1 Stefan Problem (for the circular case)

To describe the IDLA motion more precisely, we need to introduce Stefen problem, which has a time-dependent boundary condition. We have an equation which could describe the IDLA motion with a source at the origin

$$\partial_t \, p = D\nabla^2 p + \delta(\mathbf{x})S \tag{1.1}$$

where S is a constant. In a region $\Omega(t)$ with the boundary $\partial\,\Omega = \{\mathbf{d}(t)\}$.

Consider a Dirichlet boundary condition

$$p|_{\partial\,\Omega} = 0. \tag{1.2}$$

Also,

$$\hat{\mathbf{n}} \cdot \dot{\mathbf{d}} = -\alpha\, \hat{\mathbf{n}} \cdot \nabla p \tag{1.3}$$

where D and α are constants.

1.1 Find $R(t)$ by Conservation of Mass

For the circular case, solve for $p(r,t)$, with a source at origin $\delta(\boldsymbol{x})$

$$\partial_t p = \frac{D}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \delta(\boldsymbol{x})S \tag{1.4}$$

From (7.3) and (7.5), the boundary conditions are

$$p(R(t),t) = 0 \tag{1.5}$$

$$\dot{R} = -\alpha\frac{\partial p}{\partial r}(R(t),t) \tag{1.6}$$

Consider the integral for mass

$$M(t) = \int_{\Omega(t)} p\, dV = 2\pi \int_0^{R(t)} p(r,t) r dr \tag{1.7}$$

By the conservation of mass

$$\dot{M}(t) = 0 \tag{1.8}$$

$$
\begin{aligned}
\dot{M} &= 2\pi[\int_0^{R(t)} p_t r dr + p(R(t),t)R\dot{R}] \\
&= 2\pi \int_0^{R} [D\partial_r(r\partial_r p) + Sr\delta(\boldsymbol{x})]dr \\
&= 2\pi D[r\partial_r p]_0^{R(t)} + S\int_{\Omega(t)} \delta(\boldsymbol{x})dV \\
&= 2\pi DR\partial_r p(R(t),t) + S
\end{aligned}
\tag{1.9}
$$

With (7.8),

$$
\begin{aligned}
\dot{M} &= 2\pi DR(-\frac{\dot{R}}{\alpha}) + S \\
&= -\frac{D}{\alpha}\frac{d}{dt}(\pi R^2) + S \\
&= -\frac{D}{\alpha}\dot{A} + S
\end{aligned}
\tag{1.10}
$$

where

$$
A(t) = \pi R^2(t) \tag{1.11}
$$

and with (7.10)

$$
\dot{A} = \frac{S\alpha}{D} \tag{1.12}
$$

Hence,

$$
M(t) = -\frac{D}{\alpha}A(t) + St \tag{1.13}
$$

$$
A(t) = (\frac{S\alpha}{D})t = \pi R^2(t) \to R(t) = \sqrt{\frac{S\alpha}{\pi D}t} \tag{1.14}
$$

1.2 Rescale and Solve for $R(t)$

1.2.1 Rescale r and t

From equation (1.14), we have

$$
\dot{R}(t) = \sqrt{\frac{S\alpha}{\pi D}}\frac{1}{2}t^{-\frac{1}{2}} \tag{1.15}
$$

We rescale r and t as following:

$$
\hat{r} = \frac{r}{R(t)} \tag{1.16}
$$

$$
\hat{t} = t \tag{1.17}
$$

Hence,

$$\frac{\partial(\hat{r},\hat{t})}{\partial(r,t)} = \begin{pmatrix} \partial_r \hat{r} & \partial_t \hat{r} \\ \partial_r \hat{t} & \partial_t \hat{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{R} & -\frac{r\dot{R}}{R^2} \\ 0 & 1 \end{pmatrix} \tag{1.18}$$

By plugging (1.18) back into (1.4),

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} + \frac{\partial p}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial t} = \frac{\partial p}{\partial \hat{t}} - \frac{r\dot{R}}{R^2}\frac{\partial p}{\partial \hat{r}} \tag{1.19}$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial r} + \frac{\partial p}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial r} = \frac{1}{R}\frac{\partial p}{\partial \hat{r}} \tag{1.20}$$

Hence,

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) = \frac{1}{\hat{r}R}\frac{1}{R}\frac{\partial}{\partial \hat{r}}(\cancel{R}\hat{r}\frac{1}{\cancel{R}}\frac{\partial p}{\partial \hat{r}}) = \frac{1}{R^2}\frac{1}{\hat{r}}\frac{\partial}{\partial p}(\hat{r}\frac{\partial r}{\partial \hat{r}}) = \frac{\hat{\nabla}^2 p}{R^2} \tag{1.21}$$

1.2.2 Rescale $\delta(\boldsymbol{x})$

With $\delta(\alpha x) = \frac{\delta(x)}{\alpha}$, we can write

$$(\hat{x},\hat{y}) = \frac{(x,y)}{R} \tag{1.22}$$

With $\delta(\boldsymbol{x}) = \delta(x)\delta(y)$, we can write

$$\delta(R\hat{\boldsymbol{x}}) = \delta(R\hat{x})\delta(R\hat{y}) \tag{1.23}$$

Thus,

$$\delta(\boldsymbol{x}) = \frac{\delta(r)}{2\pi r} = \frac{\delta(R\,\hat{\boldsymbol{r}})}{2\pi R\,\hat{r}} = \frac{\delta(\hat{r})}{2\pi R^2 \hat{r}} = \frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.24}$$

and

$$\delta(\boldsymbol{x}) = \frac{\delta(R\hat{x})\delta(R\hat{y})}{R^2} = \frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.25}$$

1.2.3 Solve for $R(t)$

We can then plug (1.19), (1.21), and (1.25) back into (1.4),

$$\frac{\partial p}{\partial \hat{t}} - \frac{r\dot{R}}{R^2}\frac{\partial p}{\partial \hat{r}} = D\frac{\hat{\nabla}^2 p}{R^2} + S\frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.26}$$

(?)

$$\frac{\dot{R}}{R^2} = \sqrt{\frac{\pi D}{S\alpha}}\frac{\frac{1}{2}t^{1/2}}{t} = \sqrt{\frac{\pi D}{4S\alpha}}t^{-1/2} \tag{1.27}$$

From (1.6), we have

$$\dot{R} = -\alpha\frac{\partial p}{\partial r} = -\frac{\alpha}{R}\frac{\partial p}{\partial \hat{r}} \tag{1.28}$$

Consider $\frac{\partial p}{\partial \hat{t}} = 0$, neglect $\frac{r\dot{R}}{R^2}\frac{\partial p}{\partial \hat{r}}$ as $\hat{t} \to \infty$

$$0 = D\hat{\nabla}^2 p + D\delta(\hat{\boldsymbol{x}}) \tag{1.29}$$

The solution is then

$$p = -\frac{S}{2\pi D}\log(\hat{r}) + B(t) \tag{1.30}$$

With the boundary condition $p(1,t) = 0$, $B(t) = 0$.

$$\dot{R} = -\alpha(\frac{-S}{2\pi RD}) \tag{1.31}$$

$$\frac{1}{2}(R^2 - R_0^2) = \frac{\alpha S}{2\pi D}t \tag{1.32}$$

With $R_0 = 0$,

$$R(t) = \sqrt{\frac{\alpha S}{\pi D}t} \tag{1.33}$$

which is the same as the result we obtained by the conservation of mass.