$$(\mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{C}_0, \mathbb{C}_1, \ldots \in \mathbb{CAT})$$

• A class of objects $|\mathbb{C}|$

$$(C, D, E, \ldots \in |\mathbb{C}|)$$

• A class of morphisms $\mathbb{C}(C, D)$ from object C to object D

$$(f,g,h,\ldots\in\mathbb{C}(C,D))$$

 $(f,g,h,\ldots\in C\to D)$

with:

- an identity morphism $id_C \in \mathbb{C}(C, C)$
- a composition operation $-\circ \in \mathbb{C}(D,E) \to \mathbb{C}(C,D) \to \mathbb{C}(C,E)$

such that:

- $id_C \circ f = f \circ id_D = f$ (unity, left & right) for any $f \in \mathbb{C}(C, D)$
- $(f \circ g) \circ h = f \circ (g \circ h)$ (associativity) for any $h \in \mathbb{C}(C, D)$, $g \in \mathbb{C}(D, E)$ and $f \in \mathbb{C}(E, F)$

Opposite category

The opposite category \mathbb{C}^{op} consists of:

- Objects $|\mathbb{C}^{\mathsf{op}}| = |\mathbb{C}|$
- Morphisms $\mathbb{C}^{\mathsf{op}}(C, D) = \mathbb{C}(D, C)$

Terminal object

- An object $\top \in |\mathbb{C}|$
- such that for every
 - object $X \in |\mathbb{C}|$

there exists a unique morphism $\mathsf{tt} \in \mathbb{C}(X,\top).$



Cartesian product

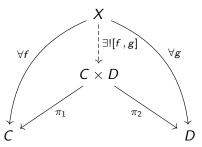
Let $C, D \in |\mathbb{C}|$.

- An object $C \times D \in |\mathbb{C}|$
- A morphism $\pi_1 \in \mathbb{C}(C \times D, C)$
- A morphism $\pi_2 \in \mathbb{C}(C \times D, D)$

such that for every

- $X \in |\mathbb{C}|$
- $f \in \mathbb{C}(X, C)$
- $g \in \mathbb{C}(X, D)$

there exists a unique morphism $[f,g] \in \mathbb{C}(X,C \times D)$ verifying



Exponential

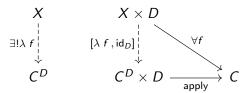
Let $C \in |\mathbb{C}|$. Let $D \in |\mathbb{C}|$ with all Cartesian products.

- An object $C^D \in |\mathbb{C}|$
- A morphism apply $\in \mathbb{C}(C^D \times D, C)$

such that for every

- $X \in |\mathbb{C}|$
- $f \in \mathbb{C}(X \times D, C)$

there exists a unique morphism $\lambda f \in \mathbb{C}(X, \mathbb{C}^D)$ verifying



Cartesian-closed category

A category \mathbb{C} with

- A terminal object T
- A Cartesian product $C \times D$ for any $C, D \in |\mathbb{C}|$
- An exponential object C^D for any $C,D\in |\mathbb{C}|$

Functor

$$(\mathcal{F},\mathcal{G},\mathcal{F}_0,\mathcal{F}_1,\ldots\in\mathbb{C}\Rightarrow\mathbb{D})$$

- An action on objects $|\mathcal{F}| \in |\mathbb{C}| \to |\mathbb{D}|$
- An action on morphisms $\mathcal{F}^{\to} \in \mathbb{C}(C, C') \to \mathbb{D}(D, D')$ such that:
 - $\mathcal{F}^{\rightarrow} \operatorname{id}_{\mathcal{C}} = \operatorname{id}_{|\mathcal{F}| \mathcal{C}}$

(identity preservation)

• $\mathcal{F}^{\rightarrow}$ $(g \circ f) = (\mathcal{F}^{\rightarrow} g) \circ (\mathcal{F}^{\rightarrow} f)$ (composition preservation)

Adjunction

Let $\mathcal{F} \in \mathbb{D} \Rightarrow \mathbb{C}$.

 \mathcal{F} is a left adjoint if for every

• Object $C \in |\mathbb{C}|$

there exists

- An object $|\mathcal{G}|C \in |\mathbb{D}|$
- A morphism $\epsilon_C \in \mathbb{C}(|\mathcal{F}|(|\mathcal{G}|(C)), C)$

such that for every

- object $D \in |\mathbb{D}|$
- morphism $f \in \mathbb{C}(|\mathcal{F}| D, C)$

there exists a unique morphism $g \in \mathbb{D}(D, |\mathcal{G}| C)$ with

$$\begin{array}{c|c}
D & |\mathcal{F}| D \\
\exists ! g \downarrow & \mathcal{F} \to g \downarrow \\
|\mathcal{G}|(C) & |\mathcal{F}|(|\mathcal{G}|(C)) \xrightarrow{\epsilon_C} C
\end{array}$$

Natural transformation $(\varphi, \psi, \varphi_0, \varphi_1, \ldots \in \mathcal{F} \stackrel{\cdot}{\Rightarrow} \mathcal{G})$

Let $\mathcal{F}, \mathcal{G} \in \mathbb{C} \Rightarrow \mathbb{D}$.

- A transformation $\varphi \in \forall C$. $\mathbb{D}(|\mathcal{F}| C, |\mathcal{G}| C)$ such that for every
 - $k \in \mathbb{C}(C, D)$

we have

$$\begin{aligned}
|\mathcal{F}| & C \xrightarrow{\varphi_{C}} |\mathcal{G}| C \\
\mathcal{F}^{\to k} \downarrow & \downarrow \mathcal{G}^{\to k} \\
|\mathcal{F}| & D \xrightarrow{\varphi_{D}} |\mathcal{G}| D
\end{aligned}$$

Adjunction, take II

Let $\mathcal{F} \in \mathbb{D} \Rightarrow \mathbb{C}$ and $\mathcal{G} \in \mathbb{C} \Rightarrow \mathbb{D}$.

 \mathcal{F} is left adjoint to \mathcal{G} ($\mathcal{F} \dashv \mathcal{G}$) if there exists

• a natural isomorphism $\varphi \in \mathbb{C}(|\mathcal{F}| D, C) \stackrel{\sim}{\to} \mathbb{D}(D, |\mathcal{G}| C)$

Conversely, \mathcal{G} is right adjoint to \mathcal{F} .

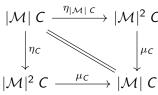
Monad

- A functor $\mathcal{M} \in \mathbb{C} \Rightarrow \mathbb{C}$
- A natural transformation $\eta \in \forall C. C \rightarrow |\mathcal{M}| C$
- A natural transformation $\mu \in \forall C. |\mathcal{M}|(|\mathcal{M}| C) \to |\mathcal{M}| C$ such that

$$|\mathcal{M}|^{3} C \xrightarrow{|\mathcal{M}|\mu_{C}} |\mathcal{M}|^{2} C$$

$$\downarrow^{\mu_{|\mathcal{M}| C}} \qquad \qquad \downarrow^{\mu_{C}}$$

$$|\mathcal{M}|^{2} C \xrightarrow{\mu_{C}} |\mathcal{M}| C$$



Kleisli category

Let \mathcal{M} be a monad over \mathbb{C} .

The Kleisli category $\mathbb{C}_{\mathcal{M}}$ consists of:

- Objects: $|\mathcal{M}|$ C for every $C \in |\mathbb{C}|$
- Morphisms: $\mathbb{C}_{\mathcal{M}}(C, C') = \mathbb{C}(C, |\mathcal{M}| C')$

Presheaf

A presheaf on a category \mathbb{C} is a functor $\hat{\mathbb{C}} \in \mathbb{C} \Rightarrow \mathbb{SET}$.

In particular, it forms a functor category:

- Objects: presheaf functors
- Morphisms: natural transformations

Further reading

- "Conceptual Mathematics", Schanuel & Lawvere
- "An introduction to Category Theory", Simmons
- "Categories for Types", Crole
- "Categories for the Working Mathematician", Mac Lane
- "Sheaves in Geometry and Logic", Mac Lane & Moerdijk