Utility Indifference Pricing Overview

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Introductory Notes

What follows below is research carried out by Vest Labs with the primary intent of it being discussed on the labs' forum. This particular session can be found here: https://vest.discourse.group/t/utility-indifference-pricing-overview/21.

Motivation

Our goal is to create a permissionless perpetual futures exchange focused on listing markets with illiquid underlyings before our centralized counterparts are able to. In order to scale to these markets quickly, we aimed to price perpetual futures (more precisely, pricing the funding rates) without trading the underlying, effectively not hedging against risk through replication. This led us to explore pricing in incomplete markets [1].

Utility indifference pricing

One framework for incomplete market pricing is utility indifference pricing. In short, determining whether or not to accept a risky payoff is based on requiring additional capital such that expected utility is kept constant. This additional capital is the indifference price, also known as the certainty equivalent. More formally by [1], let a value function be described as

$$V(x,k) = \sup_{X_T \in \mathcal{A}(x)} \mathbb{E}_{\mathbb{P}}[u(X_T + kC_T)]$$

where $\mathbb{P}$ is the real world probability measure, i.e. the distribution of prices at some terminal time T of some risky asset S, $\mathcal{A}(x)$ is the set of all attainable wealths X_T at time T given an initial endowment x, u is a utility function, C_T is the payoff of a claim contingent on the value of S at time T, and k is the number of claims. Then the indifference price $p_{C_T}(k)$ from purchasing k claims of C_T is defined as the solution to

$$V(x - p_{C_T}(k), k) = V(x_0, 0)$$

Indifference prices can be recovered effectively from a dual optimization problem over the set of equivalent martingale measures $\mathcal{Q}$. Defining the dual as $\tilde{V}(y,k) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\tilde{u}(\frac{\mathbb{Q}}{\mathbb{P}}y) - y\mathbb{E}_{\mathbb{Q}}[kC_T]]$ where $\tilde{u}(y) = \max_x[u(x) - xy]$ and using the relation $V(x,k) = \inf_{y>0}[\tilde{V}(y,k) + xy]$, one can solve

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{P}}[u(I(\frac{\mathbb{Q}}{\mathbb{P}}\hat{y}_2))] = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{P}}[u(I(\frac{\mathbb{Q}}{\mathbb{P}}\hat{y}_1))]$$

where I is the inverse of u', $\hat{y}_2$ solves $\mathbb{E}_{\mathbb{Q}}[I(\frac{\mathbb{Q}}{\mathbb{P}}\hat{y}_2)] = x - p_{C_T}(k) + \mathbb{E}_{\mathbb{Q}}[kC_T]$ and $\hat{y}_1$ solves $\mathbb{E}_{\mathbb{Q}}[I(\frac{\mathbb{Q}}{\mathbb{P}}\hat{y}_1)] = x$ given $\mathbb{Q}$. Derivations can be found in [2].

Note that the dual problem formalization also gives the general arbitrage-free price bounds

[1] https://en.wikipedia.org/wiki/Incomplete_markets

$$(\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[D\cdot C_T], \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[D\cdot C_T])$$

where D is the risk-free discount rate [4]. It follows that in a complete market where there exists a unique equivalent martingale measure, the indifference price converges to the market price.

We now explore a well-studied special case of utility indifference pricing.

Exponential utility indifference

Exponential (CARA) utility is defined as $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$ for some risk aversion parameter $\gamma > 0$. Exponential utility indifference pricing is well-studied and unifies many frameworks in incomplete market pricing. One way to formalize the pricing problem is by [5], given two agents, a buyer looking to buy a claim C_T and a seller looking to sell the claim C_T

$$\begin{aligned} \text{maximize} \quad & \mathbb{E}_{\mathbb{P}}[u_b(C_T - p(C_T)] \\ \text{subject to} \quad & \mathbb{E}_{\mathbb{P}}[u_s(x + p(C_T)] \geq u_s(x) \end{aligned}$$

where u_b is the utility function of the buyer, u_s is the utility function of the seller, and x is the initial endowment of the seller. We abuse notation slightly by taking $p^*(C_T)$ to be the optimal indifference price of a claim C_T.

The exponential utility indifference price is

$$p^*(C_T) = \frac{1}{\gamma_s}\log \mathbb{E}_{\mathbb{P}}[e^{\gamma_s \cdot C_T}]$$

where γ_s is the risk aversion of the seller.

Interestingly, the dual optimization problem is

$$p^*(C_T) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[C_T] - \frac{1}{\gamma_s}\Big(H(\mathbb{Q}||\mathbb{P}) - \inf_{\mathbb{Q}\in\mathcal{Q}} H(\mathbb{Q}||\mathbb{P})\Big)$$

where $H(\mathbb{Q}||\mathbb{P}) = \mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}]$ is the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ [3]. Note that functional form of the exponential utility indifference price is equivalent to the entropic risk measure, a convex risk measure [2]. Thus, maximizing exponential utility is dual to minimizing entropic risk.

From the dual optimization problem, we observe that taking $\lim \gamma_s \to 0$, i.e. as risk aversion vanishes, the indifference price becomes

$$p^*(C_T) = \mathbb{E}_{\hat{\mathbb{Q}}}[C_T]$$

where $\hat{\mathbb{Q}} = \arg\inf_{\mathbb{Q}\in\mathcal{Q}} H(\mathbb{Q}||\mathbb{P})$ is known as the minimal entropy martingale measure [6]. Alternatively, we observe that taking $\lim \gamma_s \to \infty$, i.e. as risk aversion grows infinitely large, the indifference price becomes

$$p^*(C_T) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[C_T]$$

otherwise known as the superhedging price [3].

[2]https://en.wikipedia.org/wiki/Entropic_risk_measure
[3]https://en.wikipedia.org/wiki/Superhedging_price

Discussion

The problem with the utility indifference pricing framework is two-fold: estimating the real world probability measure $\mathbb{P}$ and specifying a utility function u.

First, since the market is incomplete, there exists many equivalent martingale measures. We assume that calibration to market prices is not possible because market prices are unlikely to exist for illiquid assets. Hence, the choice of an equivalent martingale measure heavily depends on $\mathbb{P}$. But $\mathbb{P}$ is notoriously difficult to estimate to sufficient accuracy with finite sample points [4].
Second, calibrating the utility function u is equally difficult. How does one quantify risk aversion precisely? From exponential utility indifference pricing, we observe that the choice of risk aversion γ interpolates between the minimal entropy martingale measure, the equivalent martingale measure with the least informational difference from $\mathbb{P}$, and the superhedging price, the most conservative equivalent martingale measure.

These two issues make the utility indifference pricing framework not robust in the sense that resulting prices are extremely sensitive to parameters that are fundamentally hard to infer.

The question is, does there exist a robust approach to utility indifference pricing or are there alternative robust frameworks for incomplete market pricing that may be insensitive to the accuracy of an estimation of $\mathbb{P}$?

References

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[4] https://quant.stackexchange.com/questions/25942/why-arent-econometric-models-used-more-in-quant-finance/25957#25957