Linear Algebra Exercises

Rayhan Momin

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Exercise 1

Let x be a given $n \times 1$ vector and consider the problem

$$v(x) = \max_{y,u} \{-y'Py - u'Qu\}$$

subject to the linear constraint

$$y = Ax + Bu$$

Here:

- P is an $n \times n$ matrix and Q is an $m \times m$ matrix
- A is an $n \times n$ matrix and B is an $n \times m$ matrix
- both P and Q are symmetric and positive semidefinite

One way to solve this problem is to form the Lagrangian

$$\mathcal{L} = -y'Py - u'Qu + \lambda' [Ax + Bu - y]$$

where λ is an $n \times 1$ vector of Lagrange multipliers
Show that these conditions imply that

1. $\lambda = -2Py$
2. The optimizing choice of u satisfies $u = -(Q + B'PB)^{-1}B'PAx$
3. The function v satisfies $v(x) = -x'\tilde{P}x$ where $\tilde{P} = A'PA - A'PB(Q + B'PB)^{-1}B'PA$

My Solution to Exercise 1

1.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial(-y'Py)}{\partial y} - \frac{\partial \lambda' y}{\partial y} && = 0 && (1) \\
&= -(P+P')y - \lambda && = 0 && (2) \\
&= -2Py - \lambda && = 0 && (3) \\
\lambda &= -2Py && && (4)
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u} &= -\frac{\partial(u'Qu)}{\partial u} + \frac{\lambda' Bu}{\partial u} && = 0 && (1) \\
&= -(Q+Q')u + (\lambda' B)' && && (2) \\
&= -2Qu + B'\lambda && && (3)
\end{aligned}
$$

Recall $\lambda = -2Py$

$$
\begin{aligned}
Qu &= -B'Py && (4) \\
& && (5)
\end{aligned}
$$

Recall $y = Ax + Bu$

$$
\begin{aligned}
&= -B'P(Ax + Bu) && (6) \\
&= -B'PAx - B'PBu && (7) \\
Qu + B'PBu &= -B'PAx && (8) \\
(Q + B'PB)u &= -B'PAx && (9) \\
u &= -(Q + B'PB)^{-1}B'PAx && (10)
\end{aligned}
$$

3. Idea is to substitute in u and y into fucntion v

$$
\begin{aligned}
v(x) &= -y'Py - u'Qu && (1) \\
&= -(Ax + Bu)'P(Ax + Bu) - u'Qu && (2)
\end{aligned}
$$

Let $u = -Sx$

$$
\begin{aligned}
&= -(Ax - BSx)'P(Ax - BSx) - (Sx)'QSx && (3)\\
&= -\left[((Ax)' - (BSx)')P(Ax - BSx) + (x'S'QSx)\right] && (4)\\
&= -\left[(x'A' - x'S'B')(PAx - PBSx) + (x'S'QSx)\right] && (5)\\
&= -\left[x'A'PAx - x'S'B'PAx - x'A'PBSx + x'S'B'PBSx + x'S'QSx\right] && (6)
\end{aligned}
$$

Note that $S'B'PA$ and $A'PBS$ are symmetric $n \times n$ matrices. $B'PB - Q$ is a symmetric $m \times m$ matrix.

$$
\begin{aligned}
&= -x'\left[A'PA - S'B'PA - (A'PBS)' + S'(B'PB + Q)'S\right]x && (7)\\
&= -x'\left[A'PA - 2S'B'PA + S'(Q + B'PB)S\right]x && (8)\\
& && (9)
\end{aligned}
$$

Substitute back in $S = (Q + B'PB)^{-1}B'PA$

$$
\begin{aligned}
&= -x'\left[A'PA - 2(B'PA)'(Q + B'PB)^{-1}(B'PA) + (B'PA)'(Q + B'PB)^{-1}B'PA\right]x && (10)\\
&= -x'\underbrace{\left[A'PA - A'PB(Q + B'PB)^{-1}B'PA\right]}_{\tilde{P}}x && (11)\\
&= -x'\tilde{P}x && (12)
\end{aligned}
$$