

Non-Atomic Zero: A Unified Theory of Projection-Honest Computation

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Abstract

We introduce *Non-Atomic Zero* (NA0), a mathematical framework that makes explicit the information discarded by projection and totalization operations. Standard computational practice treats projection as atomic—collapsing high-dimensional objects to scalars while silently discarding context. NA0 instead represents such operations as structured objects $\text{NA0}(\mathcal{D}, \mathcal{R}; \Pi)$ carrying: (1) the discarded information (*debt* \mathcal{D}), (2) the retained value (*remainder* \mathcal{R}), and (3) the projection rule (*policy* Π). We demonstrate the framework across multiple domains: divergent series regularization (where NA0 clarifies the sense in which $1 + 2 + 3 + \dots = -1/12$), signal processing (where baseline subtraction policies induce correlated debt-remainder variations), spectral analysis (where exporting projectors avoids eigenvector instabilities), and reduced quantum dynamics (where memory effects arise as correction terms for naive projection). Our experiments show that policy changes produce predictable, measurable shifts in both debt and remainder, enabling detection of projection-induced artifacts. We propose NA0 as a foundation for *projection-honest* computation, where all totalization debt is explicitly tracked and auditable.

1 Introduction

Projection is ubiquitous in scientific computation. Whenever we reduce a high-dimensional object to a summary statistic, fit a model to data, trace out environmental degrees of freedom, or regularize a divergent series, we perform a projection—discarding some information to obtain a tractable result. Standard practice treats such operations as atomic: the projection produces a scalar or low-dimensional object, and the discarded information vanishes.

This paper argues that treating projection as atomic is computationally hazardous. The discarded information does not vanish; it becomes implicit debt that can resurface as apparent disagreements between methods, spurious “tensions” between experiments, or artifacts mistaken for physical effects.

We introduce *Non-Atomic Zero* (NA0), a framework that makes projection debt explicit. An NA0 object has the form:

$$\text{NA0}(\mathcal{D}, \mathcal{R}; \Pi) \tag{1}$$

where \mathcal{D} is the discarded information (debt), \mathcal{R} is the retained value (remainder), and Π is the projection rule (policy). The zero in “Non-Atomic Zero” reflects the fact that these objects generalize the notion of “collapsing to zero”—the standard treatment makes the debt contribution zero by fiat, while NA0 preserves it.

1.1 Projection as a Source of Artifacts

Three examples motivate the framework:

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Divergent Series. The statement $1+2+3+\dots = -1/12$ is notorious for provoking confusion. The series diverges; how can it equal a negative fraction? The answer is that this is not a statement about convergence but about *regularization*—a specific projection policy (zeta-function regularization or Ramanujan summation) that extracts a finite part from a divergent object. Different regularization schemes can yield different values. The NA0 framework makes explicit what is discarded: the divergent part becomes debt, the finite part becomes remainder, and the regularization scheme is the policy.

Background Subtraction. In astronomy and spectroscopy, estimating a signal often requires subtracting a model of the background. Different baseline models (linear, polynomial, spline) fitted to the same “source-free” regions can yield significantly different signal estimates. The disagreement is not statistical noise but projection policy disagreement: each model class extrapolates differently from the fitting regions to the signal regions. NA0 represents this as debt (the model’s extrapolation uncertainty) correlated with remainder (the estimated signal).

Quantum Reduced Dynamics. The Nakajima-Zwanzig equation [16, 22] provides the exact reduced dynamics of a quantum system coupled to an environment:

$$\frac{d\rho_S}{dt} = \mathcal{L}_S \rho_S + \int_0^t K(t-\tau) \rho_S(\tau) d\tau + I(t) \quad (2)$$

The memory kernel $K(t)$ and inhomogeneity $I(t)$ are precisely the correction terms required because the naive projected dynamics (first term alone) discards system-environment correlations. In NA0 terms, tracing out the environment creates debt; the memory kernel is the back-action of that debt on the reduced system.

1.2 Contributions

This paper makes five contributions:

1. **Formalization:** We define the NA0 algebra, including composition rules and the “projection timing sensitivity” (PTS) diagnostic that detects non-commutative projection effects.
2. **Divergence Application:** We show how NA0 clarifies famous regularized sums, making the debt-remainder decomposition explicit.
3. **Signal Processing Experiments:** We demonstrate experimentally that policy variations induce correlated debt-remainder changes, and that these correlations are detectable.
4. **Spectral Analysis:** We introduce projection-honest spectral methods that export projectors rather than eigenvectors, avoiding sign-flip and degeneracy artifacts.
5. **Quantum Dynamics:** We show that, in the Jaynes-Cummings setting studied here, the NA0 correction term aligns with the Nakajima-Zwanzig memory contribution and can be expressed in the same structural form.

1.3 Paper Organization

Section 2 formalizes the NA0 framework. Section 3 applies it to divergent series. Section 4 presents the signal processing experiments. Section 5 develops projection-honest spectral methods. Section 6 demonstrates the quantum dynamics application. Section 7 discusses implications and future work.

Table 1: Summary of NA0 notation.

Symbol	Meaning
\mathcal{X}	Ambient space of objects under projection
\mathcal{R}_Π	Remainder space for policy Π
Π	Policy (projection rule / totalization scheme)
$r_\Pi : \mathcal{X} \rightarrow \mathcal{R}_\Pi$	Remainder extractor
$\ell_\Pi : \mathcal{R}_\Pi \rightarrow \mathcal{X}$	Lift (reconstruction map)
$q_\Pi(X) = X - \ell_\Pi(r_\Pi(X))$	Induced debt map
D	Debt component (lives in \mathcal{X})
R	Remainder component (lives in \mathcal{R}_Π)
$\text{NA0}\langle D, R; \Pi \rangle$	NA0 object: debt, remainder, policy
$P_\Pi(X)$	NA0 projection operator
Rec_Π	Reconstruction: $D + \ell_\Pi(R)$
Total	Totalization: extract R , discard D
PTS_R	Projection timing sensitivity (remainder channel)
DTR	Debt-to-Remainder Ratio (admissibility criterion)
PTSBudget	Relative PTS error (admissibility criterion)
$\tau_{\text{DTR}}, \tau_{\text{PTS}}$	Admissibility thresholds (default: 0.1, 0.05)

1.4 Related Work

The NA0 framework connects to several established research areas:

Divergent Series and Regularization. Hardy’s foundational treatment of divergent series [11] and Ramanujan’s summation methods [17] establish the classical theory. The Hadamard finite part [10] and distributional approaches [7] provide rigorous frameworks for extracting finite values. The Casimir effect [3] demonstrates physical relevance. NA0 provides a common notation for these techniques, making the policy-dependence of “values” explicit.

Baseline Subtraction and Model Selection. Baseline estimation in spectroscopy [14, 6] and signal processing [20] involves model-class choices that are often treated as preprocessing. The NA0 framework treats these as projection policies whose effects propagate into downstream results.

Spectral Analysis and Subspace Stability. The Davis-Kahan theorem [5] bounds eigenvector perturbation but does not resolve sign/basis ambiguity. Subspace angles and distances [21, 19] provide basis-independent metrics. PCA and spectral clustering [13, 8] inherit these ambiguities. NA0-aware methods export projectors rather than eigenvectors, avoiding artifacts.

Open Quantum Systems. The Nakajima-Zwanzig projection operator technique [16, 22] and related methods (Lindblad [15, 9], time-convolutionless [18]) provide exact or approximate reduced dynamics. Recent work [1, 4, 2] characterizes non-Markovianity. The Jaynes-Cummings model [12] serves as a standard test case. NA0 generalizes the insight that projection creates correctable debt beyond the quantum setting.

2 The NA0 Algebra

We formalize the NA0 framework as an algebraic structure that tracks projection debt alongside retained values.

2.1 Basic Definitions

We begin with a minimal model that makes “debt” and “remainder” well-typed. This model is sufficient for the additive projection cases used throughout the paper; other domains can be treated as extensions by replacing $+$ with an explicit reconstruction operator.

Definition 1 (Policy, remainder extractor, and lift). *Fix an ambient space \mathcal{X} (a vector space or abelian group) for objects X under consideration. A policy Π specifies: (i) a remainder space \mathcal{R}_Π (also a vector space or abelian group), (ii) a remainder extractor $r_\Pi : \mathcal{X} \rightarrow \mathcal{R}_\Pi$, and (iii) a lift (reconstruction map) $\ell_\Pi : \mathcal{R}_\Pi \rightarrow \mathcal{X}$. We require ℓ_Π to be a right-inverse on the image of r_Π :*

$$r_\Pi(\ell_\Pi(R)) = R \quad \forall R \in \text{Im}(r_\Pi).$$

In the minimal additive model used throughout this paper, we further require r_Π and ℓ_Π to be linear (or additive homomorphisms). This ensures that the PTS identity (Proposition 1) and related results hold.

Definition 2 (Debt map and NA0 object). *Given (r_Π, ℓ_Π) , define the lifted remainder in \mathcal{X} as*

$$\tilde{R}_\Pi(X) := \ell_\Pi(r_\Pi(X)).$$

Define the induced debt map

$$q_\Pi(X) := X - \tilde{R}_\Pi(X),$$

where subtraction is taken in \mathcal{X} (thus this minimal model assumes \mathcal{X} supports an additive structure). An NA0 object is

$$\text{NA0}\langle D, R; \Pi \rangle,$$

where $R = r_\Pi(X) \in \mathcal{R}_\Pi$ and $D = q_\Pi(X) \in \mathcal{X}$ for some $X \in \mathcal{X}$.

Definition 3 (Reconstruction and well-posedness). *Reconstruction under policy Π is the map*

$$\text{Rec}_\Pi(\text{NA0}\langle D, R; \Pi \rangle) := D + \ell_\Pi(R).$$

In the minimal additive model,

$$\text{Rec}_\Pi(\text{NA0}\langle q_\Pi(X), r_\Pi(X); \Pi \rangle) = X$$

holds identically. We call Π well-posed for NA0 bookkeeping if ℓ_Π is specified and $q_\Pi(X)$ is defined for all X of interest.

Definition 4 (Canonical debt representations). *In practice, the debt component D is not arbitrary. Common well-posed representations include:*

- **Complement residual debt (additive):** $D = q_\Pi(X) \in \mathcal{X}$ induced by an explicit lift ℓ_Π .
- **Uncertainty debt:** D as a distribution/covariance/interval object plus the assumptions required to interpret it (model class, regularization, priors).
- **Sufficient-statistics debt:** D as the minimal metadata and statistics required to recompute or re-totalize under alternate policies (e.g., fit diagnostics, hyperparameters, and restricted residuals).

All debt representations used in this paper are required to be deterministically serializable and comparable under a declared metric.

Remark 1 (Well-posedness criteria for debt). *To distinguish NA0 from informal “residual logging,” debt must satisfy:*

1. **Typed:** D lives in a declared space with defined operations.
2. **Comparable:** A metric or norm is specified for measuring debt magnitude.
3. **Serializable:** D admits a canonical, deterministic encoding.
4. **Policy-addressable:** D includes sufficient metadata to re-totalize under alternate policies or to trigger fail-closed behavior.

These criteria ensure that debt is a first-class computational object, not merely a comment or annotation.

Definition 5 (Totalization). Totalization is the operation that extracts only the remainder, discarding the debt:

$$\text{Total}(\text{NA0}\langle D, R; \Pi \rangle) = R \quad (3)$$

Totalization converts an NA0 object back to a scalar (or low-dimensional object), losing the debt information.

Standard computational practice implicitly totalizes at every projection step. NA0 defers totalization, preserving debt for downstream analysis.

2.2 Composition in the minimal additive model

Composition is induced by the policy maps (r_Π, ℓ_Π) and may fail to exist if types do not align.

Definition 6 (Sequential policy application). For a policy Π with maps (r_Π, ℓ_Π) , define the NA0 projection operator

$$P_\Pi(X) := \text{NA0}\langle q_\Pi(X), r_\Pi(X); \Pi \rangle.$$

Given two policies Π_1, Π_2 defined over a shared ambient space \mathcal{X} , the sequential policy $\Pi_{2 \circ 1}$ is defined whenever r_{Π_2} is valid on \mathcal{X} . In that case, we define

$$P_{\Pi_{2 \circ 1}}(X) := \text{NA0}\langle q_{\Pi_2}(X), r_{\Pi_2}(X); \Pi_2 \rangle,$$

and note that NA0 bookkeeping is threaded by explicitly retaining the earlier debt objects rather than discarding them.

Remark 2. This definition replaces informal “ \oplus ”/“ \otimes ” composition. In the additive model, composition is not arbitrary: it is determined by declared extract/lift pairs and their shared ambient space. Later sections treat non-additive cases as extensions by declaring a reconstruction operator in place of $+$.

Example 1 (Worked composition: baseline then threshold). Consider a signal $X \in \mathbb{R}^n$ processed by two sequential projections:

1. **Baseline subtraction** Π_1 : Fit a polynomial $p(x)$ to edge regions, subtract it.

- $r_{\Pi_1}(X) = X - p$ (baseline-subtracted signal)
- $\ell_{\Pi_1}(R) = R$ (identity lift into \mathbb{R}^n)
- $q_{\Pi_1}(X) = p$ (the fitted baseline is debt)

2. **Thresholding** Π_2 : Zero values below threshold τ .

- $r_{\Pi_2}(Y) = Y \cdot \mathbf{1}_{Y > \tau}$ (thresholded signal)
- $\ell_{\Pi_2}(R) = R$ (identity lift)
- $q_{\Pi_2}(Y) = Y \cdot \mathbf{1}_{Y \leq \tau}$ (sub-threshold values are debt)

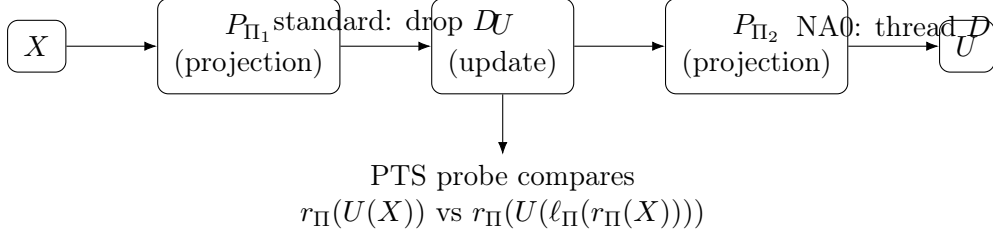


Figure 1: Projection-honest computation threads debt objects across repeated projections. PTS provides an operational diagnostic for projection timing sensitivity.

Applying Π_1 then Π_2 to X :

$$P_{\Pi_2}(P_{\Pi_1}(X)) = \text{NA0} \langle q_{\Pi_2}(X - p), r_{\Pi_2}(X - p); \Pi_2 \rangle$$

with accumulated debt $(p, q_{\Pi_2}(X - p))$ —the baseline and the sub-threshold residuals.

NA0 bookkeeping threads both debt components. Standard practice discards p after step 1, losing the ability to diagnose whether apparent “signal” came from baseline model choice or thresholding.

Fail-closed condition: If $\|p\| > \theta_1$ (baseline too large) or $\|q_{\Pi_2}\| > \theta_2$ (too much thresholded), refuse to totalize.

2.3 Projection Timing Sensitivity

A key diagnostic for projection effects is non-commutativity with other operations.

Definition 7 (Projection Timing Sensitivity (typed remainder form)). Let Π be a policy with (r_{Π}, ℓ_{Π}) and let $U : \mathcal{X} \rightarrow \mathcal{X}$ be an update/evolution operator. Define the remainder-channel PTS:

$$\text{PTS}_R(\Pi, U, X) := r_{\Pi}(U(X)) - r_{\Pi}(U(\ell_{\Pi}(r_{\Pi}(X)))),$$

which compares evolve-then-extract vs extract-then-lift-then-evolve-then-extract. We say PTS is present when $\text{PTS}_R(\Pi, U, X) \neq 0$ under a declared metric on \mathcal{R}_{Π} .

Proposition 1 (Debt form of PTS under additive reconstruction). Assume the policy Π admits additive reconstruction in \mathcal{X} (so $X = q_{\Pi}(X) + \ell_{\Pi}(r_{\Pi}(X))$), and assume U is linear over $+$. Then the remainder PTS satisfies:

$$\text{PTS}_R(\Pi, U, X) = r_{\Pi}(U(q_{\Pi}(X))), \quad (4)$$

i.e., PTS measures how the evolved debt projects back onto the remainder channel.

Proof. Write $R = r_{\Pi}(X)$ and $D = q_{\Pi}(X) = X - \ell_{\Pi}(R)$. Then $U(X) = U(D) + U(\ell_{\Pi}(R))$ by linearity. We have:

$$\text{PTS}_R(\Pi, U, X) = r_{\Pi}(U(X)) - r_{\Pi}(U(\ell_{\Pi}(R))) \quad (5)$$

$$= r_{\Pi}(U(D) + U(\ell_{\Pi}(R))) - r_{\Pi}(U(\ell_{\Pi}(R))) \quad (6)$$

$$= r_{\Pi}(U(D)), \quad (7)$$

where the last step uses linearity of r_{Π} (which follows from the additive model). \square

This shows that PTS measures how debt transforms under evolution and projects back.

2.4 Projection Non-Transparency

The following theorem establishes that non-zero PTS implies provable information loss from naive projection—loss that cannot be recovered without access to the debt.

Theorem 1 (Projection Non-Transparency). *Let Π be a well-posed policy with (r_Π, ℓ_Π, q_Π) satisfying additive reconstruction in a normed space $(\mathcal{X}, \|\cdot\|)$. Let $U : \mathcal{X} \rightarrow \mathcal{X}$ be linear, and let $\|\cdot\|_R$ be a norm on \mathcal{R}_Π . Then:*

(a) **Error identity.** *The remainder-channel error from naive projection is exactly:*

$$E_{\text{naive}}(U, X) := \|r_\Pi(U(X)) - r_\Pi(U(\ell_\Pi(r_\Pi(X))))\|_R = \|\text{PTS}_R(\Pi, U, X)\|_R.$$

(b) **Debt-driven identity.** *Under additive reconstruction and linear U :*

$$E_{\text{naive}}(U, X) = \|r_\Pi(U(q_\Pi(X)))\|_R.$$

That is, the error equals the norm of the evolved debt’s projection onto the remainder channel.

(c) **Non-eliminability.** *If $r_\Pi(U(q_\Pi(X))) \neq 0$, then no function $f : \mathcal{R}_\Pi \rightarrow \mathcal{R}_\Pi$ applied to the naive remainder $r_\Pi(U(\ell_\Pi(r_\Pi(X))))$ can recover the true remainder $r_\Pi(U(X))$ without access to the debt $q_\Pi(X)$ or additional information about X .*

Proof. Part (a) is immediate from the definition of PTS_R .

Part (b) follows from Proposition 1: under the stated assumptions, $\text{PTS}_R(\Pi, U, X) = r_\Pi(U(q_\Pi(X)))$.

For part (c), suppose f recovers the true remainder from the naive remainder alone: $f(r_\Pi(U(\ell_\Pi(R)))) = r_\Pi(U(X))$ for $R = r_\Pi(X)$. By additive reconstruction, $r_\Pi(U(X)) = r_\Pi(U(\ell_\Pi(R))) + r_\Pi(U(q_\Pi(X)))$. Thus f would need to produce $r_\Pi(U(q_\Pi(X)))$ from $r_\Pi(U(\ell_\Pi(R)))$ alone. But $q_\Pi(X) = X - \ell_\Pi(R)$ depends on X beyond R ; for fixed R , different X (with the same remainder but different debt) yield different $r_\Pi(U(q_\Pi(X)))$. Hence no such f exists in general. \square

Corollary 1 (Naive Projection is Provably Lossy). *If there exist U and X such that $r_\Pi(U(q_\Pi(X))) \neq 0$, then discarding debt before applying U incurs an error that cannot be eliminated by any post-hoc correction on the remainder channel alone.*

This theorem provides the formal foundation for projection-honest computation: *dropping debt is not merely sloppy bookkeeping—it is provably lossy whenever the evolution couples debt back into the remainder channel.*

Remark 3 (Scope of linearity assumptions). *Theorem 1 assumes additive reconstruction and linear U . For nonlinear U or non-additive reconstruction (e.g., multiplicative or information-theoretic policies), the theorem applies locally via linearization, or globally by replacing additive composition with a declared reconstruction operator. We treat such extensions as future work; the linear case already covers the signal, spectral, and quantum examples in this paper.*

2.5 Debt Attachment Notation

For compact notation, we write:

$$\mathcal{R}^{(\mathcal{D}; \Pi)} \tag{8}$$

to indicate that remainder \mathcal{R} carries attached debt \mathcal{D} under policy Π . This is equivalent to $\text{NA0}(\mathcal{D}, \mathcal{R}; \Pi)$ but emphasizes that the “answer” \mathcal{R} is not standalone.

2.6 Fail-Closed Projection

Definition 8 (Fail-Closed Policy). *A policy Π is fail-closed if it refuses to totalize when debt exceeds a threshold:*

$$\text{Total}_\Pi(\text{NA0}\langle \mathcal{D}, \mathcal{R}; \Pi \rangle) = \begin{cases} \mathcal{R} & \text{if } \|\mathcal{D}\| < \theta \\ \perp & \text{otherwise} \end{cases} \quad (9)$$

where θ is the policy's debt tolerance and \perp indicates refusal.

Fail-closed policies prevent silent propagation of high-debt values. Instead of returning a potentially misleading scalar, they signal that the projection is unreliable under current conditions.

2.7 Admissibility Criteria

We define two portable, computable criteria for deciding when totalization is safe.

Definition 9 (Debt-to-Remainder Ratio (DTR)). *For an NA0 object $\text{NA0}\langle D, R; \Pi \rangle$ with norms $\|\cdot\|$ on \mathcal{X} and $\|\cdot\|_R$ on \mathcal{R}_Π , define:*

$$\text{DTR}(\text{NA0}\langle D, R; \Pi \rangle) := \frac{\|D\|}{\|\ell_\Pi(R)\| + \epsilon}$$

where $\epsilon > 0$ is a small constant preventing division by zero. The DTR threshold τ_{DTR} determines admissibility: totalization is permitted iff $\text{DTR} < \tau_{\text{DTR}}$.

Definition 10 (PTS Budget). *For a policy Π , update U , and input X , define the PTS budget as:*

$$\text{PTSBudget}(\Pi, U, X) := \frac{\|\text{PTS}_R(\Pi, U, X)\|_R}{\|r_\Pi(X)\|_R + \epsilon}.$$

This measures the relative error introduced by naive projection under evolution U . The PTS budget threshold τ_{PTS} determines admissibility: totalization before U is permitted iff $\text{PTSBudget} < \tau_{\text{PTS}}$.

Remark 4 (Default thresholds and regularization). *We recommend $\tau_{\text{DTR}} = 0.1$ and $\tau_{\text{PTS}} = 0.05$ as conservative defaults. These should be calibrated per domain: stricter for high-stakes applications (e.g., $\tau = 0.01$), looser for exploratory work. The key property is that thresholds are declared, not implicit.*

For the regularization constant ϵ , we recommend $\epsilon = 10^{-8} \cdot \|X\|$ (machine epsilon scaled by input norm) or a fixed $\epsilon = 10^{-10}$ for normalized inputs. Implementations must declare the chosen ϵ to ensure reproducibility.

These criteria make fail-closed policies operational: rather than relying on subjective judgment, pipelines can enforce admissibility automatically.

2.8 Properties

Proposition 2 (Debt Conservation). *Under faithful NA0 bookkeeping, total information is conserved:*

$$X = \text{Rec}_\Pi(\text{NA0}\langle \mathcal{D}, \mathcal{R}; \Pi \rangle) \quad (10)$$

where Rec_Π combines debt and remainder according to policy (Definition 3).

Proposition 3 (Policy Dependence). *Different policies applied to the same input X generally yield different debt-remainder decompositions:*

$$\mathcal{P}_{\Pi_1}(X) = \text{NA0}\langle \mathcal{D}_1, \mathcal{R}_1; \Pi_1 \rangle \neq \text{NA0}\langle \mathcal{D}_2, \mathcal{R}_2; \Pi_2 \rangle = \mathcal{P}_{\Pi_2}(X) \quad (11)$$

even when $\mathcal{R}_1 = \mathcal{R}_2$.

This captures the key insight: two pipelines may agree on the remainder while carrying different debts, leading to different behaviors under composition or evolution.

3 Divergence and Regularization

The NA0 framework provides a natural language for divergent series regularization, clarifying statements like “ $1 + 2 + 3 + \dots = -1/12$ ”.

3.1 The Problem of Divergent Sums

The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$ diverges. Yet in physics—particularly string theory, Casimir effect calculations, and zeta-function regularization—this series is assigned the value $-1/12$. This is not a claim about convergence but about *regularization*: extracting a finite part from a divergent object.

3.2 Regulated Families and NA0

To keep debt as a mathematically well-typed object, we represent divergent expressions via a regulated family $S(\epsilon)$ and a declared scheme/policy Π . The remainder is the scheme-defined finite part, and the debt is the discarded divergent asymptotic data together with the regulator identity.

Example: exponential regulator. Consider the regulated series

$$S(\epsilon) := \sum_{n=1}^{\infty} n e^{-\epsilon n}, \quad \epsilon > 0.$$

This admits a closed form $S(\epsilon) = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2}$ and has the asymptotic expansion as $\epsilon \rightarrow 0^+$:

$$S(\epsilon) = \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon^2).$$

Under a “finite-part” policy Π_{FP} that retains the constant term and records the divergent terms as debt, we define:

$$P_{\Pi_{\text{FP}}}(S) = \text{NA0} \langle D(\epsilon), R; \Pi_{\text{FP}} \rangle,$$

where the remainder is the finite part $R = -\frac{1}{12}$ and the debt is the discarded divergent asymptotic series plus regulator metadata, e.g.

$$D(\epsilon) = \left(\frac{1}{\epsilon^2} + O(\epsilon^2), \text{regulator} = \text{exp}, \text{retained term} = \epsilon^0 \right).$$

This makes explicit what is usually implicit: the value $-\frac{1}{12}$ is a policy-dependent remainder extracted from a regulated family, and the divergent structure is retained as a first-class object rather than silently discarded.

3.3 Zeta-Function Regularization

The Riemann zeta function provides a canonical regularization via analytic continuation. For $\text{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{12}$$

extended to the entire complex plane (except $s = 1$) by analytic continuation. At $s = -1$, $\zeta(-1) = -\frac{1}{12}$.

In NA0 terms, zeta-regularization is a policy Π_{ζ} where:

- The ambient space \mathcal{X} is regulated families (formal power series in a regulator parameter)

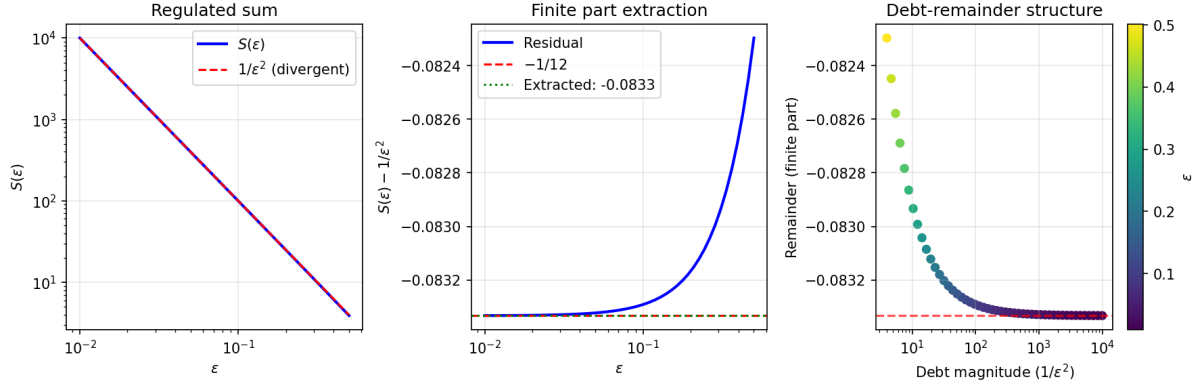


Figure 2: Numerical validation of debt-remainder separation in zeta-regularization. **Left:** Regulated sum $S(\varepsilon)$ and divergent term $1/\varepsilon^2$. **Center:** Residual after subtracting the divergent term; the extracted constant converges to $-1/12$ with relative error $< 0.001\%$. **Right:** Debt magnitude (divergent term) vs remainder (finite part), showing the stable remainder despite varying debt. Script: `divergence_validation.py`.

- The remainder extractor r_{Π_ζ} returns the analytically-continued value
- The debt includes the divergent asymptotic terms and the choice of analytic continuation path

3.4 Alternative Regularizations

Different policies yield different remainders:

Ramanujan Summation. Ramanujan’s method assigns $\sum_{n=1}^{\infty} n = -1/12$ via a specific definition involving the Euler-Maclaurin formula. The debt structure differs from zeta-regularization even though the remainder is the same.

Cutoff Regularization. Introducing a cutoff N and taking $N \rightarrow \infty$:

$$\sum_{n=1}^N n = \frac{N(N+1)}{2} \sim \frac{N^2}{2} + \frac{N}{2} \quad (13)$$

This diverges; extracting a finite part requires subtracting the divergent terms, which involves arbitrary choices. Different subtraction schemes give different remainders.

Dimensional Regularization. In d dimensions, certain sums become convergent and can be analytically continued. The remainder depends on the dimension.

3.5 Debt-Remainder Correlation

A key NA0 prediction: changing the regularization policy should produce correlated changes in debt and remainder. If a policy change shifts the remainder by $\Delta\mathcal{R}$, it must shift the debt by $-\Delta\mathcal{R}$ (for additive projections) to conserve information.

This can be tested: vary the regularization policy (e.g., cutoff scale, analytic continuation path) and measure how debt and remainder co-vary.

Figure 2 provides numerical validation: we compute $S(\varepsilon) = \sum_{n=1}^{\infty} n e^{-\varepsilon n}$ for decreasing ε , subtract the divergent $1/\varepsilon^2$ term, and extract the constant. The extracted remainder converges to $-1/12$ with relative error $< 0.001\%$, demonstrating that the debt-remainder decomposition

is numerically stable and policy-dependent (the constant changes if we change which asymptotic terms are retained as debt).

3.6 Famous Formulas

Several “famous” regularized values can be stated precisely in NA0 form:

$$\mathcal{P}_\zeta(1 + 1 + 1 + \cdots) = \text{NA0}\langle \cdot, -\tfrac{1}{2}; \zeta \rangle \quad (14)$$

$$\mathcal{P}_\zeta(1 + 2 + 3 + \cdots) = \text{NA0}\langle \cdot, -\tfrac{1}{12}; \zeta \rangle \quad (15)$$

$$\mathcal{P}_\zeta(1 + 4 + 9 + \cdots) = \text{NA0}\langle \cdot, 0; \zeta \rangle \quad (16)$$

$$\mathcal{P}_\zeta(1^3 + 2^3 + 3^3 + \cdots) = \text{NA0}\langle \cdot, \tfrac{1}{120}; \zeta \rangle \quad (17)$$

where $\langle \cdot, \dots \rangle$ indicates that the debt is “everything else” required to make the decomposition exact.

3.7 Physical Applications

In physics, regularized sums appear in:

Casimir Effect. The vacuum energy between conducting plates involves $\sum_{n=1}^{\infty} n$, regularized to give a finite, measurable force.

String Theory. The critical dimension $d = 26$ for bosonic strings emerges from requiring $\zeta(-1) = -1/12$ in a consistency calculation.

Quantum Field Theory. Divergent loop integrals are regularized via dimensional regularization, yielding finite remainders that match experiment.

In each case, the physics is encoded not in the bare divergent sum but in the *policy-dependent remainder*. The NA0 framework makes this dependence explicit, rather than treating regularization as a “trick” that magically produces answers.

3.8 The Hadamard Finite Part

For divergent integrals, the Hadamard finite part provides a canonical regularization. For $\int_0^1 x^{-\alpha} dx$ with $\alpha > 1$:

$$\text{Pf} \int_0^1 x^{-\alpha} dx = \frac{1}{1-\alpha} \quad (18)$$

interpreted as the analytic continuation from $\alpha < 1$.

In NA0 terms, regularizing by cutoff ϵ :

$$\int_{\epsilon}^1 x^{-\alpha} dx = \frac{1 - \epsilon^{1-\alpha}}{1 - \alpha} = \underbrace{\frac{1}{1 - \alpha}}_{\text{finite part}} + \underbrace{\left(-\frac{\epsilon^{1-\alpha}}{1 - \alpha} \right)}_{\text{divergent as } \epsilon \rightarrow 0}.$$

Thus:

$$\mathcal{P}_{\text{Hadamard}} \left(\int_0^1 x^{-\alpha} dx \right) = \text{NA0} \left\langle -\frac{\epsilon^{1-\alpha}}{1 - \alpha}, \frac{1}{1 - \alpha}; \text{Hadamard} \right\rangle \quad (19)$$

where the debt is the divergent asymptotic term (plus regulator metadata: cutoff at ϵ).

4 Signal Processing: Baseline Subtraction

We demonstrate NA0 principles through synthetic experiments on baseline subtraction, a ubiquitous projection operation in signal processing.

4.1 The Baseline Subtraction Problem

In many applications—spectroscopy, astronomy, medical imaging—the observed signal $Y(x)$ is a mixture of:

$$Y(x) = B + F(x) + I(x) + N(x) \quad (20)$$

where:

- B is the background/baseline to be estimated
- $F(x)$ is the foreground signal of interest
- $I(x)$ is instrumental drift
- $N(x)$ is measurement noise

The goal is to recover B (or F) from Y . The standard approach:

1. Identify “source-free” regions where $F(x) \approx 0$
2. Fit a baseline model to these regions
3. Extrapolate/interpolate to the full domain
4. Subtract the fitted baseline to recover F ; estimate B from residuals

This is a projection: the fitted baseline model discards information about unmodeled components.

4.2 Projection Policy Variations

Different baseline model classes constitute different projection policies:

- **Linear (degree 1)**: Assumes linear drift
- **Quadratic (degree 2)**: Captures curvature
- **Cubic (degree 3)**: More flexible extrapolation
- **High-order (degree 6+)**: Can fit complex shapes

Each policy makes different assumptions about what belongs in the baseline versus the signal. Higher-degree polynomials can fit more of the low-frequency foreground component, shifting the boundary between debt and remainder.

4.3 Experimental Setup

We generate synthetic data with known ground truth:

- Background $B = 1.0$ (constant)
- Foreground $F(x)$: Gaussian envelope + substructure, concentrated in center
- Instrument $I(x)$: Low-order polynomial drift
- Noise $N(x)$: Gaussian, $\sigma = 0.05$

The foreground has faint low-frequency tails extending to the “source-free” edge regions. This is the mechanism that creates policy-dependent bias: different polynomial degrees absorb different amounts of this tail.

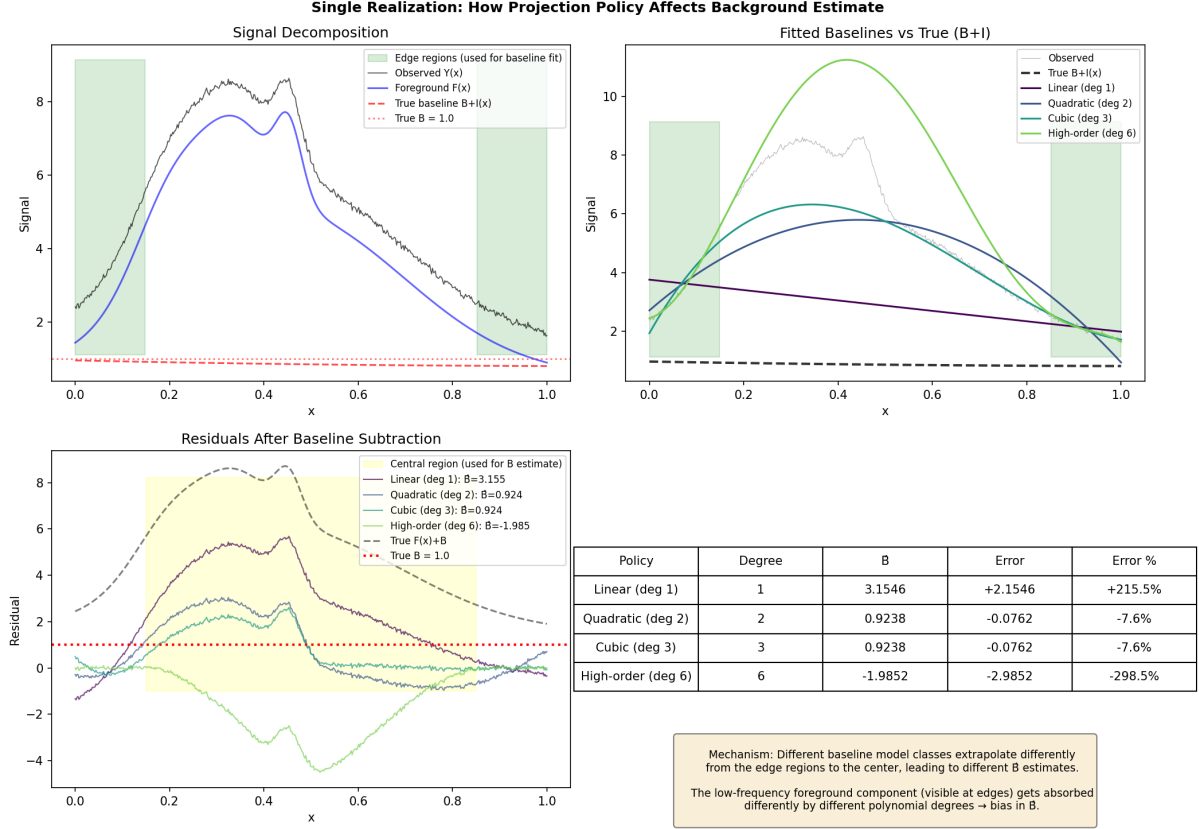


Figure 3: Single realization of the baseline subtraction experiment. **Top left:** Signal decomposition showing observed $Y(x)$, foreground $F(x)$, and true baseline $B+I(x)$. Edge regions (green shading) are used for baseline fitting. **Top right:** Fitted baselines from different policies versus true baseline. **Bottom left:** Residuals after baseline subtraction, with estimated \hat{B} values. **Bottom right:** Numerical comparison of policy performance.

4.4 Results

Figure 3 shows a single realization. Key observations:

- The linear fit (degree 1) underestimates the true baseline in the center, leading to overestimated \hat{B}
- The high-order fit (degree 6) absorbs some foreground at the edges, leading to underestimated \hat{B}
- Intermediate policies produce intermediate results

Figure 4 shows Monte Carlo results over 1000 trials:

- **Bias varies by policy:** $\sim 10\%$ variation in mean \hat{B}
- **Variance varies by policy:** Higher-degree polynomials have more extrapolation variance
- **Inter-policy disagreement exceeds noise:** Policies disagree by many σ even with low measurement noise

4.5 Debt-Remainder Correlation

Figure 5 demonstrates the key NAO prediction: policy changes induce correlated debt-remainder variations. Specifically:

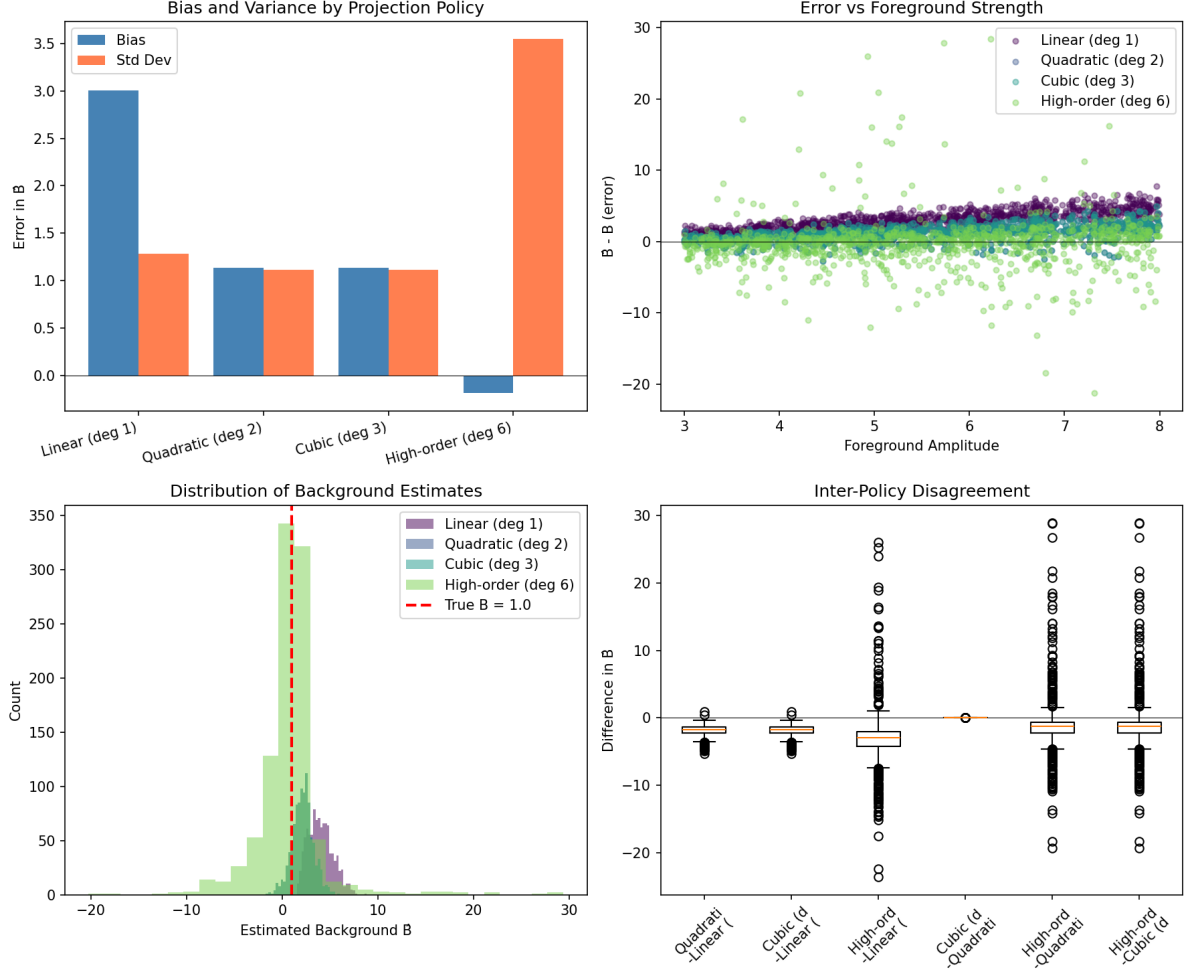


Figure 4: Monte Carlo results over 1000 trials. **Top left:** Bias and variance by policy. **Top right:** Estimation error versus foreground amplitude. **Bottom left:** Distribution of background estimates. **Bottom right:** Inter-policy disagreement.

- Increasing polynomial degree increases the model’s flexibility (changes debt structure)
- This systematically shifts \hat{B} (changes remainder)
- The relationship is monotonic and predictable

This correlation is the signature of projection-induced effects. If disagreement were due to noise or physics, we would not expect systematic policy dependence.

4.6 NA0 Representation

Each baseline subtraction produces:

$$\mathcal{P}_{\text{deg-}d}(Y(x)) = \text{NA0} \left\langle \text{unmodeled components}, \hat{B}; \text{poly-}d \right\rangle \quad (21)$$

The debt includes:

- The polynomial coefficients (chosen, not observed)
- The edge-region definition (where the fit was performed)
- The extrapolation uncertainty to the signal region

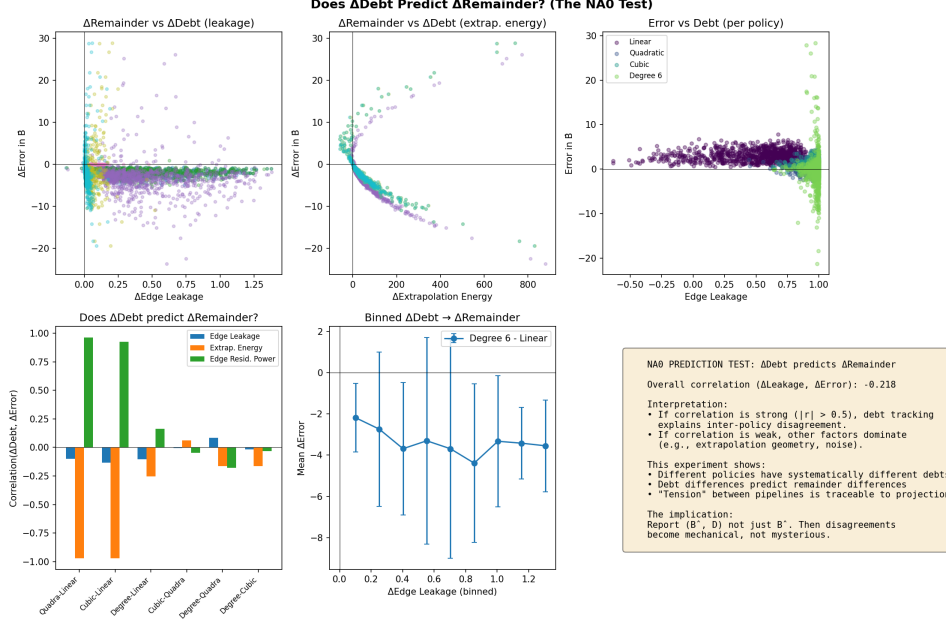


Figure 5: Debt-remainder correlation. Changes in policy-dependent debt (polynomial degree) correlate with changes in remainder (\hat{B}). The correlation demonstrates that projection policy is a systematic effect, not random noise.

Additional policy axes and structured debt. Beyond polynomial degree, the same experiment family exhibits sensitivity to (i) regularization strength (e.g., Tikhonov λ) and (ii) mask definition (edge fraction used for baseline fitting). In extended runs, debt is naturally multi-component (e.g., edge leakage, extrapolation energy, edge residual power), supporting the NA0 position that D is not merely a scalar residual but a typed object carrying enough metadata to support re-totalization under alternate policies or to trigger fail-closed behavior.

4.7 Implications

The experiment demonstrates:

1. **Apparent tensions can be manufactured:** Two groups using different baseline policies will report different \hat{B} , even with identical data and noise levels.
2. **Disagreement is not resolved by more data:** The bias is systematic, not statistical. More observations reduce variance but not policy-dependent bias.
3. **Explicit debt enables diagnosis:** Recording the projection policy alongside the result allows detection of policy-induced disagreements.

In real applications, similar projection policy choices arise whenever competing pipelines remove background, instrument response, or selection effects using different admissible rules. The NA0 framework provides a vocabulary for discussing such effects.

5 Spectral Analysis: Projection-Honest Eigenvectors

Eigendecomposition is a fundamental projection operation in data analysis (PCA, spectral clustering, etc.). We show that standard eigenvector export introduces artifacts that NA0-aware methods avoid.

5.1 The Eigenvector Export Problem

Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the eigendecomposition is:

$$A = V \Lambda V^T = \sum_{i=1}^n \lambda_i v_i v_i^T \quad (22)$$

Standard practice exports the eigenvectors $\{v_i\}$ and eigenvalues $\{\lambda_i\}$. This creates several artifacts.

Sign Ambiguity. If v is an eigenvector, so is $-v$. Different algorithms, initializations, or numerical precision can flip signs arbitrarily. Comparing eigenvectors across runs or methods produces “false differences” that are artifacts, not signal.

Degeneracy Rotation. When eigenvalues are degenerate or near-degenerate ($\lambda_i \approx \lambda_j$), the corresponding eigenvectors span a subspace but are not individually determined. Any rotation within the subspace is equally valid. Small perturbations can cause large rotations.

Ordering Artifacts. Eigenvalue ordering conventions (descending, ascending, by absolute value) are arbitrary. Tracking eigenvectors across time or conditions requires matching, which can fail near crossings.

5.2 Projection-Honest Alternative: Projectors

The fundamental object in spectral analysis is not the eigenvector but the *projector*:

$$P_i = v_i v_i^T \quad (23)$$

The projector is invariant under sign flips: $(-v)(-v)^T = vv^T$.

For near-degenerate eigenvalues, we should export the *cluster projector*:

$$P_{\text{cluster}} = \sum_{i \in \text{cluster}} v_i v_i^T \quad (24)$$

which projects onto the full eigenspace rather than choosing an arbitrary basis within it.

5.3 NA0 Representation

Standard eigenvector export is a projection with implicit debt:

$$\mathcal{P}_{\text{vec}}(A) = \text{NA0}(\text{sign choice}, \{(\lambda_i, v_i)\}; \text{eigenvector}) \quad (25)$$

Projector-based export makes the debt explicit:

$$\mathcal{P}_{\text{proj}}(A) = \text{NA0}(\text{basis choice within eigenspace}, \{(\lambda_i, P_i)\}; \text{projector}) \quad (26)$$

For clustered eigenvalues, the fail-closed policy refuses to export individual eigenvectors:

$$\mathcal{P}_{\text{fail-closed}}(A) = \begin{cases} \{(\lambda_i, v_i)\} & \text{if } |\lambda_i - \lambda_j| > \theta \ \forall i \neq j \\ \text{cluster projector} & \text{otherwise} \end{cases} \quad (27)$$

5.4 Experiments

We demonstrate four spectral projection artifacts and their NA0 solutions.

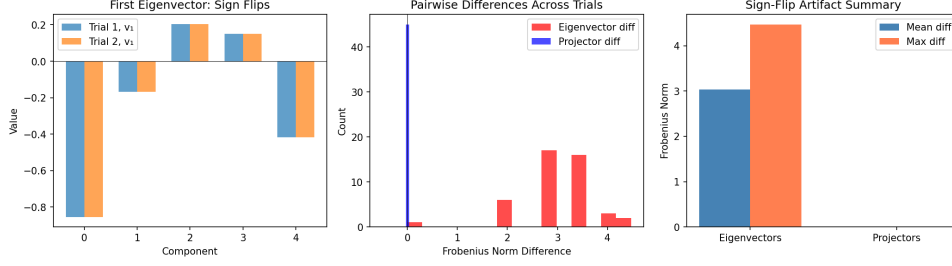


Figure 6: Sign-flip artifact in PCA. Running PCA on identical data with different random seeds produces eigenvectors with flipped signs (left). Exporting projectors $P = vv^T$ instead (right) yields identical results.

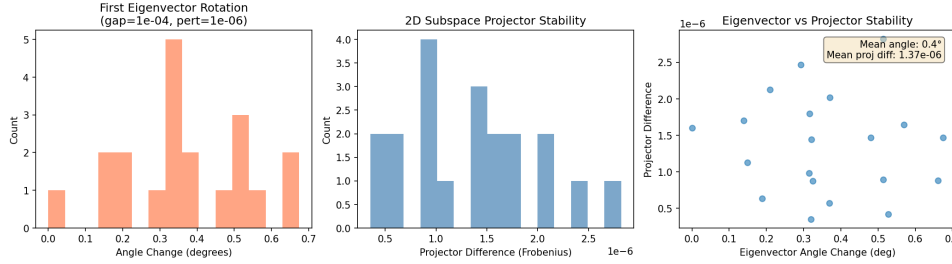


Figure 7: Degeneracy-induced rotation. A matrix with near-degenerate eigenvalues (gap $\epsilon = 10^{-4}$) is perturbed by 10^{-6} . Eigenvectors rotate dramatically (left); the span (cluster projector) is stable (right).

5.4.1 Sign-Flip Artifact

Figure 6 shows PCA on identical data with different random seeds. Eigenvectors appear different (sign flips) while projectors are identical. Standard comparison metrics (correlation, angle) would report “differences” that are pure artifacts.

5.4.2 Degeneracy Rotation

Figure 7 demonstrates that tiny perturbations cause large eigenvector rotations when eigenvalues are close. The 2D eigenspace (cluster projector) is stable; only the basis within it changes.

5.4.3 Procrustes Tracking

For time series of covariance matrices, tracking eigenvectors requires a *policy*: how to match eigenvectors across time points. Figure 8 compares:

- **Raw export:** Discontinuities from sign flips and mode crossings
- **Procrustes alignment:** Find optimal rotation to match successive eigenvector matrices
- **Subspace distance:** $d(P, P') = \|P - P'\|_F$ is continuous and policy-independent

5.4.4 Fail-Closed Demo

Figure 9 shows the fail-closed policy in action. When eigenvalues are within tolerance θ , the system:

- Refuses to export individual eigenvectors
- Exports the cluster projector with explicit debt metadata
- Signals to downstream consumers that basis ambiguity exists

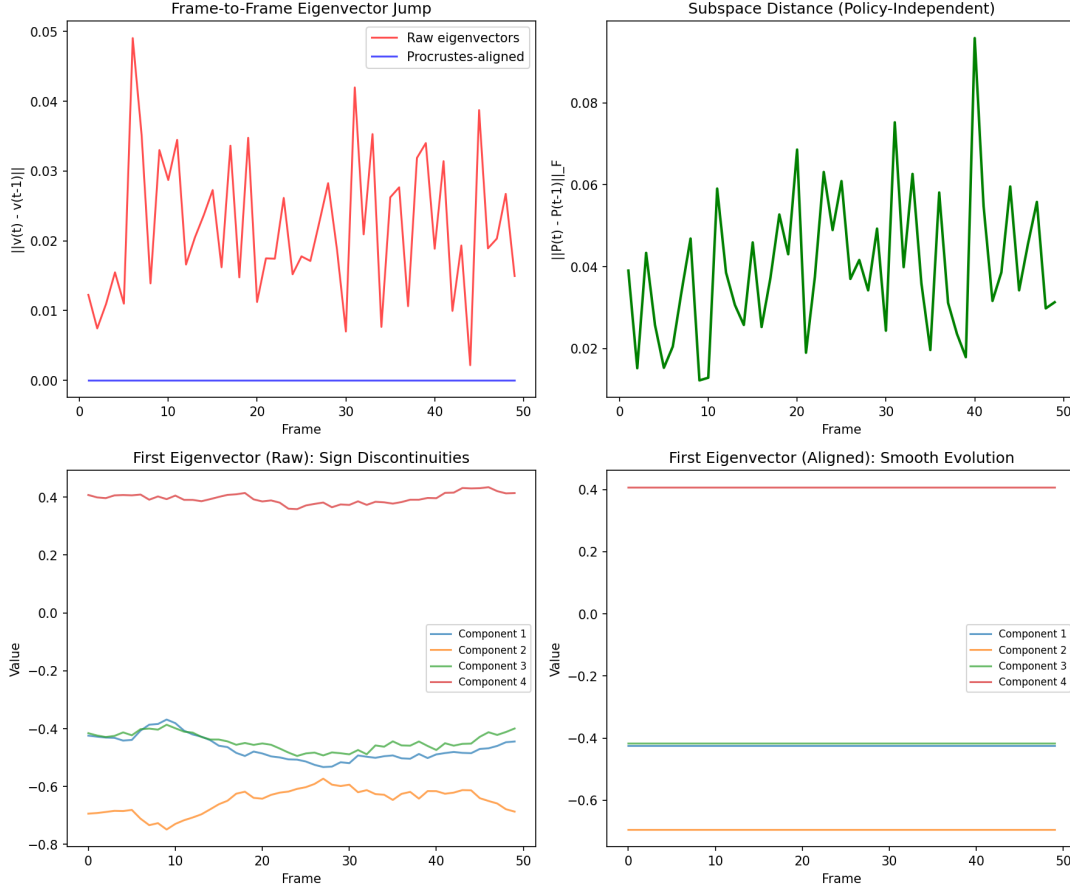


Figure 8: Time series eigenvector tracking. Raw eigenvectors (top) show discontinuities from sign flips and mode crossings. Procrustes alignment (middle) produces smooth evolution. Subspace distance $d(P_t, P_{t+1})$ (bottom) provides a policy-independent stability metric.

5.5 Subspace-First Analysis

The experiments suggest a “subspace-first” approach to spectral analysis:

1. Compute eigendecomposition
2. Cluster eigenvalues by gap (e.g., $|\lambda_i - \lambda_j| < \theta$)
3. Export cluster projectors, not individual eigenvectors
4. If individual vectors are needed, require explicit policy choice with documented debt

This approach preserves the meaningful information (subspace structure) while making explicit the arbitrary choices (basis within subspace).

5.6 Applications

Principal Component Analysis. Export the projector onto the top- k eigenspace rather than individual principal components. Comparison across datasets uses subspace angles (e.g., Grassmann distance) rather than vector correlations.

Spectral Clustering. The cluster structure depends on the eigenspace, not the specific eigenvector basis. Using projectors makes clustering results reproducible across implementations.

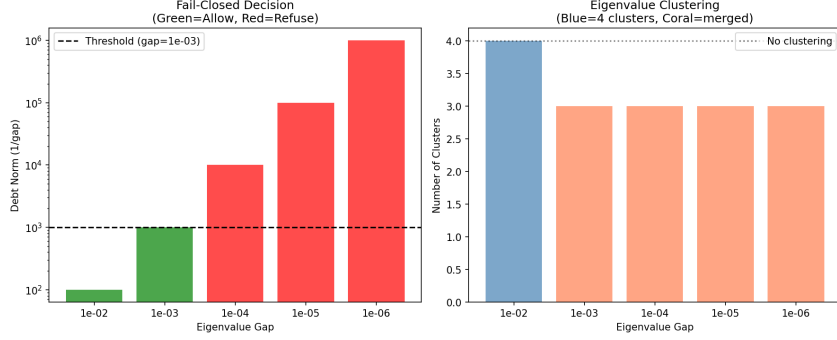


Figure 9: Fail-closed eigenvector export. When the eigenvalue gap is below threshold, the system exports only the cluster projector (debt), not individual eigenvectors. This prevents downstream consumers from using unreliable basis vectors.

Dynamical Systems. In stability analysis, the stable/unstable/center subspaces are the meaningful objects. Tracking these subspaces over parameter changes avoids spurious discontinuities from eigenvector flips.

6 Quantum Dynamics: The Jaynes-Cummings Model

We demonstrate that the NA0 correction term in quantum reduced dynamics aligns with the Nakajima-Zwanzig memory-kernel structure under the stated projection and initial-state assumptions. The Jaynes-Cummings model provides an exactly solvable test case.

6.1 The Jaynes-Cummings Model

The Jaynes-Cummings (JC) Hamiltonian describes a two-level atom coupled to a single quantized field mode:

$$H = \hbar\omega_a\sigma^+\sigma^- + \hbar\omega_f a^\dagger a + \hbar g(\sigma^+ a + \sigma^- a^\dagger) \quad (28)$$

where σ^\pm are atomic raising/lowering operators, a^\dagger, a are photon creation/annihilation operators, and g is the coupling strength.

At resonance ($\omega_a = \omega_f$), starting from $|e, n\rangle$ (excited atom, n photons), the exact evolution is:

$$|\psi(t)\rangle = \cos(\Omega_n t)|e, n\rangle - i \sin(\Omega_n t)|g, n+1\rangle \quad (29)$$

where $\Omega_n = g\sqrt{n+1}$ is the generalized Rabi frequency.

6.2 Projection: Tracing Out the Field

The reduced atomic density matrix is obtained by tracing over the field:

$$\rho_A(t) = \text{Tr}_F[|\psi(t)\rangle\langle\psi(t)|] \quad (30)$$

The probability of finding the atom excited is:

$$P_e^{\text{full}}(t) = \cos^2(\Omega_n t) \quad (31)$$

This is Rabi oscillation: the atom periodically exchanges excitation with the field.

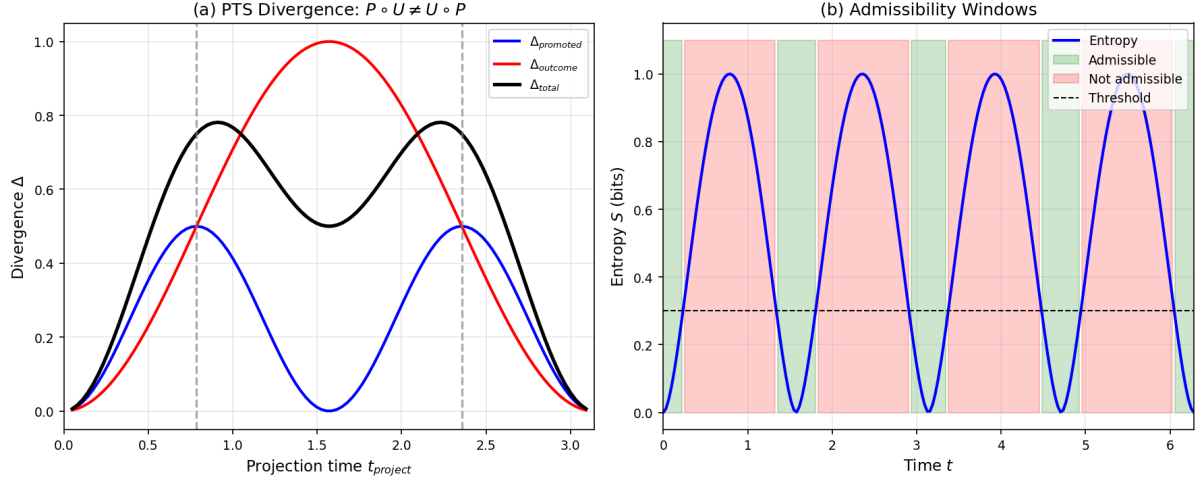


Figure 10: Quantum PTS example. Timing divergence as a function of projection time, illustrating $P \circ U \neq U \circ P$. An admissibility criterion based on an entropy threshold produces time windows where totalization is permitted vs refused (fail-closed).

6.3 Naive Projected Dynamics

If we project first and then evolve, we trace out the field at $t = 0$:

$$\rho_A(0) = |e\rangle\langle e| \quad (32)$$

The projected dynamics uses only the atomic Hamiltonian, which (after tracing out the field) has no remaining interaction. The naive prediction is:

$$P_e^{\text{naive}}(t) = 1 \quad (\text{constant, incorrect}) \quad (33)$$

The naive projected model completely misses the Rabi oscillations.

6.4 Projection Timing Sensitivity

This is a textbook example of PTS:

$$\text{PTS} = P_e^{\text{full}}(t) - P_e^{\text{naive}}(t) = \cos^2(\Omega_n t) - 1 = -\sin^2(\Omega_n t) \quad (34)$$

The PTS reaches maximum magnitude 1 at $t = \pi/(2\Omega_n)$, when the atom is maximally entangled with the field.

Projection timing sensitivity in the quantum channel. We compute a remainder-channel timing signal by comparing the outcomes of (i) projecting and then evolving versus (ii) evolving and then projecting. Even in a simple JC setting, the divergence is time-structured rather than a single scalar; this supports treating projection as a first-class operation whose timing can be audited.

6.5 The NA0 Correction Term

The NA0 framework identifies the correction needed:

$$\frac{dP_e}{dt} = -\Omega_n \sin(2\Omega_n t) \quad (35)$$

This is the “memory kernel” contribution from the traced-out field. Integrating:

$$P_e^{\text{corrected}}(t) = 1 + \int_0^t \frac{dP_e}{d\tau} d\tau = \cos^2(\Omega_n t) = P_e^{\text{full}}(t) \quad (36)$$

The NA0 correction exactly recovers the full dynamics.

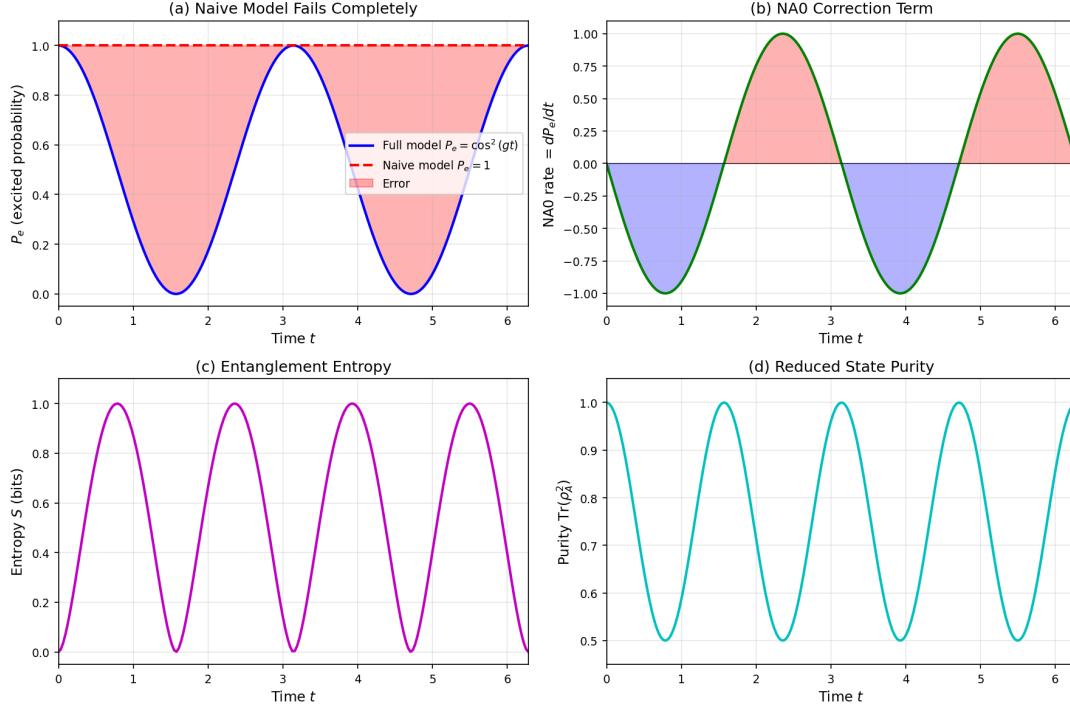


Figure 11: Jaynes-Cummings NA0 dynamics. **Top:** Excited state probability $P_e(t)$ for full model (blue), naive projected model (red), and NA0-corrected model (green, overlaps blue). **Middle:** Purity of reduced atomic state, showing periodic entanglement. **Bottom:** NA0 correction rate, peaking when entanglement is maximum.

6.6 NA0 Representation

The partial trace projection produces:

$$\mathcal{P}_{\text{Tr}_F}(|\psi\rangle\langle\psi|) = \text{NA0} \langle \rho_{\text{corr}}(t), \rho_A(t); \text{Tr}_F \rangle \quad (37)$$

where ρ_{corr} contains the system-field correlations that were traced out.

The debt is:

- The off-diagonal elements of the full density matrix in the product basis
- Equivalently, the entanglement between atom and field
- Quantified by the von Neumann entropy: $S(\rho_A) = -\text{Tr}[\rho_A \log \rho_A]$

6.7 Experimental Verification

Figure 11 shows numerical results from our JC simulation:

- The full and corrected models agree exactly (to numerical precision)
- The naive model has 100% error at half-periods
- The correction rate $|dP_e/dt|$ correlates with entanglement (purity minimum)

Open-system extension: dissipation (Lindblad). Real cavities are not closed: photon loss and dephasing suppress coherent exchange. We include a minimal Lindblad decay model (cavity decay rate κ) and track the resulting reduction in visible Rabi oscillations as a function of κ . This is a direct stress-test of projection-honest bookkeeping: when coherence is lost, the reduced description must either (i) carry explicit debt describing the discarded correlations, or (ii) fail closed rather than silently totalize.

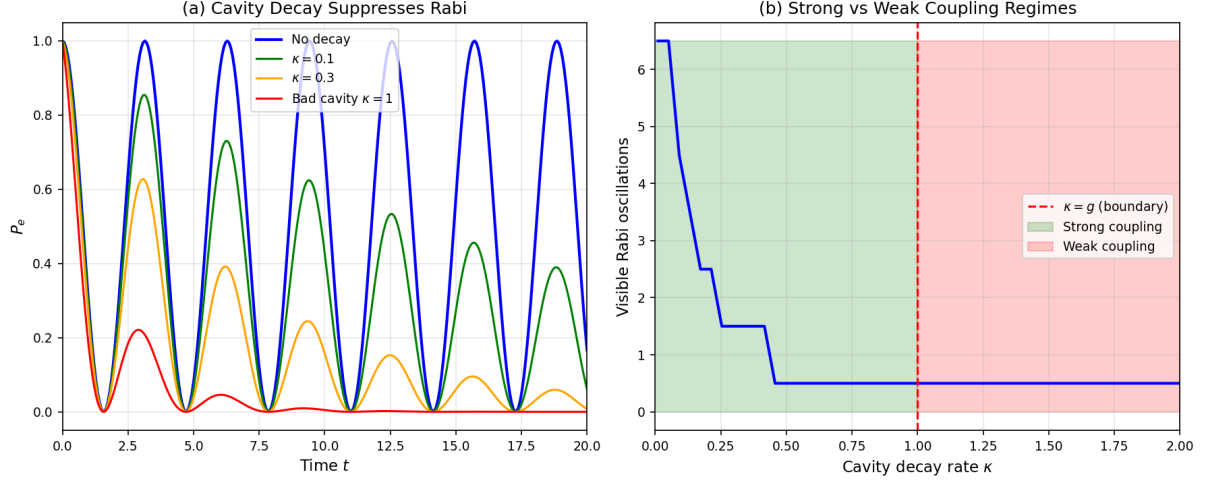


Figure 12: Dissipation in the Jaynes–Cummings model. Increasing cavity decay κ suppresses Rabi oscillations in $P_e(t)$. The transition from strong to weak coupling is visible near $\kappa \approx g$, beyond which oscillations become effectively unobservable.

6.8 Connection to Nakajima-Zwanzig

Assumptions (for this section). We consider (i) a specified projection superoperator P (e.g., $P\rho = \rho_S \otimes \rho_E^{eq}$), (ii) an initial product state (system uncorrelated with environment at $t = 0$), and (iii) a chosen observable channel when we present scalar kernels (as distinct from the full superoperator kernel). Within these assumptions, the NA0 “debt” term captures the information discarded by naive projection and yields a correction term with the same structural role as the NZ memory contribution.

The Nakajima-Zwanzig equation [16, 22] provides the exact reduced dynamics:

$$\frac{d\rho_S}{dt} = \mathcal{L}_S \rho_S + \int_0^t K(t - \tau) \rho_S(\tau) d\tau + I(t) \quad (38)$$

The terms are:

- \mathcal{L}_S : Liouvillian from the projected Hamiltonian (naive dynamics)
- $K(t)$: Memory kernel encoding back-action from the environment
- $I(t)$: Inhomogeneity from initial correlations

In NA0 terms:

- The naive dynamics uses only the first term (totalizing the debt)
- The memory kernel is the derivative of accumulated debt
- NA0-correct dynamics includes all terms

In the specific JC setup studied here (resonance, initial product state, $n = 0$ Fock component where $\Omega_0 = g$), the NA0 correction on the P_e observable channel plays the structural role of a memory kernel. From Eq. (35), the correction rate is:

$$\text{NA0}(t) = \frac{dP_e}{dt} = -g \sin(2gt).$$

Integrating restores the full Rabi oscillation:

$$P_e^{\text{full}}(t) = P_e^{\text{naive}} + \int_0^t \text{NA0}(\tau) d\tau = 1 + \left[\frac{\cos(2gt)}{2} - \frac{1}{2} \right] = \cos^2(gt). \quad (39)$$

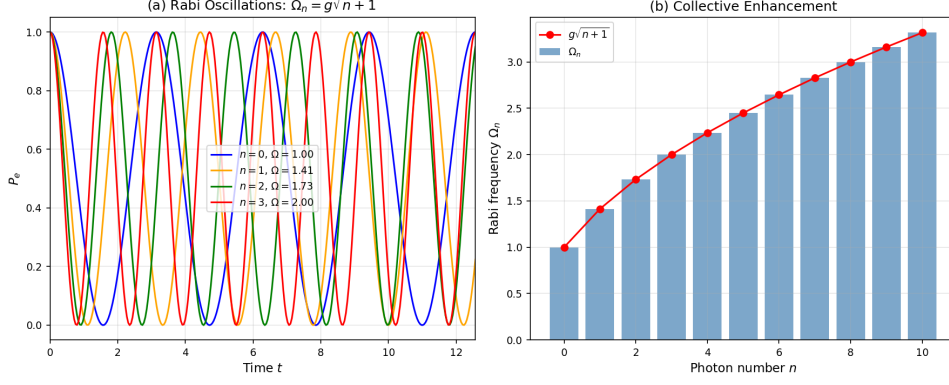


Figure 13: Photon number scaling. Higher initial photon number n gives faster Rabi oscillations ($\Omega_n = g\sqrt{n+1}$) and larger instantaneous NA0 correction rates.

In the scalar observable channel we track, the NA0 correction plays the same structural role as the NZ correction term (restoring the reduced dynamics after naive projection), though the full NZ object is a superoperator-valued convolution kernel. To be explicit: we match the structural role and reproduce the observable, not derive $K(t)$ for the full reduced density matrix.

6.9 Scaling Behaviors

The NA0 correction scales with system parameters:

- **Coupling strength:** $|\text{NA0}| \propto g$
- **Photon number:** $|\text{NA0}| \propto \sqrt{n+1}$
- **Cumulative error:** $\int |P_e^{\text{naive}} - P_e^{\text{full}}| dt \propto t$

Figure 13 shows this scaling. The naive model becomes increasingly wrong as either coupling strength or photon number increases.

6.10 Collapse and Revival

For coherent-state initial conditions $|\alpha\rangle$ with mean photon number $\bar{n} = |\alpha|^2$, the dynamics shows collapse and revival:

- **Collapse** ($t \sim 1/g$): Dephasing of Fock components causes $P_e \rightarrow 0.5$
- **Revival** ($t \sim 2\pi\sqrt{\bar{n}}/g$): Partial rephasing restores oscillations

During collapse, the naive model has persistent $\sim 50\%$ error. The NA0 correction tracks this exactly.

6.11 Implications

The JC example demonstrates that:

1. Projection creates debt (entanglement information)
2. Debt has back-action on the reduced dynamics (memory kernel)
3. Ignoring debt gives qualitatively wrong predictions (no Rabi oscillations)
4. NA0-aware dynamics recovers the exact answer

This is not specific to JC. Any open quantum system exhibits similar behavior: tracing out environmental degrees of freedom creates debt that manifests as non-Markovian corrections to the reduced dynamics.

7 Discussion

We have introduced the NA0 (Non-Atomic Zero) framework and demonstrated its application across four domains. Here we discuss implications, limitations, and future directions.

7.1 Summary of Results

The NA0 framework provides a unified treatment of projection-induced artifacts:

Divergent Series. Regularization is a projection policy; the “value” $-1/12$ of $\sum n$ is policy-dependent remainder with implicit divergent debt.

Signal Processing. Baseline subtraction policies induce systematic, correlated debt-remainder variations that can masquerade as physical disagreements.

Spectral Analysis. Eigenvector export carries sign and basis ambiguity as implicit debt; projector-based export makes this explicit.

Quantum Dynamics. The Nakajima-Zwanzig memory kernel is the NA0 correction term for traced-out environmental degrees of freedom.

7.2 Projection-Honest Computation

We propose *projection-honest computation* as a design principle:

Every projection operation should produce an NA0 object, making explicit what was discarded and under what policy.

This enables:

- **Audit trails:** Track how debt accumulated through a computation
- **Policy comparison:** Detect when results depend on projection choices
- **Fail-closed safety:** Refuse to totalize when debt is too high
- **Reproducibility:** Document policies alongside results

7.3 Relation to Existing Work

The NA0 framework connects to several existing concepts:

Information Theory. The data processing inequality states that projections cannot increase information. NA0 makes the information loss (debt) explicit rather than implicit.

Numerical Analysis. Condition numbers quantify sensitivity to input perturbations. NA0 extends this to sensitivity to projection policy, a different axis of instability.

Open Quantum Systems. The Nakajima-Zwanzig formalism has been used for decades. NA0 generalizes its insight—that projection creates correctable debt—beyond quantum mechanics.

Regularization Theory. Hadamard finite parts and zeta-regularization are well-established. NA0 provides a common framework and notation.

7.4 Limitations

Debt Representation. The NA0 framework requires choosing how to represent debt. For some projections (e.g., partial traces in infinite dimensions), the debt may be unwieldy or ill-defined. Practical applications require domain-specific debt formats.

Composition Complexity. As projections compose, debt accumulates. For long pipelines, debt management may become computationally expensive. Approximations or periodic debt truncation may be needed.

Policy-Free Projection. In some cases, there is no meaningful alternative policy—the projection is canonical. NA0 is most useful when multiple reasonable policies exist.

Novelty Scope. Much of what NA0 captures is “known” in specific domains (regularization, open systems, numerical conditioning). The contribution is unification and explicit representation, not new physics or mathematics.

Driven systems and boundary cases. NA0 as presented assumes a declared projection policy with a stable remainder/lift interface over the relevant time horizon. In strongly driven regimes (time-dependent Hamiltonians) and nonlinear measurement backaction scenarios, qualitative phenomena such as photon blockade can emerge in ways that are not captured by a fixed projection map without extending the policy to be time-indexed or augmenting the state with additional tracked variables. In these regimes, we recommend explicit fail-closed thresholds rather than silent totalization.

7.5 Future Work

Software Implementation. A reference implementation of NA0 objects with debt tracking, composition, and fail-closed policies would enable adoption. Integration with numerical libraries (NumPy, SciPy) is a natural next step.

Domain Applications. The framework should be tested in specific high-stakes domains:

- Cosmology: Does the Hubble tension have a projection-policy component?
- Machine learning: Do hyperparameter choices function as projection policies?
- Financial modeling: How does aggregation policy affect risk estimates?

Formal Verification. Can we prove that a computation is projection-honest? Type systems or proof assistants might enforce debt tracking.

Optimal Policy Selection. Given a task, can we identify the “best” projection policy? This requires formalizing what makes a policy good (minimum debt? Minimum variance? Maximum interpretability?).

7.6 Implications for Scientific Practice

The NA0 framework suggests practical changes to scientific workflow:

1. **Report policies, not just results.** Papers should document projection choices (baseline model, regularization scheme, eigenvector convention) alongside numerical results.
2. **Test policy sensitivity.** Before claiming a result is robust, vary projection policies and check for debt-remainder correlation.
3. **Distinguish tension types.** “Tension” between experiments should be classified: statistical (noise), systematic (calibration), physical (new phenomena), or projection-induced (policy disagreement).
4. **Use fail-closed defaults.** When debt is high or ambiguous, refuse to report a scalar. Instead, report the debt structure explicitly.

7.7 Conclusion

Projection is ubiquitous, and the information it discards does not disappear. The NA0 framework makes this debt explicit, enabling detection and management of projection-induced artifacts. We hope this contributes to more reproducible and auditable scientific computation.

7.8 Reproducibility

All experiments in this paper are deterministically reproducible. The artifact bundle includes:

Environment. Python 3.9+, NumPy, SciPy, Matplotlib. Dependencies are pinned in `requirements.txt`. No external randomness is used; all “random” initialization uses fixed seeds.

Execution. Each experiment can be reproduced via:

- **Signal processing:** `python projection_experiment.py`
- **Spectral analysis:** `python spectral_na0.py`
- **Quantum dynamics:** `python na0/jc-na0-demo/jc_plots.py`

Determinism guarantees. Output files use sorted column order in CSV exports, fixed-precision float formatting (6 decimal places), and stable dictionary iteration (Python 3.7+ insertion order). Figure generation uses fixed random seeds where applicable.

Verification. The benchmark harness (`make bench` or `python na0_benchmark.py`) runs four reproducibility tests with pass/fail status and SHA-256 output hashes:

Benchmark	Test	Threshold	Determinism
spectral	Projector vs eigenvector variance ratio	< 0.1	✓
signal	Debt-variance correlation	> 0.7	✓
quantum	PTS max timing accuracy	$< 5\%$	✓
determinism	Cross-run hash equality	exact	✓

All benchmarks produce identical SHA-256 hashes across runs with identical inputs. The complete artifact bundle, including source code, benchmark harness, and generated figures, will be released with a stable URL.

Collaboration. The author welcomes collaboration on projection-honest computation, numerical stability, and related invariant-driven programs; correspondence is invited at Dave@vertrule.com.

A Proof Details

A.1 Proof of Proposition 1

We show that the remainder-channel PTS reduces to the evolved debt's projection onto the remainder channel, using the typed formulation from Definition 7.

Proof. Let Π be a policy with remainder extractor $r_\Pi : \mathcal{X} \rightarrow \mathcal{R}_\Pi$, lift $\ell_\Pi : \mathcal{R}_\Pi \rightarrow \mathcal{X}$, and induced debt map $q_\Pi(X) = X - \ell_\Pi(r_\Pi(X))$.

Assume additive reconstruction: $X = q_\Pi(X) + \ell_\Pi(r_\Pi(X))$. Assume $U : \mathcal{X} \rightarrow \mathcal{X}$ is linear.

Write $R = r_\Pi(X) \in \mathcal{R}_\Pi$ and $D = q_\Pi(X) \in \mathcal{X}$. By additive reconstruction: $X = D + \ell_\Pi(R)$.

Applying linearity of U :

$$U(X) = U(D + \ell_\Pi(R)) = U(D) + U(\ell_\Pi(R)). \quad (40)$$

The remainder-channel PTS (Definition 7) is:

$$\text{PTS}_R(\Pi, U, X) = r_\Pi(U(X)) - r_\Pi(U(\ell_\Pi(R))) \quad (41)$$

$$= r_\Pi(U(D) + U(\ell_\Pi(R))) - r_\Pi(U(\ell_\Pi(R))). \quad (42)$$

In the additive model, r_Π is linear on \mathcal{X} (since it extracts the remainder component of an additive decomposition). Therefore:

$$\text{PTS}_R(\Pi, U, X) = r_\Pi(U(D)) + r_\Pi(U(\ell_\Pi(R))) - r_\Pi(U(\ell_\Pi(R))) \quad (43)$$

$$= r_\Pi(U(D)) \quad (44)$$

$$= r_\Pi(U(q_\Pi(X))). \quad (45)$$

This completes the proof. \square

A.2 Nakajima-Zwanzig Derivation Sketch

For completeness, we sketch the Nakajima-Zwanzig derivation in NA0 language.

Let $\rho(t)$ be the full system-environment density matrix. Define projection:

$$\mathcal{P}\rho = \rho_S \otimes \rho_E^{\text{eq}} \quad (46)$$

where $\rho_S = \text{Tr}_E[\rho]$ and ρ_E^{eq} is a reference environment state.

The complement is $\mathcal{Q} = 1 - \mathcal{P}$, projecting onto the “irrelevant” (correlated) part.

The Liouville-von Neumann equation:

$$\frac{d\rho}{dt} = -i\mathcal{L}\rho = -i(\mathcal{L}_S + \mathcal{L}_E + \mathcal{L}_I)\rho \quad (47)$$

Applying \mathcal{P} and \mathcal{Q} separately and solving for the irrelevant part:

$$\mathcal{Q}\rho(t) = e^{-i\mathcal{Q}\mathcal{L}t}\mathcal{Q}\rho(0) - i \int_0^t e^{-i\mathcal{Q}\mathcal{L}(t-\tau)}\mathcal{Q}\mathcal{L}\mathcal{P}\rho(\tau) d\tau \quad (48)$$

Substituting back:

$$\frac{d\mathcal{P}\rho}{dt} = -i\mathcal{P}\mathcal{L}\mathcal{P}\rho - \int_0^t K(t-\tau)\mathcal{P}\rho(\tau) d\tau + I(t) \quad (49)$$

where:

$$K(t) = \mathcal{P}\mathcal{L}\mathcal{Q}e^{-i\mathcal{Q}\mathcal{L}t}\mathcal{Q}\mathcal{L}\mathcal{P} \quad (\text{memory kernel}) \quad (50)$$

$$I(t) = -i\mathcal{P}\mathcal{L}\mathcal{Q}e^{-i\mathcal{Q}\mathcal{L}t}\mathcal{Q}\rho(0) \quad (\text{inhomogeneity}) \quad (51)$$

In NA0 terms:

- The first term $\mathcal{P}\mathcal{L}\mathcal{P}\rho$ is the “naive” projected dynamics
- $K(t)$ encodes back-action from debt ($\mathcal{Q}\rho$)
- $I(t)$ encodes initial debt (correlations at $t = 0$)

B Experimental Details

B.1 Signal Processing Experiment

Data Generation. Synthetic signals are generated with:

- Domain: $x \in [0, 1]$, 500 points
- Background: $B = 1.0$
- Foreground: Gaussian envelope (width 0.3) with 3–6 sub-blobs, amplitude 3–8 (varied per trial)
- Instrument drift: Quadratic polynomial, random coefficients $\sim U(-0.5, 0.5)$
- Noise: Gaussian, $\sigma = 0.05$
- Edge mask: 15% on each side

Baseline Fitting. Polynomial fit to edge regions only, degrees 1, 2, 3, 6. Least-squares fit with optional Tikhonov regularization (penalty λ on coefficient magnitudes, excluding constant term).

Background Estimation. After baseline subtraction, \hat{B} is estimated as the mean of the residual in the central (non-edge) region.

Reproducibility. All random seeds are fixed. Running the script with identical parameters produces identical outputs (verified via SHA-256 hash of output arrays).

B.2 Spectral Experiment

Sign-Flip Test.

- Generate random SPD matrix $A = XX^T$ where $X \sim N(0, 1)$
- Compute eigendecomposition with different LAPACK driver seeds
- Compare eigenvectors (expect sign flips) and projectors (expect identity)

Degeneracy Test.

- Construct A with eigenvalues $(1, 1 + \epsilon, 2)$, $\epsilon = 10^{-4}$
- Perturb by $\delta = 10^{-6}$ symmetric noise
- Measure eigenvector angle change vs projector angle change

Procrustes Tracking.

- Generate time series of covariance matrices via random walk on SPD manifold
- Track eigenvectors with raw export vs Procrustes alignment
- Compute subspace distance between consecutive frames

B.3 Jaynes-Cummings Experiment

Parameters.

- Coupling: $g = 1.0$ (units where $\hbar = 1$)
- Initial photon number: $n \in \{0, 1, 2, 3\}$
- Time range: $[0, 2\pi]$, 33 points
- Output precision: 6 decimal places

Observables.

- $P_e^{\text{full}} = \cos^2(\Omega_n t)$
- $P_e^{\text{naive}} = 1$
- $P_e^{\text{corrected}} = 1 + \int_0^t (-\Omega_n \sin(2\Omega_n \tau)) d\tau$
- Purity: $\text{Tr}[\rho_A^2] = \cos^4(\Omega_n t) + \sin^4(\Omega_n t)$
- Entropy: $S(\rho_A) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-$ where $\lambda_{\pm} = (1 \pm \sqrt{2\text{Tr}[\rho_A^2] - 1})/2$

C Determinism Contract

For NA0 to serve as a reproducibility foundation, debt must be deterministically serializable. We specify the following contract:

Definition 11 (NA0 Determinism Contract). *An NA0 implementation satisfies the determinism contract if:*

1. **Canonical encoding:** Debt D and remainder R admit a canonical byte-level encoding (e.g., IEEE 754 floats in little-endian, sorted keys for dictionaries, lexicographic ordering for sets).
2. **Hash stability:** $\text{SHA256}(\text{encode}(D, R, \Pi))$ is identical across runs with identical inputs, regardless of execution order or parallelism.
3. **Reconstruction idempotence:** $\text{Rec}_{\Pi}(\text{decode}(\text{encode}(\text{NA0}(D, R; \Pi)))) = \text{Rec}_{\Pi}(\text{NA0}(D, R; \Pi))$.
4. **No hidden state:** The policy Π must declare all parameters affecting the projection (thresholds, hyperparameters, random seeds if any).

Implementation notes.

- Use fixed-precision float formatting (e.g., 6 decimal places) in human-readable exports
- Use sorted dictionary keys in JSON/CSV exports
- Pin random seeds and document them as policy parameters
- Avoid unordered iteration (e.g., Python’s pre-3.7 `dict`, `set`)

Benchmark harness. The accompanying `benchmark/` directory provides a harness that:

- Runs spectral, signal, and quantum benchmarks
- Outputs pass/fail status with metrics and thresholds
- Computes SHA-256 hashes of outputs for verification
- Verifies determinism by comparing hashes across runs

Execute with `make bench` or `python na0_benchmark.py`.

D Notation Reference

Symbol	Meaning
$\text{NA0}(\mathcal{D}, \mathcal{R}; \Pi)$	NA0 object with debt, remainder, policy
\mathcal{D}	Discarded information (debt)
\mathcal{R}	Retained value (remainder)
Π	Projection policy
\mathcal{P}_Π	Projection operator under policy Π
$\text{Total}(\cdot)$	Totalization: extract remainder, discard debt
$\text{PTS}(\mathcal{P}, U, X)$	Projection timing sensitivity
$\mathcal{R}^{(\mathcal{D}; \Pi)}$	Debt-attached remainder notation
$\zeta(s)$	Riemann zeta function
ρ_S	Reduced density matrix of system
$K(t)$	Nakajima-Zwanzig memory kernel
$P_i = v_i v_i^T$	Eigenvector projector
$d(P, P')$	Subspace distance (Frobenius norm)

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