Measurement of SNC

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The most common error models for quantum computers assume the independence of errors on different qubits. However, most noise mechanisms have some correlations in space and time. We show how to improve quantum information processing for few-qubit systems when spatial correlations are present. This starts with strategies to measure the correlations. Once the correlations have been determined, we can give criteria to assess the susceptibility of candidate quantum circuits to carry out a given task. Finally, we point out ways in which the improvement of few-qubit systems can be extended to large-scale quantum computation. The basic conceptual theme of the work is the generalization of the concept of decoherence-free subspaces in order to treat the case of arbitrary correlations.

I. INTRODUCTION

The most basic justification of the pursuit of quantum computation is the existence of threshold theorems. They tell us that if a certain precision at the qubit level can be achieved, then a workable quantum computer (in principle of arbitrary size) can be made. These thresholds also give concrete goals for hardware performance in systems containing only a few qubits. However, threshold theorems generally make the key assumption that errors on different qubits occur in statistically independent fashion [1]. Some relaxation of this condition can be allowed [2], but error correction then becomes more complicated and resource-intensive. On the other hand, it is known that correlations in noise can actually be used to fight noise-induced degradation in performance, using the concept of decoherence-free subspaces (DFS) [3–5]. The resources involved in utilizing DFS appear to be less than in most error-correction schemes.

This situation raises some interesting questions. If the noise is correlated, is this fundamentally good or bad for quantum computation? If we have the choice of dealing with the situation by error correction or DFS, which is less expensive? If we do not have a DFS, but there are some correlations in the noise, is it still possible to reduce the computer's susceptibility to noise by appropriate protocols?

We investigate these questions here by looking at a small system of only two physical qubits. With very small systems such as this, quantum error correction is of course out of the question. Furthermore, the use of a DFS would prohibit any nontrivial quantum information processing. We will show, however, that signficant error *mitigation* is still possible. We will describe methods that may be used for any such system. For illustration purposes, we will use a concrete experimental example: two electron spin qubits in a Si/SiGe heterostructure [6]. This system has the great advantage that it is completely programmable, and we have some insight into the types of noise to be expected [7, 8]. The overall aim is to understand how to improve quantum information processing when noise correlations are present.

We begin in Sec. II by introducing the model and constructing the framework to describe the spatial and time correlations in the noise, limiting the dicussion to dephasing noise. In Sec. III we propose a measurement scheme to obtain these correlations, which is a simple and easily understood extension of methods used for single qubits. It can also be viewed as a concrete application of much more general scheme given in some recent papers [9–11] We apply these methods to the specific system in question to obtain the important auto- and cross-correlation functions. In Sec. IV, we generalize DFS concepts to obtain measures of decoherence that are local in Hilbert space. In Sec. V we use these measures to improve the robustness of a quantum circuit. This increase the fidelity obtained for a given quantum inofrmation processing task. In Sec. VI the concepts are applied to the model system, demonstrating that attention to noise correlations can improve quantum iinformation processing in a practical device. Sec. VII treats the extensions to many-qubit systems and and addresses the scalability of the method. In Sec. VIII we conclude with a discussion of more general error models (bit flip, quantum noise) and how to combine our method with quantum error correction,

II. NOISE CORRELATIONS

The model Hamiltonian for two qubits subject to dephasing noise is

$$H = H_0 + H_g(t) + H_n(t).$$

It consists of

$$H_0 = b_1 Z_1 + b_2 Z_2,$$

a static Hamiltonian that provides the qubit splittings b_1 and b_2 in energy units. (b_1 and b_2 need not be magnetic fields.) $H_g(t)$ is the gate Hamiltonian that is used to do qubit rotations. X_i, Y_i, Z_i are the Pauli matrices on site i.

$$H_n(t) = \delta b_1(t) Z_1 + \delta b_2(t) Z_2$$

is the noise Hamiltonian. In the case of electron spins in an inhomogeneous magnetic field, this choice of H

models random electric fields that move the qubits and vary their splittings. We use the product basis, which forms the eigenbasis for H_0. $T_2^{(1)}$, and $T_2^{(2)}$ are measures of the local noise spectra

$$S_{ij}(\omega) = \int_{-\infty}^{\infty} \langle \delta b_i(t)\ \delta b_j(0)\rangle \cos\omega t\ dt \equiv \langle \delta b_i\ \delta b_j\rangle_\omega$$

on the two qubits. In a simple theory (see, e.g., Slichter's "Magnetic Resonance" book) we just have

$$\frac{1}{T_2^{(j)}} = \frac{4}{\hbar^2}\lim_{\omega\to 0} S_{jj}(\omega)$$

as long as $T_1 >> T_2$, which is usually the case. More accurate formulas can be used, but they do not change the basic physics that T_2 comes from the noise spectrum at low frequencies. So we have measures of $S_{11}(\omega)$ and $S_{22}(\omega)$. We propose an experiment to measure

$$S_{12}(\omega) = \int_{-\infty}^{\infty} \langle \delta b_1(t)\ \delta b_2(0)\rangle \cos\omega t\ dt \equiv \langle \delta b_1\ \delta b_2\rangle_\omega .$$

This is now a correlation function of the noise at different spatial locations.

Measuring this is of interest for two reasons.

First it tells us something about the nature of the noise. For example, in semiconductor implementations, charge noise is often the dominant decoherence mechism. If this is due to defects that are far from the qubits, then the noise from the random electric field has wavelengths much longer than the separation of the qubits, and the random electric field is about the same at the two qubits. For charge qubits $\langle \delta b_1(t)\ \delta b_2(0)\rangle$ is large at small t. The opposite limit is when the defect lies between the qubits when we expect anticorrelation in the electric field.

Second, we can use the information to design noise-resistant operations, which is the focus of this paper. If $\langle \delta b_1\ \delta b_2\rangle$ is large and positive, then $\langle (\delta b_1 + \delta b_2)(\delta b_1 + \delta b_2)\rangle$ is large and $\langle (\delta b_1 - \delta b_2)(\delta b_1 - \delta b_2)\rangle$ is small. Looking back at the Hamiltonian, we see that the noise Hamiltonian for this case is approximately

$$H_n(t) \approx \delta b_1(t)(Z_1 + Z_2)$$

and it only couples to $Z_{tot} = Z_1 + Z_2$. This means that the subspace spanned by $\{|01\rangle, |10\rangle\}$ is approximately a decoherence-free subspace. Conversely, if If $\langle \delta b_1\ \delta b_2\rangle$ is large and negative, then

$$H_n(t) \approx \delta b_1(t)(Z_1 - Z_2)$$

and the subspace spanned by $\{|00\rangle, |11\rangle\}$ is approximately a decoherence-free subspace. By working "near" the appropriate subspace we can get a lower error rate. This is essentially the same idea as singlet-triplet qubits [?]. However, we are going to keep the full 2-qubit system.

In some situations this clearly generalizes to multiple qubits. We can define cross-correlation functions for any pair and it may well happen that only short-range correlations are important. Such generalizations will be discussed further in Sec. V.

III. MEASURING CORRELATIONS

In this section we suggest an experimental strattegy to determine the spatial correlations. They are determined by means of a measurement analogous to the measurement of Ramsey fringes but in different bases.

A. Experiment 1. Ramsey in the $\{|00\rangle, |11\rangle\}$ basis.

Let the north pole of a Bloch sphere be $|00\rangle$ and the south pole be $|11\rangle$. We start in the state $|00\rangle$ and then use H_g to make a $\pi/2$ rotation about the y-axis preparing the state

$$\Psi_+(t=0) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

and then let it evolve under the influence of H_0 alone. Then we have

$$\Psi_+(t) = \frac{1}{\sqrt{2}} e^{-i(b_1+b_2)t}|00\rangle + \frac{1}{\sqrt{2}} e^{i(b_1+b_2)t}|11\rangle,$$

and if we make another $\pi/2$ rotation about the y-axis and then measure the probability of being in the state $|11\rangle$ after a time t we get

$$P_+ = \cos^2[(b_1+b_2)t] = \frac{1}{2} + \frac{1}{2}\cos[2(b_1+b_2)t],$$

so the period is $\tau_+ = \pi/(b_1+b_2)$. The experiment is illustrated in Fig. 1. If we now add in H_n, the noise, we find

$$P_+ = \frac{1}{2} + \frac{1}{2} e^{-t/T_2^{(+)}} \cos[2(b_1+b_2)t],$$

in a certain time window longer than the inverse cut-off time of the noise. (At shorter times the decay is Gaussian.) Here $1/T_2^{(+)}$ is given by the integral of the Fourier transform of $4\left\langle [\delta(b_1+b_2)]^2 \right\rangle$ and a windowing function and some other factors involving the temperature, $\hbar$, etc. Omitting these prefactors and others that depend on the precise form of the power spectrum we have that

$$1/T_2^{(+)} \sim \left\langle [\delta(b_1+b_2)]^2 \right\rangle .$$

The number of oscillations observed will be N_+, which is

$$N_+ = \frac{T_2^{(+)}}{\tau_+} \sim \frac{b_1+b_2}{4\pi\left\langle [\delta(b_1+\delta b_2)]^2 \right\rangle}.$$

In the case of perfect collective dephasing N_1 diverges, a signature of a perfect DFS.

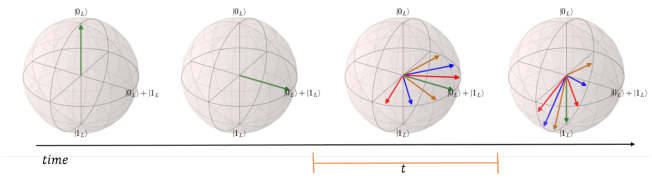

Figure III.1: A basic illustration of the Ramsey experiment in the logical qubit Bloch sphere. Both T_2 measurements can be thought of in this way with two different logical qubits defined in the two-qubit Hilbert space.

B. Experiment 2. Ramsey in the $\{|01\rangle, |10\rangle\}$ basis.

We start in the state $|01\rangle$ and then use H_g to make a $\pi/2$ rotation about the y-axis in the $\{|01\rangle, |10\rangle\}$ subspace preparing the state

$$\Psi_-(t=0) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

We have

$$\Psi_-(t) = \frac{1}{\sqrt{2}} e^{-i(b_1-b_2)t}|01\rangle + \frac{1}{\sqrt{2}} e^{i(b_1-b_2)t}|10\rangle,$$

and the probability of being in the state $|10\rangle$ after a time t is

$$P_- = \cos^2[(b_1-b_2)t] = \frac{1}{2} + \frac{1}{2}\cos[2(b_1-b_2)t],$$

so the period is $\tau_- = \pi/|b_1-b_2|$. As above, we have

$$1/T_2^{(-)} \sim 4\left\langle [\delta(b_1-b_2)]^2 \right\rangle,$$

and the number of oscillations is

$$N_- = \frac{T_2^{(-)}}{\tau_-} \sim \frac{|b_1-b_2|}{4\pi\left\langle [\delta(b_1-b_2)]^2 \right\rangle}.$$

For perfect collective dephasing $(b_1 = b_2)$, N_- diverges. The gate sequences for performing these experiments are not particularly complicated. They are shown in Fig. III.2 [[Do you want to include both the 'good' and 'bad' gate sequences here?]]

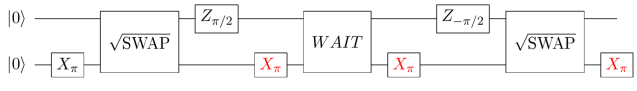

Figure III.2: A circuit for performing the Ramsey experiment described in Sec. III. The final three X rotations on the second qubit should be included to prepare Ψ_- and left out for Ψ_+.

C. Remarks.

Since

$$\frac{1}{T_2^{(\pm)}} \sim 4\left\langle [\delta(b_1 \pm b_2)]^2 \right\rangle = 4\left\langle (\delta b_1)^2 \pm 2\delta b_1 \delta b_2 + (\delta b_2)^2 \right\rangle$$

we deduce that

$$\frac{1}{T_2^{(\pm)}} = \frac{1}{T_2^{(1)}} + \frac{1}{T_2^{(2)}} \pm \frac{8}{\hbar^2}\lim_{\omega\to 0} S_{12}(\omega).$$

Thus S_{12} can also be thought of as breaking a sum rule on the T_2's - the direction of the breaking depending on whether the noise is correlated or anticorrelated. Of course this is only a rough relation. If there are differences in frequency dependences of the various S_{ij} this can modify the conclusions.

IV. LOCAL DECOHERENCE MEASURES

With the noise correlations determined, how can we use the knowledge gained to improve the performance of a quantum information processing device? DFS theory offers a simple solution for a many-qubit system. The basis states of the DFS are eigenvectors of the noise Hamiltonian H_n, the initial state of the computation lies in the DFS, and subsequent unitary operations never rotate the state out of the DFS. This in principle eliminates decoherence.

In the 2-qubit system with a 2-dimensional DFS we do not have this luxury, since no non-trivial quantum operations are possible. Still, we can hope to reduce decoherence even in this case, even if the DFS is only approximate. For a given processing task, many gate sequences are usually possible. The idea is to choose the one most resistant to correlated noise.

In the pure dephasing model we are considering, the only candidates for perfect DFSs are span$\{|00\rangle, |11\rangle\}$ and span$\{|01\rangle, |10\rangle\}$. If $T_2^{(2)}/T_2^{(1)}$ is finite, then there is no DFS. Nevertheless, if $T_2^{(2)}/T_2^{(1)} > 1$, then span$\{|01\rangle, |10\rangle\}$ is the "good" subspace and if $T_2^{(2)}/T_2^{(1)} < 1$, then span$\{|11\rangle, |00\rangle\}$ is the "good" subspace. As might be expected from symmetry, our conlusions are equally good for the two cases.

Since the subspaces are good but not perfect, this picture suggests the idea of defining a measure of decoherence for every point in the Hilbert space when noise correlations are present. We call these "local" decoherence meaasures, "local" here referring to Hilbert space, not real space.

We define two such measures.

1. The first is a geometric measure, called $d_g(|\psi\rangle)$, where $|\psi\rangle$ is any vector in the 4-dimensional 2-qubit space. Let B be an orthonormal basis for the DFS.

The projection operator onto the DFS is

$$P = \sum_{|\phi\rangle \in B} |\phi\rangle \langle\phi| ,$$

and then

$$d_g = |(I - P)\psi|^2$$

returns the square of the perpendicular Hilbert space distance from ψ to the DFS. This gives a very simple gepmetric picture of the decoherence rate of the state as depending only on the distance to the DFS. Clearly $d_g \leq 1$ and $d_g = 0$ for $|\psi\rangle$ in the DFS.

2. The second decoherence measure, called $d_c\left(|\psi\rangle\right)$, is obtained by studying the purity γ of a density matrix ρ of the 2-qubit system, defined as $\gamma = \mathrm{Tr}\left(\rho^2\right)$. $\gamma = 1$ for a pure state since then $\gamma = \mathrm{Tr}\left(\rho^2\right) = \mathrm{Tr}(\rho) = 1$. For the completely mixed state $\rho = I/D$, where D is the dimension of the Hilbert space we find $\gamma = 1/D$. Our interest is in the case $D = 4$. To use γ to form a local measure of deoherence in the Hilbert space we imagine initializing the system at time $t = 0$ in the state $\rho(0) = \rho_0 = |\psi\rangle\langle\psi|$ so that $\gamma(t = 0) = 1$ and watching γ decrease with time under the influence of the noise Hamiltonian H_n. Denote averages of $\cdot$ over noise realizations by $[\cdot]_n$. In this case, $d\rho/dt = \rho' = -i\hbar[\rho, H_n]_{av}$. We are only interested in the short-time behavior of γ so we get

$$\begin{aligned}
\gamma(\delta t) &= \mathrm{Tr}\left[\rho(\delta t)\right]^2 \\
&\approx 1 - \mathrm{Tr}\left[\delta t \rho_0' + \frac{1}{2}\delta t^2 \rho_0''\right]^2 \\
&= 1 - \frac{\delta t^2}{2}\mathrm{Tr}\rho_0\rho_0'' \\
&= 1 - \delta t^2 \mathrm{Tr}\left[\rho_0^2 H_n^2 - \rho_0 H_n \rho_0 H_n\right]_{av}
\end{aligned}$$

We identify

$$d_c(|\psi\rangle) = Tr\left[\rho_0^2 H_n^2 - \rho_0 H_n \rho_0 H_n\right]_{av}$$

as a measure of decoherence that describes how susceptible the pure state $\rho(t)$ is to mixing by the noise. Once again, $\rho_0 = |\psi\rangle\langle\psi|$ so d_c is a decoherence measure associated with a point in the Hilbert space of the computer. $d_c = 0$ if $|\psi\rangle$ is in the DFS since H_n acts as the identity operator in the DFS and $[\rho_0, H_n] = 0$. Unlike d_g, however, there is no upper bound on d_c.

The two measures differ considerably in their generality. d_g relies only on the identification of a DFS and can be thought of as an extension of the DFS concept. By contrast, to compute d_c, one needs only the noise Hamiltonian. d_g is simple to compute and to visualize. But d_c gives a more complete picture of the decoherence. It is possible that the decohrence is not even a montonic function of the distance from the approximate DFS. d_g obviously does not capture this possibility. Finally, d_c can clearly be computed also for mixed states, while d_g cannot be, at least by the above definition

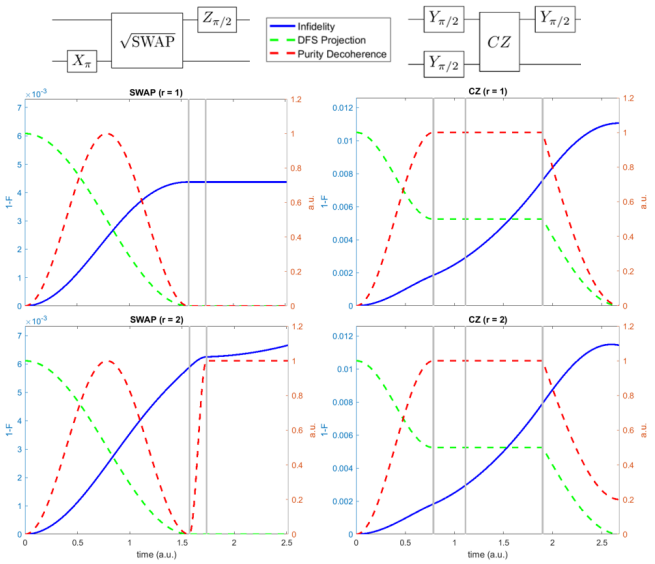

Figure IV.1: On the left, two circuits for preparing the state $\Psi = (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)/\sqrt{2}$ to perform one of the measurements described in Sec. II. On the right, state infidelity is plotted for each circuit subject to quasistatic Gaussian dephasing noise for perfectly correlated noise where the ratio between the noise strength at the two qubit locations is r. The DFS projection metric $d_g(t)$ and the purity-based decoherence $d_c(t)$ are scaled and overlaid against the infidelity to illustrate how they capture dephasing effects. Gray lines divide the time axis into intervals corresponding to each gate in the circuit.

In the course of a quantum information process, an ideal computer remains in a pure state $|\psi(t)\rangle$ that traverses a path in Hilbert space from the initial state $|\psi(t = 0)\rangle$ to the desired final state $|\psi(t = t_f)\rangle$ that encodes the answer to the computation or other process. Given this trajectory we can also compute $d_g(t)$ and $d_c(t)$. If these quantities are big on average over the interval $0 \leq t \leq t_f$, then we expect poor fidelity in the result. Of course there is a choice of gate sequences (actually an infinite number) that will take the computer from $|\psi(t = 0)\rangle$ to $|\psi(t = t_f)\rangle$. The choice is usually determined by brevity and experimental constraints. Here we suggest that one should also take into accout the minimization of decoherence. A gate sequence that minimizes $d_g(t)$ and/or $d_c(t)$ should be preferred. Of course for this small system the fidelity itself can easily be computed and used to minimize the decoherence. The use of d_g and d_c gives physical insight into this purely numerical calculation and suggests generalizations to larger systems.

As a simple example, we compute all these quantities for two gate sequences that start from an initial state $|00\rangle$ and end in the Bell state $(|01\rangle + |10\rangle)/\sqrt{2}$. We choose a simple noise model in which there is complete correlation: $\delta b_1(t) = \delta b_2(t)$. So here we are dealing with an exact DFS. The main difference between the two circuits is the nature of the entangling gate. There is an ideal

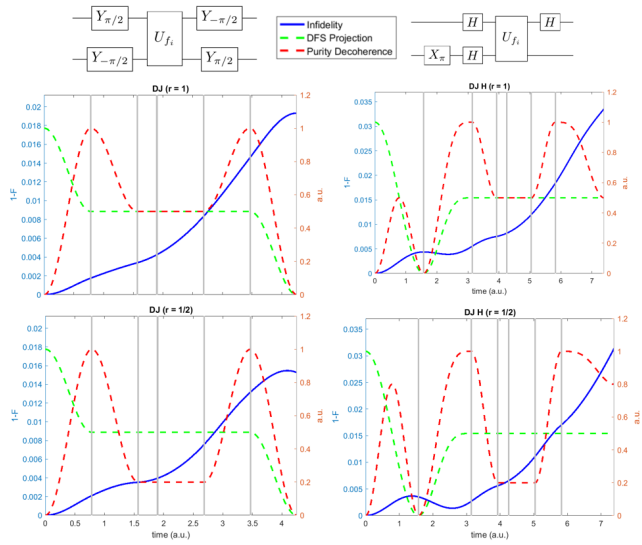

Figure IV.2: On top, two different circuits for performing the Deutsch-Jozsa algorithm. The infidelity is plotted along with $d_g(t)$ an $d_c(t)$ for the function $U_{f_i} = CNOT$.

unitary evolution for the noise-free system that is easily obtained from each gate sequence. We also compute the non-unitary dynamics of each circuit in a quasistatic noise model with $\delta\mathbf{b} \equiv (\delta b_1, \delta b_2)$ sampled from a bivariate Gaussian distribution with density

$$f(\delta b_1, \delta b_2) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\left(-\frac{1}{2}\delta\mathbf{b}^T\Sigma^{-1}\delta\mathbf{b}\right).$$

The model assumes zero mean and covariance

$$\Sigma = \begin{pmatrix} \sigma_1^2 & c\sigma_1\sigma_2 \\ c\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $\sigma_{1,2}$ is the noise strength at qubits 1 and 2 and and c is their statistical correlation. Decoherence is simulated by averaging the dynamics over several realizations of the noise. Comparison of the results of the two calculation allows us to plot the infidelity as a function of time. For the computation of $d_g(t)$ and $d_c(t)$ we need only the noise-free state. These three quantities are plotted in Fig. IV.1 as a function of time in arbitrary units, $t = 2.5$ being the end of the sequence. We also investigated the effect of changing the relative strength of the noise on the two qubits. $r = \frac{\sigma_1}{\sigma_2}$ is the ratio of the noise strength on qubit 1 to that on qubit 2. For this algorithm the initial state is not in the DFS so $d_g(t = 0) \neq 0$ and then d_g decreases. In the second gate sequence, more time is spent away from the DFS, which has the consequence that the final infidelity is larger. Interestingly $d_c\,(t = 0) = 0$ but $d_c\,(t)$ has a lengthy plateau at a high value during the course of the second algorithm. This is another way of seeing why the second algorithm is less good. This gives a consistent overall picture.
Looking in a bit more detail, we can see that at later times $t > 2$, $d_c\,(t)$ shows significant differences between the $r = 1$ and $r = 2$ cases. This is reflected in the higher final infidelity. Furthermore, near $t = 0$, $\mathrm{d}_c\,(t)$ is linear in time, while the infidelity is quadratic. We hypothesize that the infidelity is proportional to the time integral of $d_c\,(t)$. By contrast, $d_g\,(t)$ offers only a rough qualitative guide to the actual infidelity.
To show how this works for more general correlations, we plot the average of d_c over the circuit and the final infidelity as a function of noise correlation and relative field strength for the $\sqrt{\mathrm{SWAP}}$ circuit. The maximum of d_c corresponds to the maximum simulated infidelity, so this metric tells us what kinds of noise are most dangerous for a given circuit, and provides a method to discriminate between circuits once the noise has been characterized. We also include results for a comparision between

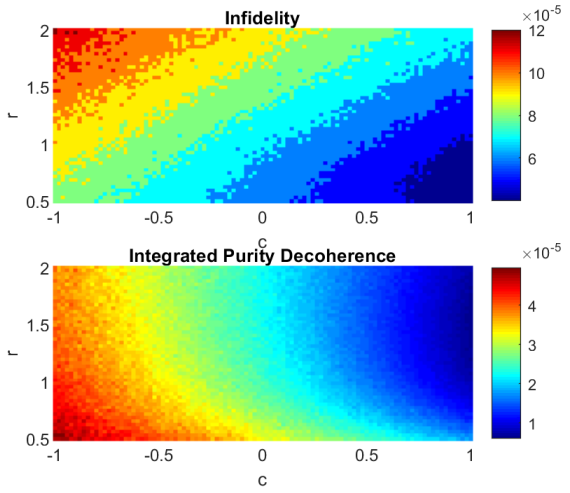

Figure IV.3: The final infidelity and averaged purity decoherence d_c for the $\sqrt{\mathrm{SWAP}}$ Ramsey gate sequence. As noted, the purity decoherence serves as a proxy for the derivative of the fidelity so the integral, or average, over the gate sequence shares the same features as the infidelity.

[[A COUPLE OF MORE FIGURES (MORE CIRCUITS?) AND COMMENTARY HERE.]]

V. EXTENSION TO MANY QUBITS

As noted above, if only a few qubits are involved, the best strategy for deciding on a gate sequence is simply to calculate the infidelities. As the number of qubits increases, the length of this computation increases exponentially, and it soon becomes impractical. The same holds for the computation of $d_g\,(t)$ and $d_c\,(t)$, since they depend on the many-body wavefunction. The question is whether we can use the physical insight gained for few-qubit systems to give a meaningful presciption for "scoring" long gate sequences in a mutli-qubit computer.
Consider an n-qubit Hilbert space $\mathcal{H}$ which has a decoherence free subspace $\mathcal{C}$. Then $\mathcal{H} = \mathcal{C} \oplus \mathcal{C}_\perp$, where we have no knowledge about the decoherence in $\mathcal{C}_\perp$. Let

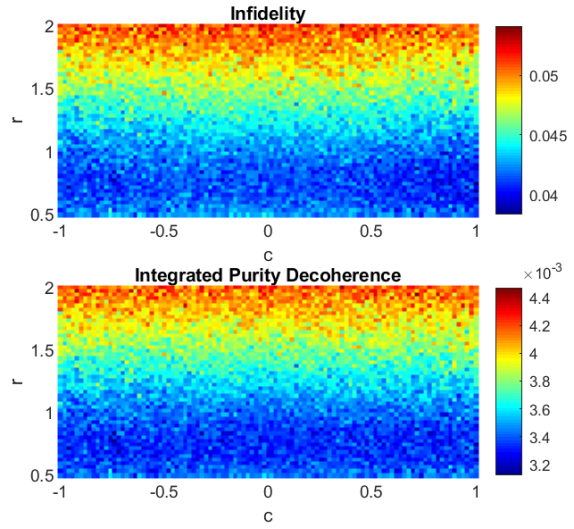

Figure IV.4: The final infidelity and integrated purity decoherence d_c for the Y gate based Deutsch-Jozsa gate sequence with $U_{f_i} = CNOT$.

D be the dimension of $\mathcal{C}$ and D_H be the dimension of $\mathcal{H}$. We conceive the computation as follows. There is an ideal evolution that consists of a sequence of unitary operations that in turn define a sequence of points Ψ_n in $\mathcal{H}$, Ψ_n being the state after n gate operations. Let us assume we have some measure of local decoherence such as d_g or d_c. If there is no correlation between the sequence defined by the algorithm and the local decoherence then it is reasonable to think of the evolution as a random walk in $\mathcal{H}$. Taking our clue from the definition of d_g, decoherence will be worse if the random walk increases the perpendicular distance of Ψ to the subspace $\mathcal{C}$. If we could compute a probability distribution $P_\perp$ for the perpendicular component of the walk steps then the expected value of the perpendicular distance to $\mathcal{C}$ after N gates would be $\langle d_N^2 \rangle^{1/2} = \sqrt{N} L$, where L is the rms step length computed using $P_\perp$. We may think of different gate sequences as having different $P_\perp$.
While it appears to be difficult to find a precise expression for $P_\perp$, there is a reasonable way to assign a "perpendicular step size" to every one or two qubit gate G. We arrange the basis of $\mathcal{H}$ so that the D basis vectors of $\mathcal{C}$ come first. We parition the unitary matrix of G into four sectors:

$$G = \begin{pmatrix} G_\parallel & M \\ M' & G_\perp \end{pmatrix},$$

where $G_\parallel$ and $G_\perp$ move the state around $\mathcal{C}$ and $\mathcal{C}_\perp$ respectively. M and M' move weight between $G_\parallel$ and $G_\perp$. A "good" gate has $M = M' = 0$ since a state in C undergoes no perpendicular motion remains in $\mathcal{C}$ and a state in $\mathcal{C}_\perp$ undergoes random motion that is not biased in the perpendicular direction. M and M' give perpendicular motion and if they are large, in some sense, then G is a bad gate, i.e., one that we expect will increase the decohrence to which the state is exposed. To quantify this we use the unitarity of G and index the blocks of G as follows. The upper left corner $G_\parallel$ is indexed by $s_{G_\parallel} = (i,j)$ such that $1 \le i \le D$ and $1 \le j, \le D$; the lower right corner $G_\perp$ is indexed by $s_{G_\perp} = (i,j)$ such that $D+1 \le i \le D_H$ and $D+1 \le j \le D_H$; the upper right corner M is indexed by $s_M = (i,j)$ such that $1 \le i \le D$ and $D+1 \le j \le D_H$; the lower left corner M' is indexed by $s_{M'} = (i,j)$ such that $D+1 \le i \le D_H$ and $1 \le j \le D$. We then define

$$B(G) = \frac{1}{4}\left(\sum_{(i,j)\in s_c} |M_{ij}|^2 + \sum_{(i,j)\in s_d} |M'_{ij}|^2 \right)^{1/2},$$

which is just a Euclidean measure of the size of the off-diagonal blocks. This may be thought of as the "badness" of G. The unitarity of G and the fact that one- and two-qubits gates are nearly diagonal imply that $0 \le B \le 1$. Table V.1 gives the badness of several common gates. A circuit A then may be thought of as a random walk with N_A gates and the analog of the integral of d_g over the total circuit is

$$d_A = \sqrt{\sum_G B^2(G)}.$$

We expect the infidelity to be roughly proportional to this quantity.
The key point is that we never need to compute any wavefunctions or other many-body quantities. Each one- and two-qubit gate has only a small number of off-diagonal elements, so the computation of B is efficient - in fact it is very fast. Thus we may score different gate sequences and choose the right one for our multi-qubit computation without significant overhead.

G	$B(G)$
X	$1/2$
Y	$1/2$
Z	0
H	$2^{-5/4}$
CNOT	$2^{-3/2}$

C	d_A	$1-F$
Bell 1	0.500	0.004
Bell 2	0.728	0.011
DJ 1	0.912	0.019
DJ 2	0.951	0.034

Table V.1: Badness $B(G)$ for some typical one and two qubit gates G. The calculation assumes the existence of a decoherence free subspace spanned by $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$.

To demonstrate the validity of this idea we consider two quantum information processing tasks achieved with distinct circuits. These are the Ramsey-type experiments of Sec. III and a Deutsch-Jozsa search algorithm.

VI. CONCLUSION

[[More general error models (bit flip, quantum noise) and how to combine our method with quantum error correction.]] Clearly, the code space should be in the

DFS as far as possible. Also, note that only relatively short circuits benefit from our method, so error correction cycles can be timed to take advantage of it.

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