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Cultivating mathematical skills: From drill-and-practice to deliberate practice

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Abstract

Contemporary theories of expertise development highlight the crucial role of deliberate practice in the development of high level performance. Deliberate practice is practice that intentionally aims at improving one's skills and competencies. It is not a mechanical or repetitive process of making performance more fluid. Instead, it involves a great deal of thinking, problem solving, and reflection for analyzing, conceptualizing, and cultivating a developing performance. This includes directing and guiding future training efforts that are then fine-tuned to dynamically evolving levels of performance. Expertise studies, particularly in music and sport, have described early forms of deliberate practice among children. These findings are made use of in our analysis of the various forms of practice in school mathematics. It is widely accepted that mathematics learning requires practice that results in effortless conducting of lower level processes (such as quick and accurate whole number arithmetic with small numbers), which relieve cognitive capacity to more complex tasks. However, the typical training of mathematical skills in educational contexts can be characterized as drill-and- practice that helps automatize basic skills, but often leads to inert routine skills instead of adaptive and flexible number knowledge. In this article we summarize findings of studies which describe students' self-initiated, deliberate practice in learning number knowledge and intervention studies applying deliberate practice in mathematics teaching, including technology-based learning environments aimed at triggering practice that goes beyond mechanical repeating of number skills.

1 Introduction

Since the constructivist turn in mathematics education, there has been a confusion concerning the role of practice in mathematics education. Many advocates of constructivism-based mathematics education have contrasted problem based discovery learning with the direct teaching of facts and practicing of procedural skills. The former was considered to represent a constructivist epistemology and the latter to represent rote learning based on behavioristic ideas (Wheatley, 1991). In the mathematics education literature, practice is often connected to automatization of procedural skills. There is a long tradition in distinguishing conceptual knowledge and procedural skills in mathematics (Brownell, 1944/1945). Conceptual knowledge is typically defined as abstracted mathematical knowledge that is rich in relationships (Hiebert & LeFevre, 1986). Learning of conceptual mathematical knowledge is considered to require conscious construction which can be characterized as deep learning. Procedural knowledge consists of series of steps, or actions, needed to accomplish a goal. It is considered to be lower level knowledge which can be learned by repeated routine practice and memorized by rote (Baroody, 2003; Hiebert & LeFevre, 1986). While practice has been considered as the key method that leads to necessary automatization of basic mathematical processes or increasing fluency in more advanced operations, in most of the scientific literature practice in mathematics learning is considered to be simple repetition (drill-and-practice) of tasks without further analyses of the quality of practice.

The conceptual versus procedural distinction is widely used and experienced as a useful way to characterize mathematical knowledge in educational contexts. However, the relationship between conceptual and procedural knowledge is not completely understood (Rittle-Johnson & Schneider, 2015). Star (2005) has criticized this dominant way of defining conceptual and procedural knowledge in mathematics education literature and has proposed that conceptual knowledge and procedural knowledge are two aspects of mathematical knowledge which both can be learned in a superficial or deep way. Particularly in more advanced levels of mathematics learning, procedural skills can also include higher level cognitive processes, for example focusing on relations between different parts of the procedures or evaluating the effectiveness of a particular procedure for a given task. Researchers have also highlighted that even though it has been theoretically useful to distinguish between conceptual and procedural knowledge, high level mathematics learning often requires a closely integrated development of these two aspects of mathematical knowledge (Rittle-Johnson & Schneider, 2015).

A review of studies about the relationship between conceptual and procedural knowledge in mathematics showed that in learning processes this relationship is bidirectional indicating that conventional models to practice procedural skills do not catch this interwoven process (Rittle-Johnson, Schneider, & Star, 2015). If procedural skills in mathematics are seen as complex cognitive processes – including higher order cognitive regulation and deep connections to conceptual knowledge – the nature of practice supporting the development of this type of procedural knowledge should also be reconsidered and aligned with attempts to develop conceptual knowledge. A promising approach to deepen the understanding of different qualitative levels of practice can be found in expertise research. The concept of deliberate practice is based on detailed analyses of the development of high level expertise and is widely accepted to be a key factor explaining exceptional performance in sport, music, and different professional fields (Ericsson, Charness, Feltovich, & Hoffman, 2006). There are few studies applying the concept of deliberate practice in formal educational contexts, but most of them focus on medical education or other fields of professional education. The aim of this article is to review the aspects of deliberate practice that could be relevant for rethinking the role and nature of practice in mathematics education.

2 Deliberate practice

The role of intensive training in expertise development has been studied for several decades, but the concept of deliberate practice is only little more than 20 years old (Lehmann & Gruber, 2014). In a detailed analyses of the learning processes of violinists during the acquisition of expertise using retrospective methods, Ericsson, Krampe, and Tesch-Römer (1993) found that today's experts differed from other individuals early on in their career: They practiced more efficiently, had more committed teachers, and had higher demands on their achievement. Experts were more often involved in laborious and systematically focused training activities over a long period of time that was solely aimed at improving performance. Such activity is called deliberate practice. In music and many other fields where the role of deliberate practice has been studied (e.g. sports) expert performance is demonstrated as exceptional procedural skills. However, expertise research has shown that in any field, expert performance is not only characterized by the fluency of externally observable procedures. High level expertise means external performance enabled by complex and integrated mental representations (Ericsson, 2016). Long-lasting deliberate practice is needed for developing fluent skills embedded in these kinds of mental representations. Thus lessons learned from the deliberate practice research could be valuable for developing approaches which support learning of mathematical procedures as embedded in rich and well-elaborated knowledge structures.

Based on Ericsson's (2016) recent work the characteristic features of deliberate practice can be summarized as follows:

1. Deliberate practice focuses typically on developing skills which are already known by other people, so that practice can be planned and supported by these knowledgeable people (teachers, mentors, coaches etc.).
2. Deliberate practice requires trying things that are beyond a person's current abilities. In other words, it demands near-maximal effort, which is not always enjoyable.
3. Deliberate practice is based on well-defined and specific goals to improve some aspects of performance. These goals are often set by a teacher or coach.
4. Deliberate practice requires a person's full conscious attention and self-control.
5. Feedback and modification of efforts is crucial for deliberate practice. In the beginning of the process modifications depend on external (teacher, coach) feedback, but gradually the person learns to monitor his or her progress.
6. Deliberate practice results in developed mental representations. More detailed and better organized mental representations improve the effectiveness of later deliberate practice.
7. Deliberate practice focuses on modifying and improving previously acquired skills and building advanced skills on the top of existing skills. It is important to guarantee that the fundamental skills are correct and well-developed, and that the orchestration of highly automatized sub-processes and full conscious processing of conceptual knowledge is functioning.

After the seminal works of Ericsson and his colleagues the term deliberate practice has been used by many other researchers. Some researchers, for example Suddendorf and his colleagues (Davis, Cullen, & Suddendorf, 2016; Suddendorf, Brinums, & Imuta, 2016) have proposed a more general definition which emphasizes the role of deliberate practice as the basic mechanism for how humans prepare themselves for the future. They define deliberate practice as repeated actions driven by the goal to improve one's future capacities. A key idea here is that it presupposes the ability to think what kind of skills oneself needs in the future (Suddendorf et al., 2016).

Deliberate practice may be quite different in different domains or even sub-domains. So far, little is known about such differences. For classical musicians, much practice is with instrumental techniques, supplemented by regular visits with a teacher. For jazz musicians, parts of deliberate practice may consist of communal practice with other musicians (Degner, Lehmann, & Gruber, 2003). In sport, the different disciplines have many differences in the nature of deliberate practice needed for exceptional achievement (Rahkamo, 2016).

3. All practice is not deliberate practice

Traditional mathematics education and the so called back-to-basics approach have seen mathematics as collection of special techniques and procedures and have highlighted the importance of extensive practice in terms of repeated exercises with these techniques (Offner, 1978). Given this, it stands to question if there is something new in the deliberate practice approach.

It is important to note that it is “deliberate“ practice that matters, not just any practice. For example, Plant, Ericsson, Hill, and Asberg (2005) found that improvement in performance in higher education did not significantly correlate with the amount of time spent studying. It did, however, relate to concentrated learning aimed at specific performance goals. Fierce academic debates have been published in which doubts were expressed about the relevance and importance of deliberate practice (e.g. in a Special Issue of “Intelligence”, concluded by a sharp reply of Ericsson, 2014). Recently, meta-analyses were published which aimed to grow such doubts. Macnamara, Hambrick, and Oswald (2014) analyzed studies in music, games, sports, education, and professions. They concluded that deliberate practice accounts for 20-30 per cent of the variance in games, music, and sports, but for less than five per cent in education and professions. Macnamara, Moreau, and Hambrick (2016) focused on deliberate practice in sports and provided additional analyses that showed that within elite individuals, deliberate practice does not explain any variance. Additionally they found that elite individuals did not begin their sport earlier in childhood than lower skill athletes. In his response, Ericsson (2016) outlines that these meta-analyses are completely flawed, because they are based only on the plain amount of practice, irrespective of the quality and intention of practice, thus completely missing the whole idea underlying the concept of “deliberate practice”. This indeed is a crucial point, even more so, if we aim

to understand the role of deliberate practice and the importance of teachers when individuals still are far away from top-level performance – as children are.
The “art of deliberate practice” obviously includes the ability and willingness to conduct highly concentrated activities which might be, to some degree, aversive in nature. For example, maximal capacity training in running is demanding and situationally unpleasant even for world-class runners, but it is undeniably a necessary part of running training. Usually, those activities do not take a long period of time, because of physiological limitations. Often they yield positive effects and help to improve other parts of the activity so that experienced self-regulated performers can accept the aversive components of practicing. However, less experienced individuals like novices tend to focus their practice on more pleasant levels of effort. For example, unexperienced musicians often practice pieces (or parts of pieces) which they have already mastered. They try to avoid errors and failures and they do not challenge their own learning. It is in response to this that trainers/mentors/guides etc. are most valuable. These early periods are also the time when it may be decided whether one continues to undergo the learning and practice activities that foster expert careers - at this phase interests, volition and motivation are formed (Gruber, Degner, & Lehmann, 2004). In the realm of mathematics education, a distinction should also be made between routine practice with existing skills and the types of deliberate practice that push students to develop their emerging skills and knowledge structures.

4 Early forms of (deliberate) practice

Expert research typically focuses on exceptional individuals in their adulthood. Detailed descriptions of the nature of deliberate practice are based on their current forms of practice and their retrospective memories about their early practice. However there are a few studies which directly focus on the nature of practice among young children and school students.
Maybe the most impressive finding showing the potential of deliberate practice in early childhood was presented in the study of Sakakibara (2014), in which a particular ability was investigated that traditionally falls into the realm of nativists (inborn abilities), absolute pitch (the ability to recognize the pitch of a tone). It was shown that every kid (aged 2 to 6 years) could acquire absolute pitch after an intensive, deliberate, and long training, which included daily training over many months.

The intervention program of Sakakibara (2014) was strictly planned and guided by adults. However, there are plenty of examples of cases in which adults have planned detailed and ambitious practice programs for children, which did not lead to expertise development. As Ericsson (2016) has emphasized, deliberate practice requires a person's full conscious attention and self-control which is only possible if the person is deeply engaged in the practice. Already in early childhood, this means that the process of deliberate practice cannot be fully initiated and controlled by adults, but children must be engaged in the activity heavily by themselves. In fact, some studies focusing on early forms of (deliberate) practice have focused on the nature of children's self-initiated practice (Côté, & Hay, 2002).

Davis and colleagues (2016) related the growth of intentional deliberate practice in preschool-aged children to the developing ability to understand that certain behavior is "future relevant". They call this ability "episodic foresight", imagining future episodes and their relevance. The idea is interesting to interpret deliberate practice "as repeated actions that are conducted with the explicit goal to improve one's future capacities" (p. 361). Referring to some research on language learning, they argue that "though adults such as parents and teachers may encourage it, deliberate practice is a voluntary process that can be self-initiated" (p. 361). One interesting question is: When and why does this happen – and when and why not? Davis and colleagues (2016) conducted two experiments and found the following (from the abstract, p. 361):

> "We present two experiments testing children's ability to selectively practice a behavior that was going to be useful in future and to reason about the role of practice in skill formation. Five-year-olds demonstrated an explicit understanding of deliberate practice both in selectively choosing to practice a future-relevant skill and in predicting skill change in others based on their practice. Four-year-olds showed some capacities, but failed to demonstrate consistent understanding of the relationship between practice and skill improvement."

Redshaw and Suddendorf (2013) studied when one knows that one does not know and how this is related to activities for finding out what is required in the future. They found that 5-, but not 4-year-olds, were appropriately seeking information they needed to solve a future problem. The results suggest that in order to monitor what the child knows and can do, and what he or she still has to learn, the child needs some capacity for metacognition. There were no mathematical tasks in these studies, but based on the findings it is possible to speculate that from the beginning of elementary school students could participate in self-initiated practice with mathematical procedures.

It is not trivial to predict whether the "episodic foresight" about future mathematical skills goes beyond the improved fluency of basic mathematical operations, a kind of improvement which can be reached with simple drill-and-practice. Studies on the development of skills in music indicate that individuals only rarely spontaneously engage in deliberate practice, although they recognise that it would improve their performance. Instead of going beyond a person's current abilities learners prefer regular activities that are motivated by inherent enjoyment or external reward (Lehmann, 2002). Suddendorf and colleagues (2016) propose that the ability to focus on demanding and not always so joyful deliberate practice instead of more pleasant training of already comprehended tasks is related to the development of executive control and inhibition. As these basic cognitive functions continue developing until adulthood, the probability to get involved in self-initiated, goal-driven, and (occasionally) unpleasant deliberate practice will increase by age.

Only a part of the development of basic cognitive skills takes place during formal and guided learning situations (Bransford et al., 2006), while many opportunities to practice and further develop recently learnt skills occur during situations that are informal and unguided (Lehtinen & Hannula, 2006; Lobato, 2012). Observations of children's activities in informal situations indicate that already young children (at the age of 3-4 years) can be spontaneously engaged in mathematically relevant practices in their everyday environments (Hannula, Mattinen & Lehtinen, 2005; Mattinen, 2006). Hannula and Lehtinen (2005) found that there were inter-individual differences in children's tendency to spontaneously focus on number of objects or events. After that several studies in different countries have confirmed the original findings and shown that the tendency of spontaneous focusing on numerosity (SFON) is a strong predictor of later development of mathematical skills. Findings of previous SFON studies revealed the unique contribution of SFON to the development of early mathematical skills (Hannula & Lehtinen, 2005; Hannula, Räsänen, & Lehtinen, 2007) and later school math achievement (Hannula, Lepola, & Lehtinen, 2010; Hannula-Sormunen, Lehtinen, & Räsänen, 2015).

Since the early studies of SFON (for a summary see, Hannula, 2015), there has been an increased interest in SFON worldwide, leading to an enrichment of the initial findings and providing new insights into this important mathematical competency (e.g., Batchelor, Inglis, & Gilmore, 2015; Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen, & Verschaffel, 2016; Edens & Potter, 2013; Gray & Reeve, 2016; Kucian et al., 2012; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016; Sella, Berteletti, Lucangeli, & Zorzi, 2016). The general explanation for these findings is that children who spontaneously focus on numerical aspects of their environment in everyday situations get much more practice of magnitude recognition, number comparison, and combining of numbers than children who only do that when formally instructed by parents or teachers. Hannula et al. (2005) provided evidence for this explanation by showing a positive correlation between children's SFON in the SFON tests and their self-initiated practice in number recognition observed during daily situations and play in day care settings. Likewise, Batchelor (2014) observed a positive association between children's task-based SFON as assessed via verbal description task and their spontaneous use of numbers as observed during parent-child play interactions. The findings of Hannula and Lehtinen (2005) also indicate that between SFON and early numerical skills there is similar reciprocal development than what Ericsson (2016) has reported between deliberate practice and developing mental representations. SFON supports the development of numerical skills and more elaborated numerical skills strengthen the SFON tendency.

SFON measured before the beginning of formal schooling has been a strong predictor of later mathematical development, but in many everyday activities numerosity is not the only mathematically relevant aspect that can be focused on. In young students' daily life there are many opportunities to focus on more complex quantitative aspects such as quantitative relations. Children can recognize and use quantitative relations without explicit guidance to do so. Based on a series of studies McMullen, Hannula-Sormunen and Lehtinen (2013; 2014) proposed that there is a similar tendency as SFON, which indicates that instead of mere numerosity children and school pupils can also focus spontaneously on quantitative relations (SFOR). The results of the longitudinal study in Finnish schools of McMullen, Hannula-Sormunen, Laakkonen and Lehtinen (2016) showed that there were substantial individual differences in students' SFOR tendencies and that SFOR tendency had a unique impact on rational number conceptual development in late primary school students during the 2-year follow-up period. These results were confirmed by a replication study in Belgium (Van Hoof et al., 2016).

In these studies SFOR tendency was particularly related to the development of conceptual understanding of rational numbers which has been difficult to support by traditional mathematics teaching and practice. The extended quantity of self-initiated practice based on spontaneous mathematical focusing tendencies may result in qualitatively different learning experiences than the organized and typically rather isolated practice with fractions and decimal numbers in formal school context. Quantitative relations experienced in everyday situations are often approximate and dynamically changing. McMullen and colleagues (2016) presented an example "… a 7-year-old child traveling with her mother to visit their grandparents in the countryside. During the boring car drive the child starts spontaneously to think about the trip in terms of quantitative relations, asking 'Are we halfway there yet?'." In this kind of situations the distances are approximated and the car is approaching half-way and after that half-way of the rest of the travel (i.e. 3 quarters). It can be assumed that intensive thinking about these kinds of "messy" relations challenge students' limited beliefs about numbers (e.g. that all numbers can be treated as natural numbers) and support the conceptual change needed in learning rational numbers. A later study (McMullen, Hannula-Sormunen & Lehtinen, submitted) shows that SFOR tendency is in a similar reciprocal relationship with rational number knowledge, which has been found between SFON and natural number knowledge.

Early spontaneous focusing on developmentally relevant features and activities is, however, not only possible in mathematics but examples can be found in other fields as well. Somewhat similar findings have been made about the development of Brazilian football players (Araújo et al. 2010). Some word-class players have acquired their superior skills mostly through self-initiated practice. Early involvement in improvised football related activities in partly messy environments resulted in skills which were highly valuable in their later careers as players on top teams. As Araújo et al. (2010, p. 169) put it "… unstructured street football allowed them to become familiar with assorted features of the game, because it is feasible for players to try many skills in different conditions without fear of ridicule or recrimination from observing coaches."

The self-initiated practice based on spontaneous focusing on mathematically relevant aspects of environment share some features of deliberate practice as described by Ericsson (2016). However, there are also important differences. Children who practice mathematical skills because they have a strong tendency to focus on numerosity or quantitative relations do not necessarily have any intention or specific goals to improve their mathematical skills. The nature of feedback they get is also different from the feedback teachers and coaches could give. In addition, the origin of the focusing tendency can be in social interaction with parents or other significant persons (Hannula, Mattinen & Lehtinen, 2005) but the very nature of the spontaneous focusing is that it takes place also without any external prompt or guidance.

5 From self-initiated practice to coached deliberate practice

The largest difference between the above described early forms of practice and the features of deliberate practice is the lack of systematic guidance in the self-initiated forms of practice. According to Ericson (2016) deliberate practice is planned and supported by knowledgeable people (teachers, mentors, coaches etc.). In principle, however, individuals should be able to perform such practice on their own. In fact, experts sometimes are able to self-regulate their own learning is such a way that they fulfil the requirements for deliberate practice (and thus may compensate the lack of a teacher) – setting up goals that have to be reached in order to improve practice, knowing the "culture" of skilled activity, monitoring the progress, and gaining specific feedback. The role teachers play in early phases of acquisition of expertise, and whether they are indispensable, is still an open question (Gruber et al., 2004).

One particularly important feature of expertise is high levels of self-regulation and monitoring, which supports both an understanding of what the expert knows (and does not know) and where the expert should turn his or her attention in terms of deliberate practice (Bonneville-Roussy & Bouffard, 2015). This self-awareness is buttressed by the highly organized and structured nature of experts' domain knowledge (Boshuizen, Schmidt, Custers, & van de Wiel, 1995). However, it is not clear how self-regulation and monitoring are related to deliberate practice at other stages of development outside of top-level expertise, particularly in childhood (Davis et al., 2016), though the role of self-initiated practice, as described above, may offer some answers. In particular, high-level experts have reorganised and restructured their knowledge in a way that enables easy access to vast amounts of knowledge as well the usage of prior professional experience, for example with cases such as patients in medicine (Boshuizen et al., 1995).

In experts, the level of self-regulation and monitoring seems to be so high that they are more easily aware of what they do not yet know or master than subjects at a lower level of performance. If the concept of deliberate practice is transferred to other phases of learning and skill acquisition than the top-expert level, the relation between self-regulation/monitoring and deliberate practice might be complex and is still little understood (Bonneville-Roussy & Bouffard, 2015). This may be in particular true for our field of interest, very early deliberate practice in childhood (Davis et al., 2016). The above description of children's and young students' self-initiated practice with numbers and quantitatvie relations can indicate an important impact on formal mathematical learning.

There is some recent research in the field of music that examines the early roots of deliberate practice and the integration of self-initiated and systematically guided practice. For example, Bonneville-Roussy and Bouffard (2015) investigated a model of how to relate self-regulation and deliberate practice, and how to disentangle their roles in the composition of practice time and in the growth of performance and musical achievement. In this framework, they proposed the term "formal practice and defined it as a goal-directed and focused period of practice that includes both self-regulation and deliberate practice strategies. This definition of formal practice is broader than the original concept of deliberate practice. They concluded that "practice time will predict musical achievement only if associated with formal practice" (p., 686). Based on a 4-month prospective study, they found that beginner musicians are more likely to 'practise informally' because they have not yet acquired the self-regulatory skills needed to practice formally according to the principles of deliberate practice, whereas expert musicians are more likely to use deliberate practice. Practice time only becomes a significant predictor of performance when the time is "used to master a specific skill" (p. 689). In the field of mathematics learning we can conclude that if individuals discover the value and relevance of practicing then there is an increased chance that a practice gradually approaching deliberate practice may occur. However, expert teachers, guides, or trainers are important in this shift from self-initiated to deliberate practice, by offering explicit teaching goals, feedback, and opportunities for gradual improvement (Lehmann, 2002).

6 Expert guidance and feedback as key factors of deliberate practice

A couple of researchers have outlined phase models of music development which might be transferred to mathematics development: In the beginning (usually in young childhood) individuals only practice a little. Then the duration, intensity, and direction of practicing evolve, strongly determined by persons in the shadows (e.g. parents), into systematic, often formal, practice patterns. The required increase in focus often constitutes a critical phase during the development. Evidence exists that outstanding musicians (including prodigies; e.g. Mozart) have received outstanding guidance and support during this period (Davidson, Howe, Moore, & Sloboda, 1996). Guided and controlled practicing and frequent teaching can be found in those individuals' careers (Lehmann, 1997). The role of parents has been stressed: They support the growth of expertise if they closely interact with the teachers and draw conclusions for home practicing.

Even for subjects at the expert level, "persons in the shadow" who guide and direct deliberate practice, seem to be of utmost importance in most cases (Gruber, Lehtinen, Palonen, & Degner, 2008; Lehmann & Kristensen, 2014). Teacher-guided instruction and deliberate practice tend to be closely related. The concept of deliberate practice implies that expert performance is acquired gradually and that substantial improvements in performance depend on teachers' or coaches' abilities (Lehmann & Ericsson, 2003). The most important reason for guidance by expert teachers is that in all complex domains a body of organized experiences in the form of knowledge and produced artefacts has been accumulated over time. Through teachers, this body is shared with learners, because teachers can foresee future skill demands. Expert teachers support learners in becoming fully encultured in a community of expert practice.

In the course of expertise development and different stages of development the kind of teaching or coaching which is needed changes (Gruber et al., 2008).

> "In the beginning, most are coached by local teachers, people who can give generously of their time and praise. Later on, however, it is essential that performers seek out more-advanced teachers to keep improving their skills. Eventually, all top performers work closely with teachers who have themselves reached international levels of achievement" (Ericsson, Prietula, & Cokely, 2007, p. 6).

Two characteristic features of deliberate practice are that it includes feedback given by a teacher and that the learner makes an effort to improve his/her skills in response to this feedback. The development of expertise requires teachers or coaches who are willing and capable of giving constructive, but also critical, feedback. But, it is fundamental for expertise development that the person is motivated to seek out such feedback and to make use of it. In course of the expertise development people also learn to understand when and if a coach's advice does not work for them (Ericsson et al., 2007).

7 How can research on deliberate practice can inform mathematics education?

Most of the research on deliberate practice has taken place in out-of-school situations such as musical training, sports, or different professional fields (Ericsson et al., 2006). Within formal education the concept of deliberate practice has been used mainly in medical education (McGaghie, Issenberg, Cohen, Barsuk, & Wayne, 2011) and recently also in the studies on teacher education and teachers' processional development (Bronkhorst, Meijer, Koster, Vermunt, 2014; Lampert, 2010). However, in the mathematics education research literature there are teaching approaches which share many of the features of deliberate practice but do not refer to this concept (Verschaffel, & Greer, 2013; Wittmann, 2011). Only a few studies have applied the concept of deliberate practice in studying school learning in different academic subjects.

A few studies have combined cognitive load theory and the deliberate practice approach in order to develop more effective and efficient instructional design (ID). These studies are based on the so called "expertise reversal effect", which show that the effects of cognitive load theory informed instructional methods become less effective as a function of increasing expertise (Kalyuga, Ayres, Chandler, & Sweller, 2003). Instructional methods which are based on attempts to reduce external load of working memory and support schema formation in the task have to be redesigned to take into account more advanced learners' cognitive processes. Principles of ID research which aim at combining cognitive load and deliberate practice approaches has been summarize as follows: (a) ID research should identify the actual effects of different instructional formats on memory structures for learners at different levels of expertise; (b) ID research should identify what instructional formats may constitute deliberate practice for learners of different levels of expertise; (c) ID research should identify what the relevant aspects of performance are in a domain and make instruction adaptive to the needs for improvement of individual learners; (d) ID research should focus on the relationship between motivation, mental effort, and different instructional formats; and (e) ID research should identify at what point in their development learners become capable of self-assessment and self-selection and what the relevant mechanisms are that support these skills (Van Gog, Ericsson, Rikers & Paas, 2005).

In geometry learning, Pachman, Sweller, and Kalyuga (2013) interpreted deliberate practice as a pedagogical process where students should practice to overcome their weaknesses. They found that even more knowledgeable students tended to choose achievable rather than difficult problems if they had the opportunity to choose. Training with these geometrical tasks resulted in minimal performance improvements. Only when a deliberate practice model was applied and these more knowledgeable students were presented with designer selected difficult problems to solve were their skills improved.

In a subsequent study Pachman, Sweller, and Kalyuga (2014) used worked examples in geometry teaching. High school students were randomly assigned to the free-choice or the deliberate practice group. The deliberate practice group was presented tasks targeting multiple weak areas, and free-choice group had the opportunity to select the tasks they solved. The deliberate practice condition resulted in better improvement for knowledgeable learners, but this condition was too demanding for less knowledgeable learners. The authors explained the results by referring to the number of weak areas in each group. For more knowledgeable learners, there were only a few weak areas and thus focusing on them during the experimental session was easier. Less knowledgeable learners were not able to successfully work with their many weak areas in the given time. These two studies on the use of deliberate practice in teaching geometry applied two aspects of deliberate practice. The main idea was focusing on weak areas in practice but teacher feedback was also included. The results show that application of deliberate practice was successful but only for more knowledgeable students. However the very short practice time (30 minutes) does not make it possible to generalize the results because extensive longitudinal work is necessary for deliberate practice.

Fluency in basic arithmetic tasks and number combination skills has proved to be crucial for later mathematical learning and weaknesses in automatization of these skills is characteristic of mathematically disabled children. The conceptual and procedural knowledge dichotomy can be seen in remedial programs developed for learning disabled students. Many remedial teaching strategies are based either on strengthening the conceptual understanding of mathematical operations and number combinations or drill-and-practice aimed automatization of arithmetic facts (Fuchs et al., 2010). In addition some remedial programs have focused on teaching strategic counting (Tournaki, 2003). Many studies have used a drill-and-practice approach to remediation with a computer application to ensure that students practiced correct responses. For example, the study of Fuchs and colleagues (2008) showed that students who received number combination drill-and-practice outperformed those in the control conditions. In a later study Fuchs and colleagues (2010) elaborated on the concept of practice. They did not refer to expertise and deliberate practice studies, but used the concept of deliberate practice to describe a model in which practice was systematically controlled for and tutor supported practice was connected with the previously taught strategies. According the results strategic counting instruction (a remedial program developed to enhance systematic use of various counting strategies in solving arithmetic tasks) with deliberate practice was superior when compared with the strategic counting instruction without deliberate practice and other control conditions (Fuchs et. al., 2010).

All the above described studies give valuable insight about the possibilities to use some of the features of deliberate practice in improving mathematics learning in school context. However the short laboratory experiment nature of these studies limits the applicability of the findings in developing the daily practices of mathematics teaching and learning in school. There are however some studies which have tried to develop models where some aspects of deliberate practice can be embedded in regular mathematics teaching.

The flexible and adaptive use of arithmetic strategies has been intensively studied during the last years (Verschaffel et al., 2009). Recent studies have shown that adaptivity with arithmetic procedures is not only about the choice between previously known strategies, but is based on rich mental numerical connections (McMullen et al., 2016; in press). In order to get students involved in long lasting intensive practice with novel situations including varying connections between numbers and operations, Lehtinen and his collaborators developed a computer application called Number Navigation Game (NNG), which is based on the empirical evidence about adaptive arithmetic strategies and principles of deliberate practice (Lehtinen et al., 2015). There are no readily-defined mathematical tasks in the game, instead the player has to create her own calculation strategies to progress in the game. The constraints in the game become gradually more demanding and require more and more advanced numerical strategies. The game gives strategic scaffolding and continuous feedback. In addition, a manual gives instructions on how the teacher can systematically scaffold the practice and give informative feedback for implementing the game in classroom. Playing through the whole game requires from 15 to 20 hours intensive work. The first study with the game showed that the quality of students' practice with the game was a unique predictor of development of adaptive number knowledge (Brezovszky et al., 2015). Findings of the study are in line with the theoretical assumption that providing extensive practice with various combinations of numbers and operations can aid students' noticing of numerical characteristics and relations, as indexed by their adaptive number knowledge (Baroody, 2003; Threlfall, 2002; 2009; Verschaffel, Luwel, Torbeyns & Van Dooren, 2009).

A randomized experimental study shows that just using the NNG as an additional tool during regular mathematics lessons without specific teacher guidance resulted in some improvement in adaptive number knowledge and arithmetic fluency (Brezovszky et al., submitted). However, when the use of the game was accompanied with well-planned teacher support and feedback in a later experiment, it resulted in stronger improvement in adaptive number knowledge and the effects also transferred to pre-algebra skills (Lehtinen et al. 2016). The findings with the NNG experiments indicate that some of the features of deliberate practice which have proved to be effective in expertise development can also be integrated in regular mathematics education. Organizing deliberate practice in the classroom in a way that students are continuously provided with tasks which are optimally challenging for each student and pushing all students to work on or beyond the border of their current skills may be too demanding for teachers in the regular classroom teaching without enabling learning environments such as well-planned digital systems.

Self-initiated and self-controlled practice is important during the early phases of expertise development and later in combination with systematically guided and controlled deliberate practice. Findings highlighting the role of early self-initiated practice (eg. Davis et al., 2016), could also inform mathematics education. A study of Hannula and her collaborators (2005) showed that it is possible to enhance children's tendency to spontaneous focusing on numerosity which subsequently results in improvement of numeric skills. Preliminary studies show that it is also possible to enhance students' tendency to spontaneous focusing on quantitative relations (McMullen et al., 2016). It could be possible to enhance students' intensive practice with mathematical procedures if formal mathematics education manage to bridge school mathematics with everyday situations and trigger students' tendency to continuously notice mathematically relevant aspects of their environment. There are some studies showing how book reading can trigger young children's mathematical thinking (Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016; van den Heuvel-Panhuizen, Elia & Robitzsch; 2014) indicating that book reading may be a potential way to enhance SFON tendency.

8 Conclusions

The debate over the primacy of procedural and conceptual knowledge in mathematics education has slowly moved towards a détente in some circles. Many researchers and educators now acknowledge that both forms of knowledge and skills are necessary for success with mathematical learning, despite the fact that routine practice with arithmetic facts has been given a bad rap in the previous discussions by many in the field of mathematics education. The argument we are making in the present article is for the inclusion of a more nuanced form of mathematical development, which includes more complex forms of practice, specifically deliberate practice. By looking towards the field of expertise development, it is possible to align the best-practices found in other domains of expertise with early mathematical development. In doing so, it may be possible to highlight the types of mathematical activities that are beneficial for all students in their development.
Namely, the following features of the concept of deliberate practice could be accounted for when thinking about teaching and learning in mathematics:

1. While deliberate practice can be done alone to the benefit of some skills, this is often on a sub-optimal level until the later stages in expertise development. Therefore, self-initiated practice should be encouraged, but with the caveat of only applying specific training when this type of practice is useful. Otherwise, coaches/teachers/mentors should be the main drivers of what type of mathematical activities and tasks are done.
2. Students should be allowed specific, constructive, and critical feedback on their activities.
3. Practice should be on the edge of students' competences, and focus on those skills which are the weakest. However, this may look different for students of different overall achievement levels, with those with the most deficits requiring more flexibility in their opportunities in order to not overwhelm.

4. Specific training on how to self-regulate one's own practice may be necessary for students to start to reach the upper levels of expertise. Giving students support in learning to push themselves by choosing more challenging activities in which they are likely to try and fail may help them begin to engage in the fruitful activities of deliberate practice.
5. Mathematics education should enhance sustainable mathematics motivation and motivational regulation which make it possible for students to also be engaged in unenjoyable, but necessary, practice.

The formation of expert-like practice activities is not a single event, but a long process in itself. The acquisition of high level competence in complex domains such as mathematics is a laborious process that needs deliberate practice over a number of years. Better understanding of the qualitatively different forms of practice, pedagogical models for integrating elements of deliberate practice in mathematics classroom, and technology-based learning environments (e.g. games) guiding students towards deliberate practice can result in mathematics education in which conceptual and procedural knowledge is learned simultaneously.

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