are called *individuals of* $\mathcal{M}$),
- for every $i \in I$, $R_i^{\mathcal{M}}$ is an n_i-ary relation on M with $n_i \geq 1$, i.e. $R_i^{\mathcal{M}} \subseteq M^{n_i}$ (the *relations of* $\mathcal{M}$),
- for every $j \in J$, $f_j^{\mathcal{M}}$ is an m_j-ary function with domain M and $m_j \geq 0$, i.e. $f_j^{\mathcal{M}} : M^{m_j} \to M$ (the *functions of* $\mathcal{M}$) and
- for every $k \in K$, $c_k^{\mathcal{M}}$ is an element of M (the *constants of* $\mathcal{M}$).

The signature of a structure $\mathcal{M}$ is determined by the number of relations, functions and constants of $\mathcal{M}$ along with the arity of the relations and functions of $\mathcal{M}$.

The structure $\mathcal{M} = (M; (R_i^{\mathcal{M}}|i \in I); (f_j^{\mathcal{M}}|j \in J); (c_k^{\mathcal{M}}|k \in K))$ has *signature*

$$\sigma(\mathcal{M}) = ((n_i|i \in I); (m_j|j \in J); K)$$

where for $i \in I$, $R_i^{\mathcal{M}}$ is n_i-ary and for $j \in J$, $f_j^{\mathcal{M}}$ is m_j-ary. (The signature σ is called the *signature of* (*the structure*) $\mathcal{M}$.)

For a structure $\mathcal{M} = (M; (R_i^{\mathcal{M}}|i \in I); (f_j^{\mathcal{M}}|j \in J); (c_k^{\mathcal{M}}|k \in K))$ with signature $\sigma(\mathcal{M}) = ((n_i|i \in I); (m_j|j \in J); K)$, we make the following assumptions:

- If the index sets I, J, K are finite, then we assume that the index sets are initial parts of the natural numbers $\mathbb{N} := \{0, 1, \ldots\}$.

- If an index set is empty, then we omit the corresponding component in the description of the structure. We replace an empty index set by – in the signature of $\mathcal{M}$,.

Henceforth, we assume that a structure $\mathcal{M}$ or a signature σ is of the form $\mathcal{M} = (M; (R_i^{\mathcal{M}}|i \in I); (f_j^{\mathcal{M}}|j \in J); (c_k^{\mathcal{M}}|k \in K))$ or $\sigma = ((n_i|i \in I); (m_j|j \in J); K)$ if $\mathcal{M}$ or σ is not further specified, respectively.

Languages The *language* $\mathcal{L} = \mathcal{L}(\sigma)$ *with signature* $\sigma = ((n_i|i \in I); (m_j|j \in J); K)$ consists of *symbols*. (The signature σ is called the *signature of* (*the language*) $\mathcal{L}(\sigma)$.) There are two types of *symbols*:

- the *logical symbols* (independent of σ) and
- the *non-logical symbols* (dependent of σ). Hereby, the non-logical symbols are names for relations, functions and constants of a structure.

Logical symbols of $\mathcal{L}(\sigma)$ are the following:

- Countably many *variables*, i. e. $v_0, v_1, \ldots$ Moreover, we fix the following alphabetical order $v_0 < v_1 < \ldots$ If the variables are not further specified we also denote variables with $x, y, z, x_0, x_1, \ldots, y_0, y_1, \ldots$
- The *connectives* $\neg$ (*negation*) and $\vee$ (*disjunction*). (The remaining common connectives $\wedge$ (conjunction), $\rightarrow$ (conditional), $\leftrightarrow$ (biconditional) will be introduced as 'abbreviations'.)
- The *existential quantifier* $\exists$. (The universal quantifier $\forall$ will be introduced as an 'abbreviation'.)
- The *equality sign* $=$.
- The *brackets* (and).
- The *comma* ,.

Non-logical symbols of $\mathcal{L}(\sigma)$ are the following:

- For every $i \in I$, the n_i-ary *relation symbol* R_i.
- For every $j \in J$, the m_j-ary *function symbol* f_j.
- For every $k \in K$, the *constant symbol* c_k.

The set of all symbols of $\mathcal{L}$ is called the *alphabet of* $\mathcal{L}$. A sequence of symbols of $\mathcal{L}$ is called a *word over* $\mathcal{L}$. If the structure $\mathcal{M}$ and the language $\mathcal{L}$ have the same signature, then

- $\mathcal{L}$ is called the *language of* $\mathcal{M}$ (and we also write $\mathcal{L} = \mathcal{L}(\mathcal{M})$), and
- $\mathcal{M}$ is called an $\mathcal{L}$*-structure*.

Henceforth, we assume the signature of a language to be of the form $\sigma = ((n_i|i \in I); (m_j|j \in J); K)$ if σ or $\mathcal{L}$ are not further specified.

Terms Let $\mathcal{L} = \mathcal{L}(\sigma)$ be a language with signature σ. The set of ($\mathcal{L}$-)*terms* is inductively defined as follows.

(T1) Every variable v_n with $n \in \mathbb{N}$ and every constant c_k with $k \in \mathbb{N}$ is a term.

(T2) If $t_0, \dots, t_{m_j-1}$ are terms, then for $j \in J$, $f_j(t_0, \dots, t_{m_j-1})$ is a term as well.

Throughout this thesis we denote terms with $t, t_0, t_1 \dots$ Terms of the form (T1) are called *atomic terms*. We denote the set of variables occurring in t with $V(t)$. If t does not contain any variables, i. e. $V(t) = \emptyset$, then t is called a *constant term*. We also write $t(x_0, \dots, x_n)$ instead of t if at most the variables $x_0, \dots, x_n$ occur in t, i. e. $V(t) \subseteq \{x_0, \dots, x_n\}$.

Interpretation of Terms Let $\mathcal{M}$ be a structure with signature σ and let $\mathcal{L}$ be the language of $\mathcal{M}$. For a constant $\mathcal{L}$-term t, its *interpretation* $t^{\mathcal{M}}$ *in* $\mathcal{M}$ is inductively defined by

(i) for $k \in K$, $(c_k)^{\mathcal{M}} := c_k^{\mathcal{M}}$,

(ii) for $j \in J$, $(f_j(t_0, \dots, t_{m_j-1}))^{\mathcal{M}} := f_j^{\mathcal{M}}(t_0^{\mathcal{M}}, \dots, t_{m_j-1}^{\mathcal{M}})$.

Let $V = \{x_0, \dots, x_n\}$ be a set of variables and $\mathcal{M}$ an $\mathcal{L}$-structure. A (*variable-*)*valuation* B *of* V *in* $\mathcal{M}$ is a function $B : V \to M$.

Let $t \equiv t(x_0, \dots, x_n) \equiv t(\overrightarrow{x})$ be an $\mathcal{L}$-term and let $B : \{x_0, \dots, x_n\} \to M$ be a valuation of those variables in the $\mathcal{L}$-structure $\mathcal{M}$. The *value* $t_B^{\mathcal{M}}$ *of* t *in* $\mathcal{M}$ regarding the valuation B is inductively defined by

(i) for $i \in \{0, \dots n\}$, $(x_i)_B^{\mathcal{M}} := B(x_i)$, and for $k \in K$, $(c_k)_B^{\mathcal{M}} := c_k^{\mathcal{M}}$,

(ii) for $j \in J$, $(f_j(t_0, \dots, t_{m_j-1}))_B^{\mathcal{M}} := f_j^{\mathcal{M}}((t_0)_B^{\mathcal{M}}, \dots, (t_{m_j-1})_B^{\mathcal{M}})$.

Let B be a valuation of $V = \{x_0, \dots, x_n\}$ in $\mathcal{M}$ with $B(x_i) = a_i$ for $i \in \{0, \dots n\}$. For $t \equiv t(x_0, \dots, x_n) \equiv t(\overrightarrow{x})$ instead of $t_B^{\mathcal{M}}$, we also write

$$t_B^{\mathcal{M}} \equiv t^{\mathcal{M}}[B(x_0), \dots, B(x_n)] \equiv t^{\mathcal{M}}[a_0, \dots, a_n] \equiv t^{\mathcal{M}}[\overrightarrow{a}].$$

Therefore the term $t \equiv t(x_0, \dots, x_n) \equiv t(\overrightarrow{x})$ can be interpreted as an n-ary function in $\mathcal{M}$, i. e.

$$f_{t(\overrightarrow{x})}^{\mathcal{M}} : M^n \to M \text{ with } f_{t(\overrightarrow{x})}^{\mathcal{M}}(\overrightarrow{a}) = t^{\mathcal{M}}[\overrightarrow{a}].$$

Formulas Let $\mathcal{L} = \mathcal{L}(\sigma)$ be a language with signature σ. The set of ($\mathcal{L}$-)*formulas* is inductively defined as follows.

(F1) (a) If t_0, t_1 are terms, then $t_0 = t_1$ is a formula.
(b) If $t_0, \dots, t_{n_i-1}$ are terms, then for $i \in I$, $R_i(t_0, \dots, t_{n_i-1})$ is a formula.

(F2) If φ is a formula, then $\neg\varphi$ is a formula as well.

(F3) If φ_0, φ_1 are formulas, then $(\varphi_0 \vee \varphi_1)$ is a formula as well.

(F4) If φ is a formula and x a variable, then $\exists x\varphi$ is a formula as well.

Throughout this thesis we denote formulas with $\varphi, \psi, \gamma, \delta, \varphi_0, \varphi_1, \ldots, \psi_0, \psi_1, \ldots$ and sets of formulas with $\Phi, \Phi_0, \Phi_1, \ldots$ Formulas of the form (F1) are called *atomic formulas.*

To improve readability, we introduce the following conventions:

- The connectives $\wedge$ (*conjunction*), $\rightarrow$ (*conditional*) and $\leftrightarrow$ (*biconditional*) are abbreviations for $(\varphi_0 \wedge \varphi_1) :\equiv \neg(\neg\varphi_0 \vee \neg\varphi_1)$, $(\varphi_0 \rightarrow \varphi_1) :\equiv (\neg\varphi_0 \vee \varphi_1)$ and $(\varphi_0 \leftrightarrow \varphi_1) :\equiv (\neg(\neg\varphi_0 \vee \varphi_1) \vee \neg(\neg\varphi_1 \vee \varphi_0))$.
- The *universal quantifier* $\forall$ is an abbreviation for $\forall x\varphi :\equiv \neg\exists x\neg\varphi$.
- Moreover, for $\neg\varphi, \exists x\varphi, \forall x\varphi$, we also permit the notation $\neg(\varphi)$, $\exists x(\varphi), \forall x(\varphi)$, respectively.
- For $(\varphi_0 * \varphi_1)$ with $* \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$, we also permit the notation $\varphi_0 * \varphi_1$. Furthermore, we also write $\{\varphi_0 * \varphi_1\}$ instead of $(\varphi_0 * \varphi_1)$ for $* \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$.
- Instead of $\neg t_0 = t_1$, we also write $t_0 \neq t_1$.
- Furthermore, for some function symbols, e. g. $+$, $\cdot$ and relation symbols, e. g. $\leq$, we utilize infix notation as well. Hereby, we sometimes omit the brackets, e. g. we write $x + y$ instead of $(x + y)$.

Free and Bounded Occurrences of Variables Let $\mathcal{L} = \mathcal{L}(\sigma)$ be a language with signature σ. *Free* and *bounded occurrences of variables in $\mathcal{L}$-formulas* are inductively defined as follows.

(i) The variable x occurs in the atomic formula $t_0 = t_1$ or $R_i(t_0, \ldots, t_{n_i-1})$ if x occurs in t_0, t_1 or $t_0, \ldots, t_{n_i-1}$, respectively. All occurrences of x are free.

(ii) The variable x occurs in $\neg\varphi$ if x occurs in φ. Then the occurrence of x is free (bounded) in $\neg\varphi$ if the respective occurrence of x is free (bounded) in φ.

(iii) The variable x occurs in the formula $(\varphi_0 \vee \varphi_1)$ if x occurs in φ_0 or φ_1. Then the occurrence of x in $(\varphi_0 \vee \varphi_1)$ is free (bounded) if the respective occurrence of x is free (bounded) in φ_0 or φ_1.

(iv) The variable x occurs in $\exists y\varphi$ if $x \equiv y$ or if x occurs in the formula φ. If $x \equiv y$, then all occurrences of x in $\exists y\varphi$ are bounded. Else an occurrence of x in $\exists y\varphi$ is free (bounded) if the respective occurrence is free (bounded) in φ.

We also write $\varphi(x_0, \ldots, x_n)$ instead of φ if at most the occurrences of the variables $x_0, \ldots, x_n$ are free.

We call a variable x a *free* (*bounded*) *variable* in φ if every occurence of x is free (bounded) in φ.

An $\mathcal{L}$-formula φ in which every occurrence of a variable is bounded, is called an ($\mathcal{L}$-)*sentence*.

Throughout this thesis we denote sentences with $\sigma, \tau, \sigma_0, \sigma_1, \ldots$ and sets of sentences with $\Sigma, \Sigma_0, \Sigma_1, \ldots$

Interpretation of Formulas Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\varphi \equiv \varphi(x_0, \ldots, x_n)$ an $\mathcal{L}$-formula and B a valuation of $\{x_0, \ldots, x_n\}$ in $\mathcal{M}$. Then the *truth value*

$$V_B^{\mathcal{M}}(\varphi) \in \{0, 1\} (= \{\text{False, True}\})$$

of φ *in* $\mathcal{M}$ *regarding the valuation* B is inductively defined by

(i) $V_B^{\mathcal{M}}(t_0 = t_1) = 1$ if and only if $(t_0)_B^{\mathcal{M}} = (t_1)_B^{\mathcal{M}}$.

(ii) For $i \in I$, $V_B^{\mathcal{M}}(R_i(t_0, \ldots, t_{n_i-1})) = 1$ if and only if $((t_0)_B^{\mathcal{M}}, \ldots, (t_{n_i-1})_B^{\mathcal{M}}) \in R_i^{\mathcal{M}}$.

(iii) $V_B^{\mathcal{M}}(\neg\psi) = 1$ if and only if $V_B^{\mathcal{M}}(\psi) = 0$.

(iv) $V_B^{\mathcal{M}}(\varphi_0 \vee \varphi_1) = 1$ if and only if $V_B^{\mathcal{M}}(\varphi_0) = 1$ or $V_B^{\mathcal{M}}(\varphi_1) = 1$ (or both).

(v) $V_B^{\mathcal{M}}(\exists y \psi) = 1$ if and only if there exists a valuation B' of $\{x_0, \ldots, x_n, y\}$ such that B' coincides with B on $\{x_0, \ldots, x_n\} \setminus \{y\}$ and $V_{B'}^{\mathcal{M}}(\psi) = 1$.

Then the truth value of an improper formula $\varphi \equiv \varphi(x_0, \ldots, x_n)$ regarding the valuation B of $\{x_0, \ldots, x_n\}$ in $\mathcal{M}$ is the following:

(i) $V_B^{\mathcal{M}}(\varphi_0 \wedge \varphi_1) = 1$ if and only if $V_B^{\mathcal{M}}(\varphi_0) = 1$ and $V_B^{\mathcal{M}}(\varphi_1) = 1$.

(ii) $V_B^{\mathcal{M}}(\varphi_0 \to \varphi_1) = 1$ if and only if $V_B^{\mathcal{M}}(\varphi_0) = 0$ or $V_B^{\mathcal{M}}(\varphi_1) = 1$ (or both).

(iii) $V_B^{\mathcal{M}}(\varphi_0 \leftrightarrow \varphi_1) = 1$ if and only if $V_B^{\mathcal{M}}(\varphi_0) = V_B^{\mathcal{M}}(\varphi_1)$.

(iv) $V_B^{\mathcal{M}}(\forall y \psi) = 1$ if and only if for all valuations B' of $\{x_0, \ldots, x_n, y\}$ such that B' coincides with B on $\{x_0, \ldots, x_n\} \setminus \{y\}$ and $V_{B'}^{\mathcal{M}}(\psi) = 1$.

Let B be a valuation of $V = \{x_0, \ldots, x_n\}$ in $\mathcal{M}$ with $B(x_i) = a_i$ for $i \in \{0, \ldots, n\}$. For $\varphi \equiv \varphi(x_0, \ldots, x_n) \equiv \varphi(\overrightarrow{x})$ instead of $V_B^{\mathcal{M}}(\varphi) = 1$, we also write

$$\mathcal{M} \vDash \varphi[B(x_0), \ldots, B(x_n)] \text{ or } \mathcal{M} \vDash \varphi[\overrightarrow{a}]$$

and say $\mathcal{M}$ *makes the formula* φ *regarding the valuation* $\overrightarrow{a}$ *true*. Accordingly, we write $\mathcal{M} \nvDash \varphi[B(x_0), \ldots, B(x_n)]$ if $V_B^{\mathcal{M}}(\varphi) = 0$.

An $\mathcal{L}$-sentence σ *is true* (*false*) *in* $\mathcal{M}$ if $V_B^{\mathcal{M}}(\sigma) = 1$ ($V_B^{\mathcal{M}}(\sigma) = 0$) regarding the valuation B of the empty set. We write $\mathcal{M} \vDash \sigma$ ($\mathcal{M} \nvDash \sigma$) and say that $\mathcal{M}$ *is a model of* σ ($\neg\sigma$) if σ is true (false) in $\mathcal{M}$.

Valid and Satisfiable Formulas An $\mathcal{L}$-sentence σ is *valid* if

$$\text{for every } \mathcal{L}\text{-structure } \mathcal{M},\ \mathcal{M} \vDash \sigma.$$

An $\mathcal{L}$-sentence φ is *satisfiable* if

$$\text{there exists an } \mathcal{L}\text{-structure } \mathcal{M}, \text{ such that } \mathcal{M} \vDash \sigma.$$

Else σ is *unsatisfiable*.

A set Σ of $\mathcal{L}$-sentences is *satisfiable* if there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M}$ is a model for every sentence in Σ. Else Σ is *unsatisfiable*.

If an $\mathcal{L}$-structure $\mathcal{M}$ is a model for every sentence in the set of sentences Σ, then we say that $\mathcal{M}$ *is a model of* Σ and write $\mathcal{M} \vDash \Sigma$, otherwise we say $\mathcal{M}$ *is no model of* Σ and write $\mathcal{M} \nvDash \Sigma$.

Let σ be an $\mathcal{L}$-sentence. An $\mathcal{L}$-sentence τ *follows from* σ if every model of σ is a model of τ, i. e.

$$\text{for every } \mathcal{L}\text{-structure } \mathcal{M},\ \mathcal{M} \vDash \sigma \Rightarrow \mathcal{M} \vDash \tau.$$

We denote this by $\sigma \vDash \tau$.

σ and τ are *equivalent* if σ follows from τ and τ follows from σ.

Let Σ be a set of $\mathcal{L}$-sentences. An $\mathcal{L}$-sentence τ *follows from* Σ if every model of Σ is a model of τ, i. e.

$$\text{for every } \mathcal{L}\text{-structure } \mathcal{M},\ \mathcal{M} \vDash \Sigma \Rightarrow \mathcal{M} \vDash \tau.$$

We denote this by $\Sigma \vDash \tau$.

Let Σ_0, Σ_1 be sets of $\mathcal{L}$-sentences. We write $\Sigma_0 \vDash \Sigma_1$ and say that Σ_1 *follows from* Σ_0 if $\Sigma_0 \vDash \sigma$ for every $\sigma \in \Sigma_1$.

Substitution If we replace every free occurrence of the variable x by the term t, then we denote the resulting formula of this *substitution* with $\varphi[t/x]$. Likewise, we extend this notion for multiple variables. If we replace every free occurrence of variables $x_0, \dots, x_n$ by the terms $t_0, \dots, t_n$, respectively, then we denote the resulting formula of this substitution with $\varphi[t_0, \dots, t_n/x_0, \dots, x_n] \equiv \varphi[\overrightarrow{t}/\overrightarrow{x}]$.

Let t be a term and φ a formula. Then t is *substitutable for* (*the variable*) x *in* φ if for any variable $y \neq x$ in t, every occurrence of y in φ is not bounded.

Deductive System A *deductive system* $\mathcal{K}$ *over* (*a language*) $\mathcal{L}$ consists of the following components:

(i) The set of ($\mathcal{K}$-)*axioms* where every axiom is an $\mathcal{L}$-formula.

(ii) The set of ($\mathcal{K}$-)*rules*. Every rule R is of the form

$$\frac{\varphi_0, \dots, \varphi_n}{\varphi}$$

where $\varphi_0, \dots, \varphi_n, \varphi$ are $\mathcal{L}$-formulas. $\varphi_0, \dots, \varphi_n$ are *premises* and φ the *conclusion of R*.

Let $\mathcal{K}$ be a deductive system and φ an $\mathcal{L}$-formula. A ($\mathcal{K}$-)*proof of* φ is a sequence $\psi_0, \ldots, \psi_n$ of $\mathcal{L}$-formulas ($n \geq 0$) where the following holds:

- $\varphi \equiv \psi_n$.
- Every formula ψ_i with $i \in \{0, \ldots, n\}$ is
  - a $\mathcal{K}$-axiom or
  - the conclusion of a $\mathcal{K}$-rule R where the premise(s) is (are) an element of $\{\psi_0, \ldots, \psi_{i-1}\}$.

The *length of the proof* $\psi_0, \ldots, \psi_n$ is $n + 1$.

An $\mathcal{L}$-formula φ is $\mathcal{K}$*-provable* if there exists a $\mathcal{K}$-proof of φ. (Otherwise, φ is $\mathcal{K}$*-unprovable*). We denote this with $\vdash_{\mathcal{K}} \varphi$ ($\nvdash_{\mathcal{K}} \varphi$).

Let $\mathcal{K}$ be a deductive system, Φ a set of $\mathcal{L}$-formulas and φ an $\mathcal{L}$-formula. A ($\mathcal{K}$-)*proof of* φ *from* Φ is a finite sequence $\psi_0, \ldots, \psi_n$ of $\mathcal{L}$-formulas ($n \geq 0$) where the following holds:

- $\varphi \equiv \psi_n$.
- Every formula ψ_i with $i \in \{0, \ldots, n\}$ is
  - a $\mathcal{K}$-axiom or
  - a formula of the set Φ or
  - the conclusion of a $\mathcal{K}$-rule R where the premise(s) is (are) an element of $\{\psi_0, \ldots, \psi_{i-1}\}$.

The *length of the proof* $\psi_0, \ldots, \psi_n$ is $n + 1$.

An $\mathcal{L}$-formula φ is $\mathcal{K}$*-provable from* Φ if there exists a $\mathcal{K}$-proof of φ from Φ. (Otherwise, φ is $\mathcal{K}$*-unprovable from* Φ.) We denote this with $\Phi \vdash_{\mathcal{K}} \varphi$ ($\Phi \nvdash_{\mathcal{K}} \varphi$).

A deductive system $\mathcal{K}$ over a language $\mathcal{L}$ is *sound* if

for all sets of $\mathcal{L}$-sentences Σ and for all $\mathcal{L}$-sentences σ, $\Sigma \vdash \sigma \Rightarrow \Sigma \vDash \sigma$.

A deductive system $\mathcal{K}$ over a language $\mathcal{L}$ is *complete* if

for all sets of $\mathcal{L}$-sentences Σ and for all $\mathcal{L}$-sentences σ, $\Sigma \vDash \sigma \Rightarrow \Sigma \vdash \sigma$.

Shoenfield-System Let $\mathcal{L} = \mathcal{L}(\sigma)$ be a language with signature σ. The *Shoenfield-system* $\mathcal{S}$ is a deductive system which is defined as follows:

(i) The set of ($\mathcal{S}$-)axioms consists of the following axioms:

(A1) $\neg\varphi \vee \varphi$.

(A2) $\varphi[t/x] \rightarrow \exists x\varphi$ if t is substitutable for x in φ.

(A3) $x = x$.

(A4) $x_0 = y_0 \wedge \ldots \wedge x_{m_j-1} = y_{m_j-1} \to f_j(x_0, \ldots, x_{m_j-1}) = f_j(y_0, \ldots, y_{m_j-1})$ for $j \in J$.

(A5) $x_0 = y_0 \wedge \ldots \wedge x_{n_j-1} = y_{n_i-1} \wedge R_i(x_1, \ldots, x_{n_i-1}) = R_i(y_1, \ldots, y_{n_i-1})$ for $i \in I$.

(A6) $x_0 = y_0 \wedge x_1 = y_1 \wedge x_0 = x_1 \to y_0 = y_1$.

(ii) The set of ($\mathcal{S}$-)*rules* consists of the following rules:

(R1) $\dfrac{\psi}{\varphi \vee \psi}$.

(R2) $\dfrac{\varphi \vee (\psi \vee \delta)}{(\varphi \vee \psi) \vee \delta}$.

(R3) $\dfrac{\varphi \vee \varphi}{\varphi}$.

(R4) $\dfrac{\varphi \vee \psi, \neg\varphi \vee \delta}{\psi \vee \delta}$.

(R5) $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$ if the occurrences of x in ψ are not free.

Throughout this thesis $\vdash$ ($\nvdash$) denotes the provability (unprovability) in $\mathcal{S}$, i. e. for a set of $\mathcal{L}$-formulas Φ,

$$\Phi \vdash \varphi \Leftrightarrow \varphi \text{ is } (\mathcal{S}\text{-})\text{provable from } \Phi \text{ or}$$
$$\Phi \nvdash \varphi \Leftrightarrow \varphi \text{ is } (\mathcal{S}\text{-})\text{unprovable from } \Phi.$$

Moreover, we also write

$$\begin{aligned} \varphi_0, \ldots, \varphi_n \vdash \varphi &\text{ instead of } \{\varphi_0, \ldots, \varphi_n\} \vdash \varphi, \\ \vdash \varphi &\text{ instead of } \emptyset \vdash \varphi \text{ or} \\ \varphi_0, \ldots, \varphi_n \nvdash \varphi &\text{ instead of } \{\varphi_0, \ldots, \varphi_n\} \nvdash \varphi, \\ \nvdash \varphi &\text{ instead of } \emptyset \nvdash \varphi. \end{aligned}$$

We write $\Sigma_0 \vdash \Sigma_1$ if $\Sigma_0 \vdash \sigma$ for every $\sigma \in \Sigma_1$, and $\Sigma_0 \nvdash \Sigma_1$ otherwise.

A set of $\mathcal{L}$-sentences Σ is *consistent* if there exists an $\mathcal{L}$-sentence σ with $\Sigma \nvdash \sigma$. Else Σ is *inconsistent*.

A set of $\mathcal{L}$-sentences Σ is (*negation-*)*complete* if for all $\mathcal{L}$-sentences σ, $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$ holds. Else Σ is *incomplete*.

Completeness and Soundness Theorem The *Completeness and Soundness Theorem* states that the Shoenfield-System $\mathcal{S}$ is complete and sound, i. e.

$$\text{for all sets of } \mathcal{L}\text{-sentences } \Sigma \text{ and for all sentences } \sigma,\ \Sigma \models \sigma \Leftrightarrow \Sigma \vdash \sigma.$$

(A version of the Completeness Theorem was first proven by Gödel in 1929 and then simplified by Leon Henkin in 1947.)

Theories An ($\mathcal{L}$-)*theory* T is a tuple $T = (\mathcal{L}, \Sigma)$ where

- $\mathcal{L} = \mathcal{L}(\sigma)$ is a language with signature σ and
- Σ is a set of $\mathcal{L}$-sentences (set of *axioms of* T).

$\mathcal{L}$ is called the *language of* T and Σ the set of *axioms of* T. Furthermore, we also denote the language of the theory T with $\mathcal{L}(T)$.

The model-class $Mod(T)$ of an $\mathcal{L}$-theory $T = (\mathcal{L}, \Sigma)$ is the set of $\mathcal{L}$-structures which are models of Σ, i. e.

$$Mod(T) := \{\mathcal{M} : \mathcal{M} \text{ is an } \mathcal{L}\text{-structure and } \mathcal{M} \vDash \Sigma\}.$$

Let $T = (\mathcal{L}, \Sigma)$ be a theory. If $\mathcal{M}$ is a model of Σ ($\mathcal{M}$ is no model of Σ), then we also call $\mathcal{M}$ a *model of* T ($\mathcal{M}$ *no model of* T) and write $\mathcal{M} \vDash T$ ($\mathcal{M} \nvDash T$) instead of $\mathcal{M} \vDash \Sigma$ ($\mathcal{M} \nvDash \Sigma$). For an $\mathcal{L}$-formula φ, if $\Sigma \vdash \varphi$ ($\Sigma \nvdash \varphi$), then we also write $T \vdash \varphi$ ($T \nvdash \varphi$). An $\mathcal{L}$-sentence σ is called a *theorem of* T if $T \vdash \sigma$.

Moreover, a theory $T = (\mathcal{L}, \Sigma)$ is *satisfiable* if Σ is satisfiable, i. e. $Mod(T) \neq \emptyset$. A theory T is *consistent*, or *inconsistent*, or *complete*, or *incomplete* if Σ is consistent, or inconsistent, or complete, or incomplete, respectively.

For a theory $T = (\mathcal{L}, \Sigma)$, we say that a finite sequence of $\mathcal{L}$-formulas $\psi_0, \ldots, \psi_n$ is an T*-proof* (*of an* $\mathcal{L}$*-formula*) φ if $\psi_0, \ldots, \psi_n$ is an $\mathcal{S}$-proof from Σ of φ. We say that an $\mathcal{L}$-formula φ is T*-provable* if φ is $\mathcal{S}$-provable from Σ.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The *theory of* $\mathcal{M}$ is

$$Th(\mathcal{M}) := \{\sigma : \mathcal{M} \vDash \sigma\}.$$

In particular, the theory $Th(\mathcal{M})$ is satisfiable and complete.

The (*syntactic*) *deductive closure of* $T = (\mathcal{L}, \Sigma)$ is

$$C_{\vdash}(T) := \{\sigma : \sigma \text{ is an } \mathcal{L}\text{-sentence and } T \vdash \sigma\}.$$

The (*semantic*) *closure of* T is

$$C_{\vDash}(T) := \{\sigma : \sigma \text{ is an } \mathcal{L}\text{-sentence and } T \vDash \sigma\}.$$

The Completeness and Soundness Theorem infers

$$C_{\vdash}(T) = C_{\vDash}(T),$$

since

$$T \vdash \sigma \Leftrightarrow T \vDash \sigma.$$

Two $\mathcal{L}$-theories $T = (\mathcal{L}, \Sigma)$ and $T' = (\mathcal{L}, \Sigma')$ are *equivalent* if their deductive closures are the same, i. e.

$$C_{\vdash}(T) = C_{\vdash}(T').$$

Extensions A language $\mathcal{L}'$ is an *extension of* (*a language*) $\mathcal{L}$ if every non-logical symbol of $\mathcal{L}$ is a symbol of $\mathcal{L}'$. We write $\mathcal{L} \subseteq \mathcal{L}'$ and say that $\mathcal{L}'$ *extends* $\mathcal{L}$ if $\mathcal{L}'$ is an extension of $\mathcal{L}$.

An $\mathcal{L}'$-theory $T' = (\mathcal{L}', \Sigma')$ is an *extension of* (*an $\mathcal{L}$-theory*) $T = (\mathcal{L}, \Sigma)$ if

(i) $\mathcal{L}'$ is an extension of $\mathcal{L}$ and

(ii) for every $\mathcal{L}$-formula φ, $T \vdash \varphi \Rightarrow T' \vdash \varphi$.

We write $T \sqsubseteq T'$ and say that T' *extends* T if T' is an extension of T.

Arithmetic Adding the addition function $+$, multiplication function $\cdot$, the successor function S ($S(n) = n+1$ for $n \in \mathbb{N}$) and the number 0 to the natural numbers $\mathbb{N}$, the structure $\mathcal{N} := (\mathbb{N}; +, \cdot, S; 0)$ is obtained. $\mathcal{N}$ has signature $\sigma(\mathcal{N}) = (-; 2, 2, 1; \{0\})$. (Note that $\mathcal{N}$ does not have any relations.) We call the structure $\mathcal{N}$ the *arithmetic* and the language $\mathcal{L}_A := \mathcal{L}(\sigma)$ the *language of arithmetic*.

We call a theory $T = (\mathcal{L}_A, \Sigma)$ an *arithmetical theory* if T is consistent and the language $\mathcal{L}(T)$ of T is the language of arithmetic $\mathcal{L}_A$.

A theory T is *arithmetically sound* if for every $\mathcal{L}_A$-sentence σ, $T \vdash \sigma$ implies $\mathcal{N} \vDash \sigma$.

If constant terms $\overline{n}$ ($n \in \mathbb{N}$) are defined by

$$\overline{0} :\equiv 0 \text{ and } \overline{n+1} :\equiv S(\overline{n}),$$

then $\overline{n}^{\mathcal{N}} = n$. (We call $\overline{n}$ a *numeral*.) Thus every natural number can be expressed by a constant term of the language $\mathcal{L}_A$.

We call $Th(\mathcal{N})$ the *true arithmetic*. (Note that for every structure $\mathcal{M}$, $Th(\mathcal{M})$ is complete. Hence in particular $Th(\mathcal{N})$ is complete.)

1.1.2 Basic Concepts of Computability Theory

In the following we introduce the basic concepts of computability theory.

Characteristic Functions The *characteristic function* of the n-ary relation $R \subseteq \mathbb{N}^n$ c_R is the n-ary function $c_R : \mathbb{N}^n \to \{0, 1\}$ such that $(m_0, \dots, m_{n-1}) \in R$ if and only if $c_R(m_0, \dots, m_{n-1}) = 0$. (Note that 0 symbolizes true and 1 false, adhering to Gödels notation.)

Primitive Recursive Functions The *basic primitive recursive functions* are the following:

- The m-ary zero function $C^m : \mathbb{N}^m \to \mathbb{N}$ ($m \geq 0$) is defined by

$$C^m(\overrightarrow{x}) = 0.$$

- The 1-ary successor function $S^1 : \mathbb{N}^1 \to \mathbb{N}$ is defined by

$$S^1(x) = x + 1.$$

- The n-ary projection function $P_i^n : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 1, 0 \leq i \leq n-1)$ is defined by

$$P_i^n(x_0, \ldots, x_{n-1}) = x_i.$$

  (Note that P_i^n returns the $(i+1)$-th variable.)

Let $g : \mathbb{N}^m \to \mathbb{N}$ $(m \geq 1)$ and $h_0, \ldots, h_{m-1} : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 1)$ be m-ary and n-ary functions, respectively. The *composition of* g *and* $h_0, \ldots, h_{m-1}$ yields the n-ary function

$$f = g(h_0, \ldots, h_{m-1}) : \mathbb{N}^n \to \mathbb{N}$$

which is defined by

$$f(\overrightarrow{x}) = g(h_0(\overrightarrow{x}), \ldots, h_{m-1}(\overrightarrow{x})).$$

Let $g : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 1)$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ be n-ary and $(n+2)$-ary functions, respectively. The *primitive recursion of* g *and* h yields the $(n+1)$-ary function

$$f = PR(g, h) : \mathbb{N}^{n+1} \to \mathbb{N}$$

which is defined by

$$f(\overrightarrow{x}, 0) = g(\overrightarrow{x}),$$

$$f(\overrightarrow{x}, y+1) = h(\overrightarrow{x}, y, f(\overrightarrow{x}, y)),$$

where $\overrightarrow{x} \in \mathbb{N}^n$ and $y \in \mathbb{N}$.

The class PRIM of *primitive recursive functions* (*p. r. functions*) is inductively defined as follows.

(i) For $0 \leq i \leq n-1, m \geq 0$, $S^1, P_i^n, C^m \in \mathsf{PRIM}$.

(ii) If the m-ary function g and the n-ary functions $h_0, \ldots, h_{m-1}$ are in PRIM, then $g(h_0, \ldots, h_{m-1}) \in \mathsf{PRIM}$.

(iii) If the n-ary function g and the $(n+2)$-ary function h are in PRIM, then $PR(g, h) \in \mathsf{PRIM}$.

Primitive Recursive Relations A relation $R \subseteq \mathbb{N}^n$ $(n \geq 0)$ is a *primitive recursive relation* (*p. r. relation*) if and only if its characteristic function c_R of R is primitive recursive. (We write $R \in \mathsf{PRIM}$.)

Note that for a relation $R \subseteq \mathbb{N}^n$ $(n \geq 0)$, we often say that $R(m_0, \ldots, m_{n-1})$ *holds* if $(m_0, \ldots, m_{n-1}) \in R$, and $R(m_0, \ldots, m_{n-1})$ *does not hold* if $(m_0, \ldots, m_{n-1}) \notin R$.

Properties and Examples of Primitive Recursion Let $f_0, \dots, f_k : \mathbb{N}^n \to \mathbb{N}$ ($n \geq 1, k \geq 0$) be functions and $C_0, \dots, C_k \subseteq \mathbb{N}^n$ be mutually exclusive relations. Then the function $f : \mathbb{N}^n \to \mathbb{N}$ is *defined by cases* $C_0, \dots, C_k$ *from functions* $f_0, \dots, f_k$ if

$$f(x_0, \dots, x_{n-1}) = \begin{cases} f_0(x_0, \dots, x_{n-1}) & \text{if } C_0(x_0, \dots, x_{n-1}), \\ f_1(x_0, \dots, x_{n-1}) & \text{if } C_1(x_0, \dots, x_{n-1}), \\ \vdots & \\ f_k(x_0, \dots, x_{n-1}) & \text{if } C_k(x_0, \dots, x_{n-1}), \\ a & \text{otherwise,} \end{cases}$$

where $a \in \mathbb{N}$.

Let $R \subseteq \mathbb{N}^{n+1}$ ($n \geq 0$) be an $(n+1)$-ary relation. Then for $\overrightarrow{x} \in \mathbb{N}^n, y \in \mathbb{N}$ the *bounded existential quantifier* is defined by

$$(\exists z < y)(R(\overrightarrow{x}, y)) :\Leftrightarrow \exists z(z < y \wedge R(\overrightarrow{x}, y)),$$

and the *bounded universal quantifier* defined by

$$(\forall z < y)(R(\overrightarrow{x}, y)) :\Leftrightarrow \exists z(z < y \wedge R(\overrightarrow{x}, y)).$$

Likewise we define $(\exists z \leq y)$ and $(\forall z \leq y)$. Those are also called bounded existential or bounded universal quantifiers, respectively.

Let $R \subseteq \mathbb{N}^{n+1}$ ($n \geq 0$) be an $(n+1)$-ary relation. The *bounded minimization operator* $(\mu z < y)$ in the n-ary function $f(x_0, \dots, x_{n-1}) = (\mu z < y)R(x_0, \dots, x_{n-1}, z)$ returns the least number $z < y$ such that $R(x_0, \dots, x_{n-1}, z)$ holds if such an $\overrightarrow{x}$ exists, or 0 otherwise:

$$\begin{aligned} f(x_0, \dots, x_{n-1}) &= (\mu z < y)R(x_0, \dots, x_{n-1}, z) \\ &= \begin{cases} \min_{0 \leq z < y} R(\overrightarrow{x}, z)) & \text{if } (\exists z < y)R(\overrightarrow{x}, z) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Analogously, we also permit $(\mu z \leq y)$ which is defined accordingly and also called the bounded minimization operator.

Primitive recursive functions and relations are closed under the following operations:

- explicit definitions,
- definition by p. r. cases and p. r. functions,
- logical connectives, i. e. $\neg, \vee, \wedge, \to, \leftrightarrow$,
- the bounded existential quantifier and the bounded universal quantifier,
- the bounded minimization operator $(\mu z < y)$.

Moreover, it can be easily shown that the following functions and relations are primitive recursive:

- The relations $=, <, \leq, >, \geq, \neq$ over $\mathbb{N}$.
- The addition function $+$ and multiplication function $\cdot$ over $\mathbb{N}$.
- The factorial function $!(x)$ over $\mathbb{N}$. (Note that we usually write $x!$ instead of $!(x)$.)
- The 2-ary relation *divides relation* $\mid(x, y)$ holds if y is divisible by x. (Note that we also use infix notation.)
- The 2-ary *exponentiation function* $exp(n, m)$ returns the product of n multiplied m-times, i.e. $exp(n, m) := \underbrace{n \cdot \ldots \cdot n}_{m\text{-times}}$. (We also write n^m instead of $exp(n, m)$.)

Partial Recursive and Recursive Functions Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be a (possibly partial) function. The *minimalization operator* μ in the n-ary function $f = \mu(g)$ is defined by

$$\begin{aligned} f(\overrightarrow{x}) &= \mu y(g(\overrightarrow{x}, y) = 0 \text{ and } \forall z < y(g(\overrightarrow{x}, z)\downarrow)) \\ &= \min\{y : g(\overrightarrow{x}, y) = 0 \text{ and } (\forall z < y)(g(\overrightarrow{x}, z)\downarrow)\} \end{aligned}$$

where $\min \emptyset \equiv \uparrow$.

The class PREK of *partial recursive functions* (*p. r. functions*) is inductively defined by

(i) For $n \geq 1$, $0 \leq i \leq n-1, m \geq 0$, $S^1, P^n_i, C^m \in \mathsf{PREK}$.

(ii) If the m-ary function g and the n-ary functions $h_0, \ldots, h_{m-1}$ are in PREK, then $g(h_0, \ldots, h_{m-1}) \in \mathsf{PREK}$.

(iii) If the n-ary function g and the $(n+2)$-ary function h are in PREK, then $PR(g, h) \in \mathsf{PREK}$.

(iv) If the $(n+1)$-ary function g is in PREK, then $f = \mu(g) \in \mathsf{PREK}$.

Throughout this thesis we denote partial recursive function with Greek letters, e.g. $\varphi, \psi, \ldots$ It should be clear from the context whether we are dealing with formulas or partial recursive functions.

A total function $f : \mathbb{N}^n \to \mathbb{N}$ ($n \geq 0$) is *recursive* if f is partial recursive. (The class of recursive functions is denoted with REK.)

A relation $R \subseteq \mathbb{N}^n$ ($n \geq 0$) is *recursive* if the characteristic function c_R of R is recursive. We write $R \in \mathsf{REK}$.

A relation $R \subseteq \mathbb{N}^n$ ($n \geq 0$) is *recursively enumerable* (*r. e.*) if there is an n-ary partial recursive function φ whose domain is R.

Enumerability, Decidability and Computability A set M is *enumerable* if there exists an algorithm $\mathfrak{A}$ which takes no input and outputs every element of M in an arbitrary order (possibly with repetitions).

A relation M is *decidable* if there exists an algorithm $\mathfrak{A}$ which takes x as input and outputs 0 if $x \in M$ and 1 if $x \notin M$. Else M is *undecidable*.

A function f is *computable* if there exists an algorithm which takes x as input and outputs $f(x)$.

Having introduced the basic concepts of enumerability, decidability and computability, we shortly state the relations between those concepts. Hereby, we limit to the case of the natural numbers.

- If $R \subseteq \mathbb{N}^n$ $(n \geq 1)$ is decidable, then R is enumerable. (The other direction does not hold.)
- If $R \subseteq \mathbb{N}^n$ and $R' \subseteq \mathbb{N}^n$ $(n \geq 1)$ are decidable, then $\overline{R} := \mathbb{N}^n \setminus R$, $R \cap R'$, $R \cup R'$ are decidable.
- If $R \subseteq \mathbb{N}^n$ and $R' \subseteq \mathbb{N}^n$ $(n \geq 1)$ are enumerable, then $R \cap R'$, $R \cup R'$ are enumerable.
- $R \subseteq \mathbb{N}^n$ $(n \geq 1)$ is decidable if and only if R and $\overline{R}$ are enumerable. (This is called the *complement lemma*.)
- $R \subseteq \mathbb{N}^n$ $(n \geq 1)$ is decidable if and only if c_R is computable.
- $R \subseteq \mathbb{N}$ is enumerable if and only if $R = \emptyset$ or R is the codomain of a computable function $f : \mathbb{N} \to \mathbb{N}$.
- $f : \mathbb{N}^n \to \mathbb{N}$ is computable if and only if $G_f \subseteq \mathbb{N}^{n+1}$ is decidable. Equivalently, $f : \mathbb{N}^n \to \mathbb{N}$ is computable if and only if G_f is enumerable where G_f is the *graph* of f defined by $G_f := \{(x, f(x)) : x \in \mathbb{N}\}$.

The *projection lemma* states the following: A set $A \subseteq \mathbb{N}$ is enumerable if and only if A is the projection of a decidable set $B \subseteq \mathbb{N}^2$, i. e.

$$x \in A \Leftrightarrow \exists y\{(x, y) \in B\}.$$

Church-Turing Thesis The Church-Turing Thesis claims that recursive functions are in fact computable functions. Moreover, the Church-Turing Thesis implies that recursive relations are decidable relations, and r. e. relations are enumerable relations.

Axiomatized Theories A theory $T = (\mathcal{L}, \Sigma)$ is *finitely axiomatized* if Σ is finite. A theory $T = (\mathcal{L}, \Sigma)$ is *finitely axiomatizable* if there exists a finitely axiomatized theory T' such that T is equivalent to T'.

A theory $T = (\mathcal{L}, \Sigma)$ is *effectively axiomatized* if Σ is a decidable set. A theory $T = (\mathcal{L}, \Sigma)$ is *effectively axiomatizable* if there exists an effectively axiomatized theory T' such that T is equivalent to T'.

A theory $T = (\mathcal{L}, \Sigma)$ is *enumerably axiomatized* if Σ is an enumerable set. A theory $T = (\mathcal{L}, \Sigma)$ is *enumerably axiomatizable* if there exists an enumerably axiomatized theory T' such that T is equivalent to T'.

We will later show that every enumerably axiomatized theory is also effectively axiomatizable.

Proves and Provability In the following, we list some properties about proves and provability:

(i) The set of proofs is decidable, i. e. one can effectively decide whether a sequence of formulas $\varphi_0, \dots, \varphi_n$ is a proof (in the Shoenfield-System). If T is an effectively axiomatized theory, then the set of T-proofs is also decidable.

(ii) The set of theorems is enumerable. This follows from (i) by the projection lemma, since a formula φ is provable if and only if there exists a proof $\varphi_0, \dots, \varphi_n$ of φ. If T is an effectively axiomatized theory, then the set of T-provable sentences is also decidable, i. e. $C_{\vdash}(T) = \{\sigma : T \vdash \sigma\}$.

1.2 Historical Background

In the early 20th century, several schools of the philosophy of mathematics ran into difficulties as they were pursuing to find a consistent foundation of mathematics by discovering various paradoxes, e. g. Russel's paradox. As a solution, the German mathematician David Hilbert together with his doctoral student Wilhelm Ackermann proposed to create a deductive system which provides solid foundations for all mathematics, known under the name Hilbert's program. Hilbert intended to ground all existing statements to a theory which is consistent and complete. Moreover, the 'Entscheidungproblem (decision problem)' that was introduced by Hilbert and Ackermann in their book 'Grundzüge der Theoretischen Logik (Principles of Mathematical Logic)' [1] published in 1928 demanded for an algorithm which decides for a given sentence in first-order logic together with a (possibly finite) number of axioms whether the sentence is valid or not. Indeed, by Gödels Completeness and Soundness Theorem, there exists a deductive system $\mathcal{K}$ such that any valid sentence is $\mathcal{K}$-provable ($\models \sigma \Leftrightarrow \vdash_{\mathcal{K}} \sigma$), so the 'Entscheidungsproblem' can be viewed as the problem to find an algorithm which decides whether a sentence is $\mathcal{K}$-provable or not. However, Hilbert's hope of completeness was destroyed by Gödel who published his incompleteness theorems in 1931, and showed that there does not exist a deductive system $\mathcal{K}$ such that any mathematical true sentence is $\mathcal{K}$-provable. This is already the case in the theory of arithmetic, i. e. in the structure $\mathcal{N} = (\mathbb{N}; +, \cdot; 0, 1)$ of the natural numbers, and thus in particular also for more powerful structures, e. g. the theory of the real numbers. In his First Theorem Gödel showed that any consistent arithmetical effectively axiomatized theory cannot be complete and in his Second Theorem he indicates that such a theory cannot prove its own consistency. Gödel's theorems were devastating for Hilbert's hope for proof of consistency in a theory and destroyed Hilbert's search for a strong and complete theory. Gödel's original proof of the incompleteness theorem is based on the liar's paradoxon which states: 'This statement is false.' By changing this to, 'This statement is unprovable.', Gödel showed that this statement can be expressed in any theory which contains the language of arithmetic. If this assertion is provable, then it is false and hence the theory is inconsistent. Thus the assertion is true and unprovable.

Gödel's Incompleteness Theorems left the 'Entscheidungsproblem' as unfinished business. Although Gödel had shown that any consistent theory of arithmetic cannot prove every arithmetical truth, it did not rule out the existence of a computable decision procedure which reveals in finite time whether a statement is valid or not. In Alan Turing's paper 'On Computable Numbers with an Application to the Entscheidungsproblem' [2], 1936, Turing succeded to define an effective procedure by inventing a simple idealized computer, the so-called Turing machine and showed that the halting problem is undecidable, meaning that there is no effective procedure or algorithm for deciding whether or not a program halts. Utilizing the undecidability of the halting problem, Gödel's theorems can be derived. Suppose that an arithmetical sound theory F which is powerful enough to reason about Turing machines, is given. For a contradiction, suppose that F is complete and consider the question whether an arbitrary Turing machine M with a blank tape as input halts. Then F could decide the halting problem since all proofs of F can be enumerated, until either a proof that M halts or a proof that M runs forever is found. Eventually, this procedure terminates because F is complete and hence it can be decided whether M halts, a contradiction to the undecidability of the halting problem.

2 Peano Arithmetic

This chapter describes some standard effectively axiomatizable theories of arithmetic. In particular the so-called first-order Peano Arithmetic PA and the important subtheory, the Robinson Arithmetic Q are depicted. Firstly, we define the notion of arithmetically definability. Secondly, we outline the arithmetical hierarchy, e. g. Δ_0-, Σ_n- and Π_n-formulas for $n \geq 0$, and show that every p. r. function can be arithmetically defined by a Σ_1-formula. Thirdly, we introduce the notion of definability in an arithmetical theory $T = (\mathcal{L}_A, \Sigma)$, and lastly show that every p. r. function can be defined in Q by a Σ_1-formula.

In this chapter we fix the language of arithmetic $\mathcal{L}_A = \mathcal{L}(\sigma)$ with signature $\sigma = (-; 2, 2, 1; \{0\})$. (We mainly reference the work of Smith [3].)

2.1 Arithmetical Definability

Recall that the arithmetic is defined by $\mathcal{N} = (\mathbb{N}; +, \cdot, S; 0)$. In the following we introduce the notion of arithmetical definability and show that every p. r. function is arithmetically defined by a Σ_1-formula.

Definition 2.1. (i) Let $\varphi(x_0, \ldots, x_{n-1})$ be an $\mathcal{L}_A$-formula. An n-ary relation $R \subseteq \mathbb{N}^n$ is *arithmetically defined by* $\varphi(x_0, \ldots, x_{n-1})$ if for all $m_0, \ldots, m_{n-1} \in \mathbb{N}$,

$$(m_0, \ldots, m_{n-1}) \in R \Leftrightarrow \mathcal{N} \vDash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}].$$

(ii) An n-ary relation $R \subseteq \mathbb{N}^n$ is *arithmetically definable* if there exists an $\mathcal{L}_A$-formula $\varphi(x_0, \ldots, x_{n-1})$ such that R is arithmetically defined by $\varphi(x_0, \ldots, x_{n-1})$.

Definition 2.2. (i) Let $\varphi(x_0, \ldots, x_n)$ be an $\mathcal{L}_A$-formula. An n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is *arithmetically defined by* $\varphi(x_0, \ldots, x_n)$ if for all $m_0, \ldots, m_{n-1}, c \in \mathbb{N}$,

$$f(m_0, \ldots, m_{n-1}) = c \Leftrightarrow \mathcal{N} \vDash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}, \overline{c}/x_0, \ldots, x_{n-1}, x_n].$$

(ii) An n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is *arithmetically definable* if there exists an $\mathcal{L}_A$-formula such that f is arithmetically defined by $\varphi(x_0, \ldots, x_n)$.

The arithmetical definability of relations correspond to the arithmetical definability of the respective characteristic functions.

Lemma 2.3. *The n-ary relation $R \subseteq \mathbb{N}^n$ is arithmetically definable if and only if the characteristic function c_R of R is arithmetically definable.*

Proof. It suffices to show the following two statements. Then the theorem immediately follows.

(i) If the n-ary relation $R \subseteq \mathbb{N}^n$ is arithmetically defined by the $\mathcal{L}_A$-formula $\varphi(x_0, \ldots, x_{n-1})$, then the characteristic function c_R of R is arithmetically defined by the $\mathcal{L}_A$-formula

$$\delta(x_0, \ldots, x_n) :\equiv (\varphi(x_0, \ldots, x_{n-1}) \wedge x_n = \overline{0}) \vee (\neg\varphi(x_0, \ldots, x_{n-1}) \wedge x_n = \overline{1}).$$

(ii) If the characteristic function c_R of an n-ary relation $R \subseteq \mathbb{N}^n$ is arithmetically defined by the $\mathcal{L}_A$-formula $\psi(x_0, \ldots, x_n)$, then R is arithmetically defined by the $\mathcal{L}_A$-formula

$$\psi[\overline{0}/x_n].$$

For a proof of (i), suppose the n-ary relation $R \subseteq \mathbb{N}^n$ is arithmetically defined by $\varphi(x_0, \ldots, x_{n-1})$. If $c_R(m_0, \ldots, m_{n-1}) = 0$, then $(m_0, \ldots, m_{n-1}) \in R$. So

$$\mathcal{N} \vDash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}]$$

since R is arithmetically defined by $\varphi(x_0, \ldots, x_{n-1})$. Evidently, $\mathcal{N} \vDash \overline{0} = \overline{0}$ and hence

$$\mathcal{N} \vDash (\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{0}) \vee (\neg\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{1}),$$

i. e.

$$\mathcal{N} \vDash \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, \overline{0}/x_0, \ldots, x_n].$$

If $c_R(m_0, \ldots, m_{n-1}) \neq 0$, then $(m_0, \ldots, m_{n-1}) \notin R$. So

$$\mathcal{N} \nvDash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}].$$

Hence

$$\mathcal{N} \nvDash (\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{0})$$

and since $\mathcal{N} \nvDash \overline{0} = \overline{1}$,

$$\mathcal{N} \nvDash (\neg\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{1}).$$

Thus

$$\mathcal{N} \nvDash (\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{0}) \vee (\neg\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{1}),$$

i. e.

$$\mathcal{N} \nvDash \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, \overline{0}/x_0, \ldots, x_n].$$

Analogously, the remaining cases $c_R(m_0, \ldots, m_{n-1}) = 1$ and $c_R(m_0, \ldots, m_{n-1}) \neq 1$ can be shown.

For a proof of (ii), suppose that the characteristic function c_R of an n-ary relation $R \subseteq \mathbb{N}^n$ is arithmetically defined by the $\mathcal{L}_A$-formula $\psi(x_0, \dots, x_n)$. Then the following holds

$$\begin{aligned}(m_0, \dots, m_{n-1}) \in R \Leftrightarrow c_R(m_0, \dots, m_{n-1}) = 0 \\ \Leftrightarrow \mathcal{N} \vDash \psi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{0}/x_0, \dots, x_{n-1}, x_n].\end{aligned}$$

As a result R is arithmetically defined by the $\mathcal{L}_A$-formula $\psi[\overline{0}/x_n]$. □

The less-than-or-equal relation $\leq$ is arithmetically definable. Throughout this thesis we employ the following abbreviations:

- Let t_1, t_2 be $\mathcal{L}_A$-terms. The *less-or-equal relation* $\leq$ is an abbreviation for

  $$t_1 \leq t_2 :\equiv \exists x(x + t_1 = t_2)$$

  where x is the alphabetically first variable which does not occur in t_1 and t_2.

- Let $\varphi(x)$ be an $\mathcal{L}_A$-formula and t an $\mathcal{L}_A$-term. The *bounded existential quantifier* $(\exists x \leq t)$ is an abbreviation for

  $$(\exists x \leq t)\varphi(x) :\equiv \exists x(x \leq t \wedge \varphi(x)).$$

  We extend this notion for multiple variables, i. e. $(\exists x_0, \dots, x_n \leq t)$ is an abbreviation for

  $$(\exists x_0, \dots, x_n \leq t)\varphi(x_0, \dots, x_n) :\equiv \exists x_0 \dots \exists x_n(x_0 \leq t \wedge \dots \wedge x_n \leq t \wedge \varphi(x_0, \dots, x_n)).$$

- Let $\varphi(x)$ be an $\mathcal{L}_A$-formula and t an $\mathcal{L}_A$-term. The *bounded universal quantifier* $(\forall x \leq t)$ is an abbreviation for

  $$(\forall x \leq t)\varphi(x) :\equiv \forall x(x \leq t \rightarrow \varphi(x)).$$

  We extend this notion for multiple variables, i. e. $(\forall x_0, \dots, x_n \leq t)$ is an abbreviation for

  $$(\forall x_0, \dots, x_n \leq t)\varphi(x_0, \dots, x_n) :\equiv \forall x_0 \dots \forall x_n(x_0 \leq t \wedge \dots \wedge x_n \leq t \rightarrow \varphi(x_0, \dots, x_n)).$$

- The *uniqueness quantifier* $\exists! x$ is an abbreviation for

  $$\exists! x\varphi(x) :\equiv \exists x(\varphi(x) \wedge \forall y(\varphi[y/x] \rightarrow y = x))$$

  where y is the first variable which does not occur freely in φ.

In the following we introduce the notion of Δ_0-, Σ_n- and Π_n-formulas for $n \geq 0$.

Definition 2.4. An $\mathcal{L}_A$-formula is a Δ_0*-formula* if every (existential or universal) quantifier is a bounded (existential or universal) quantifier.

Definition 2.5. Σ_n*-formulas* and Π_n*-formulas* are inductively defined as follows:

(i) If φ is a Δ_0-formula, then φ is a Σ_0-formula and a Π_0-formula.

(ii) If ψ is a Π_n-formula ($n \geq 0$), then $\varphi \equiv \exists x_1 \dots \exists x_m \psi$ ($m \geq 0$) is a Σ_{n+1}-formula.

(iii) If ψ is a Σ_n-formula ($n \geq 0$), then $\varphi \equiv \forall x_1 \dots \forall x_m \psi$ ($m \geq 0$) is a Π_{n+1}-formula.

A Σ_1-formula φ is of the form

$$\exists x_1 \dots \exists x_m \psi$$

where ψ is a Δ_0-formula, and a Π_1-formula φ is of the form

$$\forall x_1 \dots \forall x_m \psi$$

where ψ is a Δ_0-formula.

We extend latter definitions and call an $\mathcal{L}_A$-formula φ a Δ_0- (Σ_n-, Π_n-)*formula* if φ is equivalent to a Δ_0- (Σ_n-, Π_n-)formula by our definition. Moreover, we call an $\mathcal{L}_A$-sentence σ a Δ_0- (Σ_n-, Π_n-)sentence if σ is a Δ_0- (Σ_n-, Π_n-)formula. We denote the class of all Δ_0- (Σ_n-, Π_n-)formulas with Δ_0 (Σ_n, Π_n) and observe that

$$\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \dots$$

and

$$\Pi_0 \subset \Pi_1 \subset \Pi_2 \dots .$$

This hierarchy is called the *arithmetical hierarchy.* One can easily see that the following properties hold for the arithmetical hierarchy:

(i) $\varphi \in \Sigma_n \Leftrightarrow \neg\varphi \in \Pi_n$.

(ii) $\Delta_0 \subset \Sigma_1 \cap \Pi_1$.

(iii) $\Sigma_n \cup \Pi_n \subset \Sigma_{n+1} \cap \Pi_{n+1}$.

In this thesis we only require Δ_0, Σ_1 and Π_1, and therefore limit ourselves to the closure properties of those classes. Furthermore, we observe that the following holds:

- The class Δ_0 is closed under bounded quantifiers and every connective.
- The class Σ_1 is closed under the existential quantifier and closed under the connectives $\vee$ and $\wedge$ (but not under the universal quantifier and the negation).
- The class Π_1 is closed under the universal quantifier and the connectives $\vee$ and $\wedge$ (but not under the existential quantifier and the negation).

In the following we show that p. r. functions are arithmetically definable. For this reason we introduce Gödel's β-function and an auxiliary remainder function *rm*.

Definition 2.6. Let $rm : \mathbb{N}^2 \to \mathbb{N}$ be defined by

$$rm(c, d) = l \text{ with } 0 \leq l < d \text{ where there exists } x \in \mathbb{N} \text{ such that } x \cdot d + l = c,$$

i. e. $rm(c, d)$ returns the remainder of c divided by d.

Definition 2.7. Let the function $\beta : \mathbb{N}^3 \to \mathbb{N}$ be defined by

$$\beta(c, d, i) := rm(c, d(i + 1) + 1),$$

i. e. $\beta(c, d, i)$ returns the remainder of c divided by $d(i + 1) + 1$.

We assume that the reader is familiar with the Chinese Remainder Theorem of elementary number theory, as follows.

Theorem 2.8 (Chinese Remainder Theorem)**.** *Let $m_0, m_1, \dots, m_n$ $(n \geq 0)$ be a sequence of natural numbers where m_i, m_j $(0 \leq i < j \leq n)$ are pairwise relatively prime. Then for every sequence of natural numbers $k_0, k_1, \dots, k_n$ with $0 \leq k_i < m_i$ for $0 \leq i \leq n$, there exists $c \in \mathbb{N}$ such that $k_i = rm(c, m_i)$.*

The β-function encodes every sequence of natural numbers, i. e. for every sequence of natural numbers $k_0, \dots, k_n$ $(n \geq 0)$, there exist numbers $c, d \in \mathbb{N}$ such that $\beta(c, d, i) = k_i$ for all $0 \leq i \leq n$.

Theorem 2.9. *For every sequence of natural numbers $k_0, k_1, \dots, k_n$, there exist $c, d \in \mathbb{N}$ such that $\beta(c, d, i) = k_i$ for all $0 \leq i \leq n$.*

Proof. Let $u := max\{n+1, k_0, k_1, \dots, k_n\}$, $d := u!$ and $m_i := d(i+1)+1$ for $0 \leq i \leq n$. We claim that m_i, m_j for $0 \leq i < j \leq n$ are pairwise prime.

Suppose otherwise. Then for some prime p and some a, b such that $1 \leq a < b \leq n+1$, p divides both $da + 1$ and $db + 1$. Hence p divides $(db + 1) - (da + 1) = d(b - a)$ but $(b - a)$ is a factor of d, since d is by definition $u!$ with $u > (b - a)$. Therefore, p divides d without remainder. However, this contradicts to p dividing $da + 1$ or $db + 1$.

Since $m_i = d(i + 1) + 1$ are pairwise prime and $k_i < m_i$, the Chinese Remainder Theorem infers that there exists $c \in \mathbb{N}$ such that $rm(c, d(i+1)+1) = k_i$ for all $0 \leq i \leq n$, i. e. $\beta(c, d, i) = k_i$ for all $0 \leq i \leq n$. □

In fact, Gödel's β-function can be arithmetically defined by a Δ_0-formula.

Definition 2.10. Let *Beta* be an abbreviation for the following $\mathcal{L}_A$-formula:

$$Beta(v_0, v_1, v_2, v_3) :\equiv (\exists v_4 \leq v_0)(v_0 = (S(v_1 \cdot S(v_2)) \cdot v_4) + v_3 \wedge v_3 \leq (v_1 \cdot S(v_2))).$$

Since *Beta* only contains bounded quantification, *Beta* is a Δ_0-formula.

Lemma 2.11. *Gödel's β-function is arithmetically defined by $Beta(v_0, v_1, v_2, v_3)$.*

Proof. For $c, d, i, m \in \mathbb{N}$, we want to show:

$$\begin{aligned}\beta(c,d,i) = m &\Leftrightarrow \mathcal{N} \vDash Beta[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3] \\ &\Leftrightarrow \mathcal{N} \vDash (\exists v_4 \leq \overline{c})(\overline{c} = S(\overline{d} \cdot S(\overline{i})) \cdot v_4) + \overline{m} \wedge \overline{m} \leq (\overline{d} \cdot S(\overline{i})).\end{aligned}$$

Let $c, d, i, m \in \mathbb{N}$. Then the following holds:

$$\begin{aligned}\beta(c,d,i) = m &\Leftrightarrow rm(c, d(i+1)+1) = m \text{ with } 0 \leq m < d(i+1)+1 \\ &\Leftrightarrow rm(c, S(d \cdot S(i)) = m \text{ with } 0 \leq m \leq d \cdot S(i) \\ &\Leftrightarrow \exists x \in \mathbb{N} : x \cdot S(d \cdot S(i)) + m = c \text{ with } 0 \leq m \leq d \cdot S(i) \\ &\Leftrightarrow \exists x \in \mathbb{N} : x \cdot S(d \cdot S(i)) + m = c \text{ with } 0 \leq m \leq d \cdot S(i) \text{ and } x \leq c \\ &\Leftrightarrow \mathcal{N} \vDash (\exists v_4 \leq \overline{c})(\overline{c} = S(\overline{d} \cdot S(\overline{i})) \cdot v_4) + \overline{m} \wedge \overline{m} \leq (\overline{d} \cdot S(\overline{i})).\end{aligned}$$

□

For the following proof we summarize the recent results:

(i) Firstly, for every sequence of natural numbers $k_0, k_1, \ldots, k_n$, there exist $c, d \in \mathbb{N}$ s. t. $\beta(c,d,i) = k_i$ for all $0 \leq i \leq n$.

(ii) Secondly, Gödels β-function is arithmetically defined by the Δ_0-formula $Beta(v_0, v_1, v_2, v_3)$.

Theorem 2.12. *Every p. r. function can be arithmetically defined by a Σ_1-formula.*

Proof. By showing the following three statements, the theorem immediately follows.

A The basic primitive recursive functions can be arithmetically defined by Σ_1-formulas.

B Let $g : \mathbb{N}^m \to \mathbb{N}$ and $h_0, \ldots, h_{m-1} : \mathbb{N}^n \to \mathbb{N}$ be m-ary and n-ary functions, respectively. Let the n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ be the composition of g and $h_0, \ldots, h_{m-1}$. If g and $h_0, \ldots, h_{m-1}$ can be defined in T by Σ_1-formulas, then f can be arithmetically defined by a Σ_1-formula as well.

C Let $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ be n-ary and $(n+2)$-ary functions, respectively. Let the $(n+1)$-ary function f be the primitive recursion of g and h. If g and h can be arithmetically defined by Σ_1-formulas, then f can be arithmetically defined by a Σ_1-formula as well.

For a proof of (A), consider the following three cases:

(i) The m-ary zero function $C^m : \mathbb{N}^m \to \mathbb{N}$ $(m \geq 0)$ with $C^m(\overrightarrow{x}) = 0$ is arithmetically defined by

$$C(v_0, \ldots, v_m) :\equiv v_m = \overline{0}.$$

(ii) The 1-ary successor function $S^1 : \mathbb{N} \to \mathbb{N}$ with $S^1(x) = x + 1$ is arithmetically defined by the formula

$$S(v_0) = v_1.$$

(iii) The n-ary projection function $P_i^n : \mathbb{N}^n \to \mathbb{N}$ ($n \geq 1, 0 \leq i \leq n-1$) with $P_i^n(x_0, \ldots, x_{n-1}) = x_i$ is arithmetically defined by the formula

$$v_0 = v_0 \wedge v_1 = v_1 \wedge \ldots \wedge v_i = v_n \wedge \ldots \wedge v_{n-1} = v_{n-1}.$$

The basic primitive recursive functions are defined by Δ_0-formulas and hence defined by Σ_1-formulas.

For a proof of (B), let the m-ary function g and the n-ary functions $h_0, \ldots, h_{m-1}$ be arithmetically defined by the Σ_1-formulas $G(v_{n+1}, \ldots, v_{n+m+1})$ and $H_0(v_0, \ldots, v_n), \ldots, H_{m-1}(v_0, \ldots, v_n)$, respectively. The n-ary function f, which is the composition of g and $h_0, \ldots, h_{m-1}$ is defined by

$$f(\overrightarrow{x}) = g(h_0(\overrightarrow{x}), \ldots, h_{m-1}(\overrightarrow{x})).$$

Then the function f can be arithmetically defined by

$$\begin{aligned} F(v_{n+1}, \ldots, v_{n+m+1}) :\equiv & \exists x_0 \ldots \exists x_{m-1}(H_0[x_0/v_n] \wedge \ldots \wedge H_{m-1}[x_{m-1}/v_n] \\ & \wedge G[x_0, \ldots, x_{m-1}/v_{n+1}, \ldots, v_{n+m}]) \end{aligned}$$

where $x_0, \ldots, x_{m-1}$ are alphabetically the first n variables which do not occur free in $H_0, \ldots, H_{m-1}$ and G. Since $H_0, \ldots, H_{m-1}$ and G are Σ_1-formulas, the respective substitutions $H_0[x_0/v_n], \ldots, H_{m-1}[x_{m-1}/v_n], G[x_0, \ldots, x_{m-1}/v_{n+1}, \ldots, v_{n+m}]$ are also Σ_1-formulas. Thus by the above closure properties under connectives and the existential quantifier $F(v_{n+1}, \ldots, v_{n+m+1})$ is a Σ_1-formula.

For a proof of (C), let the n-ary function g and the $(n+2)$-ary function h be arithmetically defined by the Σ_1-formulas $G(v_0, \ldots, v_n)$ and $H(v_0, \ldots, v_{n+2})$, respectively. The $(n+1)$-ary function f which is the primitive recursion of g and h is inductively defined by

$$\begin{aligned} & f(\overrightarrow{x}, 0) = g(\overrightarrow{x}), \\ & f(\overrightarrow{x}, y+1) = h(\overrightarrow{x}, y, f(\overrightarrow{x}, y)). \end{aligned}$$

Fix $\overrightarrow{x} \in \mathbb{N}^n$ and $y, z \in \mathbb{N}$ with $f(\overrightarrow{x}, y) = z$. Then there exists a sequence of numbers $k_0, k_1, \ldots, k_y$ such that $k_0 = g(x_0, \ldots, x_{n-1})$ and if $0 \leq u < y$, then $k_{u+1} = h(x_0, \ldots, x_{n-1}, u, k_u)$ and $k_y = z$. Using the β-function, there exists $c, d \in \mathbb{N}$ such that $\beta(c, d, 0) = g(x_0, \ldots, x_{n-1})$ and if $0 \leq u < y$, then $\beta(c, d, S(u)) = h(x_0, \ldots, x_{n-1}, u, \beta(c, d, u))$ and $\beta(c, d, y) = z$. So the function f can be arithmetically defined by

$$\begin{aligned}F(v_0,\ldots,v_{n+1}) :\equiv\, & \exists x_0\exists x_1\{\exists x_2(Beta[x_0,x_1,\overline{0},x_2/v_0,v_1,v_2,v_3]\wedge G[x_2/v_n])\\ & \wedge(\forall x_3\leq v_n)(x_3\neq v_n\to\\ & \exists x_4\exists x_5\{(Beta[x_0,x_1,x_3,x_4/v_0,v_1,v_2,v_3]\\ & \wedge Beta[x_0,x_1,S(x_3),x_5/v_0,v_1,v_2,v_3])\\ & \wedge H[x_3,x_4,x_5/v_n,v_{n+1},v_{n+2}]\})\\ & \wedge Beta[x_0,x_1,v_n,v_{n+1}/v_0,v_1,v_2,v_3]\}.\end{aligned}$$

(The variables x_0 and x_1 take over the role of c and d. Moreover, the first line coincides with $\beta(c,d,0)=g(x_0,\ldots,x_{n-1})$, the second to the fifth line with $\beta(c,d,S(u))=h(x_0,\ldots,x_{n-1},u,\beta(c,d,u))$ for $0\leq u<y$, and the last line with $\beta(c,d,y)=z$.)

Since $\exists x_4\exists x_5\{\ldots\}$ is a Σ_1-formula, and unpacking the abbreviation of $\to$, $\neg x_3\neq v_{n+1}\vee\exists x_4\exists x_5\{\ldots\}$ is a Σ_1-formula as well. By the closure properties of Σ_1, $F(v_0,\ldots,v_{n+1})$ is therefore a Σ_1-formula. $\square$

Since the characteristic function c_R of a p. r. relation $R\subseteq\mathbb{N}^n$ $(n\geq 1)$ can be arithmetically defined by a Σ_1-formula $\psi(x_0,\ldots,x_n)$, the relation R can also be arithmetically defined by the Σ_1-formula $\psi[\overline{0}/x_n]$ (see proof of Lemma 2.3). As a result every p. r. function and moreover every p. r. relation can be arithmetically defined by a Σ_1-formula.

2.2 Robinson Arithmetic Q

We aim to find an arithmetical effectively axiomatized theory $T=(\mathcal{L}_A,\Sigma)$ which is negation-complete and consistent such that $C_\vdash(T)=Th(\mathcal{N})$. As a first approach, we introduce the finitely axiomatized theory, the so-called Robinson Arithmetic Q but we will soon show that Q is incomplete.

Definition 2.13. *Robinson Arithmetic* is the arithmetical theory $\mathsf{Q}:=(\mathcal{L}_A,\Sigma)$ where Σ contains the following axioms:

(Q1) $\forall v_0(\overline{0}\neq S(v_0))$.

(Q2) $\forall v_0\forall v_1(S(v_0)=S(v_1)\to v_0=v_1)$.

(Q3) $\forall v_0(v_0\neq\overline{0}\to\exists v_1(v_0=S(v_1)))$.

(Q4) $\forall v_0(v_0+\overline{0}=v_0)$.

(Q5) $\forall v_0\forall v_1(v_0+S(v_1)=S(v_0+v_1))$.

(Q6) $\forall v_0(v_0\cdot\overline{0}=\overline{0})$.

(Q7) $\forall v_0\forall v_1(v_0\cdot S(v_1)=(v_0\cdot v_1)+v_0)$.

The *induction principle* is a proof technique which is used to prove that a property $P(n)$ holds for every natural number n. This principle requires two cases to be proved. Firstly, the *base case*, requires that the property P holds for the number 0. Secondly, the *induction step*, requires that if the property P holds for a natural number n, then it holds for the next natural number $n+1$. In case both requirements are met, the property P holds for every natural number. Due to the lack of the induction principle in the axioms of Robinson arithmetic, we can show without much effort that Q is incomplete. For instance, Q cannot decide the sentence $\sigma :\equiv \forall v_0(\overline{0} + v_0 = v_0)$, i. e. $\mathsf{Q} \nvdash \sigma$ and $\mathsf{Q} \nvdash \neg\sigma$.

Theorem 2.14. Q *is incomplete.*

Proof. Consider the $\mathcal{L}_A$-sentence $\sigma :\equiv \forall v_0(0 + v_0 = v_0)$. Then one can easily show that $\mathcal{N} \vDash \mathsf{Q}$ and $\mathcal{N} \vDash \sigma$. We aim to construct an $\mathcal{L}_A$-structure $\mathcal{N}^* := (\mathbb{N} \cup \{a, b\}; +, \cdot, S; \{0\})$ where $a, b \notin \mathbb{N}$ and $a \neq b$ such that $\mathcal{N}^* \nvDash \sigma$. The function $S^{\mathcal{N}^*}$ is defined by $S^{\mathcal{N}^*}(n) = n + 1$ for every $n \in \mathbb{N}$, $S^{\mathcal{N}^*}(a) = a$ and $S^{\mathcal{N}^*}(b) = b$. Furthermore, the functions $+^{\mathcal{N}^*}$ and $\cdot^{\mathcal{N}^*}$ are defined as follows, for $n, m \in \mathbb{N}$:

$+^{\mathcal{N}^*}$	a	b	n
a	b	a	a
b	b	a	b
m	b	a	$m+n$

$\cdot^{\mathcal{N}^*}$	a	b	$n \neq 0$	0
a	b	b	b	0
b	a	a	a	0
m	a	b	$m \cdot n$	0

By definition of $+^{\mathcal{N}^*}$, $\mathcal{N}^* \vDash 0 + a \neq a$ holds and hence implies $\mathcal{N}^* \nvDash \sigma$. Again, one can easily check that $\mathcal{N}^* \vDash \mathsf{Q}$. As a result $\mathsf{Q} \nvDash \sigma$, and thus by the Completeness Theorem $\mathsf{Q} \nvdash \sigma$.

Conversely, $\mathcal{N} \nvDash \neg\sigma$ and $\mathcal{N}^* \vDash \neg\sigma$ since $\mathcal{N} \vDash \sigma$ and $\mathcal{N}^* \nvDash \sigma$ hold. As a result $\mathsf{Q} \nvDash \neg\sigma$ and thus by the Completeness Theorem $\mathsf{Q} \nvdash \neg\sigma$. To conclude, Q is incomplete, i. e. $\mathsf{Q} \nvdash \sigma$, and $\mathsf{Q} \nvdash \neg\sigma$. □

Note, that Robinson Arithmetic was studied by Raphael M. Robinson in 1952 long after Gödelian incompleteness was discovered. Despite Robinson Arithmetic being incomplete, Q has some interesting properties. Firstly, Robinson Arithmetic decides every Δ_0-sentence, i. e. $\mathsf{Q} \vdash \sigma$ or $\mathsf{Q} \vdash \neg\sigma$. Secondly, Q correctly decides every Δ_0-sentence σ, i. e. $\mathsf{Q} \vdash \sigma$ if and only if $\mathcal{N} \vDash \sigma$. Thirdly, Q proves every true Σ_1-sentence σ in $\mathcal{N}$, i. e. if $\mathcal{N} \vDash \sigma$, then $\mathsf{Q} \vdash \sigma$.

Lemma 2.15. Q *decides every Δ_0-sentence, i. e. for every Δ_0-sentence σ, $\mathsf{Q} \vdash \sigma$ or $\mathsf{Q} \vdash \neg\sigma$.*

Lemma 2.16. *For every Δ_0-sentence σ, $\mathsf{Q} \vdash \sigma$ if and only if $\mathcal{N} \vDash \sigma$.*

Lemma 2.17. *For every Σ_1-sentence σ, if $\mathcal{N} \vDash \sigma$, then $\mathsf{Q} \vdash \sigma$.*

The lemmas can be proven by induction over the structure of formulas. (A proof of those can be found in the book of Smith [3], p.71-83.)

2.3 Definability in Arithmetical Theories

We already introduced arithmetical definability by now, in the following we are going to depict definability in arithmetical theories $T = (\mathcal{L}_A, \Sigma)$. Moreover, we show that every p. r. function is defined in Q by a Σ_1-formula. Recall that an arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ is a consistent theory with the language of arithmetic $\mathcal{L}(T) = \mathcal{L}_A$.

Definition 2.18. (i) Let $\varphi(x_0, \dots, x_{n-1})$ be an $\mathcal{L}_A$-formula and $T = (\mathcal{L}_A, \Sigma)$ an arithmetical theory. An n-ary relation $R \subseteq \mathbb{N}^n$ is *defined in T by* $\varphi(x_0, \dots, x_{n-1})$ if for all $m_0, \dots, m_{n-1} \in \mathbb{N}$,

(1) if $(m_0, \dots, m_{n-1}) \in R$, then $T \vdash \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$,

(2) if $(m_0, \dots, m_{n-1}) \notin R$, then $T \vdash \neg\varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$.

(ii) Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory. An n-ary relation $R \subseteq \mathbb{N}^n$ is *definable in T* if there exists an $\mathcal{L}_A$-formula $\varphi(x_0, \dots, x_{n-1})$ such that R is defined in T by $\varphi(x_0, \dots, x_{n-1})$.

Definition 2.19. (i) Let $\varphi(x_0, \dots, x_n)$ be an $\mathcal{L}_A$-formula and $T = (\mathcal{L}_A, \Sigma)$ an arithmetical theory. An n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is *defined in T by* $\varphi(x_0, \dots, x_n)$ if for all $m_0, \dots, m_{n-1}, c \in \mathbb{N}$,

(1) if $f(m_0, \dots, m_{n-1}) = c$, then $T \vdash \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{c}/x_0, \dots, x_{n-1}, x_n]$,

(2) $T \vdash \exists! x_n \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$. (This is called the *uniqueness condition*).

(ii) Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory. An n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is *definable in T* if there exists an $\mathcal{L}_A$-formula $\varphi(x_0, \dots, x_n)$ such that f is defined in T by $\varphi(x_0, \dots, x_n)$.

Recall that a theory $T = (\mathcal{L}_A, \Sigma)$ is an extension of Q, if $\mathsf{Q} \vdash \varphi$, then $T \vdash \varphi$ for every $\mathcal{L}_A$-formula φ.

Lemma 2.20. *For any arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ which extends Q suppose that if the n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is defined in T by an $\mathcal{L}_A$-formula $\varphi(x_0, \dots, x_n)$, then for all $m_0, \dots, m_{n-1}, c \in \mathbb{N}$ the following holds.*

If $f(m_0, \dots, m_{n-1}) \neq c$, then $T \vdash \neg\varphi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{c}/x_0, \dots, x_{n-1}, x_n]$.

Proof. Fix any $m_0, \dots, m_{n-1}, c \in \mathbb{N}$ with $f(m_0, \dots, m_{n-1}) \neq c$. Then there exists a natural number $b \neq c$ with $f(m_0, \dots, m_{n-1}) = b$. Since f is defined in T by $\varphi(x_0, \dots, x_n)$, $T \vdash \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{b}/x_0, \dots, x_{n-1}, x_n]$ follows. Latter and the uniqueness condition $T \vdash \exists! x_n \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$ infer that

$$T \nvdash \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{c}/x_0, \dots, x_{n-1}, x_n].$$

Since Q and therefore every consistent extension T of Q decides a Δ_0-sentence,

$$T \vdash \neg\varphi[\overline{m_0}, \dots, \overline{m_{n-1}}, \overline{c}/x_0, \dots, x_{n-1}, x_n].$$

∎

Definability in an arithmetical theory T (which extends Q) of relations correspond to definability in T of the respective characteristic functions.

Lemma 2.21. *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory which extends Q. An n-ary relation $R \subseteq \mathbb{N}^n$ is definable in T if and only if the characteristic function c_R of R is definable in T.*

Proof. It suffices to show the following two statements. Then the theorem immediately follows.

(i) If the n-ary relation $R \subseteq \mathbb{N}^n$ is defined in T by the $\mathcal{L}_A$-formula $\varphi(x_0, \ldots, x_{n-1})$, then the characteristic function c_R of R is defined in T by the $\mathcal{L}_A$-formula

$$\delta(x_0, \ldots, x_n) :\equiv (\varphi(x_0, \ldots, x_{n-1}) \wedge x_n = \overline{0}) \vee (\neg\varphi(x_0, \ldots, x_{n-1}) \wedge x_n = \overline{1}).$$

(ii) If the characteristic function c_R of an n-ary relation $R \subseteq \mathbb{N}^n$ is defined in T by the $\mathcal{L}_A$-formula $\psi(x_0, \ldots, x_n)$, then R is defined in T by the $\mathcal{L}_A$-formula

$$\psi[\overline{0}/x_n].$$

For a proof of (i), suppose the n-ary relation $R \subseteq \mathbb{N}^n$ is defined in T by $\varphi(x_0, \ldots, x_{n-1})$. If $c_R(m_0, \ldots, m_{n-1}) = 0$, then by definition $(m_0, \ldots, m_{n-1}) \in R$. Since R is defined in T by $\varphi(x_0, \ldots, x_{n-1})$, it follows that $T \vdash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}]$. Due to the Soundness Theorem, $T \vDash \varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}]$ and since $T \vDash \overline{0} = \overline{0}$,

$$\begin{aligned}
&T \vDash (\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{0}) \vee \\
&(\neg\varphi[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \overline{0} = \overline{1}) \\
\Leftrightarrow &T \vDash \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, \overline{0}/x_0, \ldots, x_{n-1}, x_n] \\
\Leftrightarrow &T \vdash \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, \overline{0}/x_0, \ldots, x_{n-1}, x_n] \qquad (1)
\end{aligned}$$

where the last equivalence holds due to the Soundness and Completeness Theorem.

It remains to show (still with the assumption $c_R(m_0, \ldots, m_{n-1}) = 0$),

$$\begin{aligned}
&T \vdash \exists! x_n \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0 \ldots, x_{n-1}] \\
\Leftrightarrow &T \vdash \exists x_n (\delta[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \\
&\forall y(\delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, y/x_0, \ldots, x_{n-1}, x_n] \to y = x_n)) \\
\Leftrightarrow &T \vDash \exists x_n (\delta[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}] \wedge \\
&\forall y(\delta[\overline{m_0}, \ldots, \overline{m_{n-1}}, y/x_0, \ldots, x_{n-1}, x_n] \to y = x_n))
\end{aligned}$$

where y is the first variable which does not occur free in δ. Therefore, we want to show that

$$V_B^{\mathcal{M}}(\exists! x_n \delta[\overline{m_0}, \ldots, \overline{m_{n-1}}/x_0, \ldots, x_{n-1}]) = 1$$

for any model $\mathcal{M}$ of T and the valuation B of the empty set. Fix any model $\mathcal{M}$ of T, i. e. $\mathcal{M} \vDash T$ and $B' : \{x_n\} \to M$ a valuation of $\{x_n\}$ defined by $B'(x_n) = \overline{0}^{\mathcal{M}}$ (which evidently coincides with the valuation B of the empty set). Then we are done if we show that

$$V_{B'}^{\mathcal{M}}(\forall y(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n] \to y = x_n)) = 1$$

since $V_{B'}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}}/x_0,\dots,x_{n-1}]) = 1$ holds by (1). In order to show latter, let $B'' : \{x_n, y\} \to M$ be a valuation of $\{x_n, y\}$ with $B''(x_n) = \overline{0}^{\mathcal{M}}$ (which coincides with B' on $\{x_n\}$). Recall that the consistent theory T extends Q and therefore by the axioms (Q1) and (Q3) of Q, for every natural number n, $T \vdash \overline{0} \neq \overline{n}$. Then $T \vDash \overline{0} \neq \overline{n}$, $T \vdash \varphi[\overline{m_0},\dots,\overline{m_{n-1}}/x_0,\dots,x_{n-1}]$, the definition of $\delta \equiv (\varphi(x_0,\dots,x_{n-1}) \wedge x_n = \overline{0}) \vee (\neg\varphi(x_0,\dots,x_{n-1}) \wedge x_n = \overline{1})$ and $B''(x_n) = \overline{0}^{\mathcal{M}}$ imply that $V_{B''}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n]) = 1$ if and only if $B''(y) = \overline{0}^{\mathcal{M}}$.

If $B''(y) = \overline{0}^{\mathcal{M}}$, then $V_{B''}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n]) = 1$ and also $V_{B''}^{\mathcal{M}}(y = x_n) = 1$. Thus

$$V_{B''}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n] \to y = x_n) = 1.$$

Else $B''(y) \neq \overline{0}^{\mathcal{M}}$, then $V_{B''}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n]) = 0$ and hence

$$V_{B''}^{\mathcal{M}}(\delta[\overline{m_0},\dots,\overline{m_{n-1}},y/x_0,\dots,x_{n-1},x_n] \to y = x_n) = 1.$$

Analogously, the other case $c_R(m_0,\dots,m_{n-1}) = 1$ along with the uniqueness condition can be shown.

For a proof of (ii), suppose that the characteristic function c_R of an n-ary relation $R \subseteq \mathbb{N}^n$ is defined in T by the $\mathcal{L}_A$-formula $\psi(x_0,\dots,x_n)$.

If $(m_0,\dots,m_{n-1}) \in R$, then $c_R(m_0,\dots,m_{n-1}) = 0$. Since c_R is defined in T by $\psi(x_0,\dots,x_n)$,

$$T \vdash \psi[\overline{m_0},\dots,\overline{m_{n-1}},\overline{0}/x_0,\dots,x_{n-1},x_n].$$

If $R(m_0,\dots,m_{n-1})$ does not hold, then $c_R(m_0,\dots,m_{n-1}) = 1$. By Lemma 2.20,

$$T \vdash \neg\psi[\overline{m_0},\dots,\overline{m_{n-1}},\overline{0}/x_0,\dots,x_{n-1},x_n]$$

since $c_R(m_0,\dots,m_{n-1}) \neq 0$. As a result the relation R is arithmetically defined by $\psi[\overline{0}/x_n]$. $\square$

2.4 Definability in Q

We already mentioned above that the less-than-or-equal relation $\leq$ is arithmetically definable. Indeed, $\leq$ is definable in Q (by the same $\mathcal{L}_A$-formula) and therefore also definable in every arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ which extends Q. In addition, Robinson Arithmetic can prove the following sentences about the less-than-or-equal relation.

Lemma 2.22. *For Robinson Arithmetic the following holds:*

(O1) $\mathsf{Q} \vdash \forall x(\overline{0} \leq x)$.

(O2) *For any* $n \in \mathbb{N}$, $\mathsf{Q} \vdash \forall x(\{x = \overline{0} \vee \overline{1} \vee \ldots \vee x = \overline{n}\} \rightarrow x \leq \overline{n})$.

(O3) *For any* $n \in \mathbb{N}$, $\mathsf{Q} \vdash \forall x(x \leq \overline{n} \rightarrow \{x = \overline{0} \vee \overline{1} \vee \ldots \vee x = \overline{n}\})$.

(O4) *For any* $n \in \mathbb{N}$ *and any* $\mathcal{L}_A$*-formula* $\varphi(x)$*, if* $\mathsf{Q} \vdash \varphi[\overline{0}/x], \mathsf{Q} \vdash \varphi[\overline{1}/x], \ldots, \mathsf{Q} \vdash \varphi[\overline{n}/x]$*, then* $\mathsf{Q} \vdash (\forall x \leq \overline{n})\varphi(x)$.

(O5) *For any* $n \in \mathbb{N}$, $\mathsf{Q} \vdash \forall x(x \leq \overline{n} \rightarrow x \leq S(\overline{n}))$.

(O6) *For any* $n \in \mathbb{N}$, $\mathsf{Q} \vdash \forall x(\overline{n} \leq x \rightarrow (\overline{n} = x \vee S(\overline{n}) \leq x))$.

(O7) *For any* $n \in \mathbb{N}$, $\mathsf{Q} \vdash \forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

The proof of latter lemma is left to the reader.

Similar to arithmetical definability (see Theorem 2.12), we can now show that every p. r. function can be defined in Q. We would like to remind the reader of some relevant results:

(i) We showed that Gödel's β-function encodes every sequence of natural numbers, i. e. for every sequence of natural numbers $k_0, \ldots, k_n$ $(n \geq 0)$, there exists numbers $c, d \in \mathbb{N}$ such that $\beta(c, d, i) = k_i$ for all $0 \leq i \leq n$.

(ii) *Beta* is an abbreviation for

$$Beta(v_0, v_1, v_2, v_3) :\equiv (\exists v_4 \leq v_0)(v_0 = (S(v_1 \cdot S(v_2)) \cdot v_4) + v_3 \wedge v_3 \leq (v_1 \cdot S(v_2))).$$

However, Gödel's β-function cannot be defined in Q since the uniqueness condition does not necessarily hold. For this reason we define the corresponding $\mathcal{L}_A$-formula $\widetilde{Beta}$ which takes the uniqueness condition into account.

Definition 2.23. Let $\widetilde{Beta}$ be an abbreviation for the following $\mathcal{L}_A$-formula:

$$\widetilde{Beta}(v_0, v_1, v_2, v_3) :\equiv Beta(v_0, v_1, v_2, v_3) \wedge (\forall v_5 \leq v_3)(Beta[v_5/v_3] \rightarrow v_5 = v_3).$$

Since *Beta* is a Δ_0-formula and by the closure properties of Δ_0-formulas, (in particular, closure under connectives and the bounded universal quantifier,) $\widetilde{Beta}$ is a Δ_0-formula as well.

Lemma 2.24. *Gödel's* β*-function is defined in* Q *by* $\widetilde{Beta}(v_0, v_1, v_2, v_3)$.

Proof. For all $c, d, i, m \in \mathbb{N}$, it suffices to show:

(i) If $\beta(c, d, i) = m$, then $\mathsf{Q} \vdash \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3]$.

(ii) $\mathsf{Q} \vdash \exists! v_3 \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}/v_0, v_1, v_2]$.

For a proof of (i), fix any $c, d, i, m \in \mathbb{N}$ with $\beta(c, d, i) = m$. Since β is arithmetically defined by $Beta(v_0, v_1, v_2, v_3)$,

$$\mathcal{N} \vDash Beta[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3] \tag{1}$$

holds. Moreover, since for all $0 \leq n < m$, $\beta(c, d, i) \neq n$ (because β is a well-defined function),

$$\text{for all } 0 \leq n < m,\ \mathcal{N} \vDash \neg Beta[\overline{c}, \overline{d}, \overline{i}, \overline{n}/v_0, v_1, v_2, v_3]. \tag{2}$$

Putting (1) and (2) together, it can be easily seen that

$$\begin{aligned}&\mathcal{N} \vDash Beta[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3] \wedge (\forall v_5 \leq \overline{m})(Beta[\overline{c}, \overline{d}, \overline{i}, v_5/v_0, v_1, v_2, v_3] \to v_5 = \overline{m})\\ \Leftrightarrow &\mathcal{N} \vDash \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3].\end{aligned}$$

Since Q correctly decides every Δ_0-sentence,

$$\mathsf{Q} \vdash \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3]$$

follows.

For a proof of (ii), fix any $c, d, i \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ with $\beta(c, d, i) = m$. Hence by (i) $\mathsf{Q} \vdash \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3]$ and it remains to show

$$\begin{aligned}&\mathsf{Q} \vdash \forall y(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3] \to y = \overline{m})\\ \Leftrightarrow &\mathsf{Q} \vDash \forall y(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3] \to y = \overline{m})\end{aligned}$$

where y is the first variable which does not occur freely in $\widetilde{Beta}$. To show this, fix any model $\mathcal{M}$ of Q and let $B : \{y\} \to M$ be any valuation of $\{y\}$ in $\mathcal{M}$ with $B(y) = a$ (for $a \in M$ and not necessarily in $\mathbb{N}$).

In case $V_B^{\mathcal{M}}(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3]) = 0$, then $\mathsf{Q} \vDash \forall y(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3] \to y = \overline{m})$ evidently holds.

In case $V_B^{\mathcal{M}}(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3]) = 1$, then it suffices to show that $V_B^{\mathcal{M}}(y = \overline{m}) = 1$. By (O7) $\mathsf{Q} \vdash \forall y(y \leq \overline{m} \vee \overline{m} \leq y)$ and thus by the Soundness Theorem $\mathsf{Q} \vDash \forall y(y \leq \overline{m} \vee \overline{m} \leq y)$. So we have to consider two cases:

If $a \leq m$, then by definition of $\widetilde{Beta}$ and since $\mathsf{Q} \vDash \widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, \overline{m}/v_0, v_1, v_2, v_3]$, $a = m$ follows. Thus $V_B^{\mathcal{M}}(y = \overline{m}) = 1$.

If $m \leq a$, then again by definition of $\widetilde{Beta}$ and since $V_B^{\mathcal{M}}(\widetilde{Beta}[\overline{c}, \overline{d}, \overline{i}, y/v_0, v_1, v_2, v_3]) = 1$, $m = a$ follows. Thus $V_B^{\mathcal{M}}(y = \overline{m}) = 1$.

As a result we proved the uniqueness condition for $\widetilde{Beta}$.

□

Theorem 2.25. *Every p. r. function can be defined in* Q *by a* Σ_1*-formula.*

Proof. By showing the following three statements, the theorem immediately follows.

A The basic primitive recursive functions can be defined in Q by Σ_1-formulas.

B Let $g : \mathbb{N}^m \to \mathbb{N}$ and $h_0, \ldots, h_{m-1} : \mathbb{N}^n \to \mathbb{N}$ be m-ary and n-ary functions, respectively. Let the n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ be the composition of g and $h_0, \ldots, h_{m-1}$. If g and $h_0, \ldots, h_{m-1}$ can be defined in Q by Σ_1-formulas, then f can be defined in Q by a Σ_1-formula as well.

C Let $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ be n-ary and $(n+2)$-ary functions, respectively. Let the $(n+1)$-ary function f be the primitive recursion of g and h. If g and h can be defined in Q by Σ_1-formulas, then f can be defined in Q by a Σ_1-formula as well.

For a proof of (A), consider the following three cases:

(i) The m-ary zero function $C^m : \mathbb{N}^m \to \mathbb{N}$ $(m \geq 0)$ with $C^m(\overrightarrow{x}) = 0$ is defined in Q by
$$C(v_0, \ldots, v_m) :\equiv v_m = \overline{0}.$$

(ii) The 1-ary successor function $S^1 : \mathbb{N} \to \mathbb{N}$ with $S^1(x) = x + 1$ is defined in Q by the formula
$$S(v_0) = v_1.$$

(iii) The n-ary projection function $P_i^n : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 1, 0 \leq i \leq n-1)$ with $P_i^n(x_0, \ldots, x_{n-1}) = x_i$ is defined in Q by the formula
$$v_0 = v_0 \wedge v_1 = v_1 \wedge \ldots \wedge v_i = v_n \wedge \ldots \wedge v_{n-1} = v_{n-1}.$$

(The proof that those functions are defined in Q by the respective formulas is more or less trivial.) The basic primitive recursive functions are defined in Q by Δ_0-formulas and hence defined in Q by Σ_1-formulas.

For a proof of (B), let the m-ary function g and the n-ary functions $h_0, \ldots, h_{m-1}$ be defined in Q by the Σ_1-formulas $G(v_{n+1}, \ldots, v_{n+m+1})$ and $H_0(v_0, \ldots, v_n), \ldots, H_{m-1}(v_0, \ldots, v_n)$, respectively. The n-ary function f, which is the composition of g and $h_0, \ldots, h_{m-1}$ is defined by
$$f(\overrightarrow{x}) = g(h_0(\overrightarrow{x}), \ldots, h_{m-1}(\overrightarrow{x})).$$

Then f can be defined in Q by
$$\begin{aligned} F(v_{n+1}, \ldots, v_{n+m+1}) :\equiv & \exists x_0 \ldots \exists x_{m-1} (H_0[x_0/v_n] \wedge \ldots \wedge H_{m-1}[x_{m-1}/v_n] \\ & \wedge G[x_0, \ldots, x_{m-1}/v_{n+1}, \ldots, v_{n+m}]) \end{aligned}$$

where $x_0, \ldots, x_{m-1}$ are alphabetically the first n variables which do not occur freely in $H_0, \ldots, H_{m-1}$ and G. (Again, it can be easily checked that f is defined in Q by $F(v_{n+1}, \ldots, v_{n+m+1})$.) Since $H_0, \ldots, H_{m-1}$ and G are Σ_1-formulas, the respective substitutions $H_0[x_0/v_n], \dots, H_{m-1}[x_{m-1}/v_n], G[x_0, \dots, x_{m-1}/v_{n+1}, \dots, v_{n+m}]$ are also Σ_1-formulas. Thus by the above closure properties under connectives and the existential quantifier $F(v_{n+1}, \dots, v_{n+m+1})$ is a Σ_1-formula.

For a proof of (C), let the n-ary function g and the $(n+2)$-ary function h be defined in Q by the Σ_1-formulas $G(v_0, \dots, v_n)$ and $H(v_0, \dots, v_{n+2})$, respectively. The $(n+1)$-ary function f which is the primitive recursion of g and h is inductively defined by

$$\begin{aligned} &f(\overrightarrow{x}, 0) = g(\overrightarrow{x}), \\ &f(\overrightarrow{x}, y+1) = h(\overrightarrow{x}, y, f(\overrightarrow{x}, y)). \end{aligned}$$

For $\overrightarrow{x} \in \mathbb{N}^n$ and $y, z \in \mathbb{N}$ with $f(\overrightarrow{x}, y) = z$, there exists a sequence of numbers $k_0, k_1, \dots, k_y$ such that $k_0 = g(x_0, \dots, x_{n-1})$ and if $0 \leq u < y$ then $k_{u+1} = h(x_0, \dots, x_{n-1}, u, k_u)$ and $k_y = z$. Using the β-function, there exist $c, d \in \mathbb{N}$ such that $\beta(c, d, 0) = g(x_0, \dots, x_{n-1})$ and if $u < y$, then $\beta(c, d, S(u)) = h(x_0, \dots, x_{n-1}, u, \beta(c, d, u))$ and $\beta(c, d, y) = z$. Then it suffices to show that the function f can be defined in Q by

$$\begin{aligned} F(v_0, \dots, v_{n+1}) :\equiv \exists x_0 \exists x_1 \{ &\exists x_2 (\widetilde{Beta}[x_0, x_1, \overline{0}, x_2/v_0, v_1, v_2, v_3] \wedge G[x_2/v_n]) \\ &\wedge (\forall x_3 \leq v_n)(x_3 \neq v_n \rightarrow \\ &\exists x_4 \exists x_5 \{ (\widetilde{Beta}[x_0, x_1, x_3, x_4/v_0, v_1, v_2, v_3] \\ &\wedge \widetilde{Beta}[x_0, x_1, S(x_3), x_5/v_0, v_1, v_2, v_3]) \\ &\wedge H[x_3, x_4, x_5/v_n, v_{n+1}, v_{n+2}] \}) \\ &\wedge \widetilde{Beta}[x_0, x_1, v_n, v_{n+1}/v_0, v_1, v_2, v_3] \}. \end{aligned}$$

To show that f is defined in Q by $F(v_0, \dots, v_{n+1})$, we need to show that

(i) If $f(m_0, \dots, m_n) = c$, then $\mathsf{Q} \vdash F[\overline{m_0}, \dots, \overline{m_n}, \overline{c}/v_0, \dots, v_n, v_{n+1}]$.

(ii) For all $m_0, \dots, m_n \in \mathbb{N}$, $\mathsf{Q} \vdash \exists! v_{n+1} F[m_0, \dots, m_n/v_0, \dots, v_n]$.

For a proof of (i), suppose $f(m_0, \dots, m_n) = c$ for fixed $m_0, \dots, m_n, c \in \mathbb{N}$. Then evidently $\mathcal{N} \models F[\overline{m_0}, \dots, \overline{m_n}, \overline{c}/v_0, \dots, v_n, v_{n+1}]$ holds. Since $F[\overline{m_0}, \dots, \overline{m_n}, \overline{c}/v_0, \dots, v_n, v_{n+1}]$ is a Σ_1-formula (by the same arguments as in the proof of Theorem 2.12) and by Lemma 2.17, $\mathsf{Q} \vdash F[\overline{m_0}, \dots, \overline{m_n}, \overline{c}/v_0, \dots, v_n, v_{n+1}]$.

For a proof of (ii), suppose $f(m_0, \dots, m_n) = c$ for fixed $m_0, \dots, m_n, c \in \mathbb{N}$. Since the existence part follows from (i), it remains to show:

$$\begin{aligned} &\mathsf{Q} \vdash \forall y (F[\overline{m_0}, \dots, \overline{m_n}, y/v_0, \dots, v_n, v_{n+1}] \rightarrow y = \overline{c}) \\ \Leftrightarrow &\mathsf{Q} \models \forall y (F[\overline{m_0}, \dots, \overline{m_n}, y/v_0, \dots, v_n, v_{n+1}] \rightarrow y = \overline{c}) \end{aligned}$$

where y is the first variable which does not occur freely in F. Fix any model $\mathcal{M}$ of Q and let $B : \{y\} \rightarrow M$ be a valuation of $\{y\}$ in $\mathcal{M}$ with $B(y) = a \in M$. The case $V_B^{\mathcal{M}}(F[\overline{m_0}, \dots, \overline{m_n}, y/v_0, \dots, v_n, v_{n+1}]) = 0$ is trivial, so consider $V_B^{\mathcal{M}}(F[\overline{m_0}, \dots, \overline{m_n}, y/v_0, \dots, v_n, v_{n+1}]) = 1$. We distinguish between the following cases.

- If $m_n = 0$, then

$$\begin{aligned} F[\overline{m_0}, \ldots, m_{n-1}, \overline{0}, y/v_0, \ldots, v_n, v_{n+1}] \equiv & \exists x_0 \exists x_1 \{\exists x_2 (\widetilde{Beta}[x_0, x_1, \overline{0}, x_2/v_0, v_1, v_2, v_3] \\ & \wedge G[\overline{m_0}, \ldots, \overline{m_{n-1}}, x_2/v_0, \ldots, v_{n-1}, v_n]) \\ & \wedge \widetilde{Beta}[x_0, x_1, \overline{0}, y/v_0, v_1, v_2, v_3]\}. \end{aligned}$$

Note that $g(m_0, \ldots, m_{n-1}) = f(m_0, \ldots, m_{n-1}, 0) = c$. Since g and β are defined in Q by G and $\widetilde{Beta}$, respectively and since in particular the uniqueness condition holds for $\widetilde{Beta}$, $V_B^{\mathcal{M}}(y = \overline{c})$ follows.

- If $m_n > 0$, then

$$\begin{aligned} F[\overline{m_0}, \ldots, m_{n-1}, \overline{m_n}, y] :\equiv & \exists x_0 \exists x_1 \{\exists x_2 (\widetilde{Beta}[x_0, x_1, \overline{0}, x_2/v_0, v_1, v_2, v_3] \\ & \wedge G[\overline{m_0}, \ldots, \overline{m_{n-1}}, x_2/v_0, \ldots, v_{n-1}, v_n]) \\ & \wedge (\forall x_3 \leq \overline{m_n})(x_3 \neq \overline{m_n} \rightarrow \\ & \exists x_4 \exists x_5 \{(\widetilde{Beta}[x_0, x_1, x_3, x_4/v_0, v_1, v_2, v_3] \\ & \wedge \widetilde{Beta}[x_0, x_1, S(x_3), x_5/v_0, v_1, v_2, v_3]) \\ & \wedge H[\overline{m_0}, \ldots, \overline{m_{n-1}}, x_3, x_4, x_5/v_0, \ldots, v_{n-1}, v_n, v_{n+1}, v_{n+2}]\}) \\ & \wedge \widetilde{Beta}[x_0, x_1, \overline{m_{n-1}}, y/v_0, v_1, v_2, v_3]\}. \end{aligned}$$

One can easily see that by the uniqueness condition of $\widetilde{Beta}$, $V_B^{\mathcal{M}}(y = \overline{c})$ follows.

$\square$

Since the characteristic function c_R of a p. r. relation $R \subseteq \mathbb{N}^n$ $(n \geq 1)$ can be defined in Q by a Σ_1-formula $\psi(x_0, \ldots, x_n)$, the relation R can also be defined in Q by the Σ_1-formula $\psi[\overline{0}/x_n]$ (see proof of Lemma 2.21). As a result every p. r. function and moreover every p. r. relation can be defined in Q by a Σ_1-formula. Evidently, this holds for every consistent extension of Q and in particular for every arithmetical theory which extends Q.

Corollary 2.26. *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory which extends* Q. *Then any primitive recursive relation or primitive recursive function is defined in* Q *and also arithmetically defined by a Σ_1-formula.*

2.5 Peano Arithmetic PA

To compensate for Q's weakness of missing the induction principle, as far as it is possible to formulate the induction principle in first-order logic, the induction principle is formalized in the induction schema. By adding the induction schema (see (PA7)) to the axioms of Q, we obtain the theory PA.

Definition 2.27. Peano Arithmetic is the arithmetical theory $\mathsf{PA} := (\mathcal{L}_A, \Sigma)$ where Σ contains the following axioms:

(PA1) $\forall v_0(\overline{0} \neq S(v_0))$.

(PA2) $\forall v_0 \forall v_1(S(v_0) = S(v_1) \rightarrow v_0 = v_1)$.

(PA3) $\forall v_0(v_0 + \overline{0} = v_0)$.

(PA4) $\forall v_0 \forall v_1(v_0 + S(v_1) = S(v_0 + v_1))$.

(PA5) $\forall v_0(v_0 \cdot \overline{0} = \overline{0})$.

(PA6) $\forall v_0 \forall v_1(v_0 \cdot S(v_1) = (v_0 \cdot v_1) + v_0)$.

(PA7) $\{\varphi[\overline{0}/v_0] \wedge \forall v_0(\varphi(v_0) \rightarrow \varphi[S(v_0)/v_0])\} \rightarrow \forall v_0 \varphi(v_0)$ where $\varphi(v_0)$ is an $\mathcal{L}_A$-formula.

Note that Peano arithmetic is not finitely axiomatized since there exist infinitely many formulas with free variable v_0 and therefore also infinitely many instances of (PA7). Nevertheless, PA is an effectively axiomatized theory.

3 The Arithmetization of Syntax

Our goal is to construct an $\mathcal{L}_A$-sentence G which says about itself that it is unprovable. However, $\mathcal{L}_A$-sentences can only make statements about numbers and not theories, sentences or provability. As a solution Gödel's simple but powerful idea of assigning expressions in PA with code numbers is introduced. This is done by fixing a scheme of numbers and matching them to the alphabet of $\mathcal{L}_A$. As a result various syntactic relations correlate with purely numerical relations which is called the *arithmetization of syntax*.

For instance, consider the syntactic relation of being a term in $\mathcal{L}_A$ and define the corresponding numerical relation $Term(n)$ which holds when n codes a term. In the same way $Atom(n)$, $Form(n)$, $Sent(n)$ are defined which hold when n codes an atomic formula, a formula or a sentence, respectively. By defining the numerical relation $Prf(m, n)$ which holds when m is the code number in our scheme of a PA-proof of the sentence with code number n and showing that this relation is primitive recursive, a central result of this chapter is obtained. (In this chapter Smith [3] is used as the main reference.)

3.1 Gödel Numbering

Gödel numbers enable $\mathcal{L}_A$-formulas to talk indirectly about words over $\mathcal{L}_A$ (e. g. $\mathcal{L}_A$-terms, $\mathcal{L}_A$-formulas) by talking about their Gödel numbers.

Definition 3.1. Our *Gödel number scheme of* $\mathcal{L}_A$ (or *coding scheme of* $\mathcal{L}_A$) is defined by:

$$\begin{array}{cccccccccccccccc} \neg & \vee & \exists & = & (&) & 0 & S & + & \cdot & , & v_0 & v_1 & v_2 & \dots \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 2 & 4 & 6 & \dots \end{array}$$

Note that by our scheme symbols of $\mathcal{L}_A$ are assigned an even number if and only if the symbols are variables.

Definition 3.2. Let w be a word $s_0 s_1 \dots s_n$ $(n \geq 0)$ over $\mathcal{L}_A$ where every s_i is a symbol of $\mathcal{L}_A$ for $0 \leq i \leq n$. Then the *Gödel number* (*g. n.*) *of* w is calculated by firstly taking the correlated code numbers of the symbols $s_0, \dots, s_n$ and using them as exponents for the first $n+1$ prime numbers $\pi_0, \pi_1, \dots, \pi_n$ and secondly, multiplying the results, i. e.

$$\pi_0^{s_0} \cdot \pi_1^{s_1} \cdot \dots \cdot \pi_n^{s_n}.$$

For instance, the g. n. of the symbol $\vee$ is $\pi_0^3 = 2^3 = 8$ and the g. n. of the numeral $S(0)$ is $\pi_0^{15} \cdot \pi_1^9 \cdot \pi_2^{13} \cdot \pi_3^{11} = 2^{15} \cdot 3^9 \cdot 5^{13} \cdot 7^{11}$. If w is a word over $\mathcal{L}_A$, then we denote the Gödel number of w with $\ulcorner w \urcorner$. For an improper formula φ with abbreviations and notional conventions, we also denote the Gödel number $\ulcorner \psi \urcorner$ of the equivalent proper $\mathcal{L}_A$-formula ψ with $\ulcorner \varphi \urcorner$. For instance, $\ulcorner \overline{1} + \overline{0} \leq \overline{2} \urcorner = \ulcorner \exists v_0 + (+(S(0), 0), v_0) = S(S(0)) \urcorner$.

With a similar concept, super Gödel numbers enable $\mathcal{L}_A$-formulas to talk indirectly about sequences of words over $\mathcal{L}_A$.

Definition 3.3. Let p be a sequence $w_0, w_1, \dots, w_n$ of words over $\mathcal{L}_A$. Then the *super Gödel numbers* (*s. g. n.*) *of* p is calculated by firstly coding each w_i $(0 \leq i \leq n)$ by its Gödel number which yields a resulting sequence of Gödel numbers $g_0, g_1, \dots, g_n$, secondly, using the numbers $g_0, g_1, \dots, g_n$ as exponents for the first $n+1$ prime numbers $\pi_0, \pi_1, \dots, \pi_n$ and thirdly, multiplying the results, i. e.

$$\pi_0^{g_0} \cdot \pi_1^{g_1} \cdot \ldots \cdot \pi_n^{g_n}.$$

For instance, the corresponding sequence of Gödel numbers to the sequence $0, S(0)$ of words over $\mathcal{L}_A$ is $\pi_0^{13}, \pi_0^{15} \cdot \pi_1^{9} \cdot \pi_2^{13} \cdot \pi_3^{11}$. Therefore, the s. g. n. of $0, S(0)$ is

$$\pi_0^{\pi_0^{13}} \cdot \pi_0^{\pi_0^{15} \cdot \pi_1^{9} \cdot \pi_2^{13} \cdot \pi_3^{11}}.$$

If p is a sequence $w_0, w_1, \dots, w_n$ of words over $\mathcal{L}_A$, then we denote the s. g. n. of p with $\ulcorner p \urcorner$ or $\ulcorner w_0, w_1, \dots, w_n \urcorner$.

3.2 Primitive Recursive Syntactic Functions and Relations

We want to show that $Prf(m, n)$ which holds if m is the s. g. n. of a PA-proof of the formula with g. n. n is primitive recursive since by Corollary 2.26 this would imply that $Prf(m, n)$ is arithmetically defined and defined in Q by a Σ_1-formula. In the following we show that certain functions are primitive recursive in order to show that $Prf(m, n)$ is primitive recursive. In this chapter we only consider functions f with domain $\mathbb{N}^n$ $(n \geq 1)$ and codomain $\mathbb{N}$, i. e. f is of the form $f : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 0)$. Furthermore, we only consider relations of the form $R \subseteq \mathbb{N}^n$ for $n \geq 1$.

We remind the reader of the following closure properties of PRIM:

- PRIM is closed under explicit definitions.
- PRIM is closed under definition by p. r. cases from p. r. functions.
- PRIM is closed under logical connectives, i. e. $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$.
- PRIM is closed under the bounded existential quantifier and the bounded universal quantifier.
- PRIM is closed under the bounded minimization operator.

Moreover, the following functions and relations are primitive recursive:

- The relations $=, <, \leq, >, \geq, \neq$ over $\mathbb{N}$.
- The addition function $+$ and multiplication function $\cdot$ over $\mathbb{N}$.
- The factorial function $!(x)$ over $\mathbb{N}$. (Note that we usually write $x!$ instead of $!(x)$.)
- The 2-ary relation *divides relation* $\mid(x, y)$ holds if y is divisible by x. (Note that we also use infix notation.)
- The 2-ary *exponentiation function* $exp(n, m)$ returns the product of n multiplied m-times, i. e. $exp(n, m) := \underbrace{n \cdot \ldots \cdot n}_{m\text{-times}}$. (We also write n^m instead of $exp(n, m)$.)

Lemma 3.4. *The following functions and relations are primitive recursive:*

(i) The relation $Prime(n)$ holds when n is a prime number.

(ii) The function $\pi(n)$ returns the $(n+1)$-th prime, i. e. π_n. (We also write π_n instead of $\pi(n)$.)

(iii) The function $exf(n, i)$ returns the exponent of π_i in the prime factorization of n.

(iv) The function $len(n)$ returns the number of distinct prime factors of n.

(v) The concatenation function $\circ(m, n)$ returns the g. n. of the expression that results from stringing together the expression with g. n. m followed by the expression with g. n. n. (Note that for the concatenation function infix notation is also used.)

(vi) The function $num(n)$ returns the g. n. of the numeral $\overline{n}$ of $\mathcal{L}_A$.

Proof. (i) Since PRIM is closed under explicit definitions, bounded universal quantification and logical connectives, *Prime* which is defined as follows

$$Prime(n) :\Leftrightarrow n \neq 1 \wedge (\forall u \leq n)(\forall v \leq n)(u \cdot v = n \rightarrow (u = 1 \vee v = 1))$$

is primitive recursive.

(ii) The next prime after a number $k \in \mathbb{N}$ is not greater than $k! + 1$ because either $k! + 1$ is prime or it has a prime factor which must be greater than k. Let $h(k) := (\mu x \leq k! + 1)(k < x \wedge Prime(x))$. Then h is primitive recursive and returns the next prime after k. Next, let

$$\pi(0) := 2,$$
$$\pi(Sn) := h(\pi(n))$$

and therefore π is primitive recursive.

(iii) By the Fundamental Theorem of Arithmetic every number has a unique factorization into primes and therefore *exf* is well-defined. Furthermore, no exponent in the prime factorization of n can be larger than n itself. Let

$$exf(n,i) := (\mu x \leq n)\{(\pi_i^x \mid n) \wedge \neg(\pi_i^{x+1} \mid n)\}$$

and thus *exf* is primitive recursive.

(iv) Let $pf(m,n) := (Prime(m) \wedge m \mid n)$ which holds when m is a prime factor of n. In particular $pf(m,n)$ is a p. r. relation being a conjunction of p. r. relations. Let

$$p(m,n) := \begin{cases} 1 & \text{if } pf(m,n) \text{ holds,} \\ 0 & \text{otherwise,} \end{cases}$$

which is 1 when m is a prime factor of n and zero otherwise. $p(m,n)$ is defined by p. r. cases from p. r. functions and hence primitive recursive. We observe that $len(n) = p(0,n) + p(1,n) + \ldots + p(n-1,n) + p(n,n)$. To give a p. r. definition of *len*, let

$$\begin{aligned} &l(x,0) := p(0,x), \\ &l(x,y+1)) := p(S(y),x) + l(x,y). \end{aligned}$$

Lastly, let $len(n) := l(n,n)$ and hence *len* is primitive recursive.

$$\begin{aligned} m \circ n :=& (\mu x \leq f(m,n))\{\forall i < len(m))(exf(x,i) = exf(m,i)) \\ & \wedge (\forall i < len(n))(exf(x, i + len(m)) = exf(n,i))\}. \end{aligned}$$

In order to show that the function $\circ$ is p. r. it suffices to show that the μ operator can be bounded by a p. r. function $f(m,n)$. Since $len(m \circ n) = len(m) + len(n)$, the number of prime factors of $m \circ n$ is $len(m) + len(n)$ and the highest prime factor of $m \circ n$ is $\pi_{len(m)+len(n)}$. Moreover, the highest exponent of a prime factor of $m \circ n$ can be bounded by

$$max_{1 \leq i \leq len(m), 1 \leq j \leq len(n)}\{exf(n,i), exf(m,j)\} \leq m + n.$$

Thus we can define

$$f(m,n) := \pi_{len(m)+len(n)}^{(len(m)+len(n))\cdot(m+n)},$$

which is primitive recursive.

(v) The standard numeral of $S(n)$ is of the form S followed by the standard numeral for n. Consequently, let

$$\begin{aligned} &num(0) = \ulcorner 0 \urcorner = 2^{13}, \\ &num(y+1) = \ulcorner S(\urcorner \circ num(y) \circ \ulcorner) \urcorner = (2^{15} \cdot 3^9) \circ num(x) \circ 2^{11}. \end{aligned}$$

$\square$

Definition 3.5. (i) The relation *Term*(n) holds when n is the g. n. of an $\mathcal{L}_A$-term.

(ii) The relation *Form*(n) holds when n is the g. n. of a $\mathcal{L}_A$-formula.

(iii) The relation *Sent*(n) holds when n is the g. n. of a $\mathcal{L}_A$-sentence.

(iv) The relation *Prfseq*(n) holds if n is the s. g. n. of a sequence of $\mathcal{L}_A$-formulas that is PA-proof of an $\mathcal{L}_A$-formula.

(v) The relation *Prf*(m, n) holds if m is the s. g. n. of a PA-proof of the sentence with g. n. n.

Our goal is to show that *Prf* is primitive recursive. In order to do this, we firstly proof that *Term* is primitive recursive, secondly continue with *Form* and thirdly *Sent*. Then we can show that *Prfseq* and finally show that *Prf* is p. r.

We begin with showing that *Term* is p. r. and therefore introduce the auxiliary definition of a term-sequence.

Definition 3.6. A *term-sequence* of $\mathcal{L}_A$ is a sequence of $\mathcal{L}_A$-terms $t_0, \ldots, t_n$ such that each term t_k for $k \in \{0, \ldots n\}$ is in one of the forms:

(i) 0,

(ii) a variable v_j where $0 \leq j$,

(iii) $S(t_i)$ where $0 \leq i < k$,

(iv) $+(t_i, t_j)$ where $0 \leq i, j < k$,

(v) $\cdot(t_i, t_j)$ where $0 \leq i, j < k$.

Since every term must have a constructional history, a term has to be the last expression in some term-sequence.

Lemma 3.7. *The following relations are primitiv recursive:*

(i) The relation Var(n) holds when n is the g. n. of a variable of $\mathcal{L}_A$.

(ii) The relation Termseq(n) holds when n is the s. g. n. for a term-sequence of $\mathcal{L}_A$.

(iii) The relation Term(n) holds when n is the g. n. of an $\mathcal{L}_A$-term.

Proof. (i) Recall that the symbol code for a variable in our scheme is always even, i. e. of the form $2x + 2$ with $x \in \mathbb{N}$ and hence

$$Var(n) :\Leftrightarrow (\exists x \leq n)(n = 2^{2x+2}).$$

(ii) *Termseq*(n) holds if n is a s. g. n. of $l := len(n)$ terms. The terms can be decoded by utilizing *exf* where each value of $exf(n, x)$ as x runs from 0 to l is the code for either 0, or a variable, or the successor, or a sum, or a product of earlier terms. *Termseq* is primitive recursive because it can be defined as follows,

$$\begin{aligned} Termseq(n) :\Leftrightarrow (\forall x < len(n))\{ & exf(n,x) = \ulcorner 0 \urcorner \vee Var(exf(n,x)) \vee \\ & (\exists y < x)(exf(n,x) = \ulcorner S(\urcorner \circ exf(n,y) \ulcorner) \urcorner) \vee \\ & (\exists y, z < x)(exf(n,x) = \ulcorner +(\urcorner \circ exf(n,y) \circ \ulcorner , \urcorner \circ exf(n,z) \circ \ulcorner) \urcorner) \vee \\ & (\exists y, z < x)(exf(n,x) = \ulcorner \cdot(\urcorner \circ exf(n,y) \circ \ulcorner , \urcorner \circ exf(n,z) \circ \ulcorner) \urcorner). \end{aligned}$$

(iii) A term has g. n. n if and only if there is a s. g. n. x of a term-sequence which contains as last component a term which has g. n. n. Let

$$Term(n) :\Leftrightarrow (\exists x \leq f(n))(Termseq(x) \wedge n = exf(x, len(x) - 1)).$$

In order to show that the relation $Term(n)$ is primitive recursive it suffices to show that x can be bounded by a p. r. function $f(n)$. For a term t with $\ulcorner t \urcorner = n$, let $t_0, \ldots, t_k$ $(k \geq 0)$ be a term-sequence of minimal length with $t = t_k$. Then for $i \leq k$ $(i \geq 0)$, t_i is a subword of $t_k = t$. So $\ulcorner t_i \urcorner \leq \ulcorner t \urcorner = n$. Since the term-sequence has no repetitions and $|t| \leq \ulcorner t \urcorner = n$,

$$k \leq |\{w : w \text{ subword of } t\}| \leq n^n.$$

holds. Then $f(n)$ can be bounded as follows by:

$$\begin{aligned} \ulcorner t_0, \ldots, t_k \urcorner &= \pi_0^{\ulcorner t_0 \urcorner} \cdot \ldots \cdot \pi_k^{\ulcorner t_k \urcorner} \\ &\leq (\pi_k^{max_{0 \leq i \leq k}\{\ulcorner t_i \urcorner\}})^{k+1} \\ &\leq (\pi_{n^n}^n)^{n^n+1}. \end{aligned}$$

□

Repeating the same procedure, we introduce the auxiliary definition of a formula-sequence in order to show that *Form* and *Sent* are primitive recursive.

Definition 3.8. A *formula-sequence* of $\mathcal{L}_A$ is a sequence of $\mathcal{L}_A$-formulas $\varphi_0, \varphi_1, \ldots, \varphi_n$ such that each formula φ_k for $k \in \{0, \ldots n\}$ is in one of the forms:

(i) $t_0 = t_1$ where t_0 and t_1 are $\mathcal{L}_A$-terms,

(ii) $\neg\varphi_i$ where $0 \leq i < k$,

(iii) $(\varphi_i \vee \varphi_j)$ where $0 \leq i, j < k$,

(iv) $\exists x \varphi_i$ for some variable x where $0 \leq i < k$.

Since every formula must have a constructional history, a formula has to be the last expression in some formula-sequence.

Lemma 3.9. *The following relations are primitive recursive:*

(i) The relation Formseq(n) holds when n is the s. g. n. of a formula-sequence of $\mathcal{L}_A$.

(ii) The relation Form(n) holds when n is the g. n. of a $\mathcal{L}_A$-formula.

(iii) The relation Sent(n) holds when n is the g. n. of a $\mathcal{L}_A$-sentence.

Proof. (i) *Termseq*(n) holds if n is a s. g. n. of $l := len(n)$ formulas. The formulas can be decoded by utilizing *exf* where each value of $exf(n, k)$ as z runs from 0 to l is the code for a formula in the formula-sequence with s. g. n. n. *Formseq* is primitive recursive since it can be defined as follows,

$$\begin{aligned} Formseq(n) :\Leftrightarrow (\forall z < len(n))\{ &(\exists x, y < exf(n, z)) \\ &(Term(x) \wedge Term(y) \wedge exf(n, z) = (x \circ \ulcorner = \urcorner \circ y)) \\ &(\exists x < z)(exf(n, z) = \ulcorner \neg \urcorner \circ exf(n, x)) \vee \\ &(\exists x, y < z)(exf(n, z) = \ulcorner (\urcorner \circ exf(n, x) \circ \ulcorner \vee \urcorner \circ exf(n, y) \circ \ulcorner) \urcorner) \vee \\ &(\exists x < z)(exf(n, z) = \ulcorner \exists \urcorner \circ exf(n, x)). \end{aligned}$$

(ii) A formula has g. n. n if and only if there is a s. g. n. x which contains as last component a formula which has g. n. n. Let

$$Form(n) :\equiv (\exists x \leq f(n))(Formseq(x) \wedge n = exf(x, len(x) - 1)).$$

In order to show that the relation *Form*(n) is primitive recursive it suffices to show that x can be bounded by a p. r. function $f(n)$. For a formula φ with $\ulcorner \varphi \urcorner = n$, let $\varphi_0, \ldots, \varphi_k$ be a formula-sequence of minimal length with $\varphi = \varphi_k$. Then for $i \leq k$, φ_i is a subword of $\varphi_k = \varphi$. So $\ulcorner \varphi_i \urcorner \leq \ulcorner \varphi \urcorner = n$. Since the formula-sequence has no repetitions and $|\varphi| \leq \ulcorner \varphi \urcorner = n$,

$$k \leq |\{w : w \text{ subword of } \varphi\}| \leq n^n.$$

holds. Then $f(n)$ can be bounded as follows by:

$$\begin{aligned} \ulcorner \varphi_0, \ldots, \varphi_k \urcorner &= \pi_0^{\ulcorner \varphi_0 \urcorner} \cdot \ldots \cdot \pi_k^{\ulcorner \varphi_k \urcorner} \\ &\leq (\pi_k^{max_{0 \leq i \leq k}\{\ulcorner \varphi_i \urcorner\}})^{k+1} \\ &\leq (\pi_{n^n}^n)^{n^n+1}. \end{aligned}$$

(iii) Define *Bound*(c, i, n) to be true when c numbers a variable which occurs bounded in the formula with g. n. n in the $(i + 1)$-th position, as follows,

$$\begin{aligned} Bound(c, i, n) :\Leftrightarrow\, &Var(c) \wedge Form(n) \wedge \\ &(\exists x, y, z < n)\{n = x \circ \ulcorner \exists \urcorner \circ c \circ y \circ z \wedge \\ &Form(y) \wedge (len(x) \leq i \wedge i \leq (len(x) + len(y) + 1))\}. \end{aligned}$$

The middle clause ensures that the formula with g. n. x is of the form $\ldots \exists v_i \varphi \ldots$ $(i \geq 0)$ and v_i is bounded. The last clause guarantees that the position i occurs within the component $\exists v_i \varphi$. Evidently, *Bound* is p. r. since PRIM is closed under bounded quantifiers and explicit definitions. Then *Sent* can be defined as

$$Sent(n) :\Leftrightarrow Form(n) \wedge (\forall i < len(n))(Var(exf(n,i)) \rightarrow Bound(exf(n,i),i,n))$$

and therefore *Sent* is primitive recursive. The last clause ensures that every variable occurrence in the formula with g. n. n is bounded.

□

3.3 *Prf* is primitive recursive

Recall that an PA-proof is a finite sequence of formulas in which each formula is either an axiom (of PA or $\mathcal{S}$) or is obtained from previous formulas by a rule.

Theorem 3.10. *Prfseq(n) which holds if n is the s. g. n. of a sequence of $\mathcal{L}_A$-formulas that is a* PA*-proof of an $\mathcal{L}_A$-formula, is primitive recursive.*

Proof Sketch. In order to show that $Prfseq(n)$ is primitive recursive, it suffices to show that the following relations are primitive recursive:

(i) For $1 \leq i \leq 6$, $A_i(n)$ holds if n is the g. n. of an $\mathcal{S}$-axiom (Ai).

(ii) For $1 \in \{1,2,3,5\}$, $R_i(n,c)$ holds if c is the g. n. of the formula with g. n. n which follows from the premise n (Ri). $R_4(m,n,c)$ holds if c is the g. n. of the formula with g. n. n which follows from the premises m and n (Ri).

(iii) For $1 \leq i \leq 7$, $PA_i(n)$ holds if n is the g. n. of an axiom of PA (PAi).

For (i), consider for instance (A1) $(\neg\varphi \vee \varphi)$ where φ is an $\mathcal{L}_A$-formula. Since $\ulcorner\varphi\urcorner \leq \ulcorner\neg\varphi \vee \varphi\urcorner$, the following holds

$$A_1(n) :\Leftrightarrow (\exists x \leq n)(Form(x) \wedge n = \ulcorner(\neg\urcorner \circ x \circ \ulcorner\vee\urcorner \circ x \circ \ulcorner)\urcorner).$$

For (ii), consider for instance (R1) $\dfrac{\psi}{(\varphi \vee \psi)}$ where ψ and φ are $\mathcal{L}_A$-formulas. Since $\ulcorner\varphi\urcorner \leq \ulcorner\varphi \vee \psi\urcorner$, the following holds

$$R_1(n,c) :\Leftrightarrow Form(n) \wedge Form(c) \wedge (\exists x \leq c)(Form(x) \wedge c = \ulcorner(\urcorner \circ x \circ \ulcorner\vee\urcorner \circ n \circ \ulcorner)\urcorner).$$

For (iii), consider for instance (PA1) $\forall v_0(0 \neq S(v_0))$. We can define

$$PA_1(n) :\Leftrightarrow n = \ulcorner\forall v_0(0 \neq S(v_0)\urcorner$$

or if we want to be precise by unpacking the abbreviations

$$PA_1(n) :\Leftrightarrow n = \ulcorner\neg\exists v_0\neg\neg 0 = S(v_0)\urcorner.$$

The proof of the induction schema (PA7) is going to involve the idea of coding that φ has v_0 as a free variable and coding the substitution of v_0 for $\overline{0}$ or $S(v_0)$. This tiresome proof is left to the reader.

Analogously, the remaining relations can be defined and it can be shown that they are primitive recursive. Let

$$Axiom(n) := A_1(n) \vee \ldots \vee A_6(n) \vee PA_1(n) \vee \ldots \vee PA_7(n)$$

and

$$Rule(n, o) := R_1(n, o) \vee R_2(n, o) \vee R_3(n, o) \vee R_5(n, o).$$

Then *Prfseq* is primitive recursive provided $A_1, \ldots, A_6, PA_1, \ldots, PA_7, R_1, \ldots, R_5$ are primitive recursive, as follows,

$$\begin{aligned} Prfseq(n) :=& (\forall x < len(n)) \{Axiom(exf(n, x)) \vee \\ & (\exists y, z < x) R_4(exf(n, y), exf(n, z), exf(n, x)) \vee \\ & (\exists y < k) Rule(exf(n, y), exf(n, x))\}. \end{aligned}$$

□

Theorem 3.11. *The relation $Prf(m, n)$ which holds when m is the s. g. n. of a* PA*-proof of the sentence with g. n. n, is primitive recursive.*

Proof. The sentence σ with g. n. n is the last component in the PA-proof of σ so let

$$Prf(m, n) := Prfseq(n) \wedge (exf(m, len(m) - 1) = n) \wedge Sent(n).$$

□

4 First Incompleteness Theorem

Goal of this chapter is to construct a sentence G that is true in $\mathcal{N}$ if and only if it is PA-unprovable. For a formula φ with free variable v_1 and g. n. n, we aim to construct a function $\widetilde{sub}$ which on input n returns the g. n. of the formula $\varphi[\overline{\ulcorner\varphi\urcorner}/v_1] = \varphi[\overline{n}/v_1]$. Gödel showed in a time-consuming procedure that this function $\widetilde{sub}$ is primitive recursive. However, to simplify things, we instead consider the g. n. of the formula $\exists v_1(v_1 = \overline{\ulcorner\varphi\urcorner} \wedge \varphi)$ since both formulas are logically equivalent if v_1 is a free variable in φ. Again, in this chapter we fix the language of arithmetic $\mathcal{L}_A$. Moreover, we only consider functions f with domain $\mathbb{N}^n$ $(n \geq 1)$ and codomain $\mathbb{N}$, i. e. f is of the form $f : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 0)$. Furthermore, we only consider relations of the form $R \subseteq \mathbb{N}^n$ for $n \geq 1$.

Lastly, we state a generalized version of Gödel's First Theorem which applies to any arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ which satisfies specific conditions.

Definition 4.1. Let $sub(n)$ be a 1-ary function defined by

$$sub(n) := \begin{cases} \ulcorner\exists v_1(v_1 = \overline{\ulcorner\varphi\urcorner} \wedge \varphi)\urcorner & \text{if } n \text{ is the g. n. of a formula } \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this definition also applies if v_1 is not a free variable of φ.

For a formula φ with g. n. n, we call the formula $\exists v_1(v_1 = \overline{n} \wedge \varphi)$ the *indirect substitution of* v_1 *for* $\overline{\ulcorner\varphi\urcorner}$ *in* φ.

Theorem 4.2. *The function $sub(n)$ is primitive recursive.*

Proof. In case n is the g. n. of a formula φ, i. e. $Form(n)$ holds, the sub function maps n, the g. n. of φ, to the g. n. of $\exists v_1\, v_1 = \neg(\neg\overline{\ulcorner\varphi\urcorner} \vee \neg\varphi)$ (unpacking the abbreviation $\exists v_1(v_1 = \overline{\ulcorner\varphi\urcorner} \wedge \varphi)$). If $Form(n)$ holds, then let

$$sub(n) := \ulcorner\exists v_1\, v_1 = \neg(\neg\urcorner \circ num(n) \circ \ulcorner\vee\neg\urcorner \circ n \circ \ulcorner)\urcorner$$

where num is p. r. and returns the g. n. of the numeral n (see Lemma 3.4). Else, let $sub(n) = 0$. So $sub(n)$ is defined by p. r. cases from p. r. functions and thus is primitive recursive as well. $\square$

Definition 4.3. Let $Gdl(m, n)$ be a relation which holds when m is the s. g. n. of a PA-proof of $sub(n)$ (which is the g. n. of the indirect substitution of v_1 for $\overline{n}$ in the formula with g. n. n).

Theorem 4.4. *The relation $Gdl(m, n)$ is primitive recursive.*

Proof. Let $Gdl(m,n) :\Leftrightarrow Prf(m, sub(n))$. Then the relation Gdl is a composition of p. r. relations Prf, sub and therefore primitive recursive. $\square$

Definition 4.5. Corollary 2.26 ensures the existence of the following Σ_1-formulas.

(i) Let $\mathsf{Sub}(v_1, v_2)$ be the Σ_1-formula s. t. $sub(n)$ is arithmetically defined and defined in PA by $\mathsf{Sub}(v_0, v_1)$.

(ii) Let $\mathsf{Prf}(v_0, v_1)$ be the Σ_1-formula s. t. $Prf(m,n)$ is arithmetically defined and defined in PA by $\mathsf{Prf}(v_0, v_1)$.

Definition 4.6. Let $\mathsf{Gdl}(v_0, v_1) :\equiv \exists v_2(\mathsf{Prf}[v_2/v_1] \wedge \mathsf{Sub}(v_1, v_2))$ be an $\mathcal{L}_A$-formula.

Evidently, Gdl is a Σ_1-formula since Prf and Sub are Σ_1-formulas.

Proposition 4.7. *$Gdl(m,n)$ is arithmetically defined and defined in* PA *by* $\mathsf{Gdl}(v_0, v_1)$*.*

Latter proposition can be easily shown by the facts that $sub(n)$ is arithmetically defined and defined in PA by $\mathsf{Sub}(v_1, v_2)$ and $Prf(m,n)$ is arithmetically defined and defined in PA by $\mathsf{Prf}(v_0, v_1)$.

Definition 4.8. Let $\mathsf{U}(v_1) :\equiv \forall v_0 \neg \mathsf{Gdl}(v_0, v_1)$ be an $\mathcal{L}_A$-formula.

The indirect substitution of v_1 for $\overline{\ulcorner \mathsf{U} \urcorner}$ in U yields the desired formula G. Firstly, G is true if and only if it is unprovable. Secondly, it is a Π_1-sentence (and very long in its unabbreviated version).

Definition 4.9. Let $\mathsf{G} :\equiv \exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$ be an $\mathcal{L}_A$-sentence. We call G the *Gödel sentence.*

Note that G is equivalent to the direct substitution of v_1 for $\overline{\ulcorner \mathsf{U} \urcorner}$ in U. Hence

$$\mathsf{G} \equiv \exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$$

is equivalent to

$$\mathsf{U}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \equiv \forall v_0 \neg \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1].$$

Now we show that G is true in $\mathcal{N}$ if and only if G is unprovable.

Theorem 4.10. $\mathcal{N} \models \mathsf{G}$ *if and only if* $\mathsf{PA} \nvdash \mathsf{G}$*.*

Proof. By definition the following holds:

$$\begin{aligned}
\mathcal{N} \models \mathsf{G} &\Leftrightarrow \mathcal{N} \models \exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U}) \\
&\Leftrightarrow \mathcal{N} \models \forall v_0 \neg \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \\
&\Leftrightarrow \text{for all } m \in \mathbb{N}, \mathcal{N} \models \neg \mathsf{Gdl}[\overline{m}, \overline{\ulcorner \mathsf{U} \urcorner}/v_0, v_1]
\end{aligned}$$

Since *Gdl* is arithmetically defined by Gdl,

$$\begin{aligned}\mathcal{N} \vDash \mathsf{G} &\Leftrightarrow \text{for all } m \in \mathbb{N},\ \neg Gdl(m, \ulcorner \mathsf{U} \urcorner) \\ &\Leftrightarrow \text{for all } m \in \mathbb{N},\ \neg Prf(m, sub(\ulcorner \mathsf{U} \urcorner)) \\ &\Leftrightarrow \text{for all } m \in \mathbb{N},\ \neg Prf(m, \ulcorner \mathsf{G} \urcorner) \\ &\Leftrightarrow \text{there exists no } \mathsf{PA}\text{-proof of } \mathsf{G} \\ &\Leftrightarrow \mathsf{PA} \nvdash \mathsf{G}.\end{aligned}$$

As a result $\mathcal{N} \vDash \mathsf{G}$ if and only if $\mathsf{PA} \nvdash \mathsf{G}$. □

Proposition 4.11. G *is a* Π_1*-sentence.*

Proof. $\mathsf{Gdl}(v_0, v_1)$ is a Σ_1-formula and so $\mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$ is a Σ_1-formula as well. Hence its negation $\neg\mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$ is a Π_1-formula and lastly $\forall v_0 \neg\mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$ is a Π_1-sentence. As a result G is equivalent to a Π-sentence. □

4.1 The Semantic Version

Theorem 4.12 (Semantic Version of Gödel's First Incompleteness Theorem for PA)**.** *If* PA *is arithmetically sound, then for the* Π_1*-sentence* G *which is true in* $\mathcal{N}$*,* $\mathsf{PA} \nvdash \mathsf{G}$ *and* $\mathsf{PA} \nvdash \neg\mathsf{G}$*. So* PA *is negation incomplete if* PA *is arithmetically sound.*

Proof. Suppose that the theory PA is arithmetically sound, i. e. PA proves no falsehoods of $\mathcal{N}$. If G could be proved in PA, i. e. $\mathsf{PA} \vdash \mathsf{G}$, then $\mathcal{N} \nvDash \mathsf{G}$ due to Theorem 4.10. Thus PA would prove a 'false' theorem, contradicting to PA being arithmetically sound. Hence G is not provable in PA and by Theorem 4.10 $\mathcal{N} \vDash \mathsf{G}$. Since $\mathcal{N} \vDash \mathsf{G}$ and PA being arithmetically sound $\neg\mathsf{G}$ cannot be proved in PA either. To sum up, G is an undecidable sentence of PA, i. e. $\mathsf{PA} \nvdash \mathsf{G}$ and $\mathsf{PA} \nvdash \neg\mathsf{G}$. □

Finally, we showed that PA is incomplete if PA is arithmetically sound. The following corollary is a result of latter theorem.

Corollary 4.13. PA *cannot prove all true sentences of arithmetic, i. e.* $C_\vdash(\mathsf{PA}) \neq Th(\mathcal{N})$ *if* PA *is arithmetically sound.*

We aim to show that $C_\vdash(T) = Th(\mathcal{N})$ for any effectively axiomatized arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ which satisfies specific conditions. In the following we show that primitive recursively axiomatizable theories coincide with effectively axiomatizable theories T. Then it remains to prove $C_\vdash(T) = Th(\mathcal{N})$ for any primitively recursively axiomatized theory.

Definition 4.14. Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory.

(i) T is *primitive recursively axiomatized* if the set $\ulcorner \Sigma \urcorner := \{\ulcorner \sigma \urcorner : \sigma \in \Sigma\}$ is primitive recursive. T is *primitive recursively axiomatizable* if there exists a primitive recursively axiomatized theory T' such that T is equivalent to T'.

(ii) T is *recursively axiomatized* if the set $\ulcorner\Sigma\urcorner := \{\ulcorner\sigma\urcorner : \sigma \in \Sigma\}$ is recursive. T is *recursively axiomatizable* if there exists a recursively axiomatized theory T' such that T is equivalent to T'.

(iii) T is *recursive enumerably axiomatized* if the set $\ulcorner\Sigma\urcorner := \{\ulcorner\sigma\urcorner : \sigma \in \Sigma\}$ is recursive enumerable. T is *recursive enumerably axiomatizable* if there exists a recursive enumerably axiomatized theory T' such that T is equivalent to T'.

By the Church-Turing Thesis r. e. sets coincide with enumerable sets, and recursive sets with decidable sets. Thus recursively enumerably axiomatized (axiomatizable) theories correspond to enumerably axiomatized (axiomatizable) theories, and recursive axiomatized (axiomatizable) theories to effectively axiomatized (axiomatizable) theories. In the following we demonstrate that primitive recursively axiomatizable theories in fact coincide with recursive enumerably axiomatized theories and hence in particular with recursively axiomatized theories. We assume that the reader is familiar with the following projection lemma from computability theory, as follows.

Lemma 4.15 (Projection Lemma)**.** *For any r. e. set $A \subseteq \mathbb{N}$, there exists a primitive recursive set $B \subseteq \mathbb{N}^2$ such that for all x,*

(i) $x \in A \Rightarrow \exists! y(x, y) \in B$,

(ii) $x \notin A \Rightarrow \nexists y(x, y) \in B$.

Theorem 4.16 (Craig's Theorem)**.** *Let $T = (\mathcal{L}_A, \Sigma)$ be any recursive enumerably axiomatized theory. Then there exists a primitive recursively axiomatizable theory $T' = (\mathcal{L}, \Sigma)$ such that $C_\vdash(T) = C_\vdash(T')$.*

Proof. Fix $T = (\mathcal{L}_A, \Sigma)$ such that $\ulcorner\Sigma\urcorner := \{\ulcorner\sigma\urcorner : \sigma \in \Sigma\}$ is r. e. By the projection lemma fix a p. r. B such that for all $x \in \mathbb{N}$,

$$x \in \ulcorner\Sigma\urcorner \Rightarrow \exists! y(x, y) \in B,$$
$$x \notin \ulcorner\Sigma\urcorner \Rightarrow \nexists y(x, y) \in B.$$

For any $\mathcal{L}_A$-sentence σ and any number $n \geq 1$ let

$$\sigma_n :\equiv \sigma \wedge \ldots \wedge \sigma$$

be the n-fold conjunction of σ ($\sigma_1 :\equiv \sigma, \sigma_{n+1} := (\sigma_n \wedge \sigma)$) and let $\Sigma' :\equiv \{\sigma_n : (\ulcorner\sigma\urcorner, n) \in B\}$. Then Σ' is primitive recursive by the closure properties of PRIM. By definition of Σ' and choice of B

$$\sigma \in \Sigma \Leftrightarrow \exists n((\ulcorner\sigma\urcorner, n) \in B) \Leftrightarrow \exists n(\sigma_n \in \Sigma').$$

Hence for any $\sigma \in \Sigma$ there is an $\sigma' \in \Sigma'$ s. t. σ equivalent to σ'. Conversely if $\sigma' \in \Sigma$ then $\sigma' \equiv \sigma_n$ for some $\sigma \in \Sigma$. $\square$

Likewise to PA we define the 'proof'-relations in a primitive recursively axiomatized theory.

Definition 4.17. Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory. Define the following relations:

(i) Let $Prfseq_T(n)$ be a relation which holds if n is the s. g. n. of a T-proof of an $\mathcal{L}_A$-formula.

(ii) Let $Prf_T(m, n)$ be a relation which holds when m is the s. g. n. of a T-proof of the $\mathcal{L}_A$-sentence with g. n. n.

(iii) Let $Gdl_T(m, n)$ be a relation which holds when m is the s. g. n. of a T-proof of $sub(n)$.

Analogously to PA it can be shown that the 'proof'-relations are primitive recursive.

Theorem 4.18. *Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory. The following relations are primitive recursive:*

(1) The relation $Prfseq_T(n)$.

(2) The relation $Prf_T(m, n)$.

(3) The relation $Gdl_T(m, n)$.

Proof Sketch. (1) In order to show that $Prfseq_T(n)$ is primitive recursive, it suffices to show that the following relations are primitive recursive:

(i) For $1 \leq i \leq 6$, $A_i(n)$ holds if n is the g. n. of an $\mathcal{S}$-axiom (Ai).

(ii) For $1 \in \{1, 2, 3, 5\}$, $R_i(n, c)$ holds if c is the g. n. of the formula with g. n. n which follows from the premise n (Ri). $R_4(m, n, c)$ holds if c is the g. n. of the formula with g. n. n which follows from the premises m and n (Ri).

(iii) $TAxiom(n)$ holds if n is the g. n. of an axiom of T.

(i) and (ii) are similar to the proof of Theorem 3.10. For (iii), since $T = (\mathcal{L}_A, \Sigma)$ is a primitive recursively axiomatized theory, Σ is primitive recursive and therefore $TAxiom(n)$ is primitive recursive as well. Let

$$Axiom(n) := A_1(n) \vee \ldots \vee A_6(n) \vee TAxiom(n)$$

and

$$Rule(n, o) := R_1(n, o) \vee R_2(n, o) \vee R_3(n, o) \vee R_5(n, o).$$

Then $Prfseq$ is primitive recursive since

$$\begin{aligned} Prfseq(n) := (\forall x < len(n))\,\{&Axiom(exf(n, x)) \vee \\ &(\exists y, z < x) R_4(exf(n, y), exf(n, z), exf(n, x)) \vee \\ &(\exists y < k) Rule(exf(n, y), exf(n, x))\}. \end{aligned}$$

(2) Analogously to PA, the sentence σ with g. n. n is the last component in the T-proof of σ so let

$$Prf_T(m,n) := Prfseq_T(n) \land (exf(m, len(m) - 1) = n) \land Sent(n).$$

(3) Let $Gdl_T(m,n) :\Leftrightarrow Prf(m, sub(n))$. Therefore Gdl_T is primitive recursive.

□

Definition 4.19. Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory.

(i) Let $\mathsf{Prf}_T(v_0, v_1)$ be the Σ_1-formula s. t. $Prf_T(m,n)$ is arithmetically defined and defined in PA by $\mathsf{Prf}_T(v_0, v_1)$.

(ii) Let $\mathsf{Gdl}_T(v_0, v_1) :\equiv \exists v_2(\mathsf{Prf}[v_2/v_1] \land \mathsf{Sub}(v_1, v_2))$ be an $\mathcal{L}_A$-formula.

Evidently, Gdl_T is a Σ_1-formula since Prf_T and Sub are Σ_1-formulas. Moreover, $\mathsf{Gdl}_T(m,n)$ is arithmetically defined and defined in T by $\mathsf{Gdl}_T(v_0, v_1)$.

Definition 4.20. Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory and let $\mathsf{U}_T(v_1) :\equiv \forall v_0 \neg \mathsf{Gdl}_T(v_0, v_1)$ be an $\mathcal{L}_A$-formula.

Again, by the indirect substitution of the g. n. of U_T for the free variable v_1 in U_T, the desired formula G_T is constructed.

Definition 4.21. Let $\mathsf{G}_T :\equiv \exists v_1(v_1 = \overline{\ulcorner \mathsf{U}_T \urcorner} \land \mathsf{U}_T)$ an $\mathcal{L}_A$-sentence. We call G_T the *Gödel sentence of T*.

Evidently G_T is equivalent to $\mathsf{U}_T[\overline{\ulcorner \mathsf{U}_T \urcorner}/v_1]$ and $\forall v_0 \neg \mathsf{Gdl}_T[\overline{\ulcorner \mathsf{U}_T \urcorner}/v_1]$.

Theorem 4.22 (Generalized Semantic Version of Gödel's First Theorem)**.** *Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory. If T is arithmetically sound, then for the Π_1-sentence G_T which is true in $\mathcal{N}$, $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg \mathsf{G}_T$. As a result T is negation incomplete if T is arithmetically sound.*

Proof. This version can be proved in the same way as Theorem 4.12. □

Latter theorem and Craig's Theorem imply the following.

Corollary 4.23. *For any arithmetical recursively axiomatized theory $T = (\mathcal{L}_A, \Sigma)$ which is arithmetically sound, T cannot prove all true sentences of arithmetic, i. e. $C_\vdash(T) \neq Th(\mathcal{N})$.*

4.2 The Syntactic Version

Before dealing with the syntactic version of the First Incompleteness Theorem, two key notions are defined, in order to downgrade the semantic assumption of dealing with an arithmetically sound theory. Instead, the weaker assumption that the theory is consistent and ω-consistent is employed.

Definition 4.24. An arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ is *ω-incomplete* if and only if for some formula $\varphi(x)$, $T \vdash \varphi[\overline{n}/x]$ for each natural number n but $T \nvdash \forall x \varphi(x)$. A theory which is not ω-incomplete is *ω-complete.*

Definition 4.25. An arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ is *ω-inconsistent* if and only if for some formula $\varphi(x)$, $T \vdash \varphi[\overline{n}/x]$ and $T \vdash \neg\forall x \varphi(x)$. A theory which is not ω-inconsistent is *ω-consistent.*

Proposition 4.26. *If an arithmetical theory $T = (\mathcal{L}_A, \Sigma)$ theory T is ω-consistent, then T is consistent.*

Proof. For a proof via contraposition, assume that $T = (\mathcal{L}_A, \Sigma)$ is inconsistent. Then T can derive every formula. In particular, $T \vdash \exists x \varphi(x)$ and for each natural number n, $T \vdash \varphi[\overline{n}/x]$, for an $\mathcal{L}_A$-formula $\varphi(x)$. Hence T is ω-inconsistent. □

So far for the semantic version of the First Incompleteness Theorem only the result that the relation *Gdl* is arithmetically defined by Gdl was utilized. In fact *Gdl* is also defined in PA by Gdl. By using this fact without the semantic assumption that PA is sound, we show that 'PA does not prove G' if PA is consistent. By additionally assuming that PA is ω-consistent, we obtain the result that 'PA does not prove $\neg$G'.

Theorem 4.27. *If* PA *is consistent,* $\mathsf{PA} \nvdash \mathsf{G}$.

Proof. For a contradiction suppose that G is PA-provable. Then

$$\mathsf{PA} \vdash \exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$$

and there exists a s. g. n. n that codes for the proof of $\exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$. Then $Gdl(n, \ulcorner \mathsf{U} \urcorner)$ holds since $sub(\ulcorner \mathsf{U} \urcorner)$ is the g. n. of $\exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$. In fact *Gdl* is defined in PA by Gdl so

$$\mathsf{PA} \vdash \mathsf{Gdl}(\overline{n}, \overline{\ulcorner \mathsf{U} \urcorner}).$$

Since $\mathsf{PA} \vdash \mathsf{G}$ and by the Completeness and Soundness Theorem,

$$\mathsf{PA} \models \mathsf{G} \Leftrightarrow \mathsf{PA} \models \exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U}) \Leftrightarrow \mathsf{PA} \models \mathsf{U}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \Leftrightarrow \mathsf{PA} \vdash \forall v_0 \neg \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1].$$

Hence $\mathsf{PA} \vdash \forall v_0 \neg \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$ and since the universal quantification entails every instance, for $n \in \mathbb{N}$,

$$\mathsf{PA} \vdash \neg \mathsf{Gdl}(\overline{n}, \overline{\ulcorner \mathsf{U} \urcorner}).$$

This contradicts to PA being consistent. □

Last theorem deduces the following corollary:

Corollary 4.28. *If* PA *is consistent, then* PA *is ω-incomplete.*

Proof. Suppose PA is consistent. Then by Theorem 4.27, $\mathsf{PA} \nvdash \mathsf{G}$, i. e.

$$\mathsf{PA} \nvdash \forall v_0 \neg \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1].$$

Since G is unprovable, no number is the s. g. n. of a proof of G. Equivalently, no number is the s. g. n. of a proof of $\exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$. Then, for every $n \in \mathbb{N}$, $Gdl(n, \ulcorner \mathsf{U} \urcorner)$ does not hold. Since Gdl is defined in PA by Gdl,

$$\text{for every } n \in \mathbb{N}, \mathsf{PA} \vdash \mathsf{Gdl}[\overline{n}, \overline{\ulcorner \mathsf{U} \urcorner}/v_0, v_1].$$

Thus PA is ω-incomplete. $\square$

Instead of arithmetical soundness, we now require ω-consistency to show that $\mathsf{PA} \nvdash \mathsf{G}$.

Theorem 4.29. *If* PA *is ω-consistent, then* $\mathsf{PA} \nvdash \neg\mathsf{G}$*.*

Proof. For a contradiction suppose that $\neg\mathsf{G}$ is provable in PA and PA is ω-consistent. That is equivalent to

$$\mathsf{PA} \vdash \exists v_0 \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$$

because $\neg\mathsf{G} \equiv \neg\forall v_0 \neg\mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \equiv \neg\neg\exists v_0 \neg\neg\mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \equiv \exists v_0 \mathsf{Gdl}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1]$ holds. Proposition 4.26 infers that PA is consistent since PA is ω-consistent. Then G is not provable because $\neg\mathsf{G}$ is provable and PA is consistent. So for every natural number n, n is not a s. g. n. for a PA-proof of G. Equivalently, for every natural number n, n is not the s. g. n. of a PA-proof of $\exists v_1(v_1 = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U})$. Thus for each natural number n, $Gdl(n, \ulcorner \mathsf{U} \urcorner)$ does not hold. Since Gdl is defined in PA by Gdl,

$$\text{for every } n \in \mathbb{N},\ \mathsf{PA} \vdash \neg\mathsf{Gdl}[\overline{n}, \overline{\ulcorner \mathsf{U} \urcorner}/v_0, v_1].$$

However, this contradicts to PA being ω-consistent. As a result $\neg\mathsf{G}$ is unprovable, i. e. $\mathsf{PA} \nvdash \mathsf{G}$. $\square$

On the basis of Theorem 4.27 and Theorem 4.29 we formulate the syntactic version of the First Incompleteness Theorem for PA .

Theorem 4.30 (Syntactic Version of Gödel's First Incompleteness Theorem for PA)**.** *If* PA *is consistent, then for the* Π_1*-sentence* G*,* $\mathsf{PA} \nvdash \mathsf{G}$ *and if* PA *is ω-consistent* $\mathsf{PA} \nvdash \neg\mathsf{G}$*. So T is negation incomplete if* PA *is ω-consistent.*

Again, we are going to state a generalized version of Theorem 4.30 which can be proven similar to PA.

Theorem 4.31 (Generalized Syntactic Version of Gödel's First Theorem)**.** *If $T = (\mathcal{L}_A, \Sigma)$ is a consistent, p. r. axiomatized theory which extends* Q*, then for a* Π_1*-sentence* G_T *such that $T \nvdash \mathsf{G}_T$ and if T is ω-consistent $T \nvdash \neg\mathsf{G}_T$. As a result T is negation incomplete if T is ω-consistent.*

Craig's Theorem and latter infer the following.

Corollary 4.32. *For any arithmetical recursively axiomatized theory $T = (\mathcal{L}_A, \Sigma)$ which is ω-consistent and extends* Q*, T cannot prove all true sentences of arithmetic, i. e. $C_{\vdash}(T) \neq Th(\mathcal{N})$.*

5 Second Incompleteness Theorem

Before proving the Second Incompleteness Theorem, an essential statement, the so-called Formalized First Theorem, is required. Therefore, the absurdity constant '$\perp$' is introduced and as the name suggests, PA is consistent if and only if $\mathsf{PA} \nvdash \perp$. With the help of the absurdity constant, the Π_1-sentence Con is defined which is constructed so that Con is true if and only if PA is consistent.

5.1 The Formalized First Theorem

Definition 5.1. $\perp$ is an abbreviation for the $\mathcal{L}_A$-formula $\perp :\equiv \overline{0} = \overline{1}$. We call $\perp$ the *absurdity constant*.

PA and Q of course prove $\overline{0} \neq \overline{1}$. In fact, if PA and Q prove $\perp$, then both theories are inconsistent. Evidently, there are further possibilities to define the absurdity constant.

Definition 5.2. Let $Prov(n) :\Leftrightarrow \exists x Prf(x, n)$ be a relation which holds when the $\mathcal{L}_A$-sentence σ with g. n. n is a theorem in PA, i. e. $\mathsf{PA} \vdash \sigma$.

Recall that $Prf(m, n)$ is arithmetically defined and defined in PA by the Σ_1-formula $\mathsf{Prf}(v_0, v_1)$.

Definition 5.3. Let $\mathsf{Prov}(v_1) :\equiv \exists v_0 \mathsf{Prf}(v_0, v_1)$ in PA be an $\mathcal{L}_A$-formula.

Theorem 5.4. *$Prov(n)$ is arithmetically defined by $\mathsf{Prov}(v_1)$.*

Proof. If $Prov(n)$ holds, then by definition $\exists x Prf(x, n)$ holds and for some m, $Prf(m, n)$ holds. Since Prf is arithmetically defined by Prf, $\mathcal{N} \vDash \mathsf{Prf}[\overline{m}, \overline{n}/v_0, v_1]$ follows. Hence $\mathcal{N} \vDash \exists v_0 \mathsf{Prf}[\overline{n}/v_1]$, i. e. $\mathcal{N} \vDash \mathsf{Prov}[\overline{n}/v_1]$.

If $Prov(n)$ does not hold, then for all $m \in \mathbb{N}$, $Prf(m, n)$ does not hold. Since Prf is arithmetically defined by Prf for all $m \in \mathbb{N}$, $\mathcal{N} \vDash \neg\mathsf{Prf}[\overline{m}, \overline{n}/v_0, v_1]$. Thus $\mathcal{N} \vDash \forall v_0 \neg\mathsf{Prf}[\overline{n}/v_1]$, i. e. $\mathcal{N} \vDash \neg\mathsf{Prov}[\overline{n}/v_1]$.

Therefore, $Prov(n)$ is arithmetically defined by $\mathsf{Prov}(v_1)$. $\square$

It can be shown that $Prov$ is not definable in PA by Prov. (For a proof the reader may refer to the book of Smith [3], p. 183.) Since $Prov(n)$ is arithmetically defined by $\mathsf{Prov}(v_1)$, $\mathcal{N} \vDash \mathsf{Prov}[\overline{\ulcorner \varphi \urcorner}/v_1]$ if and only if $Prov(\ulcorner \varphi \urcorner)$ holds, i. e. $\mathsf{PA} \vdash \varphi$. Then the sentence $\neg\mathsf{Prov}(\overline{\ulcorner \perp \urcorner})$ is true in $\mathcal{N}$ if and only if PA does not prove $\perp$. This is equivalent to PA being consistent and therefore motivates the following definition.

Definition 5.5. Let $\mathsf{Con} :\equiv \neg\mathsf{Prov}[\overline{\ulcorner \perp \urcorner}/v_1]$ be an $\mathcal{L}_A$-formula. Con is called the *consistency sentence*.

Indeed, there are natural alternatives of consistency sentences. On modest assumptions, those formulas will be equivalent to each other.

To proof the Formalized First Theorem which states $\mathsf{PA} \vdash \mathsf{Con} \rightarrow \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$, Hilbert and Bernays isolated three conditions on Prov, the so-called *Hilbert-Bernays-Löb conditions* (*HBL*).

Theorem 5.6 (Hilbert-Bernays-Löb conditions for PA)**.** *For any $\mathcal{L}_A$-sentence σ, τ, the following holds*

(HBL1) *if* $\mathsf{PA} \vdash \sigma$, *then* $\mathsf{PA} \vdash \mathsf{Prov}[\overline{\ulcorner \sigma \urcorner}/v_1]$,

(HBL2) $\mathsf{PA} \vdash \mathsf{Prov}[\overline{\ulcorner \sigma \rightarrow \tau \urcorner}/v_1] \rightarrow (\mathsf{Prov}[\overline{\ulcorner \sigma \urcorner}/v_1] \rightarrow \mathsf{Prov}[\overline{\ulcorner \tau \urcorner}/v_1])$,

(HBL3) $\mathsf{PA} \vdash \mathsf{Prov}[\overline{\ulcorner \sigma \urcorner}/v_1] \rightarrow \mathsf{Prov}[\overline{\ulcorner \mathsf{Prov}[\overline{\ulcorner \sigma \urcorner}/v_1] \urcorner}/v_1]$.

The proof of the conditions (ii) and (iii) of latter theorem is tiresome. Nevertheless, the reader is encouraged to approach the book of Boolos ([4], p.44-49) or alternatively the book of Rautenberg ([5], p.269-284).

To improve readability, we abbreviate $\mathsf{Prov}[\overline{\ulcorner \sigma \urcorner}/v_1]$ with $\Box[\sigma]$. (Note that the abbreviation $\Box[\sigma]$ has several tasks. Firstly, we omit the corner quotes $\ulcorner\ \urcorner$ of $\ulcorner \sigma \urcorner$. Secondly, we omit $\overline{\quad}$ of $\overline{\ulcorner \sigma \urcorner}$. And thirdly, it is clear from context that we substitute the variable v_1, hence we omit $/v_1$ of $[\overline{\ulcorner \sigma \urcorner}/v_1]$.)

Then the abbreviated version of the Hilbert-Bernays-Löb conditions for PA is the following:

(HBL1) if $\mathsf{PA} \vdash \sigma$, then $\mathsf{PA} \vdash \Box[\sigma]$,

(HBL2) $\mathsf{PA} \vdash \Box[\sigma \rightarrow \tau] \rightarrow (\Box[\sigma] \rightarrow \Box[\tau])$,

(HBL3) $\mathsf{PA} \vdash \Box[\sigma] \rightarrow \Box[\Box[\sigma]]$.

Lemma 5.7. $\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \neg\Box[\mathsf{G}]$.

Proof. Firstly, rearrange $\mathsf{U}(v_1)$ by elementary logical manipulations:

$$
\begin{aligned}
\mathsf{U}(v_1) &\equiv \forall v_0 \neg \mathsf{Gdl}(v_0, v_1) && \text{definition of } \mathsf{U} \\
&\equiv \forall v_0 \neg \exists v_2 (\mathsf{Prf}[v_2/v_1] \wedge \mathsf{Sub}(v_1, v_2)) && \text{definition of } \mathsf{Gdl} \\
&\equiv \forall v_2 \forall v_0 \neg (\mathsf{Prf}[v_2/v_1] \wedge \mathsf{Sub}(v_1, v_2)) && \neg\exists v_2 \varphi \equiv \forall v_2 \neg \varphi \\
&\equiv \forall v_2 \forall v_0 (\neg \mathsf{Prf}[v_2/v_1] \vee \neg \mathsf{Sub}(v_1, v_2)) && \text{De Morgan's Law} \\
&\equiv \forall v_2 (\forall v_0 \neg \mathsf{Prf}[v_2/v_1] \vee \neg \mathsf{Sub}(v_1, v_2)) && \forall v_0 (\varphi(v_0) \vee \psi) \equiv (\forall v_0 \varphi(v_0) \vee \psi) \\
&\equiv \forall v_2 (\neg \exists v_0 \mathsf{Prf}[v_2/v_1] \vee \neg \mathsf{Sub}(v_1, v_2)) && \forall v_0 \neg \varphi \equiv \neg \exists v_0 \varphi \\
&\equiv \forall v_2 (\mathsf{Sub}(v_1, v_2) \rightarrow \neg \exists v_0 \mathsf{Prf}[v_2/v_1]) && (\psi \wedge \neg \varphi) \equiv (\varphi \rightarrow \psi) \\
&\equiv \forall v_2 (\mathsf{Sub}(v_1, v_2) \rightarrow \neg \mathsf{Prov}[v_2/v_1]) && \text{definition of } \mathsf{Prov}
\end{aligned}
$$

By definition G is the indirect substitution of v_1 for $\overline{\ulcorner \mathsf{U} \urcorner}$ in U and utilizing the Completeness Theorem,

$$\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \mathsf{U}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \tag{1}$$

holds. Above we showed that $\mathsf{U}(v_1) \equiv \forall v_2(\mathsf{Sub}(v_1, v_2) \to \neg\mathsf{Prov}[v_2/v_1])$ and by substituting $\overline{\ulcorner \mathsf{U} \urcorner}$ for v_1, we obtain

$$\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \forall v_2(\mathsf{Sub}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \to \neg\mathsf{Prov}[v_2/v_1]). \tag{2}$$

$Sub(\ulcorner \mathsf{U} \urcorner) = \ulcorner \mathsf{G} \urcorner$ holds and since Sub is defined in PA by Sub, this infers

$$\mathsf{PA} \vdash \forall v_2(\mathsf{Sub}[\overline{\ulcorner \mathsf{U} \urcorner}/v_1] \leftrightarrow v_2 = \overline{\ulcorner \mathsf{G} \urcorner}). \tag{3}$$

(2) and (3) imply

$$\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \forall v_2(v_2 = \overline{\ulcorner \mathsf{G} \urcorner} \to \neg\mathsf{Prov}[v_2/v_1]). \tag{4}$$

Since the right-hand side of the biconditional is equivalent to $\neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}]$,

$$\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]. \tag{5}$$

$\square$

As a result the Formalized First Theorem can be derived.

Theorem 5.8 (Formalized First Theorem in PA)**.** $\mathsf{PA} \vdash \mathsf{Con} \to \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$.

Proof. Elementary logic infers, for any formula φ,

$$\mathsf{PA} \models \neg\varphi \to (\varphi \to \bot)$$

and by the Completeness and Soundness Theorem

$$\mathsf{PA} \vdash \neg\varphi \to (\varphi \to \bot).$$

Latter and HBL(i) implies:

$$\mathsf{PA} \vdash \Box[\neg\varphi \to (\varphi \to \bot)].$$

Using HBL(ii),

$$\mathsf{PA} \vdash \Box[\neg\varphi] \to \Box[\varphi \to \bot] \tag{6}$$

holds.

Note that in the following we sometimes skip minor steps and use the Completeness and Soundness Theorem without explicitly stating this. Then the argumentation continues as follows:

1.	$\mathsf{PA} \vdash \mathsf{G} \to \neg\Box[\mathsf{G}]$	Lemma 5.7
2.	$\mathsf{PA} \vdash \Box[\mathsf{G} \to \neg\Box[\mathsf{G}]]$	from 1 using HBL(i)
3.	$\mathsf{PA} \vdash \Box[\mathsf{G}] \to \Box[\neg\Box[\mathsf{G}]]$	from 2, using HBL(ii)
4.	$\mathsf{PA} \vdash \Box[\neg\Box[\mathsf{G}]] \to \Box[\Box[\mathsf{G}] \to \bot]$	instance of (6) with φ as $\Box[\mathsf{G}]$
5.	$\mathsf{PA} \vdash \Box[\mathsf{G}] \to \Box[\Box[\mathsf{G}] \to \bot]$	from 3 and 4
6.	$\mathsf{PA} \vdash \Box[\mathsf{G}] \to (\Box[\Box[\mathsf{G}]] \to \Box[\bot])$	from 5, using HBL(ii)
7.	$\mathsf{PA} \vdash \Box[\mathsf{G}] \to \Box[\Box[\mathsf{G}]]$	instance of HBL(iii)
8.	$\mathsf{PA} \vdash \Box[\mathsf{G}] \to \Box[\bot]$	from 6 and 7
9.	$\mathsf{PA} \vdash \neg\Box[\bot] \to \neg\Box[\mathsf{G}]$	contraposition of 8
10.	$\mathsf{PA} \vdash \mathsf{Con} \to \neg\Box[\mathsf{G}]$	definition of Con

$\square$

5.2 The Second Theorem and Some Results

Theorem 5.9 (Gödel's Second Incompleteness Theorem for PA)**.** *If* PA *is consistent,* $\mathsf{PA} \nvdash \mathsf{Con}$.

Proof. Suppose $\mathsf{PA} \vdash \mathsf{Con}$ for a contradiction. Given the Formalized First Theorem, i. e. $\mathsf{PA} \vdash \mathsf{Con} \rightarrow \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$, this yields $\mathsf{PA} \vdash \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$. By Lemma 5.7 which states $\mathsf{PA} \vdash \mathsf{G} \leftrightarrow \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$ and due to $\mathsf{PA} \vdash \neg\mathsf{Prov}[\overline{\ulcorner \mathsf{G} \urcorner}/v_1]$, it follows that $\mathsf{PA} \vdash \mathsf{G}$. However, this contradicts to the First Incompleteness Theorem (see Theorem 4.30), assuming PA is consistent. □

Suppose that PA is an arithmetically sound theory and thus all its theorems are true, then in particular PA is consistent. Hence Con will be another true in $\mathcal{N}$ but unprovable sentence.

In the following, we show that PA is ω-incomplete if PA is consistent. Moreover, if PA is ω-consistent, PA cannot prove $\neg$Con. As a result Con is another undecidable sentence in PA, i. e. $\mathsf{PA} \nvdash \mathsf{Con}$ and $\mathsf{PA} \nvdash \neg\mathsf{Con}$ if ω-consistency of PA is assumed.

Corollary 5.10. *If* PA *is consistent, then* PA *is ω-incomplete.*

Proof. Assume PA is consistent which asserts that $\perp$ is not a theorem. Therefore, there is no number which is the s. g. n. of a proof of $\perp$, i. e. for all n, $Prf(n, \ulcorner\perp\urcorner)$ does not hold. Since Prf is defined in PA by Prf,

$$\text{for every } n,\ \mathsf{PA} \vdash \neg\mathsf{Prf}[\overline{n}, \overline{\ulcorner\perp\urcorner}/v_0, v_1] \tag{1}$$

holds. Since Con is unprovable in PA, unpacking the abbreviation

$$\mathsf{PA} \nvdash \forall v_0 \neg\mathsf{Prf}[\overline{\ulcorner\perp\urcorner}/v_1] \tag{2}$$

holds and hence by (1) and (2) PA is ω-incomplete. □

Corollary 5.11. *If* PA *is ω-consistent, then* $\mathsf{PA} \nvdash \neg\mathsf{Con}$.

Proof. For a contradiction suppose PA is ω-consistent and $\mathsf{PA} \vdash \neg\mathsf{Con}$, i. e.

$$\mathsf{PA} \vdash \exists v_0 \mathsf{Prf}[\overline{\ulcorner\perp\urcorner}/v_1]. \tag{3}$$

Since ω-consistency implies plain consistency, by the same reasoning as in the proof of Corollary 5.10,

$$\text{for every } n,\ \mathsf{PA} \vdash \neg\mathsf{Prf}[\overline{n}, \overline{\ulcorner\perp\urcorner}/v_0, v_1]. \tag{4}$$

However, (3) and (4) contradict to PA being ω-consistent and hence $\mathsf{PA} \nvdash \neg\mathsf{Con}$. □

The Second Incompleteness Theorem can be generalized for arithmetical theories $T = (\mathcal{L}_A, \Sigma)$. We define Prov_T and Con_T analogously to the case of PA.

Definition 5.12. Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory and let $Prov_T(n) :\Leftrightarrow \exists x Prf_T(x, n)$ be a relation which holds when the $\mathcal{L}_A$-sentence σ with g. n. n is a theorem in T, i. e. $T \vdash \sigma$.

Definition 5.13. Let $T = (\mathcal{L}_A, \Sigma)$ be a primitive recursively axiomatized theory.

(i) Let $\mathsf{Prov}_T(y) :\equiv \exists v_0 \mathsf{Prf}_T(v_0, y)$ be an $\mathcal{L}_A$-formula.

(ii) Let $\mathsf{Con_T} :\equiv \neg\mathsf{Prov}_T[\overline{\ulcorner \bot \urcorner}/v_1]$ be an $\mathcal{L}_A$-formula.

Note that $Prov_T(n)$ is arithmetically defined by Prov_T. We also abbreviate $\mathsf{Prov}_T(\varphi)$ by $\Box_T[\varphi]$ for an $\mathcal{L}_A$-formula φ.

Similar to PA, it can be shown that the Formalized First Theorem for T holds, i. e. $T \vdash \mathsf{Con}_T \to \neg\mathsf{Prov}_T(\overline{\ulcorner \mathsf{G}_T \urcorner})$ if the Hilbert-Bernays-Löb conditions hold for T.

Definition 5.14 (Hilbert-Bernays-Löb conditions)**.** Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory which extends Q and is primitive recursively axiomatized. T satisfies the *Hilbert-Bernays-Löb conditions* (*HBL conditions*) if for any $\mathcal{L}$-sentence σ, τ, the following conditions hold.

(i) if $T \vdash \sigma$, then $T \vdash \Box_T[\sigma]$,

(ii) $T \vdash \Box_T[\sigma \to \tau] \to (\Box_T[\sigma] \to \Box_T[\tau])$,

(iii) $T \vdash \Box_T[\sigma] \to \Box_T[\Box_T[\sigma]]$.

Theorem 5.15 (Generalized Formalized First Theorem)**.** *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory which extends Q and is primitive recursively axiomatized. Moreover, assume that T satisfies the Hilbert-Bernays-Löb conditions. Then*

$$T \vdash \mathsf{Con}_T \to \neg\mathsf{Prov}_T(\overline{\ulcorner \mathsf{G}_T \urcorner}).$$

Theorem 5.16 (Generalized Version of Gödel's Second Incompleteness Theorem)**.** *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical theory which extends Q and is primitive recursively axiomatized. Moreover, assume that T satisfies the Hilbert-Bernays-Löb conditions. Then $T \nvdash \mathsf{Con}_T$.*

6 Chaitin's Incompleteness Theorems

Certain variants of incompleteness results have received considerable attention, including the one by the American computer scientist Gregory Chaitin. Chaitin's results emerge from algorithmic complexity also known as Kolmogorov complexity. The Kolmogorov complexity of a string refers to the length of the shortest program which generates the string and stops. From the beginning it was known that Kolmogorov complexity is undecideable. However, Chaitin noticed that there is a number c such that in any consistent arithmetical theory one cannot prove that any particular string has a Kolmogorov complexity larger than c. In this chapter we are going to prove this result.

6.1 Facts from Computability Theory

We remind the reader of important facts from computability theory.

Theorem 6.1 (Existence of Standard Enumerations of the Partial Recursive Functions)**.** *There is a partial recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that the following holds:*

(i) For any partial recursive function $\psi : \mathbb{N} \to \mathbb{N}$, there is a number e such that $\psi = \varphi_e$ (where $\varphi_e = \lambda x.\varphi(e, x)$ is the e-th branch of φ. We call e the index of φ and φ a standard enumeration).

In the following we fix such a φ.

The following theorem follows from the well-known *s-m-n*-Theorem.

Theorem 6.2 (Translation Functions)**.** *For any partial recursive function $\psi(e, x)$ there is a primitive recursive function f such that for all $e \geq 0$, $\psi_e = \varphi_{f(e)}$. (The function f is called a translation function.)*

Theorem 6.3 (Recursion Theorem)**.** *For any total recursive function $f : \mathbb{N} \to \mathbb{N}$ there is an index e such that $\varphi_{f(e)} = \varphi_e$.*

6.2 Kolmogorov Complexity

Kolmogorov complexity measures the complexity of finite objects from a descriptive point of view. Therefore temporal or spatial properties of algorithms which return the desired object, can be ignored, rather the shortest length of such an algorithm is important. For instance, consider the following binary strings:

010101010101010101010101010101010101010101010101010101010101010101

100011111011010110001110000000100110111111101111101010000000000000

The first string can be simply described by a sequence of 33 couples of 01. Although the second one seems random, it is the beginning of the binary expansion of the decimal part of $\sqrt{2}$. Hence both of the strings have a simple description. Intuitively, complexity of a string x is low if there is a simple rule that describes it. Therefore Kolmogorov complexity of x can be defined as the minimum length of a program that prints x as output and stops. Based on Berry's paradox, a conceptually simpler proof of the First Incompleteness Theorem was given by Chaitin. Berry's paradox depicts the expression 'the smallest positive integer not definable in under eleven words' whereas this expression defines that integer in under eleven words. To formalize Berry's pradox, Chaitin uses the notion of Kolmogorov complexity. In the following we depict Kolmogorov complexity on the natural numbers. (For further reading, we recommend the book of Li and Vitanyi [6].)

Then the Kolmogorov complexity of natural numbers w. r. t. φ is defined as follows.

Definition 6.4. The *Kolmogorov complexity* $K(x)$ of $x \in \mathbb{N}$ is the least index e such that $\varphi_e(0) = x$:

$$K(x) = \mu e(\varphi_e(0) = x).$$

Intuitively, the index e can be viewed as the length of the description. The smaller it is, the less is the length of the description of $\varphi_e(0)$.

Proposition 6.5. *The following facts hold for K.*

(i) K is total.

(ii) For any $e \in \mathbb{N}$, $\{x : K(x) \leq e\}$ is finite.

(iii) There is no recursive function f which satisfies the inequality $K(f(m)) > m$ for all $m \in \mathbb{N}$.

(iv) K is not recursive but $C := \{(e, x) : K(x) \leq e\}$ is r. e.

Proof. (i) This holds since for any fixed x, there exists a constant function $\psi(y) = x$ for all y (which is in particular a partial recursive function) and since φ is a standard enumeration there exists an index e such that $\varphi_e = \psi$. Thus $\varphi_e(0) = x$.

(ii) For any $e \in \mathbb{N}$, $\{\varphi_0(0), \varphi_1(0), \ldots, \varphi_e(0)\}$ is finite and hence $\{x : K(x) \leq e\}$ is finite.

(iii) Assume there is such a recursive function f satisfying $K(f(m)) > m$ for all $m \in \mathbb{N}$. Then by Theorem 6.2 for the (partial) recursive function $\psi(e, x) = f(e)$ there exists a primitive recursive translation function g such that for all $e \geq 0$, $\psi_e = \varphi_{g(e)}$. By the Recursion Theorem there exists an index e such that $\varphi_{g(e)} = \varphi_e = f(e)$. Hence $K(f(e)) \leq e$ which is a contradiction.

(iv) For a contradiction assume that K is recursive. Then $f(x) = \mu y(K(y) > x)$ which satisfies $K(f(x)) > x$ for all x, would be recursive. This is a contradiction to (iii).

To show that $C := \{(e,x) : K(x) \leq e\}$ is r. e. we utilize the Church-Turing Thesis and instead prove that C is enumerable. There is an algorithm $\mathfrak{A}$ which enumerates C, as follows.

1. Set $t = 0$.
2. For all $k \leq t$.
   2.1 Run $\varphi_k(0)$ t time-steps. If $\varphi_k(0)$ stops then output $\varphi_k(0)$.
3. $t = t + 1$.
4. Go to 2.

□

We already showed that primitive recursive relations are arithmetically defined and defined in PA by a Σ_1-formula. In the following we depict r. e. relations regarding their definability properties.

Lemma 6.6 (Definability-Lemma)**.** *For any r. e. relation $R \subseteq \mathbb{N}^n$, there exists a Σ_1-formula $\varphi(x_0, \dots, x_{n-1})$ such that*

(i) $(m_0, \dots, m_{n-1}) \in R \Leftrightarrow \mathsf{PA} \vdash \varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$,

(ii) $(m_0, \dots, m_{n-1}) \notin R \Rightarrow \mathcal{N} \models \neg\varphi[\overline{m_0}, \dots, \overline{m_{n-1}}/x_0, \dots, x_{n-1}]$.

Latter lemma can be proven analogously to the definability results in Chapter 2.

By Proposition 6.5(iv) and Lemma 6.6 there is a Σ_1-formula κ for the recursive enumerable relation C such that the following holds.

Definition 6.7. Let $\kappa(v_0, v_1)$ be a Σ_1-formula such that for any $n, e \in \mathbb{N}$,

(i) $K(n) \leq e \Leftrightarrow \mathsf{PA} \vdash \kappa[\overline{n}, \overline{e}/v_0, v_1]$ and

(ii) $K(n) > e \Rightarrow \mathcal{N} \models \neg\kappa[\overline{n}, \overline{e}/v_0, v_1]$.

Theorem 6.8 (Chaitin's Incompleteness Theorem [7])**.** *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical recursively axiomatized theory which extends* PA*. Then there exists a constant $c_T \in \mathbb{N}$ such that for any $e \geq c_T$ and any $n \in \mathbb{N}$, $T \nvdash \neg\kappa[\overline{n}, \overline{e}/v_0, v_1]$.*

Proof. For a contradiction assume that $T = (\mathcal{L}_A, \Sigma)$ is an arithmetical theory which extends PA such that Σ is recursive and

$$\text{for all } e \in \mathbb{N}, \text{ there exists an } n \in \mathbb{N} \text{ such that } T \vdash \neg\kappa[\overline{n}, \overline{e}/v_0, v_1]. \tag{1}$$

Since T is recursive, the set of T-provable sentences is r. e. So, in particular,

$$\{(n,e) : T \vdash \neg\kappa[\overline{n}, \overline{e}/v_0, v_1]\}$$

is r. e. By (1) it follows that there is a computable function f such that, for any $e \in \mathbb{N}$,

$$T \vdash \neg\kappa[\overline{f(e)}, \overline{e}/v_0, v_1].$$

On the other hand,

$$T \vdash \neg\kappa[\overline{n}, \overline{e}/v_0, v_1] \Rightarrow K(n) > e.$$

(Namely, if not, then $T \vdash \neg\kappa[\overline{n}, \overline{e}/v_0, v_1]$ for some n, e such that $K(n) \leq e$. So since $\mathsf{PA} \sqsubseteq T$, $T \vdash \kappa[\overline{n}, \overline{e}/v_0, v_1]$, hence T is inconsistent contradicting to the assumption.) So $K(f(e)) > e$ for all e, contradicting to Proposition 6.5(iii). $\square$

Corollary 6.9. *Let $T = (\mathcal{L}_A, \Sigma)$ be an arithmetical recursively axiomatized theory which extends* PA*. Then $T \neq Th(\mathcal{N})$.*

Proof. By Proposition 6.5(ii) fix n such that $K(n) > c_T$. Then by Definition 6.7(ii), $\mathcal{N} \models \neg\kappa[\overline{n}, \overline{c_T}/v_0, v_1]$ but $T \not\models \neg\kappa[\overline{n}, \overline{c_T}]$ by Theorem 6.8. $\square$

Bibliography

[1] D. Hilbert and W. Ackermann. *Grundzüge der Theoretischen Logik*. Springer Berlin Heidelberg, 1972.

[2] Alonzo Church and A. M. Turing. "On Computable Numbers, with an Application to the Entscheidungsproblem." In: *The Journal of Symbolic Logic* 2.1 (1937), p. 42.

[3] Peter Smith. *An Introduction to Gödel's Theorems*. Cambridge University Press, 2009.

[4] George S. Boolos. *The Logic of Provability*. Cambridge University Press, 1994.

[5] *A Concise Introduction to Mathematical Logic*. Springer New York, 2006.

[6] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer New York, 2008.

[7] Saeed Salehi and Payam Seraji. "On constructivity and the Rosser property: a closer look at some Gödelean proofs." In: *Annals of Pure and Applied Logic* 169.10 (2018), pp. 971–980.

[8] Makoto Kikuchi. "Kolmogorov complexity and the second incompleteness theorem." In: *Archive for Mathematical Logic* 36.6 (1997), pp. 437–443.

[9] Gregory J. Chaitin. "Gödel's theorem and information." In: *International Journal of Theoretical Physics* 21.12 (1982), pp. 941–954.

[10] G. J. Chaitin. "Computational complexity and Gödel's incompleteness theorem." In: *ACM SIGACT News* 9 (1971), pp. 11–12.

[11] Shira Kritchman and Ran Raz. "The Surprise Examination Paradox and the Second Incompleteness Theorem." In: *Notices of the AMS volume 57 number 11 (December 2010), published by the American Mathematical Society* (Nov. 22, 2010).

[12] E. Mendelson. *Introduction to Mathematical Logic (Discrete Mathematics and Its Applications)*. Springer, 1997.