ON DISCRETIZING A CIRCLE

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Consider a grid (a lattice of pixels) in the plane with centers at coordinates $p = (m,n) \in \mathbb{Z}^2$.

Take a circle of radius R, centered on the origin.

A *continuous discretization* of the circle is an ordered set of distinct pixels

$$\mathcal{D}_R = (p_i)_{0 \le i \le N-1}, \tag{1}$$

such that consecutive pixels touch at their edges or corners:

$$\|p_i - p_j\|_\infty = \max(|m_i - m_j|\,,\,|n_i - n_j|) = 1, \qquad j = (i \pm 1) \bmod N. \tag{2}$$

We define the average radius of $\mathcal{D}_R$ as

$$\operatorname{Rad}\mathcal{D}_R = \frac{1}{N}\sum_{i=0}^{N-1}\sqrt{m_i^2 + n_i^2}. \tag{3}$$

In order for a discretization to be valid, it must converge to the circle in the limit as $R \to \infty$:

$$\lim_{R\to\infty} \operatorname{Rad}\mathcal{D}_R = R. \tag{4}$$

(*** Add this to the definition.) Note that it is not required for the circle to intersect the pixels in the discretization.

For $q \ge 1$, we define the q-error of the discretization to be

$$\operatorname{Err}_q \mathcal{D}_R = \left(\frac{1}{N}\sum_{i=0}^{N-1}\|p_i - R\,(\cos\phi_i, \sin\phi_i)\|_q^q\right)^{1/q}, \qquad \phi_i = \arctan(m_i, n_i). \tag{5}$$

For $q = 2$, we have

$$\begin{aligned}\|p_i - R\,(\cos\phi_i, \sin\phi_i)\|_2^2 &= (m_i - R\cos\phi_i)^2 + (n_i - R\sin\phi_i)^2 \\ &= m_i^2 + n_i^2 + R^2 - 2Rm_i\cos\phi_i - 2Rn_i\sin\phi_i.\end{aligned}$$

We use $m_i = \sqrt{m_i^2 + n_i^2}\cos\phi_i$ and $n_i = \sqrt{m_i^2 + n_i^2}\sin\phi_i$ to get

$$\|p_i - R\,(\cos\phi_i, \sin\phi_i)\|_2^2 = m_i^2 + n_i^2 - R^2. \tag{6}$$

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Hence, the L_2 error is

$$\operatorname{Err}_2 \mathcal{D}_R = \left(\frac{1}{N} \sum_{i=0}^{N-1} \left((m_i^2 + n_i^2) - R^2 \right) \right)^{1/2}. \tag{7}$$

Find a bound on this, not assuming the circle intersects the pixels.

$$m_0^2 + m_1^2 + m_2^2 + \cdots = \tfrac{1}{2} \left((m_0 - m_1)^2 + (m_1 - m_2)^2 + \cdots \right) + m_0 m_1 + m_1 m_2 + \cdots$$

$$\sum_{i=0}^{N-1} |m_i - m_j| \geq 2R? \tag{8}$$

(*** DFT? Best approximation in L^2?)

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