

# Green's function for the advection-diffusion equation

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## 1 Green's function from Bessel's equation

$$U \partial_y \phi = D \Delta \phi + \delta(\mathbf{r}) \quad (1)$$

Let  $\psi(\mathbf{r}) = e^{-Uy/2D} \phi(\mathbf{r})$ ; then  $\psi$  satisfies

$$-D \Delta \psi + \frac{U^2}{4D} \psi = \delta(\mathbf{r}). \quad (2)$$

This is a Helmholtz equation. For convenience we define the inverse length scale  $a = U/2D$ :

$$-\Delta \psi + a^2 \psi = \delta(\mathbf{r})/D. \quad (3)$$

The solution to this must be dependent on  $r$  only, so we try to solve:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{\partial \psi}{\partial r} \right) - a^2 \psi = 0. \quad (4)$$

This is a modified Bessel equation; the solution that decays as  $r \rightarrow \infty$  is

$$\psi(r) = c_1 K_0(ar). \quad (5)$$

The Green's function we seek is then of the form

$$\phi(\mathbf{r}) = e^{ay} \psi(r) = c_1 e^{ay} K_0(ar). \quad (6)$$

To find the constant  $a$ , we observe that for small  $x$

$$K_0(x) \sim -\log x, \quad x \rightarrow 0. \quad (7)$$

Hence, for the Green's function to limit to that for the Laplacian with a point source of strength  $-1/D$  as  $r \rightarrow 0$ , we require

$$\phi(\mathbf{r}) \sim -c_1 \log r \sim -\frac{1}{2\pi D} \log r, \quad r \rightarrow 0. \quad (8)$$

We obtain finally the Green's function

$$\phi(\mathbf{r}) = \frac{1}{2\pi D} e^{ay} K_0(ar), \quad a = \frac{U}{2D}. \quad (9)$$

A useful asymptotic form for this is

$$\phi(\mathbf{r}) \sim \frac{e^{a(y-r)}}{\sqrt{4\pi DU r}}, \quad r \rightarrow \infty. \quad (10)$$

In particular, this has very different limits along the  $x = 0$  axis dependent on whether we are upstream ( $y \rightarrow -\infty$ ) or downstream ( $y \rightarrow \infty$ ):

$$\phi \sim \frac{1}{\sqrt{4\pi DU |y|}} \begin{cases} 1, & y \rightarrow \infty; \\ e^{-a|y|}, & y \rightarrow -\infty. \end{cases} \quad (11)$$

## 2 The shape of contours

To find the shape of large contours  $\phi(\mathbf{r}) = c/\sqrt{4\pi DU} > 0$  we use the asymptotic form Eq. (10):

$$e^{a(y-r)} = c\sqrt{r}. \quad (12)$$

The furthest extent of the contour for  $y > 0$  is obtained by setting  $r = y = y_{\max}$  and then solving

$$1 = c\sqrt{y_{\max}} \iff y_{\max} = 1/c^2. \quad (13)$$

We let furthest extent of the countour for  $y < 0$  be  $y = -y_{\min}$ . Assuming that  $y_{\min}$  is large and that Eq. (10) is still valid, we set  $r = -y = y_{\min}$  and then solve

$$e^{-2ay_{\min}} = c^2 y_{\min}. \quad (14)$$

The solution to this is given in terms of the Lambert  $W$ -function as

$$y_{\min} = \frac{1}{2a} W(2a/c^2). \quad (15)$$

For small  $c$ , this has asymptotic expansion

$$y_{\min} \sim \frac{1}{2a} \left( \log(2a/c^2) - \log \log(2a/c^2) \right), \quad c \rightarrow 0. \quad (16)$$

This is indeed getting slowly larger as  $c \rightarrow 0$ , consistent with using the approximation Eq. (10), but the extent of the contour in  $y$  is completely dominated by  $y_{\max}$ . We can thus safely use  $y_{\min} = 0$  when computing the area.

Now that we know the limits in  $y$ , to compute the area as a function of  $y$  we use Eq. (12)

$$e^{a(y - \sqrt{x^2 + y^2})} = c (x^2 + y^2)^{1/4} \quad (17)$$

and try to solve for  $x$ . This is hard to do directly, but we expect  $y$  to be much large over most of the contour. We thus expand

$$\frac{e^{ay(1 - \sqrt{1 + (x/y)^2})}}{(x^2 + y^2)^{1/4}} = c \quad (18)$$

in large  $y$ :

$$\frac{1}{\sqrt{y}} + \frac{2ax^2}{4y^{3/2}} + O(x^2/y^{5/2}) = c \quad (19)$$

to obtain

$$x_{\max}(y) = \sqrt{2y(1 - c\sqrt{y})/a} + O(). \quad (20)$$

Now we can compute the estimated area as

$$A = 2 \int_0^{y_{\max}} x_{\max}(y) dy = \frac{64\sqrt{2}}{105\sqrt{a}} c^{-3}. \quad (21)$$

The crucial observation is that  $A \sim c^{-3}$ . Hence,

$$y_{\max} \sim c^{-2} \sim A^{2/3} \quad (22)$$

as we find in the simulations.

(\*\*\* Note that for us the area is twice the number of particles, since we have a checkerboard pattern. Also, I think  $a = 1$  is the right scaling, since  $UT = 1$  gridpoint, and  $\sqrt{2DT} = 1$  gridpoint as well.)

### 3 Using Fourier transform

(\*\*\* This is not necessary.)

We take the Fourier transform of Eq. (3):

$$k^2 \hat{\psi} + a^2 \hat{\psi} = 1/D. \quad (23)$$

Here we define the forward transform

$$\hat{\psi}(\mathbf{k}) = \int \psi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \quad (24)$$

and the inverse transform

$$\psi(\mathbf{r}) = \frac{1}{2\pi} \int \hat{\psi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}. \quad (25)$$

Solving Eq. (23) for  $\hat{\psi}$  and inverting, we have

$$\psi(\mathbf{r}) = \frac{1}{2\pi D} \int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 + a^2} d\mathbf{k}. \quad (26)$$

We can transform the  $\mathbf{k}$  integral to polar coordinates with  $\mathbf{k} = k(\cos \varphi, \sin \varphi)$  and  $\mathbf{r} = r(\cos \theta, \sin \theta)$ :

$$\psi(\mathbf{r}) = \frac{1}{2\pi D} \int_0^{2\pi} \int_0^\infty \frac{e^{ikr \cos(\varphi-\theta)}}{k^2 + a^2} k dk d\varphi. \quad (27)$$

Because of periodicity, we can translate the  $\varphi$  integral by  $\theta$  and obtain the  $\theta$ -independent form

$$\psi(r) = \frac{1}{2\pi D} \int_0^{2\pi} \int_0^\infty \frac{e^{ikr \cos \varphi}}{k^2 + a^2} k dk d\varphi. \quad (28)$$

Mathematica can carry this one out directly to give

$$\psi(r) = \frac{1}{D} K_0(ar). \quad (29)$$