Physics 449
hw3

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1) T(0.2/2) 3/5 Sad 2/10

$|\psi\rangle = \frac{4}{5}|1,0,0\rangle + \frac{3i}{5}|2,1,1\rangle$

$H\psi = E\psi \qquad \psi^* H \psi = \psi^* E \psi = \langle E \rangle$

$\Rightarrow \langle E_1 \rangle = \langle \psi_1 | E_1 | \psi_1 \rangle = E_1 \langle \psi_1 | \psi_1 \rangle = \frac{16}{25} E_1$

$\langle E_2 \rangle = \langle \psi_2 | E_2 | \psi_2 \rangle = E_2 \langle \psi_2 | \psi_2 \rangle = -\frac{9}{25} E_2$

$\langle E \rangle = \langle E_1 + E_2 \rangle = \frac{16}{25} E_1 - \frac{9}{25} E_2 = -13.6\,eV\left(\frac{16}{25}\frac{1}{1} - \frac{9}{25}\frac{1}{2}\right)$

$= -13.6\,eV \frac{16\times 2 - 9}{25 \times 2}$

$= \boxed{-13.6\,eV \frac{23}{50}}$

T(2.6)

$L^2 | E, l, m \rangle = l(l+1)\hbar^2 | E, l, m \rangle$

$\langle E, l, m | L^2 | E, l, m \rangle = l(l+1)\hbar^2 \cancel{\langle E, l, m | E, l, m \rangle}$

$\langle L_0^2 \rangle = l(l+1)\hbar^2 = 0 \qquad \langle L_1^2 \rangle = 2\hbar^2 \Rightarrow \boxed{\langle L^2 \rangle = 2\hbar^2}$

$L_z^2 | E, l, m \rangle = m\hbar | E, l, m \rangle$

$\langle E, l, m | L_z^2 | E, l, m \rangle = m\hbar \underbrace{\cancel{\langle E, l, m | E, l, m \rangle}}_{1} = m\hbar$

$\boxed{\langle L_z^2 \rangle = m\hbar}$

/b) $|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle = \frac{e^{-iE_1 t/\hbar}}{\sqrt{2}} |\psi_1\rangle + \frac{e^{-iE_2 t/\hbar}}{\sqrt{2}} |\psi_2\rangle$

The expectation value $= \langle \psi(t) | \psi(t) \rangle = \frac{\cancel{e^{iE_1 t/\hbar}}}{\sqrt{2}} \frac{\cancel{e^{-iE_1 t/\hbar}}}{\sqrt{2}} + \frac{e^{iE_2 t/\hbar}}{\sqrt{2}} \cdot \frac{e^{-iE_2 t/\hbar}}{\sqrt{2}}$

$= \boxed{1}$

2) T10.7 tritium $n=1, Z=1 \rightarrow n=1, Z=2$

$$R_{1,0} = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$R_{1,0}^{Z=1} = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}, \quad R_{1,0}^{Z=2} = 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}$$

$$\int_0^\infty r^2 dr\, R_{1,0}^{Z=1} R_{1,0}^{Z=2} = \int_0^\infty r^2 dr\; 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\; 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}$$

$$= 2^{3/2} \int_0^\infty r^2 dr\; 4 \cancel{\left(\frac{1}{a_0}\right)^3} e^{-3r/a_0}$$

$$= 2^{3/2}\; 4 \cancel{\left(\frac{1}{a_0}\right)^3} \frac{2\, \cancel{a_0^3}}{27} = 2^{3/2} \frac{8}{27}$$

$$|\langle \psi_{100}^{Z=2} | \psi_{100}^{Z=1} \rangle|^2 = \frac{8 \cdot 64}{27^2} \simeq 0.7023$$

3) T10.13

$$\frac{d\langle r \cdot p\rangle}{dt} = \frac{i}{\hbar}\langle E | [\hat{H}, \underline{r} \cdot \underline{\hat{p}}] | E \rangle = 0$$

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + V(|r|), \quad \Rightarrow \langle K \rangle = \left\langle \frac{p^2}{2\mu} \right\rangle = \frac{1}{2}\langle r \cdot \nabla V(r) \rangle$$

$$\frac{d\langle x \cdot p_x\rangle}{dt} = \frac{i}{\hbar}\langle E | [\hat{H}, \hat{x} \cdot \hat{p}] | E \rangle = 0$$

$$[\hat{H}_x, \hat{x} \cdot \hat{p}_x] = [\hat{H}, \hat{p}]\hat{x} - \hat{p}_x[\hat{x}, \hat{H}]$$

$$[\hat{H}_x, \hat{p}] = \left(\frac{\hat{p}_x^2}{2\mu} + V\right)\hat{p}_x - \hat{p}_x\left(\frac{\hat{p}_x^2}{2\mu} + V\right) \qquad \hat{p}_x \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$= \cancel{\frac{\hat{p}_x^3}{2\mu}} + V\hat{p}_x - \cancel{\frac{\hat{p}_x^3}{2\mu}} - \hat{p}_x V$$

$$= V\hat{p}_x - \hat{p}_x V$$

$$[\hat{H}_x, \hat{p}]f(x) = V\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}} - \frac{\hbar}{i}\frac{\partial V}{\partial x} - \cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}} = \boxed{-\frac{\hbar}{i}\frac{\partial V}{\partial x}}$$

$$[\hat{x}, \hat{H}_x] = \left[\hat{x}, \frac{\hat{p}_x^2}{2m} + V\right] = \left[\hat{x}, \frac{\hat{p}_x^2}{2m}\right] + [\hat{x}, V]$$

$$\hookrightarrow \left[\hat{x}, \frac{p_x^2}{2m}\right] = \hat{x}\frac{p_x^2}{2m} - \frac{p_x^2}{2m}\hat{x} = \frac{(\hat{p}_x\hat{x} + i\hbar)p_x}{2m} - \frac{p_x(\hat{x}\hat{p}_x - i\hbar)}{2m}$$

$$\left\{ [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \Rightarrow \hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i = i\hbar\delta_{ij}\right.$$

$$\Rightarrow \hat{x}\hat{p}_x = \hat{p}_x\hat{x} + i\hbar\cancel{\delta_{ij}}$$

$$\hat{p}_x\hat{x} = \hat{x}\hat{p}_x - i\hbar\cancel{\delta_{ij}}$$

$$= \frac{\hat{p}_x\cancel{\hat{x}}p_x + i\hbar p_x - p_x\cancel{\hat{x}p_x} + p_x i\hbar}{2m} = \frac{p_x}{m}i\hbar$$

$$\hookrightarrow [\hat{x}, V] = 0$$

$$\Rightarrow [\hat{x}, \hat{H}_x] = \boxed{\frac{p_x}{m}i\hbar} \qquad \Rightarrow [\hat{H}_x, \hat{x}\cdot\hat{p}_x] = -\frac{\hbar}{i}\frac{\partial V}{\partial x}\hat{x} - \hat{p}_x\frac{p_x}{m}i\hbar = 0$$

$$\Rightarrow \boxed{(+\hat{p}_x^2/m)/\cancel{i\hbar} = (\cancel{\hbar}/\cancel{i})(\partial V/\partial x)}$$

The same reasoning applied to $[\hat{H}_y, \hat{y}\cdot\hat{p}_y]$, $[\hat{H}_z, \hat{z}\cdot\hat{p}_z]$.

$$\Rightarrow +\frac{p_y^2}{m}/\cancel{i\hbar} = \frac{\cancel{\hbar}}{\cancel{i}}\frac{\partial V}{\partial y}\hat{y}$$

$$+\frac{p_z^2}{m}/\cancel{i\hbar} = \frac{\cancel{\hbar}}{\cancel{i}}\frac{\partial V}{\partial z}\hat{z}$$

$$\Rightarrow \boxed{\langle K\rangle = \left\langle\frac{p^2}{2m}\right\rangle = \frac{1}{2}\left\langle \underline{r}\cdot\vec{\nabla}V\right\rangle}$$

4) T 10.19

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V_a(|\hat{r}|) + \left(\frac{1}{4} - \frac{\hat{S}_1 \cdot \hat{S}_2}{\hbar^2}\right) V_b(|\hat{r}|)$$

$\hat{r} = \hat{r}_1 - \hat{r}_2$ $\quad V_a = \begin{cases} 0 & r<a \\ V_0 & r>a \end{cases}$ $\quad V_b = \begin{cases} 0 & r<b \\ V_0 & r>b \end{cases}$ $\quad b<a$ $\quad V_0$ large

/a) $M = m_1 + m_2$, $\quad \mu = \frac{m_1 m_2}{m_1+m_2}$, $\quad \hat{H}_{cm} = \frac{\hat{P}^2}{2M}$, $\quad \hat{H}_{rel} = \frac{\hat{p}^2}{2\mu} + V(|\hat{r}|)$

$$\hat{H}|E_{cm}, E_{rel}\rangle = (\hat{H}_{cm} + \hat{H}_{rel})|E_{cm}, E_{rel}\rangle = (E_{cm} + E_{rel})|E_{cm}, E_{rel}\rangle$$

$$\Rightarrow \hat{H} = \frac{\hat{P}^2}{2M} + \frac{\hat{p}^2}{2\mu} + V(|\hat{r}|)$$

$$\hat{S}_1 \cdot \hat{S}_2 |0,0\rangle = \frac{\hbar^2}{4}|0,0\rangle, \quad \hat{S}_1 \cdot \hat{S}_2 |1,m\rangle = -\frac{3\hbar^2}{4}|1,m\rangle$$

$$\hat{H}|0,0\rangle = \left[\frac{\hat{P}^2}{2M} + \frac{\hat{p}^2}{2\mu} + V_a(|\hat{r}|) + \left(\cancel{\frac{1}{4} + \frac{3\hbar^2}{4\hbar^2}}\right)\right]|0,0\rangle$$

$$= \left[\frac{\hat{P}^2}{2\mu} + \frac{\hat{p}^2}{2\mu} + V_a(|\hat{r}|) + V_b(|\hat{r}|)\right]|0,0\rangle$$

$$\hat{H}|1,m\rangle = \frac{\hat{P}^2}{2M} + \frac{\hat{p}^2}{2\mu} + V_a(|\hat{r}|) + \left(\cancel{\frac{1}{4} - \frac{1\hbar^2}{4\hbar^2}}\right)|1,m\rangle$$

$$= \left[\frac{\hat{P}^2}{2M} + \frac{\hat{p}^2}{2\mu} + V_a(|\hat{r}|)\right]|1,m\rangle$$

$$\Rightarrow \hat{H} = \cancel{\frac{\hat{P}^2}{2M}} + \frac{\hat{p}^2}{2\mu} + V_0.$$

kinetic energy of center of mass vanishes

$$\Rightarrow -\frac{\hbar^2}{2\mu}\frac{d^2\psi(r)}{dr^2} + \left(\frac{l(l+1)\hbar^2}{2\mu r^2} + V_0\right)\psi(r) = E\psi(r)$$

$$\psi = \frac{P(r)}{r} Y(\theta, \phi)$$

$$\Rightarrow -\frac{\hbar^2}{2\mu}\frac{d^2 P}{dr^2} + \left(\frac{l(l+1)\hbar^2}{2\mu r^2} + V_0\right)P = EP$$

$$\frac{d^2P}{dr^2} = -\frac{2\mu E}{\hbar^2}P = -k^2 P \ (r<a), \quad P = 0 \ (r>a) \rightsquigarrow$$

$\Rightarrow P(r) = A\sin(kr) + B\cos(kr) \quad (r<a)$

$\Rightarrow P_n(r) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi r}{a}\right) \quad n=1,3,5,\ldots \quad (r<a)$

$\psi_n(r) = \frac{1}{r}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi r}{a}\right)\cdot Y_{lm}(\theta,\phi) \quad (r<a)$

$\Rightarrow \boxed{\psi_1(r) = \frac{1}{r}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi r}{a}\right)\cdot Y_{00}(\theta,\phi) \quad (r<a)}$

the spin state $S=1$, the degeneracy is $\underline{3}$

/b) For a infinite potential well,

$E_{n_r,l_0} = \frac{\hbar^2}{2\mu}\left(\frac{n_r\pi}{a}\right)^2 \Rightarrow E_{n_r=1,l=1} = \frac{\hbar^2k^2}{2\mu} = \frac{\hbar^2}{2\mu}\left(\frac{4.49}{a}\right)^2 < \frac{\hbar^2\pi^2}{2\mu b^2}$

$\Rightarrow 4.49^2 < \frac{a^2\pi^2}{b^2} \Rightarrow b^2 < \frac{a^2\pi^2}{4.49^2} \Rightarrow \underline{0 < b < \frac{a\pi}{4.49}}$

The energy of the 1st excited state is given by $\frac{\hbar^2}{2\mu}\left(\frac{4.49}{a}\right)^2$.

the spin state is $l=1$, $m=0,\pm1$.

The result depends on $0<b<\frac{a\pi}{4.49}$

5)

$n=3 \quad l=2. \qquad E=17/2\, \hbar\omega$

For the 3d isotropic harmonic oscillator. $E_n=\left(n+\frac{3}{2}\right)\hbar\omega=\frac{17}{2}\hbar\omega$

$n=7$

l:	7	5	3	1	← odd parity
degeneracy $2l+1$	15	11	7	3	

(The wavefunction is written in the attached Mathematica document.)

(a) For the harmonic osillator $E_n = (n + 1/2)\hbar w$ (7.114)
$n = 0, 1, 2 \ldots$

total spin = 0

the ground state ket : $|0,0\rangle|0,0\rangle$ $E_{1s,1s} = 2 \cdot 1/2 \hbar w = \hbar w$

the 1st excited state ket: both $|2s,1s\rangle$, $|2p,1s\rangle$ kets exist in the 1st excited state
$|2s,1s\rangle$ corresponds to the lowest energy state

$$\frac{1}{\sqrt{2}}\Big(|1,0\rangle + |0,1\rangle\Big)|0,0\rangle$$

$$E_{2s,1s} = (1 + 1/2)\hbar w + 1/2 \hbar w = 2\hbar w$$

total spin = 1.
in the ground state, spin 1/2 particles cannot construct a total spin 1

in the 1st excited state.

$$\frac{1}{\sqrt{2}}\Big(|1,0\rangle - |0,1\rangle\Big)|1,1\rangle$$

$$\frac{1}{\sqrt{2}}\Big(|1,0\rangle - |0,1\rangle\Big)|1,0\rangle$$

$$\frac{1}{\sqrt{2}}\Big(|1,0\rangle - |0,1\rangle\Big)|1,-1\rangle$$

$E_{2s,1s} = 2\hbar w$.

(b) $$V(|x_1 - x_2|) = \begin{cases} -V_0 & |x_1 - x_2| < a \\ 0 & \text{elsewhere} \end{cases}$$

The energy moves down when two particles are closer. When two particles are anti-symmetric, they got closer than in other states.
For the ground state of the total spin $=0$ $|1s,1s\rangle$, the energy move down.

For $|2s,1s\rangle|0,0\rangle$, or $|2s,1s\rangle|1,m\rangle$ states, the energy is lowered but $|1s,1s\rangle|0,0\rangle$ was lowered the most

7) T12.4 $\hat{H} = \frac{\hat{p}_x^2}{2m} + b\hat{x}^4$, $E_0 = 1.060\, b^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3} = 1.060\left(\frac{b\hbar^4}{4m^2}\right)^{1/3}$

use the trial wave function.

$\psi = Ne^{-\alpha x^2}$ $\xrightarrow{\text{normalize}}$ $\psi = \left(\frac{2}{\pi}\right)^{1/4} \alpha^{1/4} e^{-\alpha x^2}$

$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$. $\hat{p}_x^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$.

$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + bx^4\psi$

$\langle\psi|\hat{H}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + bx^4\psi\right)$

$\overset{\text{mma}}{=} \frac{3bm + 8\alpha^3\hbar^2}{16m\alpha^2} = \varepsilon$

$\frac{\partial\varepsilon}{\partial\alpha} \overset{\text{mma}}{=} -\frac{3b}{8\alpha^3} + \frac{\hbar^2}{2m} = 0 \Rightarrow \alpha = \frac{3^{1/3} b^{1/3} m^{1/3}}{2^{2/3}\hbar^{2/3}}$

$\Rightarrow \varepsilon = 0.68142\frac{b^{1/3}\hbar^{4/3}}{m^{2/3}} = 1.08169\frac{b^{1/3}\hbar^{4/3}}{m^{2/3}} > E_0$ by

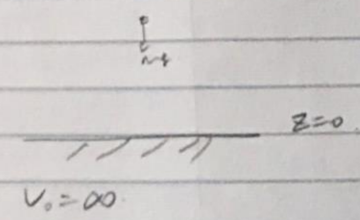
the variational principle.

The result is away from the exact one by 2%.

8) T 12.6

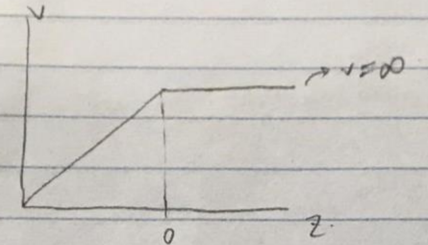
/a) $V = mgz$

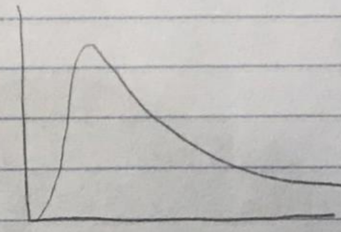
$H = \frac{\hat{p}^2}{2m} - mg\hat{z}$



(The potential energy and the ground-state wave function are sketched in MMA.)

Potential Energy of the particle : | ground-state wavefunction $\psi = Cze^{-\alpha z}$





/b) Use the trial wave function $\psi = Cze^{-\alpha z}$

normalize $\Rightarrow$ $C = 2\alpha^{3/2}$ $\rightarrow$ $\psi = 2\alpha^{3/2} z e^{-\alpha z}$

$$\langle\psi|\hat{H}|\psi\rangle = \int_{-\infty}^{\infty} dz\, \psi^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial z^2} - mgz\,\psi \right) = \frac{\alpha^3(-24gm^2+\hbar)}{m} = \mathcal{E}.$$

$$\frac{\partial \mathcal{E}}{\partial \alpha} = \frac{3gm}{2\alpha^2} + \frac{\alpha\hbar}{m} = 0 \Rightarrow \mathcal{E} \simeq 1.9656\left(\frac{g^2 m}{\hbar}\right)^{1/3} \geq \mathcal{E}_0$$

$$\alpha = -(3/2)^{1/3}\left(\frac{gm^2}{\hbar}\right)^{1/3}$$

/c) $$\langle z\rangle = \int_0^{\infty} dz\, \psi(-z)\psi = \frac{3}{2\alpha}$$

For the particle in the ground state $\alpha = -(3/2)^{1/3}\left(\frac{gm^2}{\hbar}\right)^{1/3}$ given by

$$\Rightarrow \langle z\rangle = 1.31037\left(\frac{\hbar}{gm^2}\right)^{1/3}.$$