Asymptotic level sets

11/20/18

$$\phi(x,y,t) = \int_0^t \frac{1}{4\pi D s} e^{-(x^2+(y-Us)^2)/4Ds}\, ds$$

$$x = X t^\alpha, \quad y = Y t^\beta, \quad s = \xi t^\gamma \qquad \xi = s t^{-\gamma} \quad 0<\gamma<1$$

$$h = \frac{1}{s}(x^2+(y-Us)^2) = \frac{1}{\xi t^\gamma}\left(X^2 t^{2\alpha} + (Y t^\beta - U\xi t^\gamma)^2\right)$$

Need $\alpha = 1/2$. Not gonna interact nicely with $t^{2\beta}$, $t^{2\gamma}$

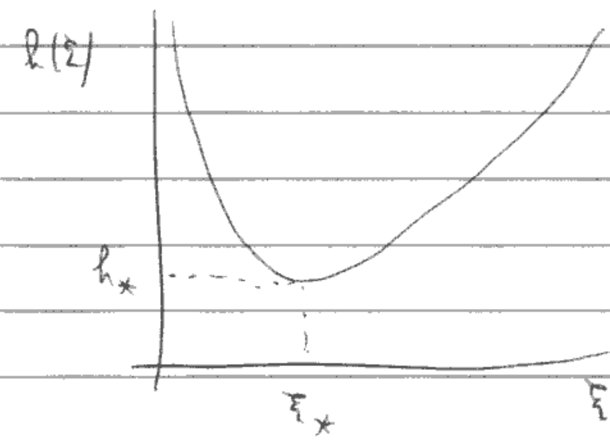
Put $\alpha = \beta = \gamma = \frac{1}{2}$?

$$h = \frac{\sqrt{t}}{\xi}\left(X^2 + (Y-U\xi)^2\right)$$

$$\phi(Xt^{1/2}, Yt^{1/2}, t) = \int_0^{t^{1/2}} \frac{1}{4\pi D\xi} e^{-\frac{t^{1/2}}{4D\xi}(X^2+(Y-U\xi)^2)}\, d\xi$$

Use ~~stationary point~~ Laplace's method.

$$h(\xi) = \frac{1}{4D\xi}\left(X^2 + (Y-U\xi)^2\right)$$



$$h'(\xi) = -\frac{1}{4D\xi^2}\left(X^2+Y^2-U^2\xi^2\right)$$

$$\xi_* = \frac{\sqrt{X^2+Y^2}}{U}$$

$$h_* = \frac{U}{2D}\left(\sqrt{X^2+Y^2} - Y\right)$$

$$h(\xi) = h_* + \underbrace{\cancel{h'(\xi_*)(\xi-\xi_*)}}_{0} + \frac{1}{2}h''(\xi_*)(\xi-\xi_*)^2 + O\left((\xi-\xi_*)^3\right)$$

$t^{1/2}$

$$\phi(Xt^{1/2}, Yt^{1/2}, t) = \int_0^\infty \frac{1}{4\pi D\xi} e^{-\sqrt{t}\,h(\xi)}\, d\xi$$

Laplace's method: $$\int_a^b g(x)\, e^{-M h(x)}\, dx \approx \sqrt{\frac{2\pi}{M|h''(x_*)|}}\, g(x_*)\, e^{-M h(x_*)} \qquad M\to\infty$$

Here: $M = \sqrt{t}$, $h(x_*)$ above, $h''_* = \dfrac{U^3}{2D\sqrt{X^2+Y^2}}$

$$\phi(Xt^{1/2}, Yt^{1/2}, t) \approx \sqrt{\frac{2\pi}{\sqrt{t}}\left(\frac{2D\sqrt{X^2+Y^2}}{U^3}\right)}\,\frac{1}{4\pi D\xi_*}\, e^{-\sqrt{t}\,h_*}$$

I think I'm just getting steady solution! Makes sense