vAMM and Virtual Liquidity Curve

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"don't conflate good vAMM with bad vAMM...the design space is HUGE and very under appreciated given the historic failures."

1 Motivation

vAMM can be a powerful tool with a properly built-in risk engine. Here, we explore how a pricing scheme defined by vAMM implicitly deploys liquidity that adapts to change in the inventory as well as volatilities of the underlying assets.

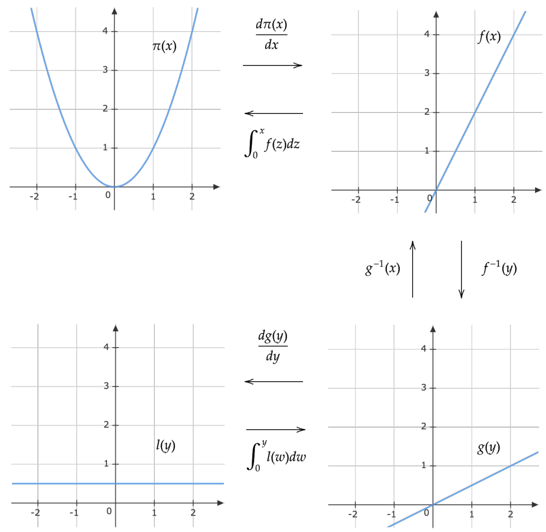
2 Derivation

Consider a vAMM with pricing function $\pi(x)$ that charges premium for a trade with size x. One such example of $\pi(x)$ is indifference pricing, where using a risk measure ρ, one defines $\pi(x)$ such that $\rho(\text{inventory} - x + \pi(x)) = \rho(\text{inventory})$, or equivalently, $\pi(x) = \rho(\text{inventory} - x) - \rho(\text{inventory})$ by cash invariance. In short, a cash $\pi(x)$ is charged such that the risk for the AMM remains the same before and after accepting the trade. This construction is slightly different from conventional CFMMs, where the invariant is defined over *non-negative* quantities of n assets, since $\pi(x)$ is defined for any real x. This is one of the factors that separates vAMMs from spot AMMs — allowing "reserves" to be negative.

We now transform the function $\pi(x)$ into a function of spread y, $l(y)$, where $\int_{y_1}^{y_2} l(y)dy$ represents the total amount of liquidity deployed within price range $(p + y_1)$ and $(p + y_2)$, where p is the index price. This will help us visualize how vAMM places virtual liquidity across different prices. Note that midprice is equal to $p + \pi'(0)$.

<u>proposition 1</u>: $l(y) = \begin{cases} 1/\pi''(\{\pi'\}^{-1}(y)) \text{ if } y \in \text{Dom}(\{\pi'\}^{-1}) \\ 0 \text{ otherwise} \end{cases}$

(proof sketch): Let $f(x) = \pi'(x)$. This represents the marginal premium charged at size x. Since premium can be thought of as the spread from index price, the inverse of $f(x)$, say $g(y) = f^{-1}(y)$, represents the total amount of liquidity consumed, when limit orders with spread y from index price are lifted. To find marginal liquidity deployed at price $(p + y)$, we differentiate $g(y)$ with respect to y, by which we obtain the function $l(y) = g'(y) = 1/f'(f^{-1}(y)) = 1/\pi''(\{\pi'\}^{-1}(y))$



Transformation when $\pi(x) = x^2$ and $l(y) = 1/2$ with midprice p

Conversely, we can also retrieve the pricing function $\pi(x)$ given liquidity curve $l(y)$ and midprice y_0.

proposition 2: $\pi(x) = \int_0^x f(z)dz$ where $f(z) = g^{-1}(z)$ and $g(y) = \int_{y_0}^y l(w)dw$

(proof sketch): We follow the steps for converting $\pi(x)$ into $l(y)$ but in reverse. Since there should be no cumulative liquidity at midprice y_0, $g(y) = \int_{y_0}^y l(w)dw$ ($g(y_0) = 0$). Let $f(z) = g^{-1}(z)$. Since there should be no premium charged for a trade with size 0, we obtain $\pi(x) = \int_0^x f(z)dz$ ($\pi(0) = 0$).

3 Example

We now apply this to indifference pricing with EVaR, explored in our previous post.

Using EVaR-indifferent pricing, we can write the premium for ith market $\pi_i(x)$ as $EVaR(\boldsymbol{X} + x\mathbf{1}_i) - EVaR(\boldsymbol{X})$, where $\boldsymbol{X} = (x_1, \cdots, x_n)^T$ represents the current inventory for the AMM and $\mathbf{1}_i$ is a vector where $\mathbf{1}_{i,j} = 0$ if $i \neq j$ and 1 if $i = j$. With an additional assumption that the underlying prices follow multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$, we can rewrite $\pi_i(x)$ as

$$\begin{aligned}
\pi_i(x) &= \boldsymbol{\mu}^T x\mathbf{1}_i + \sqrt{-2\log\alpha}\left(\sqrt{(\boldsymbol{X} + x\mathbf{1}_i)^T\boldsymbol{\Sigma}(\boldsymbol{X} + x\mathbf{1}_i)} - \sqrt{\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}}\right) \\
&= \mu_i x + \sqrt{-2\log\alpha}\left(\sqrt{(\boldsymbol{X} + x\mathbf{1}_i)^T\boldsymbol{\Sigma}(\boldsymbol{X} + x\mathbf{1}_i)} - \sqrt{\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}}\right)
\end{aligned}$$

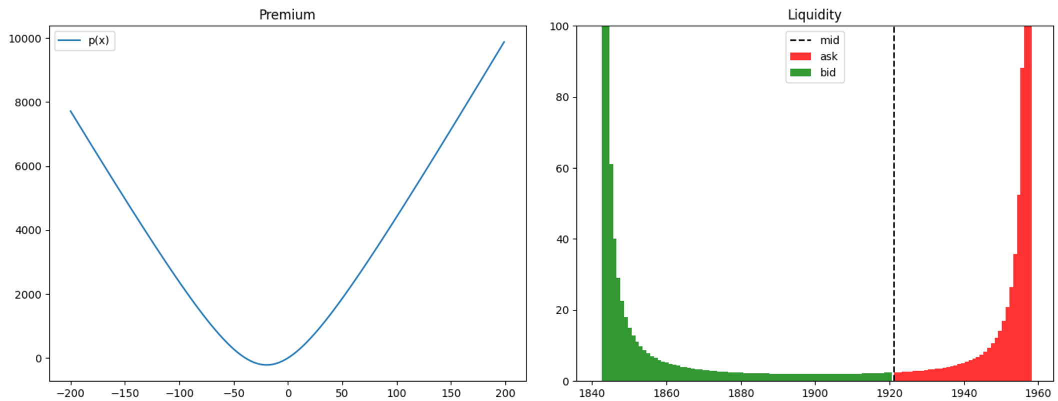
where μ_i is the expected price for the ith asset and $1 - \alpha$ is the EVaR confidence level.

Now, with some math, we get

$$
l_i(y) = \begin{cases} \frac{\sqrt{(\log\alpha)(\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}+2\boldsymbol{\Sigma}_i^T\boldsymbol{X}x+\Sigma_{ii}x^2)}}{4\Sigma_{ii}-(2\boldsymbol{\Sigma}_i^T\boldsymbol{X}+2\Sigma_{ii}x)^2/(\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}+2\boldsymbol{\Sigma}_i^T\boldsymbol{X}x+\Sigma_{ii}x^2)} & \text{if } B^2-4AC \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
x &= \frac{-B+\sqrt{B^2-4AC}}{2A} \\
A &= 2\Sigma_{ii}\{(y-\mu_i+p_i)^2+2\Sigma_{ii}+\log\alpha\} \\
B &= 4\boldsymbol{\Sigma}_i^T\boldsymbol{X}\{(y-\mu_i+p_i)^2+2\Sigma_{ii}+\log\alpha\} \\
C &= 2(y-\mu_i+p_i)^2\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}+(2\boldsymbol{\Sigma}_i^T\boldsymbol{X})^2+\log\alpha
\end{aligned}
$$

where p_i is the index price for the ith asset, $\boldsymbol{\Sigma}_i$ is the ith row of $\boldsymbol{\Sigma}$ and Σ_{ii} is the (i,i)th entry of $\boldsymbol{\Sigma}$.



Premium $\pi_1(x)$ and liquidity $l_1(y)$ with $\boldsymbol{\mu} = (p_1\exp(\sigma_{11}), p_2\exp(\sigma_{22}))^T$, $\boldsymbol{\Sigma} = \mathrm{diag}(\boldsymbol{\mu})(e^{\boldsymbol{\sigma}}-\mathbf{1})\mathrm{diag}(\boldsymbol{\mu})$, $\boldsymbol{p} = (1900, 30000)^T$, $\boldsymbol{\sigma} = \begin{pmatrix} 0.0001 & 0.00005 \\ 0.00005 & 0.000064 \end{pmatrix}$, $\mathbf{1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Note that as $x \to \infty$ $(-\infty)$, the spread per size converges to $\frac{p(x)}{x} \to \mu_i - p_i + \sqrt{-2(\log\alpha)\Sigma_{ii}}$ $(\mu_i - p_i - \sqrt{-2(\log\alpha)\Sigma_{ii}})$. This implies that unless vAMM sets a cap on open interests, it deploys infinite liquidity at prices $\mu_i + \sqrt{-2(\log\alpha)\Sigma_{ii}}$ and $\mu_i - \sqrt{-2(\log\alpha)\Sigma_{ii}}$.

In the plots above, we see that the bids are placed with negative spread so as to incentivize shorts to come in, which will help reduce the risk for the AMM who has a net short exposure in the second asset that is positively correlated with the first.

Under the same set of assumptions, we can evaluate the liquidity function for pricing with Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) in a similar manner, since both pricing functions can be written as

$$
\begin{aligned}
\pi_i^{VaR}(x) &= \mu_i x + c^{VaR}(\alpha)\left(\sqrt{(\boldsymbol{X}+x\mathbf{1}_i)^T\boldsymbol{\Sigma}(\boldsymbol{X}+x\mathbf{1}_i)} - \sqrt{\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}}\right) \text{ and} \\
\pi_i^{CVaR}(x) &= \mu_i x + c^{CVaR}(\alpha)\left(\sqrt{(\boldsymbol{X}+x\mathbf{1}_i)^T\boldsymbol{\Sigma}(\boldsymbol{X}+x\mathbf{1}_i)} - \sqrt{\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}}\right)
\end{aligned}
$$

where the coefficients $c^{VaR}(\alpha) = \Phi^{-1}(1-\alpha)$ and $c^{CVaR}(\alpha) = \varphi(\Phi^{-1}(1-\alpha))/\alpha$, where φ and Φ are PDF and CDF of standard Normal distribution.

4 Application

While we present one instance of risk measure, this construction can be applied to other risk measures with different confidence levels. In addition, converting from premium function to liquidity function makes the “addition” of multiple risk-indifference pricing more intuitive (which reminds me of the beautiful work in “A Geometric Perspective of AMMs”). While operating at the premium function level (probably) requires computing convolution of the risk measures, converting them to liquidity functions simply lets us consider the sum of the functions. If we want to consider charging an additional fee applied to the notional size (priced at the index), this is equivalent to shifting the liquidity curve away from mid price by γp, where γ is the fee level (e.g. 10 bpts).

Note that if one were to apply this market making strategy on order books, it would require them to rebalance their orders frequently, with every change in the inventory as well as changes in the volatility of the underlying assets. However, with a vAMM, we only need to change the parameters in the pricing function $\pi(x)$!

5 Discussion

While converting to liquidity function theoretically allows the aggregation of LPs with different risk tolerances as well as desired fee levels, the practicality of keeping track of the total liquidity functions is still an issue.