1 Stefan Problem: Circular Case

To describe the IDLA motion more precisely, we need to introduce Stefen problem, which has a time-dependent boundary condition. We have an equation which could describe the IDLA motion with a source at the origin

$$\partial_t\, p = D\nabla^2 p + \delta(\mathbf{x})S \tag{1.1}$$

where S is a constant. In a region $\Omega(t)$ with the boundary $\partial\,\Omega = \{\mathbf{d}(t)\}$.

Consider a Dirichlet boundary condition

$$p|_{\partial\,\Omega} = 0. \tag{1.2}$$

Also,

$$\hat{\mathbf{n}} \cdot \dot{\mathbf{d}} = -\alpha\, \hat{\mathbf{n}} \cdot \nabla p \tag{1.3}$$

where D and α are constants.

1.1 Find $R(t)$ by Conservation of Mass

For the circular case, solve for $p(r,t)$, with a source at origin $\delta(\boldsymbol{x})$

$$\partial_t p = \frac{D}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \delta(\boldsymbol{x})S \tag{1.4}$$

From (7.3) and (7.5), the boundary conditions are

$$p(R(t),t) = 0 \tag{1.5}$$

$$\dot{R} = -\alpha\frac{\partial p}{\partial r}(R(t),t) \tag{1.6}$$

Consider the integral for mass

$$M(t) = \int_{\Omega(t)} p\, dV = 2\pi \int_0^{R(t)} p(r,t) r dr \tag{1.7}$$

By the conservation of mass

$$\dot{M}(t) = 0 \tag{1.8}$$

$$\begin{aligned}
\dot{M} &= 2\pi[\int_0^{R(t)} p_t r dr + p(R(t),t)R\dot{R}] \\
&= 2\pi\int_0^{R}[D\partial_r(r\partial_r p) + Sr\delta(\boldsymbol{x})]dr \\
&= 2\pi D[r\partial_r p]_0^{R(t)} + S\int_{\Omega(t)}\delta(\boldsymbol{x})dV \\
&= 2\pi DR\partial_r p(R(t),t) + S
\end{aligned} \tag{1.9}$$

With (7.8),

$$\begin{aligned}
\dot{M} &= 2\pi DR(-\frac{\dot{R}}{\alpha}) + S \\
&= -\frac{D}{\alpha}\frac{d}{dt}(\pi R^2) + S \\
&= -\frac{D}{\alpha}\dot{A} + S
\end{aligned} \tag{1.10}$$

where

$$A(t) = \pi R^2(t) \tag{1.11}$$

and with (7.10)

$$\dot{A} = \frac{S\alpha}{D} \tag{1.12}$$

Hence,

$$M(t) = -\frac{D}{\alpha}A(t) + St \tag{1.13}$$

$$A(t) = (\frac{S\alpha}{D})t = \pi R^2(t) \rightarrow R(t) = \sqrt{\frac{S\alpha}{\pi D}t} \tag{1.14}$$

1.2 Rescale and Solve for $R(t)$

1.2.1 Rescale r and t

From equation (1.14), we have

$$\dot{R}(t) = \sqrt{\frac{S\alpha}{\pi D}}\,\frac{1}{2}t^{-\frac{1}{2}} \tag{1.15}$$

We rescale r and t as following:

$$\hat{r} = \frac{r}{R(t)} \tag{1.16}$$

$$\hat{t} = t \tag{1.17}$$

Hence,

$$\frac{\partial(\hat{r},\hat{t})}{\partial(r,t)} = \begin{pmatrix} \partial_r \hat{r} & \partial_t \hat{r} \\ \partial_r \hat{t} & \partial_t \hat{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{R} & -\frac{r\dot{R}}{R^2} \\ 0 & 1 \end{pmatrix} \tag{1.18}$$

By plugging (1.18) back into (1.4),

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} + \frac{\partial p}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial t} = \frac{\partial p}{\partial \hat{t}} - \frac{r\dot{R}}{R^2}\frac{\partial p}{\partial \hat{r}} = \frac{\partial p}{\partial \hat{t}} - \frac{\hat{r}\dot{R}}{R}\frac{\partial p}{\partial \hat{r}} \tag{1.19}$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial r} + \frac{\partial p}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial r} = \frac{1}{R}\frac{\partial p}{\partial \hat{r}} \tag{1.20}$$

Hence,

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) = \frac{1}{\hat{r}R}\frac{1}{R}\frac{\partial}{\partial \hat{r}}(\cancel{R}\hat{r}\frac{1}{\cancel{R}}\frac{\partial p}{\partial \hat{r}}) = \frac{1}{R^2}\frac{1}{\hat{r}}\frac{\partial}{\partial p}(\hat{r}\frac{\partial r}{\partial \hat{r}}) = \frac{\hat{\nabla}^2 p}{R^2} \tag{1.21}$$

1.2.2 Rescale $\delta(\boldsymbol{x})$

With $\delta(\alpha x) = \frac{\delta(x)}{\alpha}$, we can write

$$(\hat{x},\hat{y}) = \frac{(x,y)}{R} \tag{1.22}$$

With $\delta(\boldsymbol{x}) = \delta(x)\delta(y)$, we can write

$$\delta(R\hat{\boldsymbol{x}}) = \delta(R\hat{x})\delta(R\hat{y}) \tag{1.23}$$

Thus,

$$\delta(\boldsymbol{x}) = \frac{\delta(r)}{2\pi r} = \frac{\delta(R\,\hat{\boldsymbol{r}})}{2\pi R\,\hat{r}} = \frac{\delta(\hat{r})}{2\pi R^2\hat{r}} = \frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.24}$$

and

$$\delta(\boldsymbol{x}) = \frac{\delta(R\hat{x})\delta(R\hat{y})}{R^2} = \frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.25}$$

1.2.3 Solve for $R(t)$

We can then plug (1.19), (1.21), and (1.25) back into (1.4),

$$\frac{\partial p}{\partial \hat{t}} - \frac{\hat{r}\dot{R}}{R}\frac{\partial p}{\partial \hat{r}} = D\frac{\hat{\nabla}^2 p}{R^2} + S\frac{\delta(\hat{\boldsymbol{x}})}{R^2} \tag{1.26}$$

Consider $\frac{\dot{R}}{R^2}$ to see how it changes with respect to t

$$\frac{\dot{R}}{R^2} = \sqrt{\frac{\pi D}{S\alpha}}\frac{\frac{1}{2}t^{1/2}}{t} = \sqrt{\frac{\pi D}{4S\alpha}}t^{-1/2} \tag{1.27}$$

From (1.6), we have

$$\dot{R} = -\alpha\frac{\partial p}{\partial r} = -\frac{\alpha}{R}\frac{\partial p}{\partial \hat{r}} \tag{1.28}$$

Consider $\frac{\partial p}{\partial \hat{t}} = 0$, neglect $\frac{r\dot{R}}{R^2}\frac{\partial p}{\partial \hat{r}}$ as $\hat{t} \to \infty$

$$0 = D\hat{\nabla}^2 p + D\delta(\hat{\boldsymbol{x}}) \tag{1.29}$$

The solution is then

$$p = -\frac{S}{2\pi D}\log(\hat{r}) + B(t) \tag{1.30}$$

With the boundary condition $p(1,t) = 0$, $B(t) = 0$.

$$\dot{R} = -\alpha(\frac{-S}{2\pi RD}) \tag{1.31}$$

$$\frac{1}{2}(R^2 - R_0^2) = \frac{\alpha S}{2\pi D}t \tag{1.32}$$

With $R_0 = 0$,

$$R(t) = \sqrt{\frac{\alpha S}{\pi D}t} = \sqrt{\frac{\alpha S}{\pi D^2}D\,t} \tag{1.33}$$

which is the same as the result we obtained by the conservation of mass.

1.3 Solve for $p(r, t)$

We rewrite (1.26) and it becomes

$$R^2\frac{\partial p}{\partial \hat{t}} - (R\dot{R})\hat{r}\frac{\partial p}{\partial \hat{r}} = D\hat{\nabla}^2 p + S\delta(\hat{\boldsymbol{x}}) \tag{1.34}$$

Consider $\frac{\partial p}{\partial \hat{t}} \to 0$. Given (1.14) and (1.15), we know that the quantity $R\dot{R}$ is a constant. Let $R\dot{R} = Dc$,

$$\frac{1}{2}(R^2 - R_0^2) = Dct \tag{1.35}$$

With $R_0 = 0$,

$$R(t) = \sqrt{2Dct} \tag{1.36}$$

(1.34) then becomes

$$-D\,c\,\hat{r}\frac{\partial p}{\partial \hat{r}} = D\hat{\nabla}^2 p + S\delta(\hat{\boldsymbol{x}}) \tag{1.37}$$

Now we obtain an equation with respect to $\hat{r}$ only. We could then replace all $\hat{r}$ with r. Since we are solving for the circular case, the space part of the solution p would only depend on r. With $\hat{\boldsymbol{x}} \neq \mathbf{0}$ resulting in $S\delta(\hat{\boldsymbol{x}}) = 0$, the equation becomes

$$-crp' = \frac{1}{r}(rp')' \tag{1.38}$$

Let $q = rp'$, solve for q,

$$q = q_0 e^{-\frac{1}{2}cr^2} \tag{1.39}$$

Thus,

$$p' = \frac{q}{r} = \frac{1}{r} q_0 e^{-\frac{1}{2}cr^2} \tag{1.40}$$

Since $p' \to -\frac{S}{2\pi Dr}$ as $r \to 0$,

$$q_0 = -\frac{S}{2\pi D} \tag{1.41}$$

We then integrate (1.40),

$$p(r) = q_0 \int r^{-1} e^{-\frac{1}{2}cr^2} dr + p_0(t) = \frac{q_0}{2} \text{Ei}(-\frac{cr^2}{2}) + p_0 \tag{1.42}$$

where $\text{Ei}(-\frac{cr^2}{2})$ represents an exponential integral function. With $p(R) = 0$, we have

$$p_0 = -\frac{q_0}{2} \text{Ei}(-\frac{cR^2}{2}) \tag{1.43}$$

Thus,

$$p(r) = \frac{1}{2} q_0 (\text{Ei}(-\frac{cr^2}{2}) - \text{Ei}(-\frac{cR^2}{2})) \tag{1.44}$$

We could then solve for c by rescaling (1.6),

$$\dot{R} = -\frac{\alpha}{R} p'(1, \hat{t}) = -\frac{\alpha}{R}(-\frac{S}{2\pi D}) e^{-\frac{1}{2}c} \tag{1.45}$$

Hence,

$$c = (\frac{\alpha S}{2\pi D^2}) e^{-\frac{1}{2}c} \tag{1.46}$$

We could check if the quantity $\frac{\alpha S}{D^2}$ is dimensionless. The units of p and $\frac{\alpha}{L^3}$ are

$$[p] = \frac{1}{L^2} \tag{1.47}$$

$$\frac{[\alpha]}{L^3} = \frac{L}{T} \tag{1.48}$$

Thus,

$$\frac{\alpha S}{D^2} = \frac{L^4}{T} \frac{T^2}{L^4} \frac{1}{T} = 1 \tag{1.49}$$

Hence, the quantity $\frac{\alpha S}{D^2}$ is dimensionless.

2 Stefan Problem: General Case

Consider (1.1) with time dependent term

$$\partial_t p + U \partial_y p = D \Delta p + S \delta(\boldsymbol{r}) \tag{2.1}$$

with the Dirichlet boundary condition

$$p(R(\theta, t), \theta, t) = 0 \tag{2.2}$$

To rewrite the condition for the flux, we need to consider the expression for the unit normal vector $\hat{\boldsymbol{n}}$. At the boundary,

$$\boldsymbol{r}|_{\partial \Omega} = R(\theta, t) \hat{\boldsymbol{r}} \tag{2.3}$$

Hence, the tangent $\boldsymbol{t}$ would be

$$\boldsymbol{t} = \partial_\theta \boldsymbol{r} = \partial_\theta R \hat{\boldsymbol{r}} + R \partial_\theta \hat{\boldsymbol{r}} = \partial_\theta R \hat{\boldsymbol{r}} + R \hat{\boldsymbol{\theta}} \tag{2.4}$$

where we use

$$\hat{\boldsymbol{r}}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.5}$$

$$\hat{\boldsymbol{\theta}}(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{2.6}$$

The normal vector can then be written as

$$\boldsymbol{n} = \boldsymbol{t} \times \hat{\boldsymbol{z}} = \partial_\theta R (\hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}}) + R (\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{z}}) = -\partial_\theta R \hat{\boldsymbol{\theta}} + R \hat{\boldsymbol{r}} \tag{2.7}$$

Thus,

$$\hat{\boldsymbol{n}} = \frac{R \hat{\boldsymbol{r}} - \partial_\theta R \hat{\boldsymbol{\theta}}}{(R^2 + (\partial_\theta R)^2)^{\frac{1}{2}}} \tag{2.8}$$

where we use $\|\boldsymbol{n}\| = R^2 + (\partial_\theta R)^2$ We can then write down the growth of the boundary

$$\hat{\boldsymbol{n}} \cdot \partial_t \boldsymbol{r}|_{\partial\,\Omega} = -\alpha \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} p \tag{2.9}$$

with the boundary (1.52). Plug (1.52) into (1.58). At the boundary $r = R(\theta, t)$,

$$\partial_t R = -\alpha(R\partial_r p - \frac{\partial_\theta R}{R}\partial_\theta p) \tag{2.10}$$

where we use

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\partial_r + \frac{1}{r}\hat{\boldsymbol{\theta}}\partial_\theta \tag{2.11}$$

Rescale the variables with

$$m = \frac{x}{t^{1/3}},\; n = \frac{y}{t^{2/3}} \tag{2.12}$$

where the factors of $1/3$ and $2/3$ are what we obtained numerically. We can then rewrite (2.1) with the following:

$$\Delta p = \partial_x^2 p + \partial_y^2 p = t^{-2/3}\partial_m^2 p + t^{-4/3}\partial_n^2 p \tag{2.13}$$

$$U\partial_y p = Ut^{-2/3}\partial_n p \tag{2.14}$$

$$\partial_t p = \partial_t p + \partial_m p \frac{\partial m}{\partial t} + \partial_n p \frac{\partial n}{\partial t} = \partial_t p - \frac{m}{3t}\partial_m p - \frac{2y}{3t}\partial_n p \tag{2.15}$$