

# 1 Stefan Problem (for the circular case)

To describe the IDLA motion more precisely, we need to introduce Stefan problem, which has a time-dependent boundary condition. We have an equation which could describe the IDLA motion with a source at the origin

$$\partial_t p = D \nabla^2 p + \delta(\mathbf{x}) S \quad (1.1)$$

where  $S$  is a constant. In a region  $\Omega(t)$  with the boundary  $\partial \Omega = \{\mathbf{d}(t)\}$ .

Consider a Dirichlet boundary condition

$$p|_{\partial \Omega} = 0. \quad (1.2)$$

Also,

$$\hat{\mathbf{n}} \cdot \dot{\mathbf{d}} = -\alpha \hat{\mathbf{n}} \cdot \nabla p \quad (1.3)$$

where  $D$  and  $\alpha$  are constants.

## 1.1 Find $R(t)$ by Conservation of Mass

For the circular case, solve for  $p(r, t)$ , with a source at origin  $\delta(\mathbf{x})$

$$\partial_t p = \frac{D}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \delta(\mathbf{x}) S \quad (1.4)$$

From (7.3) and (7.5), the boundary conditions are

$$p(R(t), t) = 0 \quad (1.5)$$

$$\dot{R} = -\alpha \frac{\partial p}{\partial r}(R(t), t) \quad (1.6)$$

Consider the integral for mass

$$M(t) = \int_{\Omega(t)} p \, dV = 2\pi \int_0^{R(t)} p(r, t) r \, dr \quad (1.7)$$

By the conservation of mass

$$\dot{M}(t) = 0 \quad (1.8)$$

$$\begin{aligned} \dot{M} &= 2\pi \left[ \int_0^{R(t)} p_t r dr + p(R(t), t) R \dot{R} \right] \\ &= 2\pi \int_0^R [D \partial_r (r \partial_r p) + S r \delta(\mathbf{x})] dr \\ &= 2\pi D [r \partial_r p]_0^{R(t)} + S \int_{\Omega(t)} \delta(\mathbf{x}) dV \\ &= 2\pi D R \partial_r p(R(t), t) + S \end{aligned} \quad (1.9)$$

With (7.8),

$$\begin{aligned} \dot{M} &= 2\pi D R \left( -\frac{\dot{R}}{\alpha} \right) + S \\ &= -\frac{D}{\alpha} \frac{d}{dt} (\pi R^2) + S \\ &= -\frac{D}{\alpha} \dot{A} + S \end{aligned} \quad (1.10)$$

where

$$A(t) = \pi R^2(t) \quad (1.11)$$

and with (7.10)

$$\dot{A} = \frac{S\alpha}{D} \quad (1.12)$$

Hence,

$$M(t) = -\frac{D}{\alpha} A(t) + St \quad (1.13)$$

$$A(t) = \left( \frac{S\alpha}{D} \right) t = \pi R^2(t) \rightarrow R(t) = \sqrt{\frac{S\alpha}{\pi D}} t \quad (1.14)$$

## 1.2 Rescale and Solve for $R(t)$

### 1.2.1 Rescale $r$ and $t$

From equation (1.14), we have

$$\dot{R}(t) = \sqrt{\frac{S\alpha}{\pi D}} \frac{1}{2} t^{-\frac{1}{2}} \quad (1.15)$$

We rescale  $r$  and  $t$  as following:

$$\hat{r} = \frac{r}{R(t)} \quad (1.16)$$

$$\hat{t} = t \quad (1.17)$$

Hence,

$$\frac{\partial(\hat{r}, \hat{t})}{\partial(r, t)} = \begin{pmatrix} \partial_r \hat{r} & \partial_t \hat{r} \\ \partial_r \hat{t} & \partial_t \hat{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{R} & -\frac{r\dot{R}}{R^2} \\ 0 & 1 \end{pmatrix} \quad (1.18)$$

By plugging (1.18) back into (1.4),

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} + \frac{\partial p}{\partial \hat{r}} \frac{\partial \hat{r}}{\partial t} = \frac{\partial p}{\partial \hat{t}} - \frac{r\dot{R}}{R^2} \frac{\partial p}{\partial \hat{r}} = \frac{\partial p}{\partial \hat{t}} - \frac{\hat{r}\dot{R}}{R} \frac{\partial p}{\partial \hat{r}} \quad (1.19)$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial r} + \frac{\partial p}{\partial \hat{r}} \frac{\partial \hat{r}}{\partial r} = \frac{1}{R} \frac{\partial p}{\partial \hat{r}} \quad (1.20)$$

Hence,

$$\nabla^2 p = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{1}{\hat{r} R} \frac{1}{R} \frac{\partial}{\partial \hat{r}} \left( R \hat{r} \frac{1}{R} \frac{\partial p}{\partial \hat{r}} \right) = \frac{1}{R^2} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial p}{\partial \hat{r}} \right) = \frac{\hat{\nabla}^2 p}{R^2} \quad (1.21)$$

### 1.2.2 Rescale $\delta(\mathbf{x})$

With  $\delta(\alpha x) = \frac{\delta(x)}{\alpha}$ , we can write

$$(\hat{x}, \hat{y}) = \frac{(x, y)}{R} \quad (1.22)$$

With  $\delta(\mathbf{x}) = \delta(x)\delta(y)$ , we can write

$$\delta(R\hat{\mathbf{x}}) = \delta(R\hat{x})\delta(R\hat{y}) \quad (1.23)$$

Thus,

$$\delta(\mathbf{x}) = \frac{\delta(r)}{2\pi r} = \frac{\delta(R\hat{r})}{2\pi R\hat{r}} = \frac{\delta(\hat{r})}{2\pi R^2\hat{r}} = \frac{\delta(\hat{\mathbf{x}})}{R^2} \quad (1.24)$$

and

$$\delta(\mathbf{x}) = \frac{\delta(R\hat{x})\delta(R\hat{y})}{R^2} = \frac{\delta(\hat{\mathbf{x}})}{R^2} \quad (1.25)$$

### 1.2.3 Solve for $R(t)$

We can then plug (1.19), (1.21), and (1.25) back into (1.4),

$$\frac{\partial p}{\partial \hat{t}} - \frac{\hat{r}\dot{R}}{R} \frac{\partial p}{\partial \hat{r}} = D \frac{\hat{\nabla}^2 p}{R^2} + S \frac{\delta(\hat{\mathbf{x}})}{R^2} \quad (1.26)$$

Consider  $\frac{\dot{R}}{R^2}$  to see how it changes with respect to  $t$

$$\frac{\dot{R}}{R^2} = \sqrt{\frac{\pi D}{S\alpha}} \frac{t^{1/2}}{t} = \sqrt{\frac{\pi D}{4S\alpha}} t^{-1/2} \quad (1.27)$$

From (1.6), we have

$$\dot{R} = -\alpha \frac{\partial p}{\partial r} = -\frac{\alpha}{R} \frac{\partial p}{\partial \hat{r}} \quad (1.28)$$

Consider  $\frac{\partial p}{\partial \hat{t}} = 0$ , neglect  $\frac{r\dot{R}}{R^2} \frac{\partial p}{\partial \hat{r}}$  as  $\hat{t} \rightarrow \infty$

$$0 = D \hat{\nabla}^2 p + D \delta(\hat{\mathbf{x}}) \quad (1.29)$$

The solution is then

$$p = -\frac{S}{2\pi D} \log(\hat{r}) + B(t) \quad (1.30)$$

With the boundary condition  $p(1, t) = 0$ ,  $B(t) = 0$ .

$$\dot{R} = -\alpha \left( \frac{-S}{2\pi R D} \right) \quad (1.31)$$

$$\frac{1}{2}(R^2 - R_0^2) = \frac{\alpha S}{2\pi D} t \quad (1.32)$$

With  $R_0 = 0$ ,

$$R(t) = \sqrt{\frac{\alpha S}{\pi D} t} = \sqrt{\frac{\alpha S}{\pi D^2} D t} \quad (1.33)$$

which is the same as the result we obtained by the conservation of mass.

#### 1.2.4 Solve for $p(r, t)$

We rewrite (1.26) and it becomes

$$R^2 \frac{\partial p}{\partial \hat{t}} - (R\dot{R})\hat{r} \frac{\partial p}{\partial \hat{r}} = D\hat{\nabla}^2 p + S\delta(\hat{\mathbf{x}}) \quad (1.34)$$

Consider  $\frac{\partial p}{\partial \hat{t}} \rightarrow 0$ . Given (1.14) and (1.15), we know that the quantity  $R\dot{R}$  is a constant. Let  $R\dot{R} = Dc$ ,

$$\frac{1}{2}(R^2 - R_0^2) = Dct \quad (1.35)$$

With  $R_0 = 0$ ,

$$R(t) = \sqrt{2Dct} \quad (1.36)$$

(1.34) then becomes

$$-D c \hat{r} \frac{\partial p}{\partial \hat{r}} = D\hat{\nabla}^2 p + S\delta(\hat{\mathbf{x}}) \quad (1.37)$$

Now we obtain an equation with respect to  $\hat{r}$  only. We could then replace all  $\hat{r}$  with  $r$ . Since we are solving for the circular case, the space part of the solution  $p$  would only depend on  $r$ . With  $\hat{\mathbf{x}} \neq \mathbf{0}$  resulting in  $S\delta(\hat{\mathbf{x}}) = 0$ , the equation becomes

$$-crp' = \frac{1}{r}(rp')' \quad (1.38)$$

Let  $q = rp'$ , solve for  $q$ ,

$$q = q_0 e^{-\frac{1}{2}cr^2} \quad (1.39)$$

Thus,

$$p' = \frac{q}{r} = \frac{1}{r} q_0 e^{-\frac{1}{2}cr^2} \quad (1.40)$$

Since  $p' \rightarrow -\frac{S}{2\pi Dr}$  as  $r \rightarrow 0$ ,

$$q_0 = -\frac{S}{2\pi D} \quad (1.41)$$

We then integrate (1.40),

$$p(r) = q_0 \int r^{-1} e^{-\frac{1}{2}cr^2} dr + p_0(t) = \frac{q_0}{2} \text{Ei}\left(-\frac{cr^2}{2}\right) + p_0 \quad (1.42)$$

where  $\text{Ei}\left(-\frac{cr^2}{2}\right)$  represents an exponential integral function. With  $p(R) = 0$ , we have

$$p_0 = -\frac{q_0}{2} \text{Ei}\left(-\frac{cR^2}{2}\right) \quad (1.43)$$

Thus,

$$p(r) = \frac{1}{2} q_0 (\text{Ei}\left(-\frac{cr^2}{2}\right) - \text{Ei}\left(-\frac{cR^2}{2}\right)) \quad (1.44)$$

We could then solve for  $c$  by rescaling (1.6),

$$\dot{R} = -\frac{\alpha}{R} p'(1, \hat{t}) = -\frac{\alpha}{R} \left(-\frac{S}{2\pi D}\right) e^{-\frac{1}{2}c} \quad (1.45)$$

Hence,

$$c = \left(\frac{\alpha S}{2\pi D^2}\right) e^{-\frac{1}{2}c} \quad (1.46)$$

We could check if the quantity  $\frac{\alpha S}{D^2}$  is dimensionless. The units of  $p$  and  $\frac{\alpha}{L^3}$  are

$$[p] = \frac{1}{L^2} \quad (1.47)$$

$$\frac{[\alpha]}{L^3} = \frac{L}{T} \quad (1.48)$$

Thus,

$$\frac{\alpha S}{D^2} = \frac{L^4 T^2}{T L^4 T} = 1 \quad (1.49)$$

Hence, the quantity  $\frac{\alpha S}{D^2}$  is dimensionless.