

COMP3670: Introduction to Machine Learning

Question 1 Properties of Eigenvalues (5+5=10 credits)

Let \mathbf{A} be an invertible matrix.

1. Prove that all the eigenvalues of \mathbf{A} are non-zero.
2. Prove that for any eigenvalue λ of \mathbf{A} , λ^{-1} is an eigenvalue of \mathbf{A}^{-1} .

Question 2 Properties of Eigenvalues II (10 credits)

Let \mathbf{B} be a square matrix. Let \mathbf{x} be an eigenvector of \mathbf{B} with eigenvalue λ . Prove that for all integers $n \geq 1$, \mathbf{x} is an eigenvector of \mathbf{B}^n with eigenvalue λ^n .

Question 3 Distinct eigenvalues and linear independence (20+5 credits)

Let \mathbf{A} be a $n \times n$ matrix.

1. Suppose that \mathbf{A} has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, and corresponding non-zero eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Prove that $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is linearly independent.

Hint: You may use without proof the following property: If $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ is linearly dependent then there exists some p such that $1 \leq p < m$, $\mathbf{y}_{p+1} \in \text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_p\}$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_p\}$ is linearly independent.

2. Hence, or otherwise, prove that **for any matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$, there can be at most n distinct eigenvalues for \mathbf{B} .**

Question 4 Properties of Determinants (10+15=25 credits)

1. Prove $\det(A^T) = \det(A)$.
2. Prove $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

Question 5 Eigenvalues of symmetric matrices (15 credits)

1. Let \mathbf{A} be a symmetric matrix. Let \mathbf{v}_1 be an eigenvector of \mathbf{A} with eigenvalue λ_1 , and let \mathbf{v}_2 be an eigenvector of \mathbf{A} with eigenvalue λ_2 . Assume that $\lambda_1 \neq \lambda_2$. Prove that \mathbf{v}_1 and \mathbf{v}_2 are orthogonal. (Hint: Try proving $\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2$. Recall the identity $\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a}$.)

Question 6 Computations with Eigenvalues (3+3+3+3+3=15 credits)

Let $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.

1. Compute the eigenvalues of \mathbf{A} .
2. Find the eigenspace E_λ for each eigenvalue λ . Write your answer as the span of a collection of vectors.
3. Verify the set of all eigenvectors of \mathbf{A} spans \mathbb{R}^2 .
4. Hence, find an invertible matrix \mathbf{P} and a diagonal matrix \mathbf{D} such that $\mathbf{A} = \mathbf{PDP}^{-1}$.
5. Hence, find a formula for efficiently¹ calculating \mathbf{A}^n for any integer $n \geq 0$. Make your formula as simple as possible.

¹That is, a closed form formula for \mathbf{A}^n as opposed to multiplying \mathbf{A} by itself n times over.