The Osmanthus Ring Tensor
A Self–Similar Frobenius Tower

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Abstract

We introduce the *Osmanthus Ring Tensor*: a finite, commutative Frobenius algebra with generator O obeying $O^8 = 1$, parity idempotent $P_\Theta := O^4$, and trace identity $\operatorname{tr} P_\Theta = 0$. Iterating "take the span of basis elements again" yields an infinite tower $\{ R^{(n)} \}_{n \geq 0}$ in which the same three identities persist at every level, giving a self-similar algebraic object well suited to graded tensor-network constructions.

1 Level 0: the base ring

Definition 1 (Osmanthus ring $R^{(0)}$)**.** Let $R^{(0)}$ be the complex vector space with basis $\{ O^k \mid k = 0, \ldots, 7\}$ and multiplication $O^i \cdot O^j = O^{i+j \pmod 8}$.

Proposition 1. *$R^{(0)}$ is a commutative, separable Frobenius algebra. The element $P_\Theta := O^4$ satisfies $P_\Theta^2 = 1$ and $\operatorname{tr}_{R^{(0)}}(P_\Theta) = 0$.*

2 Level 1: ring of the ring

For each basis element of $R^{(0)}$ introduce a symbol $[O^k]$; let $R^{(1)} := \operatorname{span}[O^k]$ with $[O^i] \cdot [O^j] = [O^{i+j \pmod 8}]$. Define $P_\Theta^{(1)} := [P_\Theta]$.

Proposition 2. *$R^{(1)}$ is again a commutative, separable Frobenius algebra with $(P_\Theta^{(1)})^2 = 1$ and $\operatorname{tr}_{R^{(1)}} P_\Theta^{(1)} = 0$.*

3 The Frobenius tower

Inductively, set $R^{(n+1)} := \operatorname{span}\big\{ [x] \mid x \in \text{basis of } R^{(n)} \big\}$ with identical multiplication rule. Each level inherits a parity element $P_\Theta^{(n)}$ and trace cancellation $\operatorname{tr} P_\Theta^{(n)} = 0$. Thus:

Proposition 3. *For every $n \geq 0$*

$$O^8 = 1, \qquad (P_\Theta^{(n)})^2 = 1, \qquad \operatorname{tr}_{R^{(n)}} P_\Theta^{(n)} = 0.$$

4 Outlook

The self-similarity suggests applications to graded tensor networks, built-in Koszul signs, and higher-form differentials realised as iterated commutators $\operatorname{ad}^k(P_\Theta^{(n)})$.