Optimal Liquidations via Convex Optimization

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Introductory Notes

What follows below is research carried out by Vest Labs with the primary intent of it being discussed on the labs' forum. This particular session can be found here: https://research.vest.xyz/t/optimal-liquidations-via-convex-optimization/201

Motivation

Liquidation plays an integral role in maintaining the solvency of an exchange. While liquidating positions in isolated markets may be straight forward, things can get more complicated when the exchange enables cross-margining. When a portfolio with multiple open positions becomes liquidatable, the combinations of liquidation orders to bring the account back above the required margin level are not necessarily unique. Existing exchanges with permissionless liquidation mechanisms place this problem in a competitive setting, where liquidators try to liquidate positions in a profit-maximizing way. This competitive environment may not be healthy, especially for AMM-based protocols. In this short note, we design a liquidation mechanism that minimizes the negative impact on the solvency of the exchange.

Problem Formalization

We make the following simplifying assumptions: First, we only consider liquidations in the context of an AMM, which is different to that of a CLOB. Since the AMM is the counterparty to all trades, non-execution risk and price risk are subsumed by the premium charged by the AMM. Additionally, we consider an AMM that preserves a risk-indifference invariant, that is $\rho(X_t - \pi_t) = \rho(X'_t)$ where ρ is a monetary risk measure, X_t, X'_t is the AMM's liabilities before and after transactions at any time t, and π_t is the premium charged to satisfy the equation. We also assume that liquidations are atomic operations that execute at uniform prices within a block. Finally, we do not consider liquidation penalties or fees but show that an extension is straightforward.

Given these assumptions, we define an optimal liquidation order to be a minimizer of the total notional amount liquidated while satisfying the maintenance margin and the risk-indifference invariant.

We can formalize the liquidation of portfolio i as a convex optimization problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\min}. & (\mathbf{p} \otimes |\mathbf{q}_i|)^T \mathbf{w} \\ \text{subject to} & \mathbf{0} \preceq \mathbf{w} \preceq \mathbf{1} \\ & (r+\epsilon)(\mathbf{p} \otimes |\mathbf{q}_i|)^T(\mathbf{1} - \mathbf{w}) \leq c_i - \pi + \mathbf{q}_i^T(\mathbf{p} - \mathbf{p}_i) \\ & \pi = \rho(X(\mathbf{q} - \mathbf{w} \otimes \mathbf{q}_i)) - \rho(X(\mathbf{q})) \end{array}$$

where $\mathbf{w} \in \mathbb{R}^n$ is the optimal liquidation weight vector, $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^n$ are the portfolio i quantity and entry price vectors, $\mathbf{p} \in \mathbb{R}^n$ is the market price vector, $\mathbf{q} \in \mathbb{R}^n$ is the exchange total liability quantity vector. $r \in \mathbb{R}$ is the maintenance margin ratio, $\epsilon \in \mathbb{R}$ is a buffer, and $c_i \in \mathbb{R}$ is the portfolio i collateral. ρ is a convex risk measure and $X(\mathbf{q})$ is a univariate random variable where $X(\mathbf{q}) = X(\mathbf{q}, \mathbf{p}_0) = \mathbf{q}^T(\mathbf{P} - \mathbf{p}_0)$, $\mathbf{P}$ is a multivariate random variable of asset prices, and $\mathbf{p}_0$ is an associated vector of entry prices (which we omit to simplify notation). π is the premium charged by the risk-indifferent AMM for the liquidation and $\mathbf{q}_i^T(\mathbf{p}-\mathbf{p}_i)$ represents total realized and unrealized PnL. $\mathbf{p}\otimes\mathbf{q}$ denotes element-wise product of vectors $\mathbf{p}$ and $\mathbf{q}$, and $\preceq$ denotes element-wise inequality.

In words, the problem solves for an optimal partial liquidation for each position such that the portfolio's margin post-liquidation (RHS of the second constraint) is at least the maintenance margin determined by total notional size post-liquidation (LHS of the second constraint). Note that if the problem is infeasible, the portfolio will be fully liquidated.

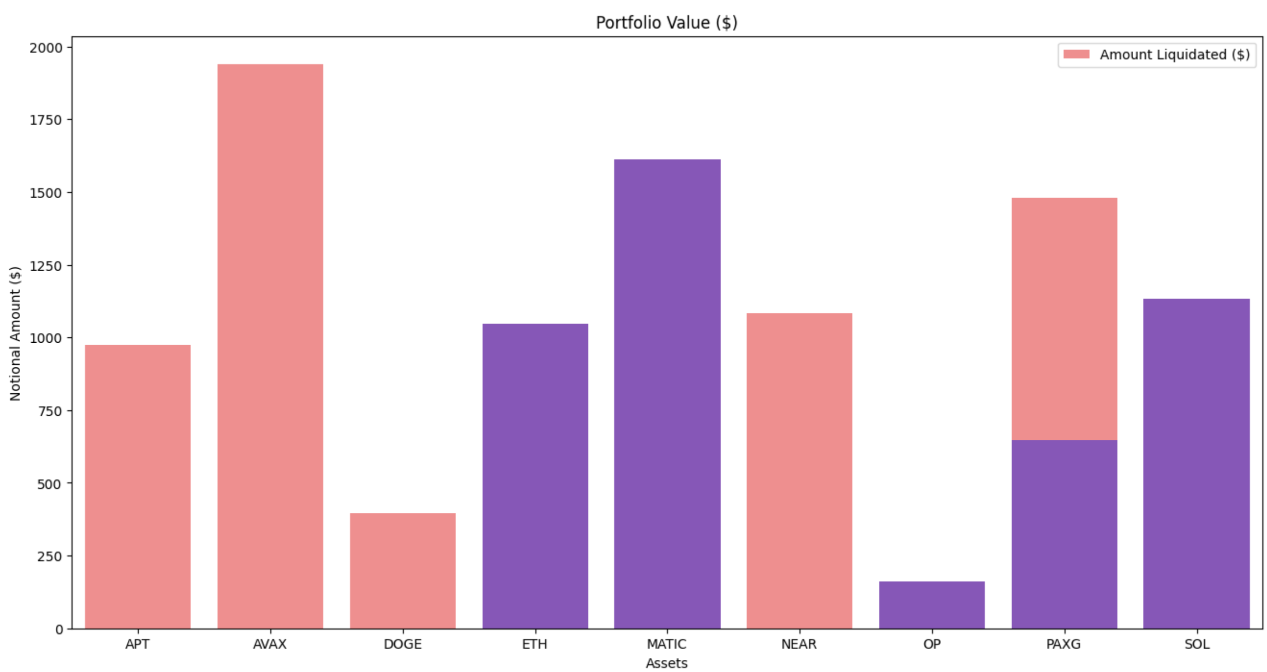
Proof of Convexity. First, note that the objective and the first constraint are both affine in $\mathbf{w}$, hence convex. For the second inequality constraint, LHS is affine in $\mathbf{w}$, and since $X(\mathbf{q}-\mathbf{w}\otimes\mathbf{q}_i)$ is affine in $\mathbf{w}$ and ρ is convex in X by definition, RHS $-\rho(X(\mathbf{q}-\mathbf{w}\otimes\mathbf{q}_i))+\mathit{constant}$ is concave.

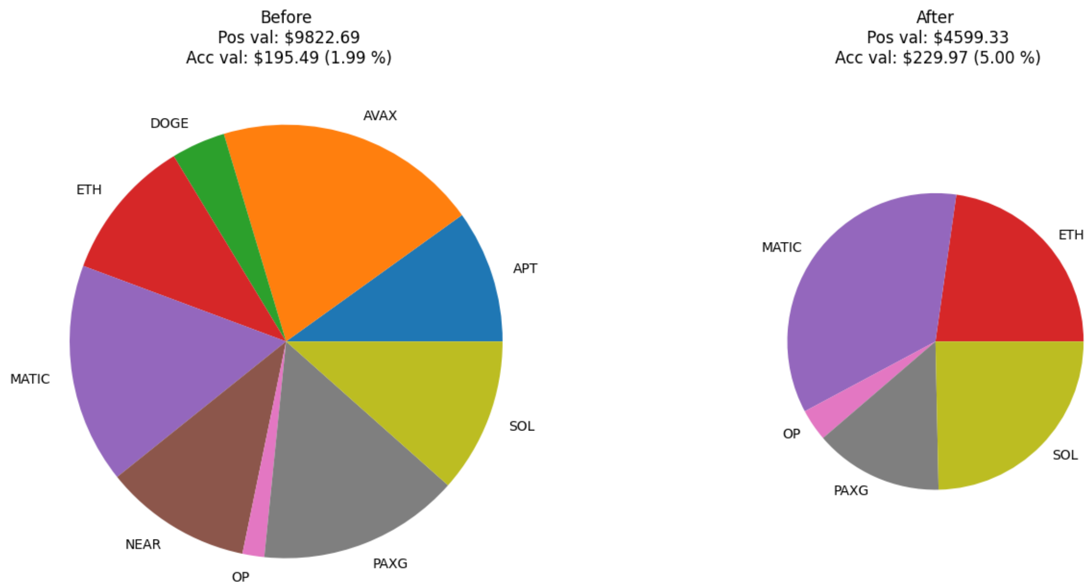
In particular, when ρ is entropic value-at-risk with confidence level $1-\alpha$ and $X(\mathbf{q})\sim N(\mathbf{q}^T(\boldsymbol{\mu}_P-\mathbf{p}_0),\mathbf{q}^T\mathbf{V}_P\mathbf{q})$, the second constraint can be represented as $\|\mathbf{A}\mathbf{w}+\mathbf{b}\|_2\leq\mathbf{c}^T\mathbf{w}+d$ with $\mathbf{A}=-\mathbf{V}_P^{1/2}diag(\mathbf{q}_i)$, $\mathbf{b}=\mathbf{V}_P^{1/2}\mathbf{q}_i$, $\mathbf{c}=((r+\epsilon)|\mathbf{q}|\otimes\mathbf{p}-\mathbf{q}_i\otimes(\mathbf{p}-\boldsymbol{\mu}_P))/\sqrt{-2\log\alpha}$, $d=(c_i-(r+\epsilon)\mathbf{p}^T|\mathbf{q}_i|+\mathbf{q}_i^T(\mathbf{p}-\mathbf{p}_i)+\sqrt{-2\log\alpha}\|\mathbf{V}_P^{1/2}\mathbf{q}\|_2)/\sqrt{-2\log\alpha}$, which turns the problem into a second order cone program.

Simulations

With this setup, we can easily solve the problem using `cvxpy`. As mentioned above, we use entropic value-at-risk with confidence level 0.99 as ρ and assume the price vector $\mathbf{P}\sim\mathcal{N}(\boldsymbol{\mu}_P,\mathbf{V}_P)$. To forecast the covariance matrix, we use GO-GARCH explored in the previous post.

We initialize the portfolio with randomly selected assets. Then, consider the moment in which the account value (margin) goes below the maintenance margin, which is set to be $r=2.5\%$ in this simulation and use the algorithm to determine an optimal weight $\mathbf{w}$ with $\epsilon=2.5\%$.





Future Research

We didn't consider liquidation penalty in this work. If we were to define it as some fraction of notional size being liquidated, we can easily extend the framework above by subtracting $r_p(\mathbf{p} \otimes \mathbf{q})^T \mathbf{w}$ from the RHS of the second constraint where r_p is the penalty rate (e.g. 2.5%), maintaining the convexity. This, however, leads to the problem being infeasible more often, leading to full liquidation.

Further topics of interest include but are not limited to:

- alternative objectives that better reflect the needs for a specific protocol
- exploration of a reward mechanism that incentivizes optimal liquidations in a decentralized setting (e.g. liquidation reward is maximized as the distance between the liquidation strategy and an optimal liquidation strategy is minimized)