CHAPTER 11

Green's Functions for Wave and Heat Equations

11.1 INTRODUCTION

In Chapter 9 we had some success in obtaining Green's functions for time-independent problems. One particularly important idea was the use of infinite space Green's functions. Here we will analyze Green's functions for the heat and wave equations. Problems with one, two, and three spatial dimensions will be considered. We will derive Green's formulas for the heat and wave equation and use them to represent the solution of nonhomogeneous problems (nonhomogeneous sources and nonhomogeneous boundary conditions) in terms of the Green's function. We will obtain elementary formulas for these infinite space Green's functions. For the wave equation, we will derive the one-dimensional infinite space Green's function by utilizing the general solution of the one-dimensional wave equation. We will derive the infinite space Green's function for the three-dimensional wave equation by making a well-known transformation to a one-dimensional wave equation. For the heat equation, we will derive the infinite space Green's function by comparing the Green's function with the appropriate solution of the initial value problem for the infinite space heat equation solved in Chapter 10.

11.2 GREEN'S FUNCTIONS FOR THE WAVE EQUATION

11.2.1 Introduction

In this section we solve the wave equation with possibly time-dependent sources,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + Q(\boldsymbol{x}, t),} \tag{11.2.1}$$

subject to the two initial conditions,

$$\boxed{u(\boldsymbol{x}, 0) = f(\boldsymbol{x})} \tag{11.2.2}$$

$$\boxed{\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = g(\boldsymbol{x}).} \tag{11.2.3}$$

If the problem is on a finite or semi-infinite region, then in general, $u(\boldsymbol{x}, t)$ will satisfy nonhomogeneous conditions on the boundary. We will determine simultaneously how to solve this problem in one, two, and three dimensions. (In one dimension $\nabla^2 = \partial^2/\partial x^2$.)

We introduce the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ as a solution due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ acting instantaneously only at $t = t_0$:

$$\boxed{\frac{\partial^2 G}{\partial t^2} = c^2 \nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0),} \tag{11.2.4}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is the Dirac delta function of the appropriate dimension. For finite or semi-infinite problems, G will satisfy the related homogeneous boundary conditions corresponding to the nonhomogeneous ones satisfied by $u(\boldsymbol{x}, t)$.

The Green's function is the response at $\boldsymbol{x}$ at time t due to a source located at $\boldsymbol{x}_0$ at time t_0. Since we desire the Green's function G to be the response due only to this source acting at $t = t_0$ (not due to some nonzero earlier conditions), we insist that the response G will be zero before the source acts ($t < t_0$):

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \qquad \text{for } t < t_0,} \tag{11.2.5}$$

known as the **causality principle** (see Section 9.2).

The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ depends on only the time after the occurrence of the concentrated source. If we introduce the **elapsed** time, $T = t - t_0$,

$$\frac{\partial^2 G}{\partial T^2} = c^2 \nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(T)$$
$$G = 0 \quad \text{for} \quad T < 0,$$

then G is also seen to be the response due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ at $T = 0$. We call this the **translation** property,

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}, t - t_0; \boldsymbol{x}_0, 0).} \tag{11.2.6}$$

11.2.2 Green's Formula for the Wave Equation

Before solving for the Green's function (in various dimensions), we will show how the solution of the nonhomogeneous wave equation (11.2.1) (with nonhomogeneous initial and boundary conditions) is obtained using the Green's function. For time-independent problems (nonhomogeneous Sturm–Liouville type or the Poisson equation), the relationship between the nonhomogeneous solution and the Green's function was obtained using Green's formula:

Sturm–Liouville operator $[L = d/dx(p\ d/dx) + q]$:

$$\int_a^b [uL(v) - vL(u)]\ dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b. \tag{11.2.7}$$

Three-dimensional Laplacian $(L = \nabla^2)$:

$$\iiint [uL(v) - vL(u)]\, d^3x = \oiint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}}\ dS, \tag{11.2.8}$$

where $d^3x = dV = dx\ dy\ dz$. There is a corresponding result for the two-dimensional Laplacian.

To extend these ideas to the nonhomogeneous wave equation, we introduce the appropriate linear differential operator:

$$\boxed{L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2.} \tag{11.2.9}$$

Using this notation, the nonhomogeneous wave equation (11.2.1) satisfies

$$L(u) = Q(\boldsymbol{x}, t), \tag{11.2.10}$$

while the Green's function (11.2.4) satisfies

$$L(G) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0). \tag{11.2.11}$$

For the wave operator L [see (11.2.9)], we will derive a Green's formula analogous to (11.2.7) and (11.2.8). We will use a *notation* corresponding to three dimensions but will make clear modifications (when necessary) for one and two dimensions. For time-dependent problems, L has both space and time variables. Formulas analogous to (11.2.7) and (11.2.8) are expected to exist, but integration will occur over both space $\boldsymbol{x}$ and time t. Since for the wave operator

$$uL(v) - vL(u) = u\frac{\partial^2 v}{\partial t^2} - v\frac{\partial^2 u}{\partial t^2} - c^2(u\nabla^2 v - v\nabla^2 u),$$

the previous Green's formulas will yield the new **"Green's formula for the wave equation"**:

$$\boxed{\begin{aligned}&\int_{t_i}^{t_f}\iiint [uL(v) - vL(u)]\, d^3x\ dt \\ &\quad = \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Bigg|_{t_i}^{t_f} d^3x - c^2\int_{t_i}^{t_f}\left(\oiint (u\nabla v - v\nabla u)\cdot\hat{\boldsymbol{n}}\ dS\right)\ dt,\end{aligned}} \tag{11.2.12}$$

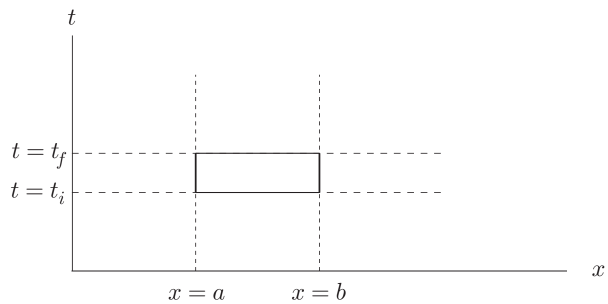

FIGURE 11.2.1 Space-time boundaries for a one-dimensional wave equation.

where $\iiint$ indicates integration over the three-dimensional space ($\int_a^b$ for one-dimensional problems) and $\oiint$ indicates integration over its boundary ($|_a^b$ for one-dimensional problems). The terms on the right-hand side represent contributions from the boundaries: the spatial boundaries for all time and the temporal boundaries ($t = t_i$ and $t = t_f$) for all space. These space-time boundaries are illustrated (for a one-dimensional problem) in Fig. 11.2.1.

For example, *if* both u and v satisfy the usual type of homogeneous boundary conditions (in space, for all time), then $\oiint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}}\, dS = 0$, but

$$\iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right)\Big|_{t_i}^{t_f}\ d^3x$$

may not equal zero due to contributions from the "initial" time t_i and "final" time t_f.

11.2.3 Reciprocity

For time-independent problems, we have shown that the Green's function is symmetric, $G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x})$. We proved this result using Green's formula for two different Green's functions [$G(\boldsymbol{x}, \boldsymbol{x}_1)$ and $G(\boldsymbol{x}, \boldsymbol{x}_2)$]. The result followed because the boundary terms in Green's formula vanished.

For the wave equation there is a somewhat analogous property. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ satisfies

$$\frac{\partial^2 G}{\partial t^2} - c^2\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0), \tag{11.2.13}$$

subject to the causality principle,

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \qquad \text{for } t < t_0. \tag{11.2.14}$$

G will be nonzero for $t > t_0$. To utilize Green's formula (to prove reciprocity), we need a second Green's function. If we choose it to be $G(\boldsymbol{x}, t; \boldsymbol{x}_A, t_A)$, then the contribution $\int_{t_i}^{t_f} \oiint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}}\ dS\ dt$ on the spatial boundary (or infinity) vanishes, but the contribution

$$\iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right)\Bigg|_{t_i}^{t_f} \; d^3x$$

on the time boundary will not vanish at both $t = t_i$ and $t = t_f$. In time, our problem is an initial value problem, not a boundary value problem. If we let $t_i \leq t_0$ in Green's formula, the "initial" contribution will vanish.

For a second Green's function, we are interested in varying the source time t, $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$, what we call the **source-varying Green's function**. From the translation property,

$$\boxed{G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, -t; \boldsymbol{x}_1, -t_1),} \tag{11.2.15}$$

since the elapsed times are the same $[-t - (-t_1) = t_1 - t]$. By causality, these are zero if $t_1 < t$ (or, equivalently, $-t < -t_1$):

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = 0, \qquad t > t_1. \tag{11.2.16}$$

We call this the **source-varying causality principle**. By introducing this Green's function, we will show that the "final" contribution from Green's formula may vanish.

To determine the differential equation satisfied by the source-varying Green's function, we let $t = -\tau$, in which case, from (11.2.15),

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, \tau; \boldsymbol{x}_1, -t_1).$$

This is the ordinary (variable-response position) Green's function, with τ being the time variable. It has a concentrated source located at $\boldsymbol{x} = \boldsymbol{x}_1$ when $\tau = -t_1$ $(t = t_1)$:

$$\left(\frac{\partial^2}{\partial \tau^2} - c^2\nabla^2 \right) G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).$$

Since $\tau = -t$, from the chain rule $\partial/\partial\tau = -\partial/\partial t$, but $\partial^2/\partial\tau^2 = \partial^2/\partial t^2$. Thus, the wave operator is symmetric in time, and therefore

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2 \right) G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = L[G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).} \tag{11.2.17}$$

A reciprocity formula results from Green's formula (11.2.12) using two Green's functions, one with varying response time,

$$u = G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0), \tag{11.2.18}$$

and one with varying source time,

$$v = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t). \tag{11.2.19}$$

Both satisfy partial differential equations involving the *same* wave operator, $L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2$. We integrate from $t = -\infty$ to $t = +\infty$ in Green's formula (11.2.12) (i.e., $t_i = -\infty$ and $t_f = +\infty$). Since both Green's functions satisfy the same homogeneous boundary conditions, Green's formula (11.2.12) yields

$$\int_{-\infty}^{\infty} \iiint [u\delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1) - v\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)]\ d^3x\ dt$$
$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right)\Big|_{-\infty}^{+\infty}\ d^3x. \tag{11.2.20}$$

From the causality principles, u and $\partial u/\partial t$ vanish for $t < t_0$, and v and $\partial v/\partial t$ vanish for $t > t_1$. Thus, the r.h.s. of (11.2.20) vanishes. Consequently, using the properties of the Dirac delta function, u at $\boldsymbol{x} = \boldsymbol{x}_1$, $t = t_1$ equals v at $\boldsymbol{x} = \boldsymbol{x}_0$, $t = t_0$:

$$\boxed{G(\boldsymbol{x}_1, t_1; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}_0, t_1; \boldsymbol{x}_1, t_0),} \tag{11.2.21}$$

the **reciprocity formula** for the Green's function for the wave equation. Assuming that $t_1 > t_0$, the response at $\boldsymbol{x}_1$ (at time t_1) due to a source at $\boldsymbol{x}_0$ (at time t_0) is the same as the response at $\boldsymbol{x}_0$ (at time t_1) due to a source at $\boldsymbol{x}_1$, *as long as the elapsed times from the sources are the same*. In this case it is seen that interchanging the source and location points has no effect, what we called Maxwell reciprocity for time-independent Green's functions.

11.2.4 Using the Green's Function

As with our earlier work, the relationship between the Green's function and the solution of the nonhomogeneous problem is established using the appropriate Green's formula, (11.2.12). We let

$$u = u(\boldsymbol{x}, t) \tag{11.2.22}$$
$$v = G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t) = G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t), \tag{11.2.23}$$

where $u(\boldsymbol{x}, t)$ is the solution of the nonhomogeneous wave equation satisfying

$$L(u) = Q(\boldsymbol{x}, t)$$

subject to the given initial conditions for $u(\boldsymbol{x}, 0)$ and $\partial u/\partial t(\boldsymbol{x}, 0)$, and where $G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)$ is the source-varying Green's function satisfying (11.2.17):

$$L[G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)$$

subject to the source-varying causality principle

$$G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t) = 0 \qquad \text{for} \quad t > t_0.$$

G satisfies homogeneous boundary conditions, but u may not. We use Green's formula (11.2.12) with $t_i = 0$ and $t_f = t_{0+}$; we integrate just beyond the appearance of a concentrated source at $t = t_0$:

$$\int_0^{t_{0+}} \iiint [u(\boldsymbol{x}, t)\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0) - G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)] d^3x \ dt$$
$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right)\bigg|_0^{t_{0+}} \ d^3x - c^2 \int_0^{t_{0+}} \left[\oiint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}} \ dS \right] \ dt.$$

At $t = t_0+$, $v = 0$ and $\partial v/\partial t = 0$, since we are using the source-varying Green's function. We obtain, using the reciprocity formula (11.2.21),

$$\begin{aligned} u(\boldsymbol{x}_0, t_0) = &\int_0^{t_0+} \iiint G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t)Q(\boldsymbol{x}, t) \ d^3x \ dt \\ &+ \iiint \left[\frac{\partial u}{\partial t}(\boldsymbol{x}, 0)G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, 0) - u(\boldsymbol{x}, 0)\frac{\partial}{\partial t}G(\boldsymbol{x}_0, t_0; x, 0) \right] \ d^3x \\ &-c^2 \int_0^{t_0+} \left[\oiint \ (u(\boldsymbol{x}, t)\nabla G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t) - G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t)\nabla u(\boldsymbol{x}, t)) \cdot \hat{\boldsymbol{n}} \ dS \right] dt. \end{aligned}$$

It can be shown that t_{0+} may be replaced by t_0 in these limits. If the roles of $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are interchanged (as well as t and t_0), we obtain a representation formula for $u(\boldsymbol{x}, t)$ in terms of the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$:

$$\boxed{\begin{aligned} u(\boldsymbol{x}, t) = &\int_0^t \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)Q(\boldsymbol{x}_0, t_0) \ d^3x_0 \ dt_0 \\ &+ \iiint \left[\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0, 0)G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) - u(\boldsymbol{x}_0, 0)\frac{\partial}{\partial t_0}G(\boldsymbol{x}, t; x_0, 0) \right] \ d^3x_0 \\ &-c^2 \int_0^t \left[\oiint (u(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) - G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} u(\boldsymbol{x}_0, t_0)) \cdot \hat{\boldsymbol{n}} \ dS_0 \right] dt_0. \end{aligned}} \tag{11.2.24}$$

Note that $\nabla_{\boldsymbol{x}_0}$ means a derivative with respect to the source position. Equation (11.2.24) expresses the response due to the three kinds of nonhomogeneous terms: source terms, initial conditions, and nonhomogeneous boundary conditions. In particular, the initial position $u(\boldsymbol{x}_0, 0)$ has an influence function

$$-\frac{\partial}{\partial t_0}G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$$

(meaning the source time derivative evaluated initially), while the influence function for the initial velocity is $G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$.

Furthermore, for example, if u is given on the boundary, then G satisfies the related homogeneous boundary condition; that is, $G = 0$ on the boundary. In this case the boundary term in (11.2.24) simplifies to

$$-c^2 \int_0^t \left[\oiint u(\boldsymbol{x}_0, t_0) \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\boldsymbol{n}}\ dS_0 \right] dt_0.$$

The influence function for this nonhomogeneous boundary condition is

$$-c^2 \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\boldsymbol{n}}.$$

This is $-c^2$ times the source outward normal derivative of the Green's function.

11.2.5 Green's Function for the Wave Equation

We recall that the Green's function for the wave equation satisfies (11.2.4) and (11.2.5):

$$\boxed{\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)} \tag{11.2.25}$$

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for} \quad t < t_0, \tag{11.2.26}$$

subject to homogeneous boundary conditions. We will describe the Green's function in a different way.

11.2.6 Alternate Differential Equation for the Green's Function

Using Green's formula, the solution of the wave equation with homogeneous boundary conditions and with no sources, $Q(\boldsymbol{x}, t) = 0$, is represented in terms of the Green's function by (11.2.24),

$$u(\boldsymbol{x}, t) = \iiint \left[\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0, 0) G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) - u(\boldsymbol{x}_0, 0) \frac{\partial}{\partial t_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) \right] d^3\boldsymbol{x}_0.$$

From this we see that G is also the influence function for the initial condition for the derivative $\frac{\partial u}{\partial t}$, while $-\frac{\partial G}{\partial t_0}$ is the influence function for the initial condition for u. If we solve the wave equation with the initial conditions $u = 0$ and $\frac{\partial u}{\partial t} = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$, the solution is the Green's function itself. Thus, the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ satisfies the ordinary wave equation with no sources,

$$\boxed{\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = 0,} \tag{11.2.27}$$

subject to homogeneous boundary conditions and the specific concentrated initial conditions at $t = t_0$:

$$G = 0 \tag{11.2.28}$$

$$\frac{\partial G}{\partial t} = \delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{11.2.29}$$

The Green's function for the wave equation can be determined directly from the initial value problem (11.2.27)–(11.2.29) rather than from its defining differential equation (11.2.4) or (11.2.26). Exercise 11.2.9 outlines another derivation of (11.2.27)–(11.2.29) in which the defining equation (11.2.4) is integrated from t_{0-} until t_{0+}.

11.2.7 Infinite Space Green's Function for the One-Dimensional Wave Equation and d'Alembert's Solution

We will determine the infinite space Green's function by solving the one-dimensional wave equation, $\frac{\partial^2 G}{\partial t^2} - c^2 \frac{\partial^2 G}{\partial x^2} = 0$, subject to initial conditions (11.2.28) and (11.2.29). In Chapter 12 (briefly mentioned in Chapter 4) it is shown that there is a remarkable general solution of the one-dimensional wave equation,

$$G = f(x - ct) + g(x + ct), \tag{11.2.30}$$

where $f(x-ct)$ is an arbitrary function moving to the right with velocity c and $g(x+ct)$ is an arbitrary function moving to the left with velocity $-c$. It can be verified by direct substitution that (11.2.30) solves the wave equation. For ease, we assume $t_0 = 0$ and $x_0 = 0$. Since from (11.2.28), $G = 0$ at $t = 0$, it follows that in this case $g(x) = -f(x)$, so that $G = f(x - ct) - f(x + ct)$. We calculate $\frac{\partial G}{\partial t} = -c\frac{df(x-ct)}{d(x-ct)} - c\frac{df(x+ct)}{d(x+ct)}$. In order to satisfy the initial condition (11.2.29), $\delta(x) = \frac{\partial G}{\partial t}|_{t=0} = -2c\frac{df(x)}{dx}$. By integration, $f(x) = -\frac{1}{2c}H(x)+k$, where $H(x)$ is the Heaviside step function (and k is an unimportant constant of integration):

$$\boxed{G(x,t;0,0) = \frac{1}{2c}[H(x+ct) - H(x-ct)] = \begin{cases} 0 & |x| > ct \\ \frac{1}{2c} & |x| < ct. \end{cases}} \tag{11.2.31}$$

Thus, **the infinite space Green's function for the one-dimensional wave equation is an expanding rectangular pulse moving at the wave speed c**, as is sketched in Fig. 11.2.2. Initially (in general, at $t = t_0$), it is located at one point $x = x_0$. Each end spreads out at velocity c. In general,

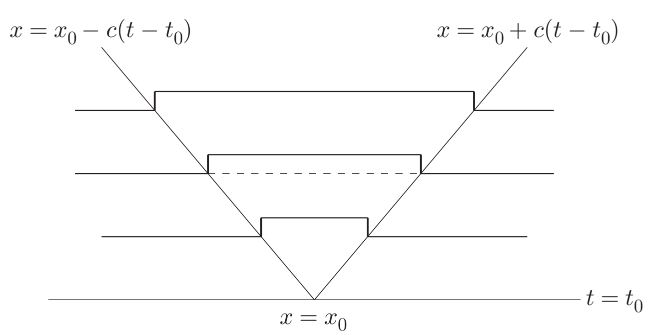

FIGURE 11.2.2 Green's function for the one-dimensional wave equation.

$$\boxed{G(x,t;x_0,t_0) = \frac{1}{2c}\{H[(x-x_0)+c(t-t_0)] - H[(x-x_0)-c(t-t_0)]\}.} \tag{11.2.32}$$

D'Alembert's solution. To illustrate the use of this Green's function, consider the initial value problem for the wave equation without sources on an infinite domain $-\infty < x < \infty$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{11.2.33}$$

$$u(x,0) = f(x) \tag{11.2.34}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x). \tag{11.2.35}$$

In the formula (11.2.24), the boundary contribution[1] vanishes since $G = 0$ for x sufficiently large (positive or negative); see Fig. 11.2.2. Since there are no sources, $u(x,t)$ is caused only by the initial conditions:

$$u(x,t) = \int_{-\infty}^{\infty} \left[g(x_0)G(x,t;x_0,0) - f(x_0)\frac{\partial}{\partial t_0}G(x,t;x_0,0) \right] \, dx_0.$$

We need to calculate $\frac{\partial}{\partial t_0}G(x,t;x_0,0)$ from (11.2.32). Using properties of the derivative of a step function [see (9.3.32)], it follows that

$$\frac{\partial}{\partial t_0}G(x,t;x_0,t_0) = \frac{1}{2}\left[-\delta\left(x-x_0+c(t-t_0)\right) - \delta\left(x-x_0-c(t-t_0)\right)\right]$$

and thus,

$$\frac{\partial}{\partial t_0}G(x,t;x_0,0) = \frac{1}{2}[-\delta(x-x_0+ct) - \delta(x-x_0-ct)].$$

Finally, we obtain the solution of the initial value problem:

$$\boxed{u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0) \, dx_0.} \tag{11.2.36}$$

This is known as **d'Alembert's solution** of the wave equation. It can be obtained more simply by the method of characteristics (see Chapter 12). There we will discuss the physical interpretation of the one-dimensional wave equation.

[1]The boundary contribution for an infinite problem is the limit as $L \to \infty$ of the boundaries of a finite region, $-L < x < L$.

Related problems. Semi-infinite or finite problems for the one-dimensional wave equation can be solved by obtaining the Green's function by the method of images. In some cases, transform or series techniques may be used. Of greatest usefulness is the method of characteristics.

11.2.8 Infinite Space Green's Function for the Three-Dimensional Wave Equation (Huygens' Principle)

We solve the infinite space Green's function using (11.2.27)–(11.2.29). The solution should be spherically symmetric and depend only on the distance $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. Thus, the Green's function satisfies the spherically symmetric wave equation, $\frac{\partial^2 G}{\partial t^2} - c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial G}{\partial \rho}) = 0$. Through an unmotivated but very well-known transformation, $G = \frac{h}{\rho}$, the spherically symmetric wave equation simplifies:

$$0 = \frac{1}{\rho}\frac{\partial^2 h}{\partial t^2} - \frac{c^2}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial h}{\partial \rho} - h\right) = \frac{1}{\rho}\left(\frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial \rho^2}\right).$$

Thus, h satisfies the one-dimensional wave equation. In Chapter 12 it is shown that the general solution of the one-dimensional wave equation can be represented by the sum of left- and right-going waves moving at velocity c. Consequently, we obtain the exceptionally significant result that **the general solution of the spherically symmetric wave equation** is

$$\boxed{G = \frac{f(\rho - ct) + g(\rho + ct)}{\rho}.} \tag{11.2.37}$$

$f(\rho - ct)$ is spherically expanding at velocity c, while $g(\rho + ct)$ is spherically contracting at velocity c. To satisfy the initial condition (11.2.28), $G = 0$ at $t = 0$, $g(\rho) = -f(\rho)$, and hence $G = \frac{f(\rho - ct) - f(\rho + ct)}{\rho}$. We calculate $\frac{\partial G}{\partial t} = -\frac{c}{\rho}\left[\frac{df(\rho - ct)}{d(\rho - ct)} + \frac{df(\rho + ct)}{d(\rho + ct)}\right]$. Thus, applying the initial condition (11.2.29) yields

$$\delta(\boldsymbol{x} - \boldsymbol{x}_0) = \left.\frac{\partial G}{\partial t}\right|_{t=0} = -\frac{2c}{\rho}\frac{df(\rho)}{d\rho}.$$

Here $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is a three-dimensional delta function. The function f will be constant, which we set to zero away from $\rho = 0$. We integrate in three-dimensional space over a sphere of radius R and obtain

$$1 = -2c \int_0^R \frac{1}{\rho}\frac{df}{d\rho} 4\pi \rho^2 \, d\rho = 8\pi c \int_0^R f \, d\rho,$$

after an integration by parts using $f = 0$ for $\rho > 0$ (One way to justify integrating by parts is to introduce the even extensions of f for $\rho < 0$ so that $\int_0^R = \frac{1}{2}\int_{-R}^R$). Thus,

$f = \frac{1}{4\pi c}\delta(\rho)$, where $\delta(\rho)$ is the one-dimensional delta function that is even and hence satisfies $\int_0^R \delta(\rho)\,d\rho = \frac{1}{2}$. Consequently,

$$\boxed{G = \frac{1}{4\pi c\rho}[\delta(\rho - ct) - \delta(\rho + ct)].} \tag{11.2.38}$$

However, since $\rho > 0$ and $t > 0$, the latter Dirac delta function is always zero. To be more general, t should be replaced by $t - t_0$. In this way we obtain the **infinite space Green's function for the three-dimensional wave equation**:

$$\boxed{G(x, t; x_0, t_0) = \frac{1}{4\pi c\rho}[\delta(\rho - c(t - t_0))],} \tag{11.2.39}$$

where $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. The Green's function for the three-dimensional wave equation is a spherical shell impulse spreading out from the source ($x = x_0$ and $t = t_0$) at radial velocity c with an intensity decaying proportional to $\frac{1}{\rho}$.

Huygens' principle. We have shown that a concentrated source at $\boldsymbol{x}_0$ (at time t_0) influences the position $\boldsymbol{x}$ (at time t) only if $|\boldsymbol{x} - \boldsymbol{x}_0| = c(t - t_0)$. The distance from source to location equals c times the time. The point source emits a wave moving in all directions at velocity c. At time $t - t_0$ later, the source's effect is located on a spherical shell a distance $c(t - t_0)$ away. This is part of what is known as **Huygens' principle**.

EXAMPLE

To be more specific, let us analyze the effect of sources, $Q(\boldsymbol{x}, t)$. Consider the wave equation with sources in infinite three-dimensional space with zero initial conditions. According to Green's formula (11.2.24),

$$u(\boldsymbol{x}, t) = \int_0^t \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) Q(\boldsymbol{x}_0, t_0)\ d^3x_0\ dt_0 \tag{11.2.40}$$

since the "boundary" contribution vanishes. Using the infinite three-dimensional space Green's function,

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi c}\int_0^t \iiint \frac{1}{\rho}\delta[\rho - c(t - t_0)] Q(\boldsymbol{x}_0, t_0)\ d^3x_0\ dt_0, \tag{11.2.41}$$

where $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. The only sources that contribute satisfy $|\boldsymbol{x} - \boldsymbol{x}_0| = c(t - t_0)$. The effect at $\boldsymbol{x}$ at time t is caused by all received sources; the velocity of propagation of each source is c.

11.2.9 Two-Dimensional Infinite Space Green's Function

The two-dimensional Green's function for the wave equation is not as simple as the one- and three-dimensional cases. In Exercise 11.2.12 the two-dimensional Green's function is derived by the method of descent by using the three-dimensional solution with a two-dimensional source. The signal again propagates with velocity c, so that the solution is zero before the signal is received, that is for the elapsed time $t - t_0 < \frac{r}{c}$, where $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ in two dimensions. However, once the signal is received, it is largest (infinite) at the moment the signal is first received, and then the signal gradually decreases:

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = \begin{cases} 0 & \text{if } r > c(t - t_0) \\ \frac{1}{2\pi c} \frac{1}{\sqrt{c^2(t-t_0)^2 - r^2}} & \text{if } r < c(t - t_0) \end{cases} .} \tag{11.2.42}$$

11.2.10 Summary

For the wave equation in any dimension, information propagates at velocity c. The Green's functions for the wave equation in one and three dimensions are different. Huygens' principle is valid only in three dimensions in which the influence of a concentrated source is felt only on the surface of the expanding sphere propagating at velocity c. In one dimension, the influence is felt uniformly inside the expanding pulse. In two dimensions, the largest effect occurs on the circumference corresponding to the propagation velocity c, but the effect diminishes behind the pulse.

EXERCISES 11.2

11.2.1. **(a)** Show that for $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0), \partial G/\partial t = -\partial G/\partial t_0$.

(b) Use part (a) to show that the response due to $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$ is the time derivative of the response due to $\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = f(\boldsymbol{x})$.

11.2.2. Express (11.2.24) for a one-dimensional problem.

11.2.3. If $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0$ for $\boldsymbol{x}$ on the boundary, explain why the corresponding term in (11.2.24) vanishes (for any $\boldsymbol{x}$).

11.2.4. For the one-dimensional wave equation, sketch $G(x, t; x_0, t_0)$ as a function of

(a) x with t fixed (x_0, t_0 fixed)

(b) t with x fixed (x_0, t_0 fixed)

11.2.5. **(a)** For the one-dimensional wave equation, for what values of x_0 (x, t, t_0 fixed) is $G(x, t; x_0, t_0) \neq 0$?

(b) Determine the answer to part (a) using the reciprocity property.

11.2.6. **(a)** Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x, t), \qquad -\infty < x < \infty$$

with $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = 0$.

***(b)** What space-time locations of the source $Q(x, t)$ influence u at position x_1 and time t_1?

11.2.7. Reconsider Exercise 11.2.6 if $Q(x,t) = g(x)e^{-i\omega t}$.

***(a)** Solve for $u(x,t)$. Show that the influence function for $g(x)$ is an outward-propagating wave.

(b) Instead, determine a particular solution of the form $u(x,t) = \psi(x)e^{-i\omega t}$. (See Exercise 8.3.13.)

(c) Compare parts (a) and (b).

11.2.8. ***(a)** In three-dimensional infinite space, solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + g(\boldsymbol{x})e^{-i\omega t}$$

with zero initial conditions, $u(\boldsymbol{x},0) = 0$ and $\frac{\partial u}{\partial t}(\boldsymbol{x},0) = 0$. From your solution, show that the influence function for $g(\boldsymbol{x})$ is an outward-propagating wave.

(b) Compare with Exercise 9.5.10.

11.2.9. Consider the Green's function $G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$ for the wave equation. From (11.2.24) we easily obtain the influence functions for $Q(\boldsymbol{x}_0,t_0)$, $u(\boldsymbol{x}_0,0)$, and $\partial u/\partial t_0(\boldsymbol{x}_0,0)$. These results may be obtained in the following alternative way:

(a) For $t > t_{0+}$ show that

$$\frac{\partial^2 G}{\partial t^2} = c^2\nabla^2 G, \tag{11.2.43}$$

where (by integrating from t_{0-} to t_{0+})

$$G(\boldsymbol{x},t_{0+};\boldsymbol{x}_0,t_0) = 0 \tag{11.2.44}$$

$$\frac{\partial G}{\partial t}(\boldsymbol{x},t_{0+};\boldsymbol{x}_0,t_0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0). \tag{11.2.45}$$

From (11.2.32), briefly explain why $G(\boldsymbol{x},t;\boldsymbol{x}_0,0)$ is the influence function for $\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0,0)$.

(b) Let $\phi = \partial G/\partial t$. Show that for $t > t_{0+}$,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\nabla^2 \phi \tag{11.2.46}$$

$$\phi(\boldsymbol{x},t_{0+};\boldsymbol{x}_0,t_0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0) \tag{11.2.47}$$

$$\frac{\partial \phi}{\partial t}(\boldsymbol{x},t_{0+};\boldsymbol{x}_0,t_0) = 0. \tag{11.2.48}$$

From (11.2.46)–(11.2.48), briefly explain why $\frac{-\partial G}{\partial t_0}(\boldsymbol{x},t;\boldsymbol{x}_0,0)$ is the influence function for $u(\boldsymbol{x}_0,0)$.

11.2.10. Consider

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad x > 0\\
u(x,0) &= f(x)\\
\frac{\partial u}{\partial t}(x,0) &= g(x)\\
u(0,t) &= h(t).
\end{aligned}$$

(a) Determine the appropriate Green's function using the method of images.

***(b)** Solve for $u(x,t)$ if $Q(x,t)=0$, $f(x)=0$, and $g(x)=0$.

(c) For what values of t does $h(t)$ influence $u(x_1,t_1)$? Briefly interpret physically.

11.2.11. Reconsider Exercise 11.2.10:

(a) if $Q(x,t)\neq 0$, but $f(x)=0$, $g(x)=0$, and $h(t)=0$

(b) if $f(x)\neq 0$, but $Q(x,t)=0$, $g(x)=0$, and $h(t)=0$

(c) if $g(x)\neq 0$, but $Q(x,t)=0$, $f(x)=0$, and $h(t)=0$

11.2.12. Consider the Green's function $G(\boldsymbol{x},t;\boldsymbol{x}_1,t_1)$ for the *two*-dimensional wave equation as the solution of the following *three*-dimensional wave equation:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\nabla^2 u + Q(\boldsymbol{x},t)\\ u(\boldsymbol{x},0) &= 0\\ \frac{\partial u}{\partial t}(\boldsymbol{x},0) &= 0\\ Q(\boldsymbol{x},t) &= \delta(x-x_1)\delta(y-y_1)\delta(t-t_1).\end{aligned}$$

We will solve for the two-dimensional Green's function by this **method of descent** (descending from three dimensions to two dimensions).

***(a)** Solve for $G(\boldsymbol{x},t;\boldsymbol{x}_1,t_1)$ using the general solution of the three-dimensional wave equation. Here, the source $Q(\boldsymbol{x},t)$ may be interpreted either as a point source in two dimensions or as a line source in three dimensions. [*Hint*: $\int_{-\infty}^{\infty}\cdots dz_0$ may be evaluated by introducing the three-dimensional distance ρ from the point source,

$$\rho^2=(x-x_1)^2+(y-y_1)^2+(z-z_0)^2.]$$

(b) Show that G is a function only of the elapsed time $t-t_1$ and the two-dimensional distance r from the line source,

$$r^2=(x-x_1)^2+(y-y_1)^2.$$

(c) Where is the effect of an impulse felt after a time τ has elapsed? Compare to the one- and three-dimensional problems.

(d) Sketch G for $t-t_1$ fixed.

(e) Sketch G for r fixed.

11.2.13. Consider the three-dimensional wave equation. Determine the response to a unit point source moving at the constant velocity $\boldsymbol{v}$:

$$Q(\boldsymbol{x},t)=\delta(\boldsymbol{x}-\boldsymbol{v}t).$$

11.2.14. Solve the wave equation in infinite three-dimensional space without sources, subject to the initial conditions

(a) $u(\boldsymbol{x},0)=0$ and $\frac{\partial u}{\partial t}(\boldsymbol{x},0)=g(\boldsymbol{x})$. The answer is called **Kirchhoff's formula**, although it is due to Poisson (according to Weinberger [1995]).

(b) $u(\boldsymbol{x},0) = f(\boldsymbol{x})$ and $\frac{\partial u}{\partial t}(\boldsymbol{x},0) = 0$. [*Hint*: Use (11.2.24).]

(c) Solve part (b) in the following manner. Let $v(\boldsymbol{x},t) = \frac{\partial}{\partial t}u(\boldsymbol{x},t)$, where $u(\boldsymbol{x},t)$ satisfies part (a). [*Hint:* Show that $v(\boldsymbol{x},t)$ satisfies the wave equation with $v(\boldsymbol{x},0) = g(\boldsymbol{x})$ and $\frac{\partial v}{\partial t}(\boldsymbol{x},0) = 0$.]

11.2.15. Derive the one-dimensional Green's function for the wave equation by considering a three-dimensional problem with $Q(\boldsymbol{x},t) = \delta(x - x_1)\delta(t - t_1)$. [*Hint:* Use polar coordinates for the y_0, z_0 integration centered at $y_0 = y$, $z_0 = z$.]

11.3 GREEN'S FUNCTIONS FOR THE HEAT EQUATION

11.3.1 Introduction

We are interested in solving the heat equation with possibly time-dependent sources,

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u + Q(\boldsymbol{x},t),} \tag{11.3.1}$$

subject to the initial condition $u(\boldsymbol{x},0) = g(\boldsymbol{x})$. We will analyze this problem in one, two, and three spatial dimensions. In this subsection we do not specify the geometric region or the possibly nonhomogeneous boundary conditions. There can be three nonhomogeneous terms: the source $Q(\boldsymbol{x},t)$, the initial condition, and the boundary conditions.

We define the **Green's function** $G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$ as the solution of

$$\boxed{\frac{\partial G}{\partial t} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)} \tag{11.3.2}$$

on the same region with the related homogeneous boundary conditions. Since the Green's function represents the temperature response at $\boldsymbol{x}$ (at time t) due to a concentrated thermal source at $\boldsymbol{x}_0$ (at time t_0), we will insist that this Green's function is zero before the source acts:

$$\boxed{G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = 0 \qquad \text{for} \qquad t < t_0,} \tag{11.3.3}$$

the **causality principle**.

Furthermore, we show that only the elapsed time $t - t_0$ (from the initiation time $t = t_0$) is needed:

$$\boxed{G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = G(\boldsymbol{x},t - t_0;\boldsymbol{x}_0,0),} \tag{11.3.4}$$

the **translation property**. Equation (11.3.4) is shown by letting $T = t - t_0$, in which case the Green's function $G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$ satisfies

$$\frac{\partial G}{\partial T} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(T) \quad \text{with} \quad G = 0 \quad \text{for} \quad T < 0.$$

This is precisely the response due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ at $T = 0$, implying (11.3.4).

We postpone until later subsections the actual calculation of the Green's function. For now, we will assume that the Green's function is known and ask how to represent the temperature $u(\boldsymbol{x}, t)$ in terms of the Green's function.

11.3.2 Non-Self-Adjoint Nature of the Heat Equation

To show how this problem relates to others discussed in this book, we introduce the linear operator notation,

$$L = \frac{\partial}{\partial t} - k\nabla^2, \tag{11.3.5}$$

called the **heat** or **diffusion operator**. In previous problems the relation between the solution of the nonhomogeneous problem and its Green's function was based on Green's formulas. We have solved problems in which L is the Sturm–Liouville operator, the Laplacian, and most recently the wave operator.

The heat operator L is composed of two parts. ∇^2 is easily analyzed by Green's formula for the Laplacian [see (11.2.8)]. However, as innocuous as $\partial/\partial t$ appears, it is much harder to analyze than any of the other previous operators. To illustrate the difficulty presented by first derivatives, consider

$$L = \frac{\partial}{\partial t}.$$

For second-order Sturm–Liouville operators, elementary integrations yielded Green's formula. The same idea for $L = \partial/\partial t$ will not work. In particular,

$$\int [uL(v) - vL(u)]\, dt = \int \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) dt$$

cannot be simplified. There is no formula to evaluate $\int [uL(v) - vL(u)]\, dt$. The operator $L = \partial/\partial t$ is not self-adjoint. Instead, by standard integration by parts,

$$\int_a^b uL(v)\ dt = \int_a^b u\frac{\partial v}{\partial t}\ dt = uv\Big|_a^b - \int_a^b v\frac{\partial u}{\partial t}\ dt,$$

and thus

$$\int_a^b \left(u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t} \right) dt = uv\Big|_a^b. \tag{11.3.6}$$

For the operator $L = \partial/\partial t$, we introduce the **adjoint operator**,

$$\boxed{L^* = -\frac{\partial}{\partial t}.} \tag{11.3.7}$$

From (11.3.6),

$$\boxed{\int_a^b [uL^*(v) - vL(u)]\ dt = -uv\bigg|_a^b.} \tag{11.3.8}$$

This is analogous to Green's formula.[2]

11.3.3 Green's Formula for the Heat Equation

We now return to the nonhomogeneous heat problem:

$$\boxed{L(u) = Q(\boldsymbol{x}, t)} \tag{11.3.9}$$

$$\boxed{L(G) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0),} \tag{11.3.10}$$

where

$$\boxed{L = \frac{\partial}{\partial t} - k\nabla^2.} \tag{11.3.11}$$

For the nonhomogeneous heat equation, our results are more complicated since we must introduce the **adjoint heat operator**,

$$\boxed{L^* = -\frac{\partial}{\partial t} - k\nabla^2.} \tag{11.3.12}$$

By a direct calculation,

$$uL^*(v) - vL(u) = -u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} + k(v\nabla^2 u - u\nabla^2 v),$$

and thus Green's formula for the heat equation is

$$\boxed{\begin{aligned}&\int_{t_i}^{t_f}\iiint [uL^*(v) - vL(u)]\ d^3x\ dt \\ &\quad = -\iiint uv\bigg|_{t_i}^{t_f} d^3x + k\int_{t_i}^{t_f} \oiint (v\nabla u - u\nabla v)\cdot\hat{\boldsymbol{n}}\ dS\ dt.\end{aligned}} \tag{11.3.13}$$

[2] For a first-order operator, typically there is only one "boundary condition," $u(a) = 0$. For the integrated-by-parts term to vanish, we must introduce an *adjoint boundary condition*, $v(b) = 0$.

We have integrated over all space and from some time $t = t_i$ to another time $t = t_f$. We have used (11.3.6) for the $\partial/\partial t$-terms and Green's formula (11.2.8) for the ∇^2 operator. The "boundary contributions" are of two types, the spatial part (over $\oiint$[3]) and a temporal part (at the initial t_i and final t_f times). If both u and v satisfy the same homogeneous boundary condition (of the usual types), then the spatial contributions vanish:

$$\oiint (v\nabla u - u\nabla v)\cdot \hat{\boldsymbol{n}}\, dS = 0.$$

Equation (11.3.13) will involve initial contributions (at $t = t_i$) and final contributions ($t = t_f$).

11.3.4 Adjoint Green's Function

In order to eventually derive a representation formula for $u(\boldsymbol{x}, t)$ in terms of the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$, we must consider summing up various source times. Thus, we consider the **source-varying Green's function,**

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, -t; \boldsymbol{x}_1, -t_1),$$

where the translation property has been used. This is precisely the procedure we employed when analyzing the wave equation [see (11.2.15)]. By causality, these are zero if $t > t_1$:

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = 0 \qquad \text{for } t > t_1. \tag{11.3.14}$$

Letting $\tau = -t$, we see that the source-varying Green's function $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$ satisfies

$$\left(-\frac{\partial}{\partial t} - k\nabla^2\right) G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1), \tag{11.3.15}$$

as well as the source-varying causality principle (11.3.14). The heat operator L does not occur. Instead, the adjoint heat operator L^* appears:

$$\boxed{L^*[G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).} \tag{11.3.16}$$

We see that $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$ is the Green's function for the adjoint heat operator (with the source-varying causality principle). Sometimes it is called the **adjoint Green's function**, $G^*(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1)$. However, it is unnecessary to ever calculate or use it since

$$G^*(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1) = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t), \tag{11.3.17}$$

and both are zero for $t > t_1$.

[3]For infinite or semi-infinite geometries, we consider finite regions in some appropriate limit. The boundary terms at infinity will vanish if u and v decay sufficiently fast.

11.3.5 Reciprocity

As with the wave equation, we derive a reciprocity formula. Here, there are some small differences because of the occurrence of the adjoint operator in Green's formula, (11.3.13). In (11.3.13) we introduce

$$u = G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \tag{11.3.18}$$

$$v = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t), \tag{11.3.19}$$

the latter having been shown to be the source-varying or adjoint Green's function. Thus, the defining properties for u and v are

$$\begin{array}{ll} L(u) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0), & L^*(v) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1) \\ u = 0 \quad \text{for } t < t_0, & v = 0 \quad \text{for } t > t_1. \end{array}$$

We integrate from $t = -\infty$ to $t = +\infty$ [i.e., $t_i = -\infty$ and $t_f = +\infty$ in (11.3.13)], obtaining

$$\begin{aligned} &\int_{-\infty}^{\infty} \iiint [G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1) - G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)] \; d^3x \; dt \\ &\quad = -\iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)\Big|_{t=-\infty}^{t=\infty} d^3x, \end{aligned}$$

since u and v both satisfy the same homogeneous boundary conditions, so that

$$\oiint (v\nabla u - u\nabla v) \cdot \hat{\boldsymbol{n}} \; dS$$

vanishes. The contributions also vanish at $t = \pm\infty$ due to causality. Using the properties of the Dirac delta function, we obtain **reciprocity**:

$$\boxed{G(\boldsymbol{x}_1, t_1; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}_0, t_1; \boldsymbol{x}_1, t_0).} \tag{11.3.20}$$

As we have shown for the wave equation [see (11.2.21)], interchanging the source and location positions does not alter the responses if the elapsed times from the sources are the same. In this sense the Green's function for the heat (diffusion) equation is symmetric because of Green's formula.

11.3.6 Representation of the Solution Using Green's Functions

To obtain the relationship between the solution of the nonhomogeneous problem and the Green's function, we apply Green's formula (11.3.13) with u satisfying (11.3.1) subject to nonhomogeneous boundary and initial conditions. We let v equal the source-varying or adjoint Green's function, $v = G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)$. Using the defining differential equations (11.3.9) and (11.3.10), Green's formula (11.3.13) becomes

$$\begin{aligned}\int_0^{t_0+} \iiint [u\delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(t-t_0) - G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)]\ d^3x\ dt \\ = \iiint u(\boldsymbol{x}, 0)G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, 0)d^3x \\ + k\int_0^{t_0+} \oiint [G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)\nabla u - u\nabla G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] \cdot \hat{\boldsymbol{n}}\ dS\ dt,\end{aligned}$$

since $G = 0$ for $t > t_0$. Solving for u, we obtain

$$\begin{aligned}u(\boldsymbol{x}_0, t_0) = \int_0^{t_0} \iiint G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)\ d^3x\ dt \\ + \iiint u(\boldsymbol{x}, 0)G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, 0)\ d^3x \\ + k\int_0^{t_0} \oiint [G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)\nabla u - u\nabla G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] \cdot \hat{\boldsymbol{n}}\ dS\ dt.\end{aligned}$$

It can be shown that the limits t_{0+} may be replaced by t_0. We now (as before) interchange $\boldsymbol{x}$ with $\boldsymbol{x}_0$ and t with t_0. In addition, we use reciprocity and derive

$$\boxed{\begin{aligned}u(\boldsymbol{x}, t) = \int_0^{t} \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)Q(\boldsymbol{x}_0, t_0)\ d^3x_0\ dt_0 \\ + \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)u(\boldsymbol{x}_0, 0)\ d^3x_0 \\ + k\int_0^{t} \oiint [G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} u - u(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)] \cdot \hat{\boldsymbol{n}}\ dS_0\ dt_0.\end{aligned}}$$

(11.3.21)

Equation (11.3.21) illustrates how the temperature $u(\boldsymbol{x}, t)$ is affected by the three nonhomogeneous terms. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ is the influence function for the source term $Q(\boldsymbol{x}_0, t_0)$ as well as for the initial temperature distribution $u(\boldsymbol{x}_0, 0)$ (if we evaluate the Green's function at $t_0 = 0$, as is quite reasonable). Furthermore, nonhomogeneous boundary conditions are accounted for by the term $k\int_0^t \oiint (G\nabla_{\boldsymbol{x}_0} u - u\nabla_{\boldsymbol{x}_0} G) \cdot \hat{\boldsymbol{n}}\ dS_0\ dt_0$. Equation (11.3.21) illustrates the causality principle; at time t, the sources and boundary conditions have an effect only for $t_0 < t$. Equation (11.3.21) generalizes the results obtained by the method of eigenfunction expansion in Section 8.2 for the one-dimensional heat equation on a finite interval with zero boundary conditions.

EXAMPLE

Both u and its normal derivative seem to be needed on the boundary. To clarify the effect of the nonhomogeneous boundary conditions, we consider an example in which the temperature is specified along the entire boundary:

$$u(\boldsymbol{x}, t) = u_B(\boldsymbol{x}, t) \qquad \text{along the boundary.}$$

The Green's function satisfies the related homogeneous boundary conditions, in this case

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for all } \boldsymbol{x} \text{ along the boundary.}$$

Thus, the effect of this imposed temperature distribution is

$$-k \int_0^t \oiint u_B(\boldsymbol{x}_0, t_0) \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\boldsymbol{n}}\, dS_0\, dt_0.$$

The influence function for the nonhomogeneous boundary conditions is minus k times the normal derivative of the Green's function (a dipole distribution).

One-dimensional case. It may be helpful to illustrate the modifications necessary for one-dimensional problems. Volume integrals $\iiint d^3x_0$ become one-dimensional integrals $\int_a^b dx_0$. Boundary contributions on the closed surface $\oiint dS_0$ become contributions at the two ends $x = a$ and $x = b$. For example, if the temperature is prescribed at both ends, $u(a, t) = A(t)$ and $u(b, t) = B(t)$, then these nonhomogeneous boundary conditions influence the temperature $u(\boldsymbol{x}, t)$:

$$-k \int_0^t \oiint u_B(\boldsymbol{x}_0, t_0) \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\boldsymbol{n}}\, dS_0\, dt_0 \text{ becomes}$$
$$-k \int_0^t \left[B(t_0) \frac{\partial G}{\partial x_0}(x, t; b, t_0) - A(t_0) \frac{\partial G}{\partial x_0}(x, t; a, t_0) \right] dt_0.$$

This agrees with results that could be obtained by the method of eigenfunction expansions (Chapter 9) for nonhomogeneous boundary conditions.

11.3.7 Alternate Differential Equation for the Green's Function

Using Green's formula, we derived (11.3.21), which shows the influence of sources, nonhomogeneous boundary conditions, and as is initial condition for the heat equation. The Green's function for the heat equation is not only the influence function for the sources, but also the influence function for the initial condition. If there are no sources, if the boundary conditions are homogeneous, and if the initial condition is a delta function, then the response is the Green's function itself. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ may be determined directly from the diffusion equation with no sources:

$$\boxed{\frac{\partial G}{\partial t} = k \nabla^2 G,} \tag{11.3.22}$$

subject to homogeneous boundary conditions and the concentrated initial conditions at $t = t_0$:

$$G = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{11.3.23}$$

rather than its defining differential equation (11.3.2).

11.3.8 Infinite Space Green's Function for the Diffusion Equation

If there are no boundaries and no sources, $\frac{\partial u}{\partial t} = k\nabla^2 u$ with initial conditions $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$, then (11.3.21) represents the solution of the diffusion equation in terms of its Green's function:

$$u(x,t) = \iiint u(x_0,0)G(x,t;x_0,0)\,d^3x_0 = \iiint f(x_0)G(x,t;x_0,0)\,d^3x_0. \tag{11.3.24}$$

Instead of solving (11.3.22), we note that this initial value problem was analyzed using the Fourier transform in Chapter 10, and we obtained the one-dimensional solution (10.4.6):

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0)\frac{1}{\sqrt{4\pi kt}}e^{-(x-x_0)^2/4kt}dx_0. \tag{11.3.25}$$

By comparing (11.3.24) and (11.3.25), we are able to determine the one-dimensional infinite space Green's function for the diffusion equation:

$$G(x,t;x_0,0) = \frac{1}{\sqrt{4\pi kt}}e^{-(x-x_0)^2/4kt}. \tag{11.3.26}$$

Due to translational invariance, the more general Green's function involves the elapsed time:

$$\boxed{G(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}e^{-(x-x_0)^2/4k(t-t_0)}.} \tag{11.3.27}$$

For n dimensions ($n = 1, 2, 3$), the solution was also obtained in Chapter 10 (10.6.95), and hence the **n-dimensional infinite space Green's function for the diffusion equation** is

$$\boxed{G(\boldsymbol{x},t;\boldsymbol{x_0},t_0) = \left[\frac{1}{4\pi k(t-t_0)}\right]^{\frac{n}{2}} e^{-|\boldsymbol{x}-\boldsymbol{x_0}|^2/4k(t-t_0)}.} \tag{11.3.28}$$

This Green's function shows the symmetry of the response and source positions as long as the elapsed time is the same. As with one-dimensional problems discussed in Section 10.4, the influence of a concentrated heat source diminishes exponentially as one moves away from the source. For small times (t near t_0) the decay is especially strong.

EXAMPLE

In this manner we can obtain the solution of the heat equation with sources on an infinite domain:

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u + Q(\boldsymbol{x}, t)} \tag{11.3.29}$$

$$u(x, 0) = f(x).$$

According to (11.3.21) and (11.3.28), the solution is

$$\boxed{\begin{aligned} u(\boldsymbol{x}, t) = \int_0^t \int_{-\infty}^{\infty} \left[\frac{1}{4\pi k(t - t_0)}\right]^{n/2} & e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/4k(t-t_0)} Q(\boldsymbol{x}_0, t_0)\, d^n x_0 \; dt_0 \\ + \int_{-\infty}^{\infty} \left(\frac{1}{4\pi kt}\right)^{n/2} & e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/4kt} f(\boldsymbol{x}_0) \; d^n x_0. \end{aligned}} \tag{11.3.30}$$

If $Q(\boldsymbol{x}, t) = 0$, this simplifies to the solution obtained in Chapter 10 using Fourier transforms directly without using the Green's function.

11.3.9 Green's Function for the Heat Equation (Semi-Infinite Domain)

In this subsection we obtain the Green's function needed to solve the nonhomogeneous heat equation on the semi-infinite interval in one dimension ($x > 0$), subject to a nonhomogeneous boundary condition at $x = 0$:

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t),} \quad x > 0, \tag{11.3.31}$$

$$\text{BC:} \quad \boxed{u(0, t) = A(t)} \tag{11.3.32}$$

$$\text{IC:} \quad \boxed{u(x, 0) = f(x).} \tag{11.3.33}$$

Equation (11.3.21) can be used to determine $u(x, t)$ if we can obtain the Green's function. The Green's function $G(x, t; x_0, t_0)$ is the response due to a concentrated source:

$$\frac{\partial G}{\partial t} = k\frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\delta(t - t_0).$$

The Green's function satisfies the corresponding *homogeneous* boundary condition,

$$G(0, t; x_0, t_0) = 0,$$

and the causality principle,

$$G(x, t; x_0, t_0) = 0 \qquad \text{for } t < t_0.$$

The Green's function is determined by the method of images (see Section 9.5.8). Instead of a semi-infinite interval with one concentrated positive source at $x = x_0$, we consider an infinite interval with an additional negative source (the image source) located at $x = -x_0$. By symmetry, the temperature G will be zero at $x = 0$ for all t. The Green's function is thus the sum of two infinite space Green's functions:

$$\boxed{\begin{aligned} G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} \Bigg\{ &\exp\left[-\frac{(x - x_0)^2}{4k(t - t_0)}\right] \\ &- \exp\left[-\frac{(x + x_0)^2}{4k(t - t_0)}\right] \Bigg\}. \end{aligned}} \tag{11.3.34}$$

We note that the boundary condition at $x = 0$ is automatically satisfied.

11.3.10 Green's Function for the Heat Equation (on a Finite Region)

For a one-dimensional rod, $0 < x < L$, we have already determined in Chapter 9 the Green's function for the heat equation by the method of eigenfunction expansions. With zero boundary conditions at both ends,

$$\boxed{G(x, t; x_0, t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi x_0}{L} e^{-k(n\pi/L)^2(t - t_0)}.} \tag{11.3.35}$$

We can obtain an alternative representation for this Green's function by utilizing the method of images. By symmetry (see Fig. 11.3.1), the boundary conditions at $x = 0$ and at $x = L$ are satisfied if positive concentrated sources are located at $x = x_0 + 2Ln$ and negative concentrated sources are located at $x = -x_0 + 2Ln$ (for all integers n, $-\infty < n < \infty$). Using the infinite space Green's function, we have an alternative representation of the Green's function for a one-dimensional rod:

$$\boxed{\begin{aligned} G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} \sum_{n=-\infty}^{\infty} \Bigg\{ &\exp\left[-\frac{(x - x_0 - 2Ln)^2}{4k(t - t_0)}\right] \\ &- \exp\left[-\frac{(x + x_0 - 2Ln)^2}{4k(t - t_0)}\right] \Bigg\}. \end{aligned}} \tag{11.3.36}$$

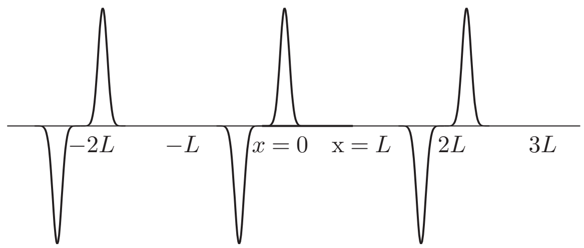

FIGURE 11.3.1 Multiple image sources for the Green's function for the heat equation for a finite one-dimensional rod.

Each form has its own advantage. The eigenfunction expansion, (11.3.35), is an infinite series that converges rapidly if $(t - t_0)k/L^2$ is large. It is thus most useful for $t \gg t_0$. In fact, if $t \gg t_0$,

$$G(x, t; x_0, t_0) \approx \frac{2}{L} \sin \frac{\pi x_0}{L} \sin \frac{\pi x}{L} e^{-k(\pi/L)^2(t-t_0)}.$$

However, if the elapsed time $t - t_0$ is small, then many terms of the infinite series are needed.

Using the method of images, the Green's function is also represented by an infinite series, (11.3.36). The infinite space Green's function (at fixed t) exponentially decays away from the source position,

$$\frac{1}{\sqrt{4\pi k(t - t_0)}} e^{-(x-x_0)^2/4k(t-t_0)}.$$

It decays in space very sharply if t is near t_0. If t is near t_0, then only sources near the response location x are important; sources far away will not be important (if t is near t_0); see Fig. 11.3.1. Thus, the image sources can be neglected if t is near t_0 (and if x or x_0 is neither near the boundaries 0 or L, as is explained in Exercise 11.3.8). As an approximation,

$$G(x, t; x_0, t_0) \approx \frac{1}{\sqrt{4\pi k(t - t_0)}} e^{-(x-x_0)^2/4k(t-t_0)};$$

if t is near t_0, the Green's function with boundaries can be approximated (in regions away from the boundaries) **by the infinite space Green's function**. This means that for small times the boundary can be neglected (away from the boundary).

To be more precise, the effect of every image source is much smaller than the actual source if $L^2/k(t - t_0)$ is large. This yields a better understanding of a "small time" approximation. The Green's function may be approximated by the infinite space Green's function if $t - t_0$ is small (i.e., if $t - t_0 \ll L^2/k$, where L^2/k is a ratio of physically measurable quantities). Alternatively, this approximation is valid for a "long rod" in the sense that $L \gg \sqrt{k(t - t_0)}$.

In summary, the image method yields a rapidly convergent infinite series for the Green's function if $L^2/k(t - t_0) \gg 1$, while the eigenfunction expansion yields a rapidly convergent infinite series representation of the Green's function if $L^2/k(t-t_0) \ll 1$. If $L^2/k(t-t_0)$ is neither small nor large, then the two expansions are competitive, but both require at least a moderate number of terms.

EXERCISES 11.3

11.3.1. Show that for the Green's functions defined by (11.3.2) with (11.3.3),

$$G^*(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t).$$

11.3.2. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad x > 0 \\ u(0,t) &= A(t) \\ u(x,0) &= f(x). \end{aligned}$$

(a) Solve if $A(t) = 0$ and $f(x) = 0$. Simplify this result if $Q(x,t) = 1$.

(b) Solve if $Q(x,t) = 0$ and $A(t) = 0$. Simplify this result if $f(x) = 1$.

***(c)** Solve if $Q(x,t) = 0$ and $f(x) = 0$. Simplify this result if $A(t) = 1$.

***11.3.3.** Determine the Green's function for

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad x > 0 \\ \frac{\partial u}{\partial x}(0,t) &= A(t) \\ u(x,0) &= f(x). \end{aligned}$$

11.3.4. Consider (11.3.34), the Green's function for (11.3.31). Show that the Green's function for this semi-infinite problem may be approximated by the Green's function for the infinite problem if

$$\frac{xx_0}{k(t-t_0)} \gg 1 \qquad \text{(i.e., } t - t_0 \text{ small).}$$

Explain physically why this approximation fails if x or x_0 is near the boundary.

11.3.5. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ u(x,0) &= f(x) \\ \frac{\partial u}{\partial x}(0,t) &= A(t) \\ \frac{\partial u}{\partial x}(L,t) &= B(t). \end{aligned}$$

(a) Solve for the appropriate Green's function using the method of eigenfunction expansion.

(b) Approximate the Green's function of part (a). Under what conditions is your approximation valid?

(c) Solve for the appropriate Green's function using the infinite space Green's function.

(d) Approximate the Green's function of part (c). Under what conditions is your approximation valid?

(e) Solve for $u(x, t)$ in terms of the Green's function.

11.3.6. Determine the Green's function for the heat equation subject to zero boundary conditions at $x = 0$ and $x = L$ by applying the method of eigenfunction expansions directly to the defining differential equation. [*Hint*: The answer is given by (11.3.35).]

11.3.7. Derive the two-dimensional infinite space Green's function by taking two-dimensional transforms.

11.3.8. Derive the three-dimensional infinite space Green's function by taking three-dimensional transforms.