low-energy limit

$$f = -\frac{\mu}{2\pi\hbar^2}\int d^3r'\, \underbrace{e^{-i\bar{k}_f\cdot\bar{r}'}}_{\approx 1}\, V(r')\,\psi(r')$$

$\therefore$ f ind. of $\bar{k}_f$ (s-wave)

$$\psi(r) = e^{ikz} + \psi^{sc} = 1 + \psi^{sc}$$

$$\psi = \rho/r, \quad \frac{d^2\rho}{dr^2} + \cancel{k^2\rho}^{\,0} = \frac{2m}{\hbar^2} V\rho$$

scattering length

large r, $\rho = r - a \Rightarrow \psi = 1 - a/r$

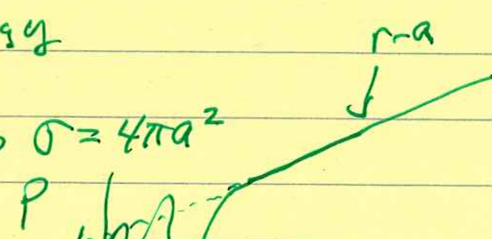
$$\cancel{f = -\frac{\mu}{2\pi\hbar^2}\int d^3r'}\ V\psi$$

$$f = -\frac{1}{4\pi}\int d^3r' \underbrace{\frac{2mV}{\hbar^2}\psi}_{=\nabla^2\psi}$$

$$= -\frac{1}{4\pi}\int d^2S\ \hat{r}\cdot\nabla\psi = -\frac{1}{4\pi} r^2\, 4\pi\left(-\frac{a}{r^2}\right) = a$$

$\therefore$ $f = a$ at zero energy

$\therefore$ ~~$\sigma = 4\pi a$~~ $\frac{d\sigma}{d\Omega} = a \Rightarrow \sigma = 4\pi a^2$



large r $\psi = \frac{\sin kr}{kr} + \frac{f e^{ikr}}{r} = A\frac{\sin(kr+\delta)}{r}$

$\cancel{= A\frac{\sin kr \cos\delta}{r} + A\frac{\cos kr \sin\delta}{r}}$

$= A\left(\frac{e^{i(kr+\delta)} - e^{-ikr-i\delta}}{2ir}\right)$

$\frac{e^{ikr} - e^{-ikr}}{2ikr} + \frac{f e^{ikr}}{r} =$

e^{-ikr} parts $\quad -\frac{1}{2ikr} = -\frac{Ae^{-i\delta}}{2ir} \Rightarrow A = \frac{e^{i\delta}}{k}$

$e^{ikr} \quad \frac{1}{2ikr} + \frac{2if}{2ir} = \frac{Ae^{i\delta}}{2ir} = \frac{e^{2i\delta}}{2ikr}$

$\therefore \quad f = \frac{e^{2i\delta} - 1}{2ik} = \frac{e^{i\delta}\sin\delta}{k}$

$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{\sin^2\delta}{k^2} \Rightarrow \sigma = \frac{4\pi\sin^2\delta}{k^2}$

Check $k \to 0$ $\quad \delta = -ka \quad \sigma = 4\pi a^2$ ✓

getting δ from numerical integration

calc $\quad r\psi = A\sin(kr+\delta)$

$$\frac{(r\psi)'}{r\psi} = \frac{kA\cos(kr+\delta)}{A\sin(kr+\delta)} = k\cot(kr+\delta)$$