Green's function for the advection-diffusion equation

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1 Green's function from Bessel's equation

$$U\,\partial_y\phi = D\,\Delta\phi + \delta(\boldsymbol{r}) \tag{1}$$

Let $\psi(\boldsymbol{r}) = \mathrm{e}^{-Uy/2D}\phi(\boldsymbol{r})$; then ψ satisfies

$$-\,D\,\Delta\psi + \frac{U^2}{4D}\,\psi = \delta(\boldsymbol{r}). \tag{2}$$

This is a Hemholtz equation. For convenience we define the inverse length scale $a = U/2D$:

$$-\,\Delta\psi + a^2\,\psi = \delta(\boldsymbol{r})/D. \tag{3}$$

The solution to this must be dependent on r only, so we try to solve:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\psi}{\partial r}\right) - a^2\psi = 0. \tag{4}$$

This is a modified Bessel equation; the solution that decays as $r \to \infty$ is

$$\psi(r) = c_1\,K_0(ar). \tag{5}$$

The Green's function we seek is then of the form

$$\phi(\boldsymbol{r}) = \mathrm{e}^{ay}\,\psi(r) = c_1\,\mathrm{e}^{ay}\,K_0(ar). \tag{6}$$

To find the constant a, we observe that for small x

$$K_0(x) \sim -\log x, \qquad x \to 0. \tag{7}$$

Hence, for the Green's function to limit to that for the Laplacian with a point source of strength $-1/D$ as $r \to 0$, we require

$$\phi(\boldsymbol{r}) \sim -c_1 \log r \sim -\frac{1}{2\pi D} \log r, \qquad r \to 0. \tag{8}$$

We obtain finally the Green's function

$$\phi(\boldsymbol{r}) = \frac{1}{2\pi D} \, \mathrm{e}^{ay} \, K_0(ar), \qquad a = \frac{U}{2D}. \tag{9}$$

A useful asymptotic form for this is

$$\phi(\boldsymbol{r}) \sim \frac{\mathrm{e}^{a(y-r)}}{\sqrt{4\pi D U r}}, \qquad r \to \infty. \tag{10}$$

In particular, this has very different limits along the $x = 0$ axis dependent on whether we are upstream ($y \to -\infty$) or downstream ($y \to \infty$):

$$\phi \sim \frac{1}{\sqrt{4\pi D U |y|}} \begin{cases} 1\,, & y \to \infty; \\ \mathrm{e}^{-2a|y|}\,, & y \to -\infty. \end{cases} \tag{11}$$

2 The shape of contours

To find the shape of large contours $\phi(\boldsymbol{r}) = c/\sqrt{4\pi D U} > 0$ we use the asymptotic form Eq. (10):

$$\mathrm{e}^{a(y-r)} = c\,\sqrt{r}. \tag{12}$$

The furthest extent of the contour for $y > 0$ is obtained by setting $r = y = y_{\max}$ and then solving

$$1 = c\,\sqrt{y_{\max}} \quad \Longleftrightarrow \quad y_{\max} = 1/c^2. \tag{13}$$

We let furthest extent of the countour for $y < 0$ be $y = -y_{\min}$. Assuming that $y_{\min}$ is large and that Eq. (10) is still valid, we set $r = -y = y_{\min}$ and then solve

$$\mathrm{e}^{-4ay_{\min}} = c^2\, y_{\min}. \tag{14}$$

The solution to this is given in terms of the Lambert W-function as

$$y_{\min} = \frac{1}{4a}\, W(4a/c^2). \tag{15}$$

For small c, this has asymptotic expansion

$$y_{\min} \sim \frac{1}{4a}\left(\log(4a/c^2) - \log\log(4a/c^2)\right), \qquad c \to 0. \tag{16}$$

This is indeed getting slowly larger as $c \to 0$, consistent with using the approximation Eq. (10), but the extent of the contour in y is completely dominated by $y_{\max}$. We can thus safely use $y_{\min} = 0$ when computing the area.

Now that we know the limits in y, to compute the area as a function of y we use Eq. (12)

$$\mathrm{e}^{a(y-\sqrt{x^2+y^2})} = c\,(x^2+y^2)^{1/4} \tag{17}$$

and try to solve for x. This is hard to do directly, but we expect y to be much larger than x over most of the contour. We thus expand

$$\frac{\mathrm{e}^{ay(1-\sqrt{1+(x/y)^2})}}{(x^2+y^2)^{1/4}} = c \tag{18}$$

in small x/y:

$$\frac{\mathrm{e}^{ay\left(-\frac{1}{2}(x/y)^2+\mathrm{O}(x^4/y^4)\right)}}{\sqrt{y}\,(1+\mathrm{O}(x^2/y^2))^{1/4}} = c\,. \tag{19}$$

We then expand the small argument of the exponential,

$$\frac{1-\frac{1}{2}a(x^2/y)+\mathrm{O}(x^4/y^3)}{\sqrt{y}\,(1+\mathrm{O}(x^2/y^2))^{1/4}} = c\,, \tag{20}$$

or

$$\frac{1}{\sqrt{y}}\left(1-\tfrac{1}{2}a(x^2/y)+\mathrm{O}(x^4/y^3)\right)(1+\mathrm{O}(x^2/y^2)) = c\,. \tag{21}$$

Keeping only the dominant error term, we have

$$1-\tfrac{1}{2}a(x^2/y)+\mathrm{O}(x^2/y^2) = c\,\sqrt{y}. \tag{22}$$

We solve this to obtain

$$x^2 = \frac{2y}{a}\,(1-c\sqrt{y}\,)+\mathrm{O}(x^2/y^2). \tag{23}$$

This means that

$$x^2 = \mathrm{O}(y\,,\,y(1-c\sqrt{y})), \tag{24}$$

i.e., x is of order $y^{1/2}$, except near $y_{\max}$ where it is smaller by a factor $(1-c\sqrt{y}) \geq 0$. This still implies that $x = \mathrm{O}(y^{1/2})$, so we take that as an estimate. We obtain finally

$$x_{\max}(y) = \sqrt{2y\left(1-c\sqrt{y}\,\right)/a} + \mathrm{O}(y^{-1}) \tag{25}$$

for the shape of the contour as a function of y. Now we can compute the estimated area as

$$A = 2\int_0^{y_{\max}} x_{\max}(y)\,\mathrm{d}y = \frac{64\sqrt{2}}{105\sqrt{a}}\,c^{-3}. \tag{26}$$

The crucial observation is that $A \sim c^{-3}$. Hence,

$$y_{\max} \sim c^{-2} \sim A^{2/3} \tag{27}$$

as we find in the simulations.

(*** Note that for us the area is twice the number of particles, since we have a checkerboard pattern. Also, I think $a = 1$ is the right scaling, since $UT = 1$ gridpoint, and $\sqrt{2DT} = 1$ gridpoint as well.)

3 Using Fourier transform

(*** This is not necessary.)

We take the Fourier transform of Eq. (3):

$$k^2\,\hat{\psi} + a^2\,\hat{\psi} = 1/D. \tag{28}$$

Here we define the forward transform

$$\hat{\psi}(\boldsymbol{k}) = \int \psi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r} \tag{29}$$

and the inverse transform

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi}\int \hat{\psi}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}. \tag{30}$$

Solving Eq. (28) for $\hat{\psi}$ and inverting, we have

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi D}\int \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2+a^2}\,\mathrm{d}\boldsymbol{k}. \tag{31}$$

We can transform the $\boldsymbol{k}$ integral to polar coordinates with $\boldsymbol{k} = k\,(\cos\varphi, \sin\varphi)$ and $\boldsymbol{r} = r\,(\cos\theta, \sin\theta)$:

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi D}\int_0^{2\pi}\int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr\cos(\varphi-\theta)}}{k^2+a^2}\,k\,\mathrm{d}k\,\mathrm{d}\varphi. \tag{32}$$

Because of periodicity, we can translate the φ integral by θ and obtain the θ-independent form

$$\psi(r) = \frac{1}{2\pi D}\int_0^{2\pi}\int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr\cos\varphi}}{k^2+a^2}\,k\,\mathrm{d}k\,\mathrm{d}\varphi. \tag{33}$$

Mathematica can carry this one out directly to give

$$\psi(r) = \frac{1}{D}\,K_0(ar). \tag{34}$$