A Short Introduction to Theoretical Mechanics

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"Seek simplicity and distrust it."... Alfred North Whitehead

Chapter 1

Introduction

1.1 The Laws of Motion

The three laws that govern motion are usually attributed to Isaac Newton. They are:

1. Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

2. Rate of change motion is proportional to the impressed force, and is in the direction in which the force acts.

3. To every action there is always an equal an opposite reaction.

The bodies referred to in the First Law should be taken to be particles. Particles are idealizations of objects that have no size. The Laws of Motion can, in most cases, be generalized to systems of interacting particles, although care must be taken in some specific instances.

The tendency of a particle to continue in a state of rest or uniform motion is called *inertia*. *Mass* is the quantitative measure of inertia of a body and it is given numerical value by comparison to a standard. An *inertial reference frame* is one in which Newton's First Law holds strictly. Such a frame is also an idealization rather than a physically realizable frame.

Changes in position are measured in terms of distances and times. Distances and times are given quantitative meaning by comparison to standards.

The concept of motion in the Second Law can be given quantitative measure as the momentum of a particle. If m is the mass and $\mathbf{v}$ is the rate of change of position, then the *momentum* $\mathbf{p}$ is given by $\mathbf{p} = m\mathbf{v}$. Newton's Second Law then can be expressed as a vector equation,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt}.$$

Conceptually, forces are pushes or pulls. The fundamental forces of classical mechanics are either the gravitational force or the electromagnetic force or some manifestations (friction, viscosity, contact, etc.) of these which may obscure their true origins. Forces are given quantitative meaning through Newton's Second Law which connects force to quantitative measures of mass, length and time.

Newton's three Laws of Motion are based on experiment and cannot be proved or derived.

1.2 Vectors

There are two aspects to motion. The first is called *kinematics* and the second is called *dynamics*. Kinematics is purely the description of motion independent of the laws of physics which govern it. The latter study is dynamics. We are particularly interested in the dynamics of particles and systems, but before we turn to dynamics, we have to develop a mathematical language for describing motion.

We will be concerned with two kinds of quantitites. The first, called *scalar* quantities, are characterized by magnitude only and are represented by ordinary real numbers. Time and temperature are scalar quantities. Other quantitites have the combined characteristics of magnitude and direction. These are represented by mathematical objects called *vectors*. Relative position, velocity, force and acceleration are vector quantities.

In three-dimensional space, you can think of a vector as a directed line segment having length (magnitude) and direction. You may also think of it in terms of an ordered set of three scalar *components*. In texts, vectors are usually represented in bold face.

$$\mathbf{a} = (a_1, a_2, a_3)$$

The essential characteristics of vectors are the following:

1.2.1 Equality of Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal if and only if their respective components are equal, i.e. $\mathbf{a} = \mathbf{b}$ means that $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

1.2.2 Vector Addition

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

1.2.3 Vector Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

1.2.4 Null Vector

There exists a null vector $\mathbf{0}$ such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.

1.2.5 Commutative Law of Addition

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

1.2.6 Associative Law of Addition

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

In component form this can be written:

$$(a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3))$$

$$= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, (a_3 + b_3) + c_3)$$

1.2.7 Multiplication of a Vector by a Scalar

There are several kinds of multiplication defined for vectors. Let p be a scalar and let $\mathbf{a}$ be a vector. Then,

$$p\mathbf{a} = p(a_1, a_2, a_3) = (pa_1, pa_2, pa_3)$$

1.2.8 Scalar Multiplication of Vectors; Dot Product

There are two kinds of multiplication defined for vectors with vectors. The first is called the *dot product.*

$$\mathbf{a} \cdot \mathbf{b} = |a||b| \cos\theta = \sum a_i b_i = a_i b_i.$$

Here, $|a|$ is the *magnitude* of $\mathbf{a}$, i.e.,

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i a_i},$$

and θ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$. Very often, for brevity, an index such as i in the expression $a_i a_i$ that is repeated exactly twice will imply a summation on that index. In such cases the summation sign $\sum$ is not written and the index i is said to be a dummy index because when you actually write out the expression, it is replaced by $1s$, $2s$ and $3s$. Such is the case for

$$a_i a_i \equiv \sum a_i a_i \equiv a_1^2 + a_2^2 + a_3^2 \equiv a_j a_j.$$

The dot product of two vectors is itself a scalar; hence this form of multiplication is described as "scalar multiplication" of vectors.

The following are theorems for the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$p(\mathbf{a} \cdot \mathbf{b}) = (p\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (p\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})p$$

1.2.9 Vector Multiplication of Vectors; Cross Product

The second kind of multiplication produces a vector. Again θ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$. We then have,

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = |a||b| \sin\theta \hat{\mathbf{n}},$$

where $\mathbf{n}$ is a vector of unit length and direction perpendicular to the plane which contains both $\mathbf{a}$ and $\mathbf{b}$. The direction is further defined by a *right hand rule*, i.e., the direction of $\hat{\mathbf{n}}$ is the direction the thumb of our right hand would be pointing if you pointed the fingers of your right hand along the direction of $\mathbf{a}$ and curled them toward the direction of $\mathbf{b}$. In truth, the cross product is not a true vector, but rather is a *pseudovector.* It does have magnitude and components like a vector, but its direction is ambiguous and has to be defined artificially by a right-hand-rule. Physical quantities, such as the magnetic field, that are represented by pseudovectors do not have a true direction, but are given one artificially by a right-hand-rule.

We may also write,

$$c_i = \sum_{j,k} \delta_{ijk} a_j b_k \equiv \delta_{ijk} a_j b_k$$

where, again, the repeated indices j and k indicate a sum and are dummies. The symbol δ_{ijk} is a $3 \times 3 \times 3$ matrix called the *Levi-Civita tensor.* If i, j and k have the values 1, 2 and 3 respectively or any even permutation of (123), then $\delta_{ijk} = +1$. If the values of i, j, and k respectively are an odd permutation of (123) then $\delta_{ijk} = -1$. Otherwise (such as in the case of two indices being the same), $\delta_{ijk} = 0$. (Think of the indices as labeled beads on a loop of string. An *even* permutation is one you can get by simply moving the beads counterclockwise or clockwise along the loop to change their order as they sit in your hand. Thus (312) and (231) are even permutations of (123). An *odd* permutation, on the other hand, is one for which you would have to exchange the positions of two adjacent indices. Thus (213) and (132) are odd permutations of (123).) While this form takes a little getting used to, it can be a powerful way of writing and manipulating vectors. Please note that it is the ith *component* and not the full pseudovector that is given by the subscript form.

We have the following theorems:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$p(\mathbf{a} \times \mathbf{b}) = (p\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (p\mathbf{b}) = (\mathbf{a} \times \mathbf{b})p$$

1.2.10 Unit Vectors

We define a special set of vectors that have unit length. Usually (almost always) we will choose three vectors that are mutually orthogonal. Thus,

$$\hat{\mathbf{e}}_1 = (1, 0, 0)$$

$$\hat{\mathbf{e}}_2 = (0, 1, 0)$$
$$\hat{\mathbf{e}}_3 = (0, 0, 1)$$

However, observe that in this instance the subscripts are used to number the three unit vectors rather than specify components of the vectors. With this in mind and given that the three unit vectors are ordered by a right-hand-rule, i.e., that the direction of $\hat{\mathbf{e}}_3$ is the direction of $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$,, we have

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$$

where δ_{ij} is a matrix whose values are 1 if $i = j$ but 0 otherwise. Using the convention that a repeated index implies summation on that index, we may also write,

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \delta_{ijk} \hat{\mathbf{e}}_k.$$

An arbitrary vector may be written in terms of unit vectors,

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3.$$

1.2.11 Derivatives of Vectors

The derivative of a vector with respect to a scalar is defined by:

$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t}.$$

One may easily show that

$$\frac{d}{dt}(p\mathbf{a}) = p\frac{d\mathbf{a}}{dt} + \frac{dp}{dt}\mathbf{a}$$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{b}$$

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}.$$

As long as the direction and length of the unit vectors is unchanging, we may write,

$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\hat{\mathbf{e}}_1 + \frac{da_2}{dt}\hat{\mathbf{e}}_2 + \frac{da_3}{dt}\hat{\mathbf{e}}_3 \equiv \frac{da_i}{dt}\hat{\mathbf{e}}_i.$$

1.2.12 Directional Derivative, $d\psi_{PQ}$

1. $\psi(x_1, x_2, x_3) = c$ where c is a constant defines a surface in Cartesian coordinates. If the constant c is changed by a small amount, a second surface near the first is defined. An example of such a surface would be the sphere defined by

$$x^2 + y^2 + z^2 = a^2.$$

   In the example, a is the radius of the spherical surface. Changing a creates a nested set of spherical surfaces.

2. Suppose P and Q are two points that are on separate surfaces that are close together so that P and Q are close together. The *chain rule* of differentiation gives us an estimate of the change in ψ as one moves from P to Q.

$$d\psi_{PQ} = \frac{\partial\psi}{\partial x_1}dx_1 + \frac{\partial\psi}{\partial x_2}dx_2 + \frac{\partial\psi}{\partial x_3}dx_3.$$

   Using subscript notation and summation convention, this can be written,

$$d\psi_{PQ} \equiv \partial_i\psi dx_i \equiv \nabla_i\psi dx_i.$$

   We may also define a vector $d\mathbf{s}$ which connects P to Q,

$$d\mathbf{s} = dx_1\hat{\mathbf{e}}_1 + dx_2\hat{\mathbf{e}}_2 + dx_3\hat{\mathbf{e}}_3 = dx_i\hat{\mathbf{e}}_i$$

   and define the *gradient* of ψ as,

$$\nabla\psi = \frac{\partial\psi}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial\psi}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial\psi}{\partial x_3}\hat{\mathbf{e}}_3 \equiv \nabla_i\psi\hat{\mathbf{e}}_i.$$

   Thus,

$$d\psi_{PQ} = \nabla\psi \cdot d\mathbf{s}.$$

3. Observe that if P and Q were in the *same* surface, $d\psi_{PQ} = 0$ for a $d\mathbf{s}$ connecting them. This means that the gradient would have to be perpendicular to the surface at each point on the surface.

1.2.13 Transformation of Vectors

Consider two sets of orthogonal unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$. The second set is rotated in a completely general way relative to the first so that there is an angle α_{11} between $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}_1$, an angle α_{12} between $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}_2$, etc. We may use either of these two sets of unit vectors to express an arbitrary vector $\mathbf{a}$,

$$\mathbf{a} \equiv a_i \hat{\mathbf{e}}_i \equiv a'_i \hat{\mathbf{e}}'_i.$$

We may form a dot product of these expressions with one of the unit vectors of the second set to form,

$$a_i(\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i) = a'_i(\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}'_i).$$

But $\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}'_i = \delta_{ji}$. Thus

$$a'_j = \cos(\alpha_{ji}) a_i.$$

If we define a matrix S of the cosines of the angles between the primed and unprimed axes, the elements of the matrix are defined to be $S_{ij} = \cos(\alpha_{ij})$, i.e.,

$$a'_j = S_{ji} a_i.$$

Using a similar argument one can also show that

$$a_j = S_{ij} a'_i.$$

The matrix S is called the *transformation matrix*, the elements of the matrix S_{ij} are called the *transformation coefficients* and the set of $\cos\alpha_{ij}$ are called the *direction cosines* of the primed axis set relative to the unprimed set. It is also easily shown that

$$\hat{\mathbf{e}}'_i = S_{ij} \hat{\mathbf{e}}_j,$$

where, again, the repeated index implies summation.

1.2.14 Index notation for Cartesian vectors

The symbol δ_{ij} is called the Kronecker delta. It is a 3x3 matrix whose elements are 1 if $i = j$ and 0 if $i \neq j$. The symbol δ_{ijk} is called the Levi-Civita tensor. It is a 3x3x3 matrix whose elements are 1 if (ijk) stand for an even permutation of (123), -1 if (ijk) stand for an odd permutation of (123), and 0 if any two indices are the same. (123) are labels for the axes of a right-handed, Cartesian coordinate system.

Unless otherwise indicated, it is assumed that there is a summation on repeated indices.

$\mathbf{a} \cdot \mathbf{b} = a_i b_i$		(dot product)
$\mathbf{c} = \mathbf{a} \times \mathbf{b}$	; $c_i = \delta_{ijk} a_j b_k$	(cross product)
$det(a_{ij}) = a_{1i} a_{2j} a_{3k} \delta_{ijk}$		
$det(a_{ij}) = a_{i1} a_{j2} a_{k3} \delta_{ijk}$		(determinant)
$\nabla \cdot \mathbf{a} = \nabla_i a_i$		(divergence)
$\mathbf{c} = \nabla \times \mathbf{a}$	; $c_i = \delta_{ijk} \nabla_j a_k$	(curl)
$\nabla \psi = \hat{\mathbf{e}}_i \nabla_i \psi$		(gradient)
$\nabla^2 \psi = \nabla_i \nabla_i \psi$		(Laplacian)

The rules are:

1. Repeated indices mean "sum". The repeated index is a dummy and may have *any* symbol as an index that is otherwise unused in a particular term.

2. No more than two identical indices are allowed in any term.

3. The operator $\nabla_i \equiv \partial / \partial x_i$ operates on everything to its right.

Example: $\nabla_i a_j b_k = a_j \nabla_i b_k + b_k \nabla_i a_j$

4. The operator ∇_i operating on a Cartesian coordinate, x_j yields 1 if $i = j$, but yields 0 if $i \neq j$.

$$\nabla_i x_j = \delta_{ij}, \text{ where}$$
$$\delta_{ij} = 0 \quad \text{if i}\neq\text{j}$$
$$= 1 \quad \text{if i=j} \qquad \text{(Kronecker delta)}$$

5. Products of Kronecker deltas may be reduced if an index is repeated (summation).

$$\delta_{ij} \delta_{kj} = \delta_{ik}$$
$$but\ \delta_{ij} \delta_{ij} = \delta_{jj} = 3 \qquad \text{(trace of } \delta_{ij}\text{)}$$

6. Relationships among elements of the Levi-Civita tensor.

$$\begin{aligned}
&\delta_{ijk} = -\delta_{jik} = -\delta_{ikj} = -\delta_{kji} && \text{(odd permutations)}\\
&\delta_{ijk} = \delta_{kij} = \delta_{jki} && \text{(even permutations)}
\end{aligned}$$

7. Products of Levi-Civita tensors may be reduced if an index is repeated (summation) with the following pattern:

$$\delta_{ijk}\delta_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Example:

$$\begin{aligned}
(\mathbf{a} \times \nabla) \cdot \mathbf{r} \quad &= \delta_{ijk} a_j \nabla_k x_i \\
&= \delta_{ijk} a_j \delta_{ki} \\
&= \delta_{iji} a_j \\
&= 0
\end{aligned}$$

1.3 Position, Velocity, Acceleration

The position $\mathbf{r}$ of a particle relative to a chosen origin is a vector quantity. Like any other vector, it can be written,

$$\mathbf{r} = x_i \hat{\mathbf{e}}_i.$$

The magnitude of $\mathbf{r}$ is a scalar quantity given by $r = \sqrt{x_i x_i}$. The time derivative of $\mathbf{r}$ is the *velocity* $\mathbf{v}$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} = \frac{d}{dt}(x_i \hat{\mathbf{e}}_i) = \dot{x}_i \hat{\mathbf{e}}_i + x_i \frac{d\hat{\mathbf{e}}_i}{dt}.$$

In a coordinate system that is fixed in space, the unit vectors are constant in magnitude and in direction. In this case the time derivatives of the unit vectors vanish and we may write,

$$\mathbf{v} = \dot{x}_i \hat{\mathbf{e}}_i.$$

The magnitude of the vector $\mathbf{v}$ is the *speed* v. In a fixed coordinate system,

$$v = \sqrt{\dot{x}_i \dot{x}_i}.$$

The time derivative of the velocity is the *acceleration*,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} \equiv \ddot{\mathbf{r}}.$$

In a fixed coordinate system,

$$\mathbf{a} = \frac{d\dot{x}_i}{dt}\hat{\mathbf{e}}_i \equiv \ddot{x}_i\hat{\mathbf{e}}_i.$$

Sometimes it is convenient to use a set of unit vectors that is *not* fixed in space. While the unit vectors continue to have constant unit length, their directions may be changing. As a result the unit vectors have time derivatives that do not vanish. There are two circumstances where this commonly occurs.

First, consider a particle moving along a curved path through space. Let $\hat{\boldsymbol{\tau}}$ be a unit vector tangent to the particle's path. As the particle moves from point to point along the curved path, $\hat{\boldsymbol{\tau}}$ changes direction. At two times separated by Δt,

$$\hat{\boldsymbol{\tau}}(t + \Delta t) = \hat{\boldsymbol{\tau}}(t) + d\hat{\boldsymbol{\tau}}.$$

You can think of the three vectors in this expression forming a small triangle with an angle $d\theta$ between $\hat{\boldsymbol{\tau}}(t + \Delta t)$ and $\hat{\boldsymbol{\tau}}(t)$. Then,

$$\mathbf{v} = v\hat{\boldsymbol{\tau}}$$

and,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\hat{\boldsymbol{\tau}} + v\frac{d\hat{\boldsymbol{\tau}}}{dt}.$$

If Δt is very small, $d\theta$ is very small and $d\hat{\boldsymbol{\tau}}$ is (almost) perpendicular to $\hat{\boldsymbol{\tau}}$. Let $\hat{\mathbf{n}}$ be a unit vector along $d\hat{\boldsymbol{\tau}}$ in the limit that $\Delta t \to 0$. The magnitude of $d\hat{\boldsymbol{\tau}}$ is $(1)d\theta$. Hence, $d\hat{\boldsymbol{\tau}} = d\theta\hat{\mathbf{n}}$. Dividing by dt, we have,

$$\frac{d\hat{\boldsymbol{\tau}}}{dt} = \frac{d\theta}{dt}\hat{\mathbf{n}} = \dot{\theta}\hat{\mathbf{n}}$$

and,

$$\mathbf{a} = \frac{dv}{dt}\hat{\boldsymbol{\tau}} + v\dot{\theta}\hat{\mathbf{n}}.$$

For Δt small, the particle can be thought of as moving along a path that can be approximated as a short segment of a circle with radius ρ. Then $v = \rho\dot{\theta}$ and,

$$\mathbf{a} = \frac{dv}{dt}\hat{\boldsymbol{\tau}} + \frac{v^2}{\rho}\hat{\mathbf{n}}.$$

In general the *radius of curvature* ρ will be a function of the position of the particle along its trajectory. The formula is instantly applicable to particles moving in circles for which ρ is the radius of the circle. We observe that acceleration has a tangential component dv/dt long its tangent and a *centripetal* component v^2/ρ towards the center of the arc of its trajectory. Compare this with the form of $\mathbf{a} = \ddot{x}_i\hat{\mathbf{e}}_i$ in a fixed coordinate system. Both are correct. The two are just different ways of describing the same thing.

Second, consider a frame of reference F' with a set of unit vectors $\hat{\mathbf{e}}_i'$. Assume that the set is rotating in some arbitrary fashion. To characterize the rotation, we will invent a pseudovector which we will call the *angular velocity* $\boldsymbol{\omega}$. We will imagine the instantaneous rotation to be about some axis along which we will put a unit vector $\hat{\boldsymbol{\lambda}}$. To give a direction to $\boldsymbol{\omega}$ we will define a positive rotation along $\hat{\boldsymbol{\lambda}}$ by a right-hand-rule, i.e., the rotation is positive if the thumb of your right hand points along $\hat{\boldsymbol{\lambda}}$ when your fingers curl forward in the direction of the rotation.

As the result of an infinitesimal rotation of the system of unit vectors, the unit vector $\hat{\mathbf{e}}_1$ changes direction such that

$$\hat{\mathbf{e}}_1'(t + \Delta t) = \hat{\mathbf{e}}_1'(t) + d\hat{\mathbf{e}}_1'.$$

Since Δt is small, $d\hat{\mathbf{e}}_1'$ is perpendicular to $\hat{\mathbf{e}}_1'$ so that we may write,

$$d\hat{\mathbf{e}}_1' = (ds_2)\hat{\mathbf{e}}_2' + (ds_3)\hat{\mathbf{e}}_3'$$

where $ds_2 = (1)d\theta_3$ and $ds_3 = (1)d\theta_2$. Dividing by dt,

$$\frac{d\hat{\mathbf{e}}_1'}{dt} = \frac{d\theta_3}{dt}\hat{\mathbf{e}}_2' + \frac{d\theta_2}{dt}\hat{\mathbf{e}}_e' = \dot{\theta}_3\hat{\mathbf{e}}_2' + \dot{\theta}_2\hat{\mathbf{e}}_3'.$$

If we now identify the $\dot{\theta}s$ to be the components of a pseudovector $\boldsymbol{\omega}$ *in the primed set*, then

$$\frac{d\hat{\mathbf{e}}_1'}{dt} = \omega_3'\hat{\mathbf{e}}_2' - \omega_2'\hat{\mathbf{e}}_3' = \boldsymbol{\omega} \times \hat{\mathbf{e}}_1' = \delta_{1jk}\hat{\mathbf{e}}_j'\omega_k'.$$

The negative sign in front of ω_2' arises because a right-hand screw twisted in the sense of increasing θ_2 is a negative rotation about $\hat{\mathbf{e}}_2$ by the right-hand-rule that gives directions to rotations. We can derive similar expressions for $\hat{\mathbf{e}}_2'$ and $\hat{\mathbf{e}}_3'$. All three such relationships are summarized by,

$$\frac{d\hat{\mathbf{e}}_i'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}_i' = \delta_{ijk}\hat{\mathbf{e}}_j'\omega_k'.$$

Now consider two frames F and F' each with origins O and O' and sets of unit vectors $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ respectively. The unprimed frame is fixed in space and its unit vectors are constant. Assume that F' is moving relative to F with arbitrary translation and rotation. Let $\mathbf{r}$ be the position of a particle relative to O, $\mathbf{r}'$ be the position of the same particle but with respect to O'. Finally, let $\mathbf{R}$ be the position of O' with respect to O. Then

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}.$$

The vectors $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{R}$ can all be expressed either in terms of $\hat{\mathbf{e}}_i$ or $\hat{\mathbf{e}}'_i$ with different sets of components in the two systems related by a transformation. One of the several ways we can write $\mathbf{r} = \mathbf{r}' + \mathbf{R}$ in terms of unit vectors is,

$$x_i\hat{\mathbf{e}}_i = x'_i\hat{\mathbf{e}}'_i + X_i\hat{\mathbf{e}}_i.$$

Be careful to note that we have chosen to express $\mathbf{r}'$ in terms of $\hat{\mathbf{e}}'_i$ but have expressed $\mathbf{r}$ and $\mathbf{R}$ in terms of $\hat{\mathbf{e}}_i$. We could also have written $\mathbf{r}' = x^*_i\hat{\mathbf{e}}_i$ where the asterisk is used to indicate components of $\mathbf{r}'$ in the unprimed system, but it would not have served our purpose as you will see below.

If we differentiate with respect to time,

$$\dot{x}_i\hat{\mathbf{e}}_i = \dot{x}'_i\hat{\mathbf{e}}'_i + x'_i\frac{d\hat{\mathbf{e}}'_i}{dt} + \dot{X}_i\hat{\mathbf{e}}_i.$$

Using,

$$\frac{d\hat{\mathbf{e}}'_i}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}'_i,$$

we have,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_o.$$

If we differentiate again with respect to time,

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}'}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}' + \boldsymbol{\omega} \times \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{V}_o}{dt}.$$

Using,

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}$$

$$\frac{d\mathbf{v}'}{dt} = \frac{d}{dt}(\dot{x}'_i\hat{\mathbf{e}}'_i) = \mathbf{a}' + \boldsymbol{\omega} \times \mathbf{v}'$$

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega'_i\hat{\mathbf{e}}'_i) = \dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times \boldsymbol{\omega}) \equiv \dot{\boldsymbol{\omega}}$$

$$\frac{d\mathbf{r}'}{dt} = \frac{d}{dt}(x_i'\hat{\mathbf{e}}_i') = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'$$

$$\frac{d\mathbf{V}}{dt} = \mathbf{A}_0,$$

we then have

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0.$$

In this form, we have the acceleration expressed as a series of terms. The first term on the right is the acceleration of the particle relative to O', the second is called the *Coriolus term*, the third is the *transverse acceleration*, the fourth is the *centripetal acceleration* and the fifth is the acceleration of O' relative to O. We will return to this equation later, but derive it here to emphasize that when the unit vectors are themselves changing with time, care must be taken if motion is to be correctly described in such a system.

1.4 Problems

1. Let

$$\mathbf{a} = (1, 1, 1),$$

$$\mathbf{b} = (1, 2, 3),$$

   and the matrix A,

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

   Evaluate explicitly

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i,$$

$$c_i = \delta_{ijk} a_j b_k,$$

$$c_i = A_{ij} a_j,$$

$$c_i = A_{ji} a_j,$$

$$A_{ii},$$

$$d = a_i A_{ij} b_j.$$

2. To gain experience in using the summation convention for Cartesian vectors, prove the following:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3. A vector $\mathbf{a}$ has components $(1, 1, 1)$ in a Cartesian system. What are its components in a system obtained by a $+60^\circ$ rotation about the x_2-axis? Note that the sense of positive rotation is given by a right-hand-rule.

Chapter 2

Generalized Coordinate Systems

2.1 Generalized Coordinate Systems

Cartesian coordinates are the familiar rectangular coordinates x_i. They have the dimensions of $[length]$ so that speed has the dimensions of $[length]/[time]$ and acceleration has the dimensions of $[length]/[time]^2$. But what if you wish to use plane polar coordinates, r and θ? Theta does not have the dimension of length. How then should we write down an expression for acceleration in terms of polar coordinates? What we attempt below is to deal with so-called generalized coordinates so that we see how to write down correct expressions for positions, velocities and accelerations. We will try to do it in a general way so that we do it once and for all for plane polar coordinates, cylindrical coordinates, spherical coordinates, parabolic coordinates and some coordinate systems you haven't even made up yet.

Unless otherwise indicated, it is assumed that there is a summation on repeated indices.

2.1.1 Definitions

$\hat{\mathbf{e}}_i$	(unit vectors)
$\mathbf{a} = a_i\hat{\mathbf{e}}_i$	(physical components, a_i)
q^i, dq^i	(generalized coordinate, differential)
$d\mathbf{s}$	(differential line element)

$\mathbf{b}_i = \frac{\partial \mathbf{r}}{\partial q^i}$ (basis vectors, $\mathbf{b}_i$)
$\mathbf{a} = A^i \mathbf{b}_i$ (contravariant components, A^i)
$\mathbf{b}^i = \nabla q^i$ (reciprocal basis vectors, $\mathbf{b}^i$)
$\mathbf{a} = A_i \mathbf{b}^i$ (covariant components, A_i)
$g_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j$ (covariant metric tensor, g_{ij})
$g^{ij} = \mathbf{b}^i \cdot \mathbf{b}^j$ (contravariant metric tensor, g^{ij})
$\Gamma^k_{ij} = \mathbf{b}^k \cdot \frac{\partial \mathbf{b}_i}{\partial q^j}$ (Christoffel symbols, Γ^k_{ij})
$T = 1/2m(ds/dt)^2$ (kinetic energy)
V (potential energy)
$F_i = -\nabla_i V$ (covariant force component, F_i)
$L = T - V$ (Lagrangian)

2.1.2 Theorems

$d\mathbf{s} = dq^i \mathbf{b}_i$
$ds^2 = g_{ij} dq^i dq^j$
$\mathbf{b}^i \cdot \mathbf{b}_j = \delta^i_j$
$a_i = \sqrt{g_{(ii)}} A^i$ (no sum on (ii))
$g_{ij} g^{jk} = \delta^k_i$
$A_i = g_{ij} A^j$
$A^i = g^{ik} A_k$
$\mathbf{b}_i = g_{ij} \mathbf{b}^j$
$\mathbf{b}^i = g^{ij} \mathbf{b}_j$
$\frac{\partial \mathbf{b}_i}{\partial q^j} = \frac{\partial \mathbf{b}_j}{\partial q^i}$
$\Gamma^k_{ij} = g^{kl} \frac{1}{2} (\partial g_{il}/\partial q^j + \partial g_{jl}/\partial q^i - \partial g_{ij}/\partial q^l)$ (Christoffel symbols)

If $\mathbf{a} = a_i \hat{\mathbf{e}}_i = A^i \mathbf{b}_i = A_i \mathbf{b}^i$ is acceleration:

$A^k = (\ddot{q}^k + \dot{q}^i \dot{q}^j \Gamma^k_{ij})$
$A_l = \frac{d}{dt}(g_{lk} \dot{q}^k) - \frac{1}{2}(\partial g_{ij}/\partial q^l) \dot{q}^i \dot{q}^j$
$a_i = \sqrt{g_{(ii)}} A^i$ (no sum on (ii))
$a_i = \sqrt{g_{(ii)}} g^{ij} A_j$ (no sum on (ii))
$F_k = m A_k = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k}) - \frac{\partial T}{\partial q^k}$

As a practical matter, we use this machinery in the following manner. We write down an arbitrary d$\mathbf{s}$ from which we form $ds^2 \equiv d\mathbf{s} \cdot d\mathbf{s}$. We divide the expression for ds^2 by dt^2 to form $(ds/dt)^2$ and from this we form $T = (1/2)m(ds/dt)^2$. We then operate on T according to the prescription,

$$A_k = \frac{1}{m}\left(\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k}) - \frac{\partial T}{\partial q^k}\right)$$

to obtain A_i. From the A_i we obtain the A^i and from the A^i we obtain the physical components of acceleration, a_i. These expressions tell us how to write down correctly the acceleration in our particular coordinate system. Since $F_k = mA_k$, we may also obtain the generalized forces if we know the acceleration *a priori*. In systems where a potential energy function, V, exists and we can write,

$$F_k = -\frac{\partial V}{\partial q^k}$$

we can dispense with most of the machinery by defining $L = T - V$ and operating on L according to the following prescription:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^k}) - \frac{\partial L}{\partial q^k} = 0$$

These latter equations are *Lagrange's Equations* and they are the differential equations of motion of the system.

In problems with particle motion in electric and magnetic fields, the Lagrangian, L, becomes:

$$L = T - V - e\psi + (e/c)(\mathbf{A} \cdot \mathbf{v})$$

where e is the charge on the particle, ψ is the electric potential, $\mathbf{A}$ is the magnetic vector potential and $\mathbf{v}$ is the velocity of the particle.

Example: Plane polar coordinates

$\hat{\mathbf{e}}_1 = \hat{\mathbf{r}}$	$\hat{\mathbf{e}}_2 = \hat{\boldsymbol{\theta}}$
$x_1 = r\cos\theta$	$r = \sqrt{(x_1^2 + x_2^2)}$
$x_2 = r\sin\theta$	$\theta = \arctan(x_2/x_1)$
$\partial r/\partial x_1 = x_1/\sqrt{(x_1^2 + x_2^2)} = \cos\theta$	$\partial r/\partial x_2 = x_2/\sqrt{(x_1^2 + x_2^2)} = \sin\theta$
$\partial\theta/\partial x_1 = -x_2/(x_1^2 + x_2^2) = -\sin\theta/r$	$\partial\theta/\partial x_2 = x_1/(x_1^2 + x_2^2) = \cos\theta/r$
$\partial x_1/\partial r = \cos\theta$	$\partial x_1/\partial\theta = -r\sin\theta$
$\partial x_2/\partial r = \sin\theta$	$\partial x_2/\partial\theta = r\cos\theta$

2.1.3 An Example of Generalized Coordinates

Coordinates are used to specify the position of a particle in space. In a three-dimensional space, we need three such coordinates. The three Cartesian coordinates (x_1, x_2, x_3) are an example. Spherical coordinates, cylindrical coordinates and plane polar coordinates are alternatives that are sometimes used. Consider, for example, spherical coordinates (r, θ, ϕ). Observe that while the Cartesian coordinates all have dimensions of length, only r in the set of spherical coordinates has dimensions of length. Newton's Second Law of Motion can easily be expressed in Cartesian coordinates as

$$F_i = m\frac{d^2 x_i}{dt^2} = m\ddot{x}_i$$

where the second derivative of x_i with respect to time will have the dimensions of acceleration if Cartesian coordinates are used. But, how should we express the Second Law if we use coordinates that may not even have dimensions of length? In what follows we shall try to solve this problem once and for all for all admissible *generalized coordinates*. The development is somewhat abstract and general, and it will probably be useful to keep a concrete example in mind.

Consider spherical coordinates,

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\theta = \arctan(\frac{\sqrt{x_1^2 + x_2^2}}{x_3})$$

$$\phi = \arctan(\frac{x_2}{x_1}).$$

Think of each of these as an example of the form $\psi(x_1, x_2, x_3) = c$ discussed in connection with the directional derivative. If r is a constant, then a spherical surface is defined. If θ is constant, then a conical surface is defined. If ϕ is constant, then a half-plane is defined. These surfaces intersect at a point whose coordinates are (r, θ, ϕ). Define unit vectors to lie along the lines of intersection of these surfaces and pointing in the direction of increasing coordinate. For example, $\hat{\mathbf{r}}$ lies along the line of intersection of the cone and the plane and points away from the origin, i.e. in the direction of increasing r if θ and ϕ are held constant to define the constant surfaces (cone and plane).

Similarly, $\hat{\boldsymbol{\theta}}$ lies along the intersection of sphere and plane in the direction of increasing θ. Finally, $\hat{\boldsymbol{\phi}}$ lies along the intersection of sphere and cone in the direction of increasing ϕ. See Fig. 2.1.

Observe the following:

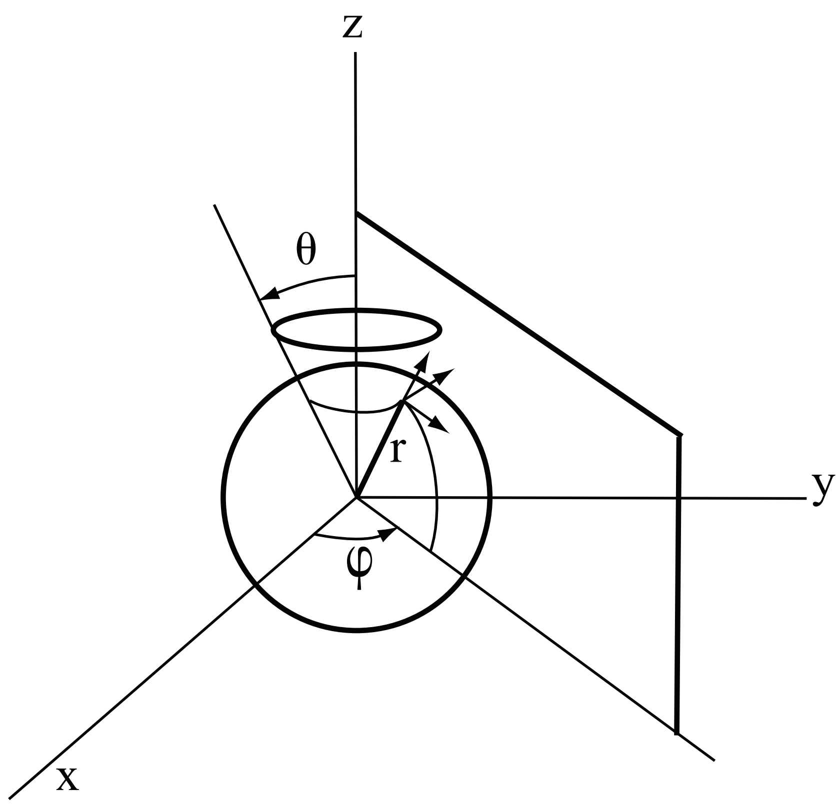
1. $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ are unit vectors. They have unit length. In general, we will denote unit vectors associated with generalized coordinates as $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$,$\hat{\mathbf{e}}_3$.

2. (r, θ, ϕ) are an example of generalized coordinates. In general, we will denote generalized coordinates as (q^1, q^2, q^3).

3. A small displacement of the particle is denoted $d\mathbf{s}$. If θ and ϕ are held constant in spherical coordinates, a small displacement resulting from a change in r would be
$$d\mathbf{s}(1) = dr\hat{\mathbf{r}}.$$
Similarly, if r and ϕ are held constant,
$$d\mathbf{s}(2) = rd\theta\hat{\boldsymbol{\theta}}.$$
If r and θ are held constant,
$$d\mathbf{s}(3) = r\sin\theta d\phi\hat{\boldsymbol{\phi}}.$$

4. A completely arbitrary displacement in spherical coordinates would be a vector sum of these three,
$$d\mathbf{s} = d\mathbf{s}(1) + d\mathbf{s}(2) + d\mathbf{s}(3) = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}.$$
If we form the dot product, $d\mathbf{s}\cdot d\mathbf{s}$, we obtain
$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$
In general, we denote this quantity as $ds^2 = g_{ij}dq^i dq^j$, where a double sum is intended.

5. The velocity of the particle is immediately found in spherical coordinates to be,
$$\mathbf{v} = \frac{d\mathbf{s}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}.$$



.

Figure 2.1: The intersection of constant surfaces (sphere, cone, and half-plane) that define the spherical coordinates of a point.

The square of the speed is,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2.$$

In general, this latter quantity will be written,

$$v^2 = g_{ij}\dot{q}^i\dot{q}^j.$$

6. Finally, the kinetic energy of the particle in spherical coordinates may be written,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2).$$

   In general, we will write,

$$T = \frac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j.$$

   The pattern which is illustrated here for finding the kinetic energy will turn out to be very important.

2.1.4 Generalized Coordinates

Imagine a set of generalized coordinates, $q^i(x_1, x_2, x_3)$. When each is set equal to a constant, a set of surfaces are defined for which the point of intersection, P, has coordinates (q_p^1, q_p^2, q_p^3). For spherical coordinates, the surface of constant r is a sphere, the surface of constant θ is a cone, and the surface of constant ϕ is a half-plane. The cone of angle θ_p intersects the sphere of radius r_p in a circle. The half-plane of constant ϕ_p intersects the circle at a point. The generalized coordinates that uniquely define this point, P, are (r_p, θ_p, ϕ_p).

Now, define a set of unit vectors, $\hat{\mathbf{e}}_i$, which are tangent to the lines of intersection of the various surfaces taken two at a time in the same way that the unit vectors in spherical coordinates were defined. Thus defined, the unit vectors must form a linearly independent set for the coordinates system to be "admissible." The unit vectors are not necessarily orthogonal, although, like the unit vectors in spherical coordinates, they often are. Unlike the unit vectors in a Cartesian system, the directions of the unit vectors depend on the point at which they are defined. For example, in spherical coordinates, the unit vector $\hat{\mathbf{r}}$ has a different direction for different points in space. Its direction is given by the tangent line along the intersection of the cone and the half-plane, but as θ or ϕ varies, this tangent line varies in space.

An arbitrary vector may be expressed as a linear combination of admissible unit vectors,

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 = a_i\hat{\mathbf{e}}_i.$$

The set of numbers, a_i, are said to be the *physical components* of $\mathbf{a}$. The set of numbers, q^i, are said to be the *generalized coordinates* of a particle in the system, but they do not necessarily have dimensions of length. The coordinates must uniquely define the position of the particle.

2.1.5 Generalized Basis Vectors

We now introduce a new set of basis vectors that have the same direction as the unit vectors, but are not necessarily of unit length nor dimensionless. Consider a point of intersection of the level surfaces such that q_p^2 and q_p^3 are held fixed and only q^1 is allowed to vary. This defines the line of intersection of the 2-surface and the 3-surface which we can describe as a path $\mathbf{s}(1)$. A tangent vector to this path, $\mathbf{b}_1$ is obtained by differentiation,

$$\mathbf{b}_1 \equiv \frac{\partial \mathbf{s}(1)}{\partial q^1}.$$

The partial derivative indicates that q^2 and q^3 are being held constant. The faster q^1 changes in space along $s(1)$, the smaller will be the magnitude of $\mathbf{b}_1$. Two other generalized basis vectors, $\mathbf{b}_2$ and $\mathbf{b}_3$, are defined in a similar way.

In this context, $d\mathbf{s} = d\mathbf{s}(1)+d\mathbf{s}(2)+d\mathbf{s}(3)$ represents a differential change, $d\mathbf{r}$, to the position of a particle at position, $\mathbf{r}$. Thus, the partial derivatives imply that,

$$\frac{\partial \mathbf{r}}{\partial q^1} = \frac{\partial \mathbf{s}(1)}{\partial q^1},$$
$$\frac{\partial \mathbf{r}}{\partial q^2} = \frac{\partial \mathbf{s}(2)}{\partial q^2},$$
$$\frac{\partial \mathbf{r}}{\partial q^3} = \frac{\partial \mathbf{s}(3)}{\partial q^3}.$$

Therefore, we may write,

$$\mathbf{b}_i = \frac{\partial \mathbf{r}}{\partial q^i}.$$

If these new basis vectors are "admissible," then we may use them as a basis for expressing an arbitrary vector, $\mathbf{a}$,

$$\mathbf{a} = \mathbf{b}_1 A^1 + \mathbf{b}_2 A^2 + \mathbf{b}_3 A^3 = \mathbf{b}_i A^i.$$

The coefficients, A^i, are called the *contravariant components* of the vector $\mathbf{a}$. They are denoted with a superscript to establish a pattern for the Einstein summation convention: A repeated index, once as a subscript and once as a superscript, denotes summation. Unless otherwise indicated, this convention applies henceforth throughout this chapter.

2.1.6 Theorem

Theorem: $d\mathbf{s} = \mathbf{b}_i dq^i$.

Proof: Let $d\mathbf{s} = d\mathbf{s}(1) + d\mathbf{s}(2) + d\mathbf{s}(3)$ be the infinitesimal displacement from (q^1, q^2, q^3) to $(q^1 + dq^1, q^2 + dq^2, q^3 + dq^3)$. By the chain rule,

$$d\mathbf{s} = \frac{\partial \mathbf{s}(i)}{\partial q^i} dq^i.$$

(Sum on i.) But, by definition, $\mathbf{b}_i = \partial \mathbf{s}(i)/\partial q^i$, so $d\mathbf{s} = \mathbf{b}_i dq^i$. QED.

2.1.7 The Metric Tensor

The inner (dot) product of two arbitrary vectors, $\mathbf{a}$ and $\mathbf{c}$ may now be computed using the generalized basis vectors:

$$\mathbf{a} \cdot \mathbf{c} = (\mathbf{b}_i A^i) \cdot (\mathbf{b}_j C^j) = (\mathbf{b}_i \cdot \mathbf{b}_j) A^i C^j.$$

Thus, any inner product is characterized by the nine products, $\mathbf{b}_i \cdot \mathbf{b}_j$. These may be grouped together into a 3×3 matrix G called the *metric tensor* with components g_{ij} such that,

$$g_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j.$$

Observe that the matrix is symmetric, $g_{ij} = g_{ji}$. (A tensor is a generalization of a vector. Vectors and tensors are defined by their transformation patterns when one changes from one coordinate system to another. In our present application, the tensor property of the metric tensor is not important.)

2.1.8 Theorem

Theorem: $ds^2 = g_{ij} dq^i dq^j$.

Proof: Since we have shown that we may write $d\mathbf{s} = \mathbf{b}_i dq^i$, it follows that,

$$d\mathbf{s} \cdot d\mathbf{s} = ds^2 = (\mathbf{b}_i \cdot \mathbf{b}_j) dq^i dq^j = g_{ij} dq^i dq^j.$$

QED.

As a practical matter, we obtain the metric tensor by inspection of the form of ds^2. For example, an arbitrary displacement in spherical coordinates is written,

$$d\mathbf{s} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + (r\sin\theta)\hat{\boldsymbol{\phi}}.$$

Taking the dot product with itself, we have,

$$ds^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2.$$

Comparing to the form $ds^2 = g_{ij} dq^i dq^j$, we obtain, by inspection,

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

We are now in a position to establish the relationship between the unit vectors and our generalized basis vectors. The length of one of the basis vectors is obtained from the inner product of the vector with itself,

$$|\mathbf{b}_i| = \sqrt{\mathbf{b}_i \cdot \mathbf{b}_i} = \sqrt{g_{(ii)}}.$$

(No sum on (ii).) Since the unit vectors are already known to have the same direction as the basis vectors, we immediately have

$$\hat{\mathbf{e}}_i = \frac{\mathbf{b}_i}{|\mathbf{b}_i|} = \frac{\mathbf{b}_i}{\sqrt{g_{(ii)}}}.$$

2.1.9 Theorem

Theorem: $a_i = \sqrt{g_{(ii)}} A^i$. (No sum on (ii).)

Proof:

$$\mathbf{a} = \mathbf{b}_i A^i = (\sqrt{g_{(ii)}}\hat{\mathbf{e}}_i) A^i = (\sqrt{g_{(ii)}} A^i)\hat{\mathbf{e}}_i = a_i \hat{\mathbf{e}}_i.$$

Thus, $a_i = \sqrt{g_{(ii)}} A^i$. QED.

2.1.10 Reciprocal Basis Vectors

Set each of the generalized coordinates, $q^i(x_1, x_2, x_3) = q^i(x_i) = c$, where each constant, c, defines a different surface. The gradients of these functions of the Cartesian coordinates are vectors that are normal to the respective surfaces at the point (q_p^1, q_p^2, q_p^3), i.e.,

$$\nabla q^i = \mathbf{b}^i.$$

The vectors, $\mathbf{b}^i$, are called *reciprocal basis vectors.* Two sets of basis vectors, $\mathbf{b}_i$ and $\mathbf{b}^i$, are "reciprocal" if they satisfy the relationship, $\mathbf{b}_i \cdot \mathbf{b}^j = \delta_i^j$. This relationship says that $\mathbf{b}^1$ is orthogonal to both $\mathbf{b}_2$ and $\mathbf{b}_3$. It also says that if $\mathbf{b}^1$ has a large magnitude, then its reciprocal, $\mathbf{b}_1$, has a small magnitude. However, we must show that the reciprocal relationship is, indeed, satisfied.

2.1.11 Theorem

Theorem: $\mathbf{b}_i \cdot \mathbf{b}^j = \delta_i^j$.

Proof: The proof is a simple application of the chain rule. The generalized coordinates are independent of one another. Hence,

$$\mathbf{b}_i \cdot \mathbf{b}^j = \frac{\partial \mathbf{r}}{\partial q^i} \cdot \nabla q^j = \frac{\partial x^k}{\partial q^i} \frac{\partial q^j}{\partial x^k} = \frac{\partial q^j}{\partial q^i} = \delta_i^j.$$

QED.

An arbitrary vector, $\mathbf{a}$, can be written in terms of reciprocal basis vectors just as well as it can be written in terms of unit vectors,

$$\mathbf{a} = A_1 \mathbf{b}^1 + A_2 \mathbf{b}^2 + A_3 \mathbf{b}^3 = A_i \mathbf{b}^i.$$

The set of numbers, A_i, are called the *covariant components* of $\mathbf{a}$. The reciprocal basis vectors do not necessarily have unit length nor are they necessarily parallel to the unit vectors, but they must be linearly independent if the coordinate system is to be admissible. The covariant components of $\mathbf{a}$ are not the same as the physical components of $\mathbf{a}$ nor are they the same as the contravariant components.

2.1.12 Theorem

Theorem: $A^i = \mathbf{b}^i \cdot \mathbf{a}$.

Proof: We use the relationship, $\mathbf{b}^i \cdot \mathbf{b}_j = \delta^i_j$.

$$\mathbf{b}^i \cdot \mathbf{a} = \mathbf{b}^i \cdot (A^j \mathbf{b}_j) = (\mathbf{b}^i \cdot \mathbf{b}_j) A^j = \delta^i_j A^j = A^i.$$

QED. In similar fashion, we can also show $A_i = \mathbf{b}_i \cdot \mathbf{a}$.

Define $g^{ij} \equiv \mathbf{b}^i \cdot \mathbf{b}^j$. By doing so, we maintain a parallelism between the reciprocal basis vectors and the basis vectors. The elements, g^{ij}, form the matrix G^{-1} and are called the contravariant components of the metric tensor to distinguish them from the covariant components of the metric tensor, g_{ij}. Observe that G^{-1}, like G, is symmetric, i.e. $g^{ij} = g^{ji}$.

The covariant and contravariant forms of the metric tensor are useful in relating covariant and contravariant components of vectors. Note that the pattern of notation is consistent: an index repeated once as a subscript and once as a superscript denotes summation.

2.1.13 Theorem

Theorem: $A_i = g_{ij} A^j$.

Proof:

$$A_i = \mathbf{b}_i \cdot \mathbf{a} = \mathbf{b}_i \cdot (\mathbf{b}_j A^j) = (\mathbf{b}_i \cdot \mathbf{b}_j) A^j = g_{ij} A^j.$$

QED. This important operation is called "lowering the index."

2.1.14 Theorem

Theorem: $A^i = g^{ij} A_j$.

Proof:

$$A^i = \mathbf{b}^i \cdot \mathbf{a} = \mathbf{b}^i \cdot (\mathbf{b}^j A_j) = (\mathbf{b}^i \cdot \mathbf{b}^j) A_j = g^{ij} A_j.$$

QED. This important operation is called "raising the index."

The covariant and contravariant forms of the metric tensor are inverses of each other,

$$G^{-1} G = I,$$

or,

$$g_{ij} g^{jk} = \delta^k_i.$$

If you apply the combination $G^{-1}G$ in succession to an arbitrary vector, G will first "lower the index" and G^{-1} will turn around and "raise the index," returning you to where you began, so that the combination of the two, one after the other, acts exactly as the identity operation. In practice, this property is used to find the contravariant components of the metric tensor after the covariant components are extracted from the form, $ds^2 = g_{ij}dq^i dq^j$.

2.1.15 Theorem

Theorem: $\mathbf{b}_i = g_{ik}\mathbf{b}^k$.

Proof:

1. In an admissible generalized coordinate system, the reciprocal basis vectors must form a linearly independent set of vectors. Thus, the set of basis vectors, $\mathbf{b}_i$, can be expressed as a linear combination of the reciprocal basis vectors,
$$\mathbf{b}_i = a_{ij}\mathbf{b}^j.$$

2. We have previously established that,
$$g_{ik} = g_{ki} = \mathbf{b}_k \cdot \mathbf{b}_i$$
so,
$$\begin{aligned} g_{ik} &= \mathbf{b}_k \cdot (a_{ij}\mathbf{b}^j) = a_{ij}(\mathbf{b}_k \cdot \mathbf{b}^j) \\ &= a_{ij}\delta^j_k = a_{ik}. \end{aligned}$$

3. We conclude,
$$\mathbf{b}_i = g_{ij}\mathbf{b}^j.$$

QED.

2.1.16 Theorem

Theorem: $\mathbf{b}^i = g^{ij}\mathbf{b}_j$.

Proof:

1. We have established that,
$$g_{ij}g^{kj} = \delta^k_i$$
$$\mathbf{b}_i = g_{ij}\mathbf{b}^j.$$

2. Hence, multiplying both sides of the latter by g^{ki} (and forming a sum indicated by the repeated index, i), we have,

$$g^{ki}\mathbf{b}_i = g^{ki}(g_{ij}\mathbf{b}^j) = \delta^k_j\mathbf{b}^j = \mathbf{b}^k.$$

3. We conclude,

$$\mathbf{b}^i = g^{ij}\mathbf{b}_j.$$

   QED.

2.1.17 Theorem

Theorem: $\partial\mathbf{b}_i/\partial q^j = \partial\mathbf{b}_j/\partial q^i$.

Proof: This symmetry of pattern merely reflects the fact that the order in which partial derivatives are taken does not matter:

$$\frac{\partial\mathbf{b}_i}{\partial q^j} = \frac{\partial}{\partial q^j}\frac{\partial\mathbf{r}}{\partial q^i} = \frac{\partial}{\partial q^i}\frac{\partial\mathbf{r}}{\partial q^j} = \frac{\partial\mathbf{b}_j}{\partial q^i}.$$

QED.

Since each of these nine partial derivatives, $\partial\mathbf{b}_i/\partial q^j$ are themselves vectors, we can express each as some linear combination of either the unit vectors, reciprocal basis vectors or basis vectors. We choose to express them as a linear combination of basis vectors,

$$\frac{\partial\mathbf{b}_i}{\partial q^j} = \Gamma^k_{ij}\mathbf{b}_k.$$

The numbers which form the coefficients of the sum generated by the dummy index k are called Christoffel symbols of the first kind.

2.1.18 Theorem

Theorem:

$$\Gamma^l_{ij} = g^{lk}\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

Proof:

1. We have, by definition,

$$\mathbf{b}_i \cdot \mathbf{b}_k = g_{ik}$$

$$\mathbf{b}_j \cdot \mathbf{b}_k = g_{jk}$$

$$\mathbf{b}_i \cdot \mathbf{b}_j = g_{ij}.$$

2. Differentiate each of these expressions in turn to form,

$$\frac{\partial g_{ik}}{\partial q^j} = \mathbf{b}_i \cdot \frac{\partial \mathbf{b}_k}{\partial q^j} + \frac{\partial \mathbf{b}_i}{\partial q^j} \cdot \mathbf{b}_k$$

$$\frac{\partial g_{jk}}{\partial q^i} = \mathbf{b}_j \cdot \frac{\partial \mathbf{b}_k}{\partial q^i} + \frac{\partial \mathbf{b}_j}{\partial q^i} \cdot \mathbf{b}_k$$

$$\frac{\partial g_{ij}}{\partial q^k} = \mathbf{b}_i \cdot \frac{\partial \mathbf{b}_j}{\partial q^k} + \frac{\partial \mathbf{b}_i}{\partial q^k} \cdot \mathbf{b}_j$$

and form the combination,

$$\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}.$$

3. We have proved as an earlier theorem that

$$\frac{\partial \mathbf{b}_k}{\partial q^j} = \frac{\partial \mathbf{b}_j}{\partial q^k}$$

and,

$$\frac{\partial \mathbf{b}_k}{\partial q^i} = \frac{\partial \mathbf{b}_i}{\partial q^k}.$$

Thus, when we form the combination

$$\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k},$$

four terms add out and two combine. We are left with the result,

$$\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} = 2\frac{\partial \mathbf{b}_i}{\partial q^j} \cdot \mathbf{b}_k.$$

4. We may now use our assumption that

$$\frac{\partial \mathbf{b}_i}{\partial q^j} = \Gamma^l_{ij}\mathbf{b}_l.$$

Then,

$$\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} = 2[\Gamma^l_{ij}\mathbf{b}_l] \cdot \mathbf{b}_k = 2\Gamma^l_{ij}(\mathbf{b}_l \cdot \mathbf{b}_k) = 2\Gamma^l_{ij} g_{lk}.$$

5. It is tempting to solve for Γ^l_{ij} by dividing both sides of this expression by $2g_{lk}$, but one must remember that the repeated index l indicates a sum of terms, not an isolated term. In fact, what we are dealing with here are twenty-seven equations, one for each of the combinations of i, j, and k! The correct way to proceed is to multiply both sides by g^{mk}, thus introducing an additional sum indicated by the dummy index k. We then use a result that we have already proved,

$$g^{mk} g_{lk} = \delta^m_l.$$

Thus,

$$2g^{mk}\Gamma^l_{ij} g_{lk} = 2\delta^m_l \Gamma^l_{ij} = 2\Gamma^m_{ij}.$$

6. We conclude,

$$\Gamma^m_{ij} = g^{mk}\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right).$$

QED.

In principle, once one knows the covariant and contravariant forms of the metric tensor, the Christoffel symbols can be calculated. In three dimensional space there are twenty-seven of them. You will be relieved to know that for our purposes this is a formal result essential to the derivation of Lagrange's equations that follow, but not one that we will use as a practical tool.

2.2 Application to Acceleration

The mathematics we have introduced can now be extended in several directions. It was used by Albert Einstein as the original language of his General Theory of Relativity and a simple modification of it is also the basis of the most elegant formalism of the Special Theory of Relativity (see Chapter 10). The mathematical formalism can also be used to give general expressions for the divergence, curl and Laplacian in generalized coordinates. However, our immediate purpose is to apply the formalism to a particular vector quantity, the acceleration of a particle, and thus to reformulate Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, into a new form. The advantage of this new form is that it is done in terms of generalized coordinates, so that the reader gets to choose the most convenient coordinates for a problem without undue concern about how to write down acceleration properly in terms of the coordinates that have been chosen. Newton's Second Law is a vector equation which, when resolved into components, becomes a set of ordinary differential equations. The equations which result from the reformulation of Newton's Second Law are called Lagrange's Equations, but they are completely equivalent ordinary differential equations which describe the motion of a particle. Our immediate purpose is to show the connection between Newton's Second Law and Lagrange's equations. We shall first find the contravariant components of acceleration, then "lower the index" to find the covariant components. Finally, we will write the covariant form of Newton's Second Law and show that for a large class of problems, the covariant components of Newton's Second Law are Lagrange's equations.

2.2.1 Contravariant Components of Acceleration

(a) We begin with the expression for an arbitrary displacement: $d\mathbf{s} = dq^i\mathbf{b}_i$.

(b) We divide both sides by dt. Observe that this is a division and not the process of differentiation, but the outcome is to turn $d\mathbf{s}$ into a velocity,

$$\mathbf{v} = \frac{d\mathbf{s}}{dt} = \dot{q}^i\mathbf{b}_i.$$

(c) To obtain acceleration, we must differentiate $\mathbf{v}$ with respect to time, but we must note that as the particle moves, the basis vectors change with its position and must therefore be considered functions of time. Hence,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{q}^i \mathbf{b}_i + \dot{q}^i \frac{d\mathbf{b}_i}{dt}.$$

(d) To compute $d\mathbf{b}_i/dt$, we can apply the chain rule,

$$\frac{d\mathbf{b}_i}{dt} = \frac{\partial \mathbf{b}_i}{\partial q^j}\frac{dq^j}{dt} = \dot{q}^j \frac{\partial \mathbf{b}_i}{\partial q^j}.$$

From this, we conclude,

$$\mathbf{a} = \ddot{q}^i \mathbf{b}_i + \dot{q}^i \dot{q}^j \frac{\partial \mathbf{b}_i}{\partial q^j} = \ddot{q}^i \mathbf{b}_i + \dot{q}^i \dot{q}^j \Gamma^k_{ij} \mathbf{b}_k,$$

or,

$$\mathbf{a} = [\ddot{q}^k + \dot{q}^i \dot{q}^j \Gamma^k_{ij}]\mathbf{b}_k = A^k \mathbf{b}_k.$$

From this expression, we identify the contravariant components of acceleration as,

$$A^k = \ddot{q}^k + \dot{q}^i \dot{q}^j \Gamma^k_{ij}.$$

2.2.2 Covariant Components of Acceleration

1. To obtain the covariant components of acceleration, we lower the index using $A_l = g_{lk} A^k$.

$$A_l = g_{lk} A^k = g_{lk} \ddot{q}^k + g_{lk} \Gamma^k_{ij} \dot{q}^i \dot{q}^j.$$

2. We may use the expression which we have just derived for the Christoffel symbols to write,

$$g_{lk}\Gamma^k_{ij} = \frac{1}{2} g_{lk} g^{kn} \left(\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^n} \right).$$

Observe,

$$g_{lk} g^{kn} = \delta^n_l,$$

so that we can write,

$$g_{lk}\Gamma^k_{ij} = \frac{1}{2}\delta^n_l\left(\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^n}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^l}\right).$$

3. Thus,

$$A_l = g_{lk}\ddot{q}^k + \frac{1}{2}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^l}\right)\dot{q}^i\dot{q}^j.$$

In principle, the covariant components of the acceleration can be obtained from this expression once the covariant form of the metric tensor is known. Again, this is a formal result that is essential to our derivation, but not one that we will use as a practical tool.

2.2.3 A Better Form for the Covariant Components of Acceleration

We now begin to change the form of the covariant components of acceleration to put them into a form that is a practical tool. Consider the first term in the expression we have just derived for A_l, namely,

$$g_{lk}\ddot{q}^k.$$

Observe that this term arises in the expression,

$$\frac{d}{dt}(g_{lk}\dot{q}^k) = g_{lk}\ddot{q}^k + \frac{\partial g_{lk}}{\partial q^m}\dot{q}^m\dot{q}^k,$$

where we have used the chain rule in the second term on the right to obtain

$$\frac{d}{dt}(g_{lk}) = \frac{\partial g_{lk}}{\partial q^m}\dot{q}^m.$$

A renaming of dummy indices (in the second term below) immediately yields,

$$\frac{d}{dt}(g_{lk}\dot{q}^k) = g_{lk}\ddot{q}^k + \frac{\partial g_{lj}}{\partial q^i}\dot{q}^i\dot{q}^j.$$

We may now write,

$$A_l = \frac{d}{dt}(g_{lk}\dot{q}^k) - \frac{\partial g_{lj}}{\partial q^i}\dot{q}^i\dot{q}^j$$

$$+\frac{1}{2}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^l}\right)\dot{q}^i\dot{q}^j.$$

Observe that by simply switching dummy indices and using the fact that $g_{il} = g_{li}$, we can show that

$$\frac{\partial g_{lj}}{\partial q^i}\dot{q}^i\dot{q}^j = \frac{\partial g_{il}}{\partial q^j}\dot{q}^j\dot{q}^i = \frac{\partial g_{jl}}{\partial q^i}\dot{q}^i\dot{q}^j.$$

These equalities may be used to eliminate the middle three terms and to reduce our expression for A_l to

$$A_l = \frac{d}{dt}(g_{lk}\dot{q}^k) - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^l}\dot{q}^i\dot{q}^j.$$

2.2.4 Newton's Second Law in Covariant Form

Finally, we arrive at the point of our entire excursion into differential geometry! The covariant components of Newton's Second Law are $F_k = mA_k$. In some texts, the covariant components of force, F_k, are called generalized forces. In terms of physical components, of course, Newton's Second Law is the familiar $f_k = ma_k$.

1. Observe that

   $$ds^2 = g_{ij}dq^i dq^j$$

   and that, therefore, we may write the kinetic energy, T, of a particle as,

   $$T = \frac{1}{2}m(\frac{ds}{dt})^2 = \frac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j.$$

2. Observe further that if we take the view of generalized coordinates and generalized velocities as independent variables for purposes of taking partial derivatives,

   $$\frac{\partial T}{\partial q^k} = \frac{1}{2}m\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j,$$

   $$\frac{\partial T}{\partial \dot{q}^k} = mg_{ik}\dot{q}^i,$$

and,

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k}) = m\frac{d}{dt}(g_{ik}\dot{q}^i) = m\frac{d}{dt}(g_{ki}\dot{q}^i)$$

(The factor of one-half disappears in the latter two expressions because there is a double sum in the expression for T. If you have trouble seeing this, you should write out a short double sum so that you see it works.)

3. We may therefore write,

$$F_l = mA_l = m\frac{d}{dt}(g_{lk}\dot{q}^k) - \frac{1}{2}m\frac{\partial g_{ij}}{\partial q^l}\dot{q}^i\dot{q}^j = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^l}) - \frac{\partial T}{\partial q^l}.$$

Newton's Second Law, written in covariant form is,

$$F_l = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^l}) - \frac{\partial T}{\partial q^l}.$$

This is a perfectly general result. The primary difficulty with it is knowing how to write down F_l.

2.2.5 Lagrange's Equations

If we narrow ourselves to a class of problems that satisfies two additional conditions we arrive at a remarkable result. If the system we are describing is "conservative" so that we may define a potential energy function, V, such that

$$F_l = -\frac{\partial V}{\partial q^l}$$

and V depends only on the generalized coordinates and not on the generalized velocities, $\dot{q}^i$, then we may simplify the covariant form of Newton's Second Law. We may write,

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^l}) - \frac{\partial T}{\partial q^l} = -\frac{\partial V}{\partial q^l}$$

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}^l} - \frac{\partial (T-V)}{\partial q^l} = 0.$$

(Since V does not depend on the generalized velocities, it may formally be included in the first term without making any difference.) We now define the so-called "Lagrangian", L,

$$L \equiv T - V$$

and write

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^l}) - \frac{\partial L}{\partial q^l} = 0.$$

These are Lagrange's equations and they are very important in theoretical mechanics.

1. Lagrange's equations are just the components of Newton's Second Law for conservative systems written in a different and probably unfamiliar form.

2. The equations that are obtained by operating on the Lagrangian according to the prescription of Lagrange's equations are ordinary differential equations. They are the "equations of motion" of the system. Solutions of the equations are models of the motion of the system.

3. Much of the intervening machinery used to derive the equations disappears and does not return. For example, we do not directly use (for our purposes) $\mathbf{b}_i$, $\mathbf{b}^i$, Γ^k_{ij}, or g^{ij}. These were introduced in order to derive Lagrange's equations and to show their connection to Newton's Second Law.

4. T and V are usually easy to write down correctly and once written down lead to the differential equations of the system in a very methodical way.

2.3 Problems

1. Do the following:

   - Draw a diagram (in two dimensions) of a particle whose position is specified by plane polar coordinates (r, θ). Show the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ on your diagram and clearly label them.

   - Express an arbitrary differential displacement $d\mathbf{s}$ in terms of these coordinates and show this displacement on your diagram. Find $ds^2 = d\mathbf{s} \cdot d\mathbf{s}$ in terms of dr and $d\theta$ and extract from ds^2 the metric tensor g_{ij}. Display the metric tensor as a 2×2 matrix. Using ds^2, write down the kinetic energy of the particle in terms of r and θ and their time derivatives.

- Determine g^{ij} and display as a 2×2 matrix.
- Determine the covariant, contravariant and physical components of the acceleration of the particle. Show how these might appear on a diagram. (The diagram is only to be qualitative, i.e. it need only indicate directions of the vectors and whether the vectors have unit length or not.)
- Let a particle move on a circle of constant radius b. What is the radial component of acceleration (physical)? What is the transverse component?
- A bug crawls outward with constant speed v_0 along the spoke of a wheel which is rotating with constant angular speed ω. Find the radial and transverse components of the physical acceleration as functions of time. Assume $r = 0$ at $t = 0$. (Ans: $a_r = -v_0 t\omega^2$, $a_\theta = 2v_0\omega$)

2. Cylindrical coordinates are defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Calculate the metric tensor, physical velocity and physical acceleration in cylindrical coordinates.

3. Do the following:

   - Beginning with Newton's Second Law,

   $$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k}) - \frac{\partial T}{\partial q^k} = F_k,$$

   show that if F_k can be expressed in the form,

   $$F_k = -\frac{\partial V}{\partial q^k}(q^1, q^2, q^3),$$

   then,

   $$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^k}) - \frac{\partial L}{\partial q^k} = 0$$

   - For the one-dimensional harmonic oscillator, we can write $F_x = -kx$. Show that $V = \frac{1}{2}kx^2$. Define the function $L = T - V$ and obtain the differential equation of the simple harmonic oscillator, $m\ddot{x} + kx = 0$.

4. Obtain the differential equations of motion for a tether ball. (Spherical coordinates. Observe that the tether line has constant length. The system has only two degrees of freedom and there will only be two differential equations of motion. The length of the tether appears as a constant in the Lagrangian.)

5. Obtain the differential equations of motion for a projectile on a flat earth. (Cartesian coordinates)

6. Obtain the differential equations of motion for the earth's motion about the sun. (Plane polar coordinates)

7. Obtain the differential equations of motion for a particle moving without friction on the inside of a cone. The cone is parameterized in cylindrical coordinates as $z = \alpha r$. Write the equations in terms of r, $\dot{r}$ and $\ddot{r}$ by eliminating z.

8. Obtain the differential equations of motion for a particle sliding without friction from the top of a hemispherical dome. (Plane polar coordinates. The particle does not separate from the dome. The system has only one degree of freedom since the radius of the motion appears as a constant in the Lagrangian.)

9. Consider an inclined plane consisting of a right-triangular shaped block that is free to slide on its bottom along a frictionless horizontal surface. If the incline slopes toward the right, the left-hand edge of the block is vertical. Define a coordinate X that keeps track of the position of this vertical side. Now, put another block on the inclined plane so that it slides down the plane (without friction) under the influence of gravity. Keep track of the second block's position with a coordinate x measured from the top of the inclined plane. Obtain the differential equations of the motions of both blocks and solve them directly for the accelerations $(\ddot{x}, \dot{X})$ of each. (Unusual, nonorthogal coordinates. This problem introduces a problem not found in the previous ones. Here there are two masses. In such a case, the Lagrangian is written simply as the sum of the the two independent kinetic energies of the two objects minus the sum of the two independent potential energies. The system has two degrees of freedom.)

Chapter 3

Differential Equations

3.1 Solutions to differential equations

Newton's Second Law of motion in one dimension, $F = m\ddot{x}$, is a second order differential equation. In the simplest cases or in the lowest order of approximation, this equation will sometimes reduce to a linear equation. Linear equations are the most studied of differential equations and have the most systematic methods for solution. Hence, textbook examples typically center on problems that can be reduced to linear equations.

In those problems where the force can be derived as the derivative of a potential function, $F_x = -dV/dx$, the potential may often be such that it can be expanded in a Taylor's series, usually about some minimum point. If the series converges rapidly enough, it can be truncated after one or a few terms. If the series takes the form:

$$V(x) = V_0 + (dV/dx)_{x0}(x - x_0) + (1/2)(d^2V/dx^2)_{x0}(x - x_0)^2 + \cdots$$

then, we get force terms of the type

$$F = k_1(x - x_0) + k_2(x - x_0)^2 + \cdots$$

which, in turn, lead to a non-linear differential equation of the form

$$\ddot{x} + \alpha x + \beta x^2 + \gamma x^3 + \cdots = 0.$$

If all but the linear term (αx) are discarded, one has the equation for the simple harmonic oscillator. Thus, the simple harmonic oscillator sometimes becomes the simplest approximation to a more general problem.

In two- and three-dimensional problems, constants of the motion, such as the total energy or a component of angular momentum, may be used to reduce a multiple-dimensioned problem to a one-dimensional differential equation of the type derived above. For example, in the case of periodic motion under a central force, one can convert from a time-dependent differential equation $(\ddot{r}, \dot{r}, \ddot{\theta}, \dot{\theta})$ to an orbit equation relating r and θ (in plane polar coordinates). We use the substitutions:

$$x = 1/r,$$

$$\dot{x} = (dx/d\theta)\dot{\theta} = -(1/r^2)\dot{r}$$

and use the constant of the motion,

$$\ell = mr^2\dot{\theta}.$$

We then have,

$$\dot{r} = -(\ell/m)(dx/d\theta)$$

and

$$\ddot{r} = -(\ell^2 x^2/m^2)(d^2x/d\theta^2)$$

The resulting equation may be a one-dimensional equation that can be solved using the methods described here.

3.2 Equations that may be either linear or non-linear

3.2.1 Equation: $F(x) = m\ddot{x}$ where F is only a function of position and may be derived from a potential function, $F(x) = -dV/dx$.

Method of solution: Direct integration, separation of variables

Solution: Use the chain rule to write $\ddot{x} = d\dot{x}/dt = (d\dot{x}/dx)(dx/dt) = (dv/dx)v = 1/2d(v^2)/dx$. Then,

$$F(x) = d(1/2mv^2)/dx = -dV/dx.$$

Both sides are perfect differentials, so $T = -V + constant$. This is *energy conservation*, $E = T + V$. From energy conservation, it may be possible to solve for $v = dx/dt = f(x)$ or invert to solve for $x = g(v)$.

If one can write $dx/dt = f(x)$, then,

$$t(x) = \int (1/f(x))dx.$$

In conservative systems, this will take the form,

$$t = (m/2)^{1/2} \int (E - V(x))^{-1/2} dx.$$

3.2.2 Equation: $F(t) = m\ddot{x}$ where F is only a function of time.

Method of solution: Direct integration, separation of variables

Solution:

$$F(t) = m(dv/dt)$$

$$v(t) = (1/m) \int F(t)dt$$

$$x(t) = \int v(t)dt$$

3.2.3 Equation: $F(v) = m\ddot{x}$ where F is only a function of velocity.

Method of solution: Direct integration, separation of variables

Solution: Alternative 1:

$$t = m \int (1/F(v))dv$$

We may then possibly be able to solve for v(t) and integrate again: $x(t) = \int v(t)dt$

Alternative 2:

$$\ddot{x} = (dv/dt) = (dv/dx)(dx/dt) = v(dv/dx)$$

$$x(v) = m \int (v/F(v))dv$$

We may the possibly be able to solve for $v(x) = dx/dt$. Then

$$t(x) = \int (1/v(x))dx$$

We may then possibly be able to invert the expression for $t(x)$ to solve for $x(t)$.

3.3 Linear Differential Equations

The most general *linear* first-order differential equation is:

3.3.1 Equation: $\frac{dy}{dx} + g(x)y = h(x)$.

Method of solution: Integrating factor.

Solution: Observe that

$$\frac{d}{dx}(ye^{\int g(x')dx'}) = (\frac{dy}{dx} + yg(x))e^{\int g(x')dx'}.$$

Thus, multiply the given differential equation by the factor (called the *integrating factor*),

$$e^{\int g(x')dx'},$$

which turns the left-hand-side of the equation into a perfect differential that can be integrated immediately. Then,

$$\frac{d}{dx}(ye^{\int g(x')dx'}) = h(x)e^{\int g(x')dx'}.$$

Integrating both sides, we have,

$$ye^{\int g(x')dx'} = \int h(x)e^{\int g(x')dx'}dx + C,$$

where C is the constant of integration. From this expression, one may solve for $y(x)$ if $g(x)$ can be integrated. Formally,

$$y(x) = e^{-\int g(x')dx'}(\int h(x)e^{\int g(x')dx'}dx + C).$$

However, one should *not* remember the formula itself as a solution. Rather, an equation of this type should be solved by actually carrying out the steps just illustrated,

1. Compute the integrating factor.
2. Multiply the differential equation by the integrating factor.
3. Integrate both sides of the equation, remembering that the left-hand-side is a perfect differential.
4. Solve the integrated equation for $y(x)$.

The first-order equation may occur in mechanics in the following way. If a frictional force proportional to the square of velocity exists, Newton's Second Law may take the form,

$$m\ddot{x} = h(x) - g(x)\dot{x}^2.$$

Observe that

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = m\dot{x}\frac{d\dot{x}}{dx} = m\dot{x}\frac{d\dot{x}}{dt}\frac{dt}{dx} = m\frac{\dot{x}}{\dot{x}}\ddot{x} = m\ddot{x}.$$

Thus,

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = h(x) - g(x)\dot{x}^2.$$

If we now define $y \equiv \dot{x}^2$, we have,

$$\frac{dy}{dx} + \frac{2g(x)}{m}y = \frac{2h(x)}{m}$$

which is of the form of interest. The solution yields velocity as a function of position, but only for problems in which the frictional force is proportion to the square of speed.

The most general *linear* second-order differential equation is:

$$\ddot{x} + f(t)\dot{x} + g(t)x = h(t)$$

3.3.2 Equation: $m\ddot{x} = constant$ (freely falling particle)

Method of solution: Direct integration

Solution: If the constant is -mg (freely falling particle)

$$\dot{x} = \dot{x}_0 - gt$$

$$x = x_0 + \dot{x}_0 t - (1/2)gt^2$$

3.3.3 Equation: $m\ddot{x} + kx = 0$ (harmonic oscillator)

Method of solution: Substitution of the form Ae^{mt}

Solution:

$$x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = A\cos(\omega_0 t - \alpha)$$

where $\omega_0 = \sqrt{(k/m)}$ and c_1 and c_2 or A and α are determined from initial conditions.

3.3.4 Equation: $m\ddot{x} + b\dot{x} = -mg = constant$ (particle falling in a resistive medium)

Method of solution: Integrating factor

Solution:

$$\dot{x} = (mg/b)[e^{-bt/m}(1 + b\dot{x}_0/mg) - 1]$$

$$x = x_0 + (mg/b)[(m/b)(1 - e^{-bt/m})(1 + b\dot{x}_0/mg) - t]$$

3.3.5 Equation: $m\ddot{x} + b\dot{x} + kx = 0$ (particle oscillating in resistive medium)

Method of solution: Substitution of general form, Ae^{mt}

Solution: The general solution is:

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$$

where

$$\alpha_1 = -(b/2m) + \sqrt{(b/2m)^2 - k/m}$$

$$\alpha_2 = -(b/2m) - \sqrt{(b/2m)^2 - k/m}$$

In those cases where α_1 and α_2 are complex numbers, the solution is oscillatory, but damped. It may then be written,

$$x = e^{-(b/2m)t} A\cos(\omega_1 t - \alpha)$$

where

$$\omega_1 = \sqrt{k/m}\sqrt{(1 - b^2/4mk)}$$

3.3.6 Equation: $m\ddot{x} + kx = F_0 \cos(\omega t)$ (driven harmonic oscillator)

Method of solution: Substitute a trial solution of the form $A\cos(\omega t)$

Solution: The solution is the sum of a particular solution plus the solution to the homogeneous equation (right hand side set to 0 , see above). The solution to the homogeneous equation is called the *transient solution*. The transient solution will contain two constants that are determined by the initial conditions.

The particular solution is of the form: $A\cos(\omega t)$. Direct substitution leads to the following:

$$\dot{x} = -[(F_0\omega)/m(\omega_0^2 - \omega^2)]\sin(\omega t)$$

$$x = [(F_0)/m(\omega_0^2 - \omega^2)]\cos(\omega t)$$

where $\omega_0^2 = k/m$.

3.3.7 Equation: $m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$ (damped, driven harmonic oscillator)

(Also, $m\ddot{y} + b\dot{y} + ky = F_0 \sin(\omega t)$).

Method of solution: These two equations can be combined and solved simultaneously by defining the complex number $z = x + iy$. By multiplying the second of the two by $i = \sqrt{-1}$ and adding, we form the complex equation,

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\omega t}$$

After solving for z we can recover the solutions for x and y as the real and imaginary parts of z.

The solution is the sum of a particular solution plus the solution to the homogeneous equation (right-hand side set to 0, see above). The solution to the homogeneous equation is called the transient solution. The transient solution will contain two constants that are determined by initial conditions.

Solution: The particular solution is of the form:

$$z = Be^{i(\omega t - \beta)}$$

where,

$$B = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 b^2)}},$$

$$tan\beta = \frac{(\omega b)}{m(\omega_0^2 - \omega^2)},$$

and, $\omega_0^2 = k/m$.

3.3.8 Equation: $m\ddot{x} + b\dot{x} + kx = a_0 + \sum F_n \cos(n\omega t - \alpha_n)$ (damped, periodically driven harmonic oscillator)

Method of solution: The right-hand side of the equation is a Fourier series representation of a periodic driving force. The constants a_0, F_n, and α_n are assumed known from the Fourier series. A complex number $z = x + iy$ is defined and the equation above is replaced with an equation in z. The solution for x will be the real part of the solution for z. The equation becomes:

$$m\ddot{z} + b\dot{z} + kz = a_0 + \sum F_n e^{i(n\omega t - \alpha_n)}$$

The solution is the sum of a particular solution plus the solution to the homogeneous equation (right-hand side set to 0, see above). The solution to the homogeneous equation is called the transient solution. The transient solution will contain two constants that are determined by initial conditions.

We proceed to obtain the particular solution by substituting a trial solution of the form,

$$z = B_0 + \sum B_n e^{i(n\omega t - \gamma_n)}$$

Solution:

$$x = (a_0/k) + \sum B_n \cos(n\omega t - \gamma_n)$$

where,

$$B_n = \frac{F_n}{\sqrt{m^2(\omega_0^2 - n^2\omega^2)^2 + \omega^2 n^2 b^2}}$$

and,

$$tan\beta_n = \frac{bn\omega}{m(\omega_0^2 - n^2\omega^2)}$$

and,

$$\gamma_n = \alpha_n + \beta_n$$

and $\omega_0^2 = k/m$.

The solution is simply a sum of the solutions that would result if each term in the Fourier series expansion of the driving force were present alone!

This important *superposition principle* is valid for systems which obey linear differential equations. The superposition principle breaks down for non-linear differential equations.

3.3.9 Equation: $g_{ij}\ddot{q}^j + k_{ij}q^j = 0$ (multi-particle, harmonic oscillation)

Method of solution: If the kinetic energy and potential energy in a conservative, multi-particle system can be expressed as:

$$T = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$$

$$V = \frac{1}{2}k_{ij}q^iq^j$$

then the above differential equations follow. The equations can also be written in matrix form,

$$G\ddot{Q} + KQ = 0$$

A trial solution of the form $Q = X\cos(\omega t - \alpha)$ can be substituted. Q and X are column vectors. Solutions of the equation,

$$det(K - \omega^2 G) = 0$$

(the *secular equation*) yields the *eigenfrequencies*, ω_i. When these are substituted one by one into the matrix equation,

$$(K - \omega^2 G)X = 0$$

one obtains the individual *eigenvectors*, X_i.

Solution:

$$Q = a_1X_1\cos(\omega_1 t - \alpha_1) + \cdots + a_fX_f\cos(\omega_f t - \alpha_f)$$

where $a_i,\ \alpha_i$ are determined by initial conditions.

3.4 Nonlinear Differential Equations

3.4.1 Equation: $\ddot{x} + \alpha x + \gamma x^3 = 0$

Method of solution: If,

1. force on the particle is derivable from a potential, V(x)
2. V(x) is symmetric about the Taylor's series point of expansion,
3. the motion is periodic,
4. $\dot{x} = 0$ at $t = 0$

then, the lowest-order, non-linear approximation is $\ddot{x} + \alpha x + \gamma x^3 = 0$.

Since we assume that the solution will be periodic in time, we use symmetry to eliminate terms in the solution of the type $\sin(\omega t)$ and assume a solution of the form,

$$x(t) = b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \cdots$$

We assume that b_3 is smaller than b_1 and discard any terms of higher order. We substitute the form into the differential equation and gather the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$ and separately set them to zero.

Solution:

$$x(t) \approx b_1 \cos(\omega t - \phi) + (\gamma b_1^3/32\alpha) \cos(3\omega t - 3\phi)$$

$$\omega^2 \approx \alpha + (3/4)\gamma b_1^2$$

b_1 and ϕ are determined from initial conditions.

3.4.2 Equation: $\ddot{x} + \alpha x - \beta x^2 + \gamma x^3 = 0$

Method of solution:

If,

1. force on the particle is derivable from a potential, V(x),
2. V(x) is asymmetric about the Taylor's series point of expansion,
3. the motion is periodic,

4. $\dot{x} = 0$ at $t = 0$,

5. $\beta > 0$.

then, the third-order, non-linear approximation is $\ddot{x} + \alpha x - \beta x^2 + \gamma x^3 = 0$

Since we assume that the solution will be periodic in time, we use symmetry to eliminate terms of the type $\sin(n\omega t)$ and assume a solution of the form,

$$x(t) = b_0 + b_1 \cos(\omega t) + b_2 \cos(2\omega t) + b_3 \cos(3\omega t) + ...$$

We have assumed that the oscillation assumes its maximum amplitude at $t = 0$. We assume that $b_1 > b_2 > b_3$ and discard any terms of higher order. We substitute the form into the differential equation, gather the coefficients of $\cos(\omega t)$, $\cos(2\omega t)$, $\cos(3\omega t)$ and separately set them to zero.

Solution:

$$x(t) \approx (\beta b_1^2/2\alpha) + b_1 \cos(\omega t) - (\beta b_1^2/6\alpha) \cos(2\omega t) + ((\gamma b_1^3/32\alpha)$$

$$+(\beta^2 b_1^3/48\alpha^2)) \cos(3\omega t)$$

where,

$$\omega^2 \approx \alpha + (3/4)\gamma b_1^2 - (5\beta^2 b_1^2/6\alpha)$$

3.4.3 Equation: $\ddot{x} + \alpha x \pm \gamma x^3 = 0$

Method of solution:

If,

1. force is derivable from a potential, V(x),

2. and if, the motion is periodic,

3. and if, the potential is symmetric, let $x = Ay(at)$

4. or if, the potential is asymmetric, let $x = B + Ay(at)$

Solution: The resulting equation may possibly be put into one of the forms for which Jacobian elliptic functions are known solutions.

1. $(y')^2 = (1-y^2)(1-k^2y^2)$
$y'' + y(1+k^2) - 2k^2y^3 = 0$
$y = \mathrm{sn}u, y' = (\mathrm{cn}u)(\mathrm{dn}u)$

2. $(y')^2 = (1-y^2)(1-k^2+ky^2)$
$y'' + y(1-2k^2) + 2k^2y^3 = 0$
$y = \mathrm{cn}u, y' = (-\mathrm{sn}u)(\mathrm{dn}u)$

3. $(y')^2 = (1-y^2)(y^2-1+k^2)$
$y'' + y(k^2-2) + 2y^3 = 0$
$y = \mathrm{dn}u, y' = -k^2(\mathrm{sn}u)(\mathrm{cn}u)$

4. $(y')^2 = (1+y^2)(1+(1-k^2)y^2)$
$y'' - y(2-k^2) - 2y^3(1-k^2) = 0$
$y = \mathrm{tn}u, y' = (\mathrm{dn}u)/(\mathrm{cn}^2u)$

5. $(y')^2 = (y^2-1)(y^2-k^2)$
$y'' + y(1+k^2) - 2y^3 = 0$
$y = 1/\mathrm{sn}u, y' = -(\mathrm{cn}u)(\mathrm{dn}u)/(\mathrm{sn}^2u)$

6. $(y')^2 = (y^2-1)[(1-k^2)y^2+k^2]$
$y'' + y(1-2k^2) - 2y^3(1-k^2) = 0$
$y = 1/\mathrm{cn}u, y' = (\mathrm{sn}u)(\mathrm{dn}u)/(\mathrm{cn}^2u)$

7. $y' = \sqrt{(1+y^2)^2 - 4k^2y^2}$
$y'' + 2y(2k^2-1) - 2y^3 = 0$
$y = (\mathrm{dn}u)(\mathrm{tn}u)$

8. $(y')^2 = 4y(1-y)(1-k^2y)$
$y'' - 2 + 4y(k^2+1) - 6k^2y^2 = 0$
$y = \mathrm{sn}^2u$

9. $(y')^2 = 4y(1-y)(1-k^2+k^2y)$
$y'' - 2(1-k^2) - 4y(2k^2-1) + 6k^2y^2 = 0$
$y = \mathrm{cn}^2u$

10. $(y')^2 = 4y(1-y)(y-1+k^2)$
$y'' - 2(k^2-1) - 4y(2-k^2) + 6y^2 = 0$
$y = \mathrm{dn}^2u$

11. $(y')^2 = 4y(y-1)(y-k^2)$
$y'' - 2k^2 + 4y(k^2+1) - 6y^2 = 0$
$y = 1/(\mathrm{sn}^2u)$

12. $(y')^2 = 4y(y-1)[(1-k^2)y+k^2]$
$y'' + 2k^2 - 4y(2k^2-1) - 6y^2(1-k^2) = 0$
$y = 1/(\mathrm{cn}^2u)$

13. $(y')^2 = 4y[(1+y)^2 - 4k^2y]$

$$y'' - 2 - 8y(1-2k^2) - 6y^2 = 0$$
$$y = (\mathrm{dn}^2 u)(\mathrm{tn}^2 u)$$

14. $$(y')^2 = 4y(1+y)[1+(1-k^2)y]$$
$$y'' - 2 - 4(2-k^2)y - 6(1-k^2)y^2 = 0$$
$$y = \mathrm{tn}^2 u$$

3.4.4 Equation: $\ddot{x} + \omega_0^2 x = f(x, \dot{x})$

Method of solution: (Averaging Method) If the function, f, can be considered a small perturbation on otherwise periodic motion, then we can assume a solution of the form:

$$x(t) = A(t)\cos(\omega_0 t + \phi(t))$$

Assuming that A and ϕ vary only slowly with time, they may be considered constant over one cycle of the motion. When this assumption is made, we can write differential equations for A and ϕ as follows:

$$\dot{A} = -(1/\omega_0)\langle f(x, \dot{x})\sin(\omega_0 t + \phi)\rangle$$

$$\dot{\phi} = -(1/\omega_0)\langle [f(x, \dot{x})\cos(\omega_0 t + \phi)]/A\rangle$$

In performing the integrations, the form for $x(t)$ is substituted, but A and ϕ are considered constant in the time average over one cycle of the motion. The resulting differential equations for A and ϕ are then solved and substituted into the original assumed form to yield the solution.

3.5 Numerical Solutions to Differential Equations

Consider the second-order differential equation,

$$\ddot{x} = f(x, \dot{x}, t)$$

We can define a new variable, $z(t)$, such that

$$\dot{x} = dx/dt = z(t)$$

Then, our original equation can be reduced to solving two first-order differential equations:

$$dz/dt = f(x, z, t)$$

$$dx/dt = z$$

Note that the right-hand sides are free of explicit "dotted" variables.

This scheme can be generalized for multi-dimensional systems:

$$dy_i/dt = f_i(y_1, ..., y_n, t)$$

The underlying idea for solving the initial value problem is always this: Rewrite the dy's and dt as Δy_i and Δt. Then,

$$\Delta y_i = f_i(y_1, ..., y_n, t)\Delta t$$

i.e.

$$y_i(t + \Delta t) = y_i(t) + f_i(y_1, ..., y_n, t)\Delta t$$

In the limit of taking Δt to be small (but not too small!), a good approximation to the underlying differential equation is achieved. Literal implementation of this scheme is called *Euler's Method*.

Solving differential equations by numerical methods is something of an art. The idea of the basic Euler method has been modified and improved in a variety of ways to improve accuracy, numerical stability, and speed. Runge-Kutta methods, the Bulirsch-Stoer method, and predictor-corrector methods are some of these variations. None can be said to be "the best." The choice of method depends on the nature of the equation, the nature of the boundary conditions, etc.

There are now a number of computer software packages that solve differential equations by these or other methods. Some packages allow the user to choose among several available methods. In others, the solution occurs in a "black box."

3.6 Problems

1. Computer Project 1 (see Appendix)

2. Computer Project 2 (see Appendix)

Chapter 4

One Dimensional Motion

4.1 Definitions and Theorems

Consider a particle which moves in a one-dimensional region of space under the influence of a force F. Consider two points, a and b in the region of space.

4.1.1 Definition

Work,

$$W_{ab} = \int_a^b F dx$$

4.1.2 Definition

Kinetic Energy,

$$(1/2)mv^2,$$

where m is the mass of the particle and v is its speed.

4.1.3 Theorem

Let a and b refer to any two points on the trajectory of a particle. The work done between these two points by the force $F(x, v, t)$ is equal to the increase in the kinetic energy of the particle. (Note: The force may depend explicitly on position, velocity and time.)

Proof: By definition,

$$W_{ab} = \int_a^b F dx.$$

But Newton's Second Law in one dimension allows us to replace F with mdv/dt. Moreover, we can replace dx with $(dx/dt)dt$, so that we have,

$$W_{ab} = \int_a^b F dx = \int_a^b m\frac{dv}{dt}\frac{dx}{dt}dt = \int_a^b mv\frac{dv}{dt}dt$$
$$= \int_a^b \frac{d}{dt}(\frac{1}{2}mv^2)dt = \int_a^b d(\frac{1}{2}mv^2).$$

Since the argument of the integral has now been reduced to a perfect differential, we may integrate by inspection,

$$W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = T_b - T_a.$$

QED.

4.1.4 Definition

Potential Energy,

$$V(x,t).$$

If the force is a function of x and t but not the velocity v, and if it is possible to write

$$F = -\frac{\partial V(x,t)}{\partial x},$$

then $V(x,t)$ is the *potential energy* of the particle. (Note: Since the derivative of a constant is zero, any constant may be added to $V(x,t)$ without changing the derived force.)

4.1.5 Theorem

If the force $F(x)$ has no explicit time or velocity dependence, then

$$F(x) = -\frac{dV(x)}{dx},$$

and,

$$W_{ab} = -V_b + V_a.$$

Proof:

$$W_{ab} = \int_a^b F dx = -\int_a^b \frac{dV}{dx}dx = -\int_a^b dV = -V_b + V_a.$$

QED.

4.1.6 Theorem

If $F(x)$ has no explicit time or velocity dependence, then,

$$E \equiv T_a + V_a = T_b + V_b,$$

i.e., the energy, $E = T + V$ is conserved and $F(x)$ is said to be a *conservative force.*

Proof: Under the given conditions, we have both that,

$$W_{ab} = T_b - T_a,$$

and,

$$W_{ab} = -V_b + V_a.$$

If we subtract the second from the first, we conclude,

$$T_b + V_b = T_a + V_a.$$

QED.

4.1.7 Theorem

If the force $F(x,t)$ has explicit time dependence, we may still define $E = T + V$, but

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}.$$

Proof: Since V depends both on position and on time, we have,

$$F = -\frac{\partial V}{\partial x}$$

and, using the chain rule,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial t}dt = -Fdx + \frac{\partial V}{\partial t}dt.$$

We can no longer write,

$$W_{ab} = \int_a^b Fdx = -\int_a^b \frac{dV}{dx}dx = -V_b + V_a$$

because the middle step no longer holds. However, we can still define $E = T + V$. Then,

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}(\frac{1}{2}m\dot{x}^2) + \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial t} \\ &= m\dot{x}\ddot{x} - (m\ddot{x})\dot{x} + \frac{\partial V}{\partial t}.\end{aligned}$$

Hence,

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}.$$

QED.

4.2 Problems

1. If the force acting on a particle is conservative, i.e., a function only of position and expressible as

$$F_x = -\frac{dV}{dx},$$

show for the one-dimensional case that a formal solution of the equation of motion is,

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{(2/m)[E - V(x)]}},$$

where x_0 is the value of x at t_0 and E is the total energy of the particle, $E = T + V$.

2. How long would it take an object of mass m that is released at rest from the position of the earth, r_e, to fall into the sun? The sun represents a fixed center of force that attracts the object according to the inverse square law $F(r) = -k/r^2$. Show that the time required for the object to reach the origin at the sun is

$$t = \pi(\frac{mr_e^3}{8k})^{1/2}.$$

What did dimensional analysis tell you to expect (see Appendix B)? If the object is the earth, how long does it take?

3. A particle of mass m is initially at rest. A constant force F_0 acts on the particle for a time t_0. The force then increases linearly with time such that after an additional interval t_0 the force is equal to $2F_0$. Show that the total distance the particle goes in the total time $2t_0$ is,

$$\frac{13}{6}F_0 t_0^2/m.$$

(Hint: Watch your constants of integration.) (G. R. Fowles, *Analytical Mechanics*, Holt, Rinehart, Winston, 1970.)

4. A boy runs and then slides on some slushy ice. It is hypothesized that friction varies as the square root of the speed, $F(v) = -cv^{1/2}$. The initial speed of the boy is v_0 at time $t = 0$. What does dimensional analysis give for a length scale for the problem? Find the values of v and x as function of the time t. Partial answer:

$$x = v_0 t - (\frac{cv_0^{1/2}}{2m})t^2 + \frac{c^2}{12m^2}t^3.$$

Show that the boy cannot travel farther than

$$x_{max} = \frac{2m}{3c}v_0^{3/2}.$$

Use some reasonable values of m, v_0 and x_{max} to estimate c in SI units.

5. A particle moves upward in a medium in which the frictional force is proportional to the square of the velocity. What does dimensional analysis give for a length scale? Find the velocity as a function of height. (Hint: Solve the differential equation for $\dot{x}^2$ as a function of x.) Find the maximum height to which the particle will rise for a given initial velocity.

6. Show that the time average (over one period) of the kinetic energy of a harmonic oscillator is equal to the time average of the potential energy. Hint: By definition,

$$T_{avg} \equiv \frac{\int_{period} T dt}{\int_{period} dt}.$$

7. A linear oscillator consists of a mass m attached to a spring of force constant k. The oscillator experiences a viscous damping proportional to the velocity. The oscillator is initially quiescent. At $t = 0$, a force of constant value F_0 is applied. Before calculating, describe as best you can what you think the motion will be. Find the motion of the oscillator. Show your final answer as one expression with arbitrary constants expressed in terms of initial conditions. Don't forget the particular solution *and* the homogeneous solution. Convince the reader that your answer is correct.

8. Find the Fourier series expansion of the function,

$$F(t) = \frac{F_0}{\pi}\omega t - F_0$$

$$0 \leq \omega T < 2\pi.$$

Ans:

$$F(t) = -\sum_1^\infty \frac{2F_0}{\pi n} \sin n\omega t.$$

9.
   - Write down a particular solution, with any constants expressed explicitly in terms of ω, k,m,b, and F_0, to the equation

$$m\ddot{x} + b\dot{x} + kx = -\sum_1^\infty \frac{2F_0}{\pi n} \sin n\omega t.$$

   - What is the solution to the homogeneous equation in terms of two arbitrary constants? (You need not evaluate the constants in terms of initial conditions.)

10. Find the one-dimensional motion of a particle if its potential energy is given by

$$V(x) = -k_0 x$$

$$x < 0$$

$$V(x) = \frac{1}{2}k_1 x^2$$

$$x > 0$$

Let the initial displacement be $x_2 > 0$ and the initial velocity be zero. Before calculating, describe the motion you expect. Where will the particle move slowly, where more quickly? Where are the turning points (i.e. points where the velocity vanishes as the particle changes direction). What is the period of the motion? The period is just twice the time it takes to get between turning points. Partial answer:

$$Period = \frac{\pi}{\omega} + \frac{2m\omega x_2}{k_0}$$

11. The potential energy of a particle moving in one dimension is

$$V(x) = \frac{1}{2}k_1 x^2 + \frac{k_2}{x},$$

with $k_1 > 0$, $k_2 > 0$, and $x > 0$. What do you guess the frequency will be for small oscillations. Show that the motion is periodic. What is the equilibrium position? What is the frequency of the motion if the amplitude of the vibrations is very small?

12. In a particular problem dealing with the orbit (i.e., periodic motion) of a particle, the approximate differential equation of motion (obtained by doing a Taylor's expansion of the potential function) is found to be

$$(1 + a^2)\frac{d^2x}{d\theta^2} + 3(x - x_1) - \frac{3}{x_1}(x - x_1)^2 + \frac{4}{x_1^2}(x - x_1)^3 = 0.$$

In this equation a and x_1 are *known* constants. The variable x is defined, $x = 1/r$ where r and θ are the plane polar coordinates of the particle. Using results from our summary of solutions to differential equations, show that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{r_1 b_1^2}{2} + b_1 \cos(\lambda\theta) - \frac{r_1 b_1^2}{6}\cos(2\lambda\theta) + \frac{b_1^3 r_1^2}{16}\cos(3\lambda\theta).$$

- What is λ^2 equal to in terms of r_1 and b_1?
- Which is the constant in this result to be specified by initial conditions?
- Why are there not *two* such arbitrary constants?

13. Since both the Coulomb force and the force of gravity obey inverse square laws, they share another feature in common. Show by analogy that if Gauss' Law for the Coulomb force is

$$\epsilon_0 \int_A \mathbf{E} \cdot d\mathbf{A} = \int_V \rho_c dV,$$

then, there ought to be a Gauss' Law for gravity,

$$\int_A \mathbf{g} \cdot d\mathbf{A} = -4\pi G \int_V \rho_m dV.$$

Assuming the earth to be uniform, show that if a straight hole were drilled from the North Pole to the South pole, a particle dropped into the hole would execute simple harmonic motion. Before calculating, guess what the period will be, in minutes. What does dimensional analysis give for the time scale of the problem? Find the period of this motion. Let a be the earth's radius, G be the universal gravitation constant, and M be the earth's mass. Ans: $Period = 2\pi(a^3/GM)^{1/2}$.

14. Assume that the sun is at the center of a uniform, spherical dust cloud of density ρ. Find the law of force which binds the planet to the sun if the cloud extends beyond the limits of the orbits. (Ans:

$$\frac{\mathbf{F}}{m} = -\frac{GM_{sun}}{r^2}\hat{\mathbf{r}} - \frac{4}{3}\pi G\rho r\hat{\mathbf{r}}.$$

15. A pendulum swings with an amplitude of 20 degrees. What is the percentage error in taking the period using the simple harmonic approximation? How many minutes a day error does this make?

(Begin with the conservation of energy equation for a simple pendulum. Show that by making the definitions and substitutions,

$$k^2 \equiv E/(2mg\ell),$$
$$y \equiv (1/k)\sin(\theta/2),$$
$$u \equiv \sqrt{g/\ell}t,$$

one may transform the energy equation into the form,

$$(\frac{dy}{du})^2 = (1-y^2)(1-k^2y^2).$$

This equation has a known solution called the Jacobian elliptic function, $y = sn(u)$. Show that when θ has its maximum value, $y_{max} = 1$. Then show that the period of the pendulum is given by,

$$P_t = 4\sqrt{\ell/g}\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 4\sqrt{\ell/g}K(k).$$

$K(k)$ is called an "elliptic integral of the first kind" and tables of values are available in the Chemical Rubber Company's *Standard Mathematical Tables*. In particular, $K(0.17365) = 1.5828$. Assume $\sqrt{\ell/g} = 1$.)

16. A particle is placed at the top of a smooth hemisphere which is sitting convex side up on a table. As the particle slides down the sphere, will it leave the surface before hitting the table on which the hemisphere is sitting? If so, at what angle? The condition for the particle and hemisphere to be considered separated is for the force exerted on the particle by the hemisphere to vanish. Find this force as a function of angle. (It is not particularly helpful to use the Lagrangian formulation here. Simply resolve forces and use Newton's Second Law in its more familiar form, $F = ma$.)

17. Computer Project 3 (see Appendix)

18. Computer Project 4 (See Appendix)

19. Computer Project 5 (See Appendix)

20. Computer Project 6 (See Appendix)

21. Computer Project 7 (See Appendix)

Chapter 5

Motion of a Particle in Two and Three Dimensions

5.1 Definitions and Theorems

Consider a particle which moves in a two or three-dimensional region of space under the influence of a force $\mathbf{F}$. Consider two points, a and b in the region of space.

5.1.1 Definition

The *net force* is the vector sum of all forces acting on the particle.

5.1.2 Definition

Then we generalize the definition of work to be

$$W_{ab} \equiv \int_a^b \mathbf{F} \cdot d\mathbf{s}.$$

5.2 Theorem

Let a and b refer to any two points on the trajectory of a particle. The work done between these two points by the net force $\mathbf{F}(\mathbf{x}, \mathbf{v}, t)$ is equal to the increase in the kinetic energy of the particle. (Note: The force may depend explicitly on position, velocity and time.)

Proof: By definition,

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s}.$$

But Newton's Second Law in one dimension allows us to replace $\mathbf{F}$ with $md\mathbf{v}/dt$. Moreover, we can replace $d\mathbf{s}$ with $(d\mathbf{s}/dt)dt$, so that we have,

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s} = \int_a^b m\frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{s}}{dt}dt = \int_a^b mv\frac{dv}{dt}dt.$$

It is useful in the last step above to observe that

$$\mathbf{v} \cdot \mathbf{v} = v^2$$

so that differentiating both sides with respect to t gives,

$$2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2v\frac{dv}{dt}.$$

We then have,

$$W_{ab} = \int_a^b \frac{d}{dt}(\frac{1}{2}mv^2)dt = \int_a^b d(\frac{1}{2}mv^2).$$

Since the argument of the integral has now been reduced to a perfect differential, we may integrate by inspection,

$$W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = T_b - T_a.$$

QED.

5.2.1 Definition

A region Σ is *simply connected* if every closed path in Σ can be continuously deformed into a point without any portion of the path passing out of Σ.

5.2.2 Definition

If in a simply connected region

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0,$$

then $\mathbf{F}$ is said to be *conservative* in Σ.

5.2.3 Theorem

A necessary and sufficient condition for $\mathbf{F}$ to be conservative is that for two points, a and b,

$$\int_a^b \mathbf{F} \cdot d\mathbf{s}$$

be independent of the path connecting the two points.

Proof: Let us assume that $\mathbf{F}$ is conservative. By definition,

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0.$$

We may divide this path integral which closes on itself into two arbitrary pieces and write,

$$(Path1) \int_a^b \mathbf{F} \cdot d\mathbf{s} + (Path2) \int_b^a \mathbf{F} \cdot d\mathbf{s} = 0.$$

We may reverse the order of integration in the second (thus changing its sign) so that,

$$(Path1) \int_a^b \mathbf{F} \cdot d\mathbf{s} - (Path2) \int_a^b \mathbf{F} \cdot d\mathbf{s} = 0$$

and conclude,

$$(Path1) \int_a^b \mathbf{F} \cdot d\mathbf{s} = (Path2) \int_a^b \mathbf{F} \cdot d\mathbf{s} = 0.$$

Thus we have shown that if $\mathbf{F}$ is conservative, the work done in going from point a to point b is independent of the path. Since the steps of the proof are reversible, we also conclude that if the work done in going from point a to point b is independent of the path, then the force, $\mathbf{F}$, is conservative. QED.

5.2.4 Theorem

A necessary and sufficient condition for $\mathbf{F}$ to be conservative in a simply connected region Σ is that

$$\nabla \times \mathbf{F} = 0$$

in Σ.

Proof: If we assume that $\mathbf{F}$ is conservative, we may apply Stokes Theorem as follows,

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0 = \int_\Sigma (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} d\sigma.$$

In this expression, Σ, is a simply connected region in which there is a differential patch of area, $d\sigma$. The unit vector, $\hat{\mathbf{n}}$, is oriented perpendicular to $d\sigma$. Let the path of integration implied in $\oint \mathbf{F} \cdot d\mathbf{s}$ shrink until it just bounds $d\sigma$. Then,

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} d\sigma = 0.$$

Since $d\sigma$ and $\hat{\mathbf{n}}$ are arbitrary, we must conclude that $\nabla \times \mathbf{F} = 0$ in Σ. Again, the steps of the proof are reversible and the vanishing of the curl is a necessary and sufficient condition for the force to be conservative. QED.

5.2.5 Definition

Potential Energy, $V(\mathbf{x}, t)$. If the force is a function of $\mathbf{x}$ and t but not the velocity $\mathbf{v}$, and if it is possible to write

$$F = -\nabla V(\mathbf{x}, t)$$

then $V(\mathbf{x}, t)$ is the *potential energy* of the particle. (Note: Since the derivative of a constant is zero, any constant may be added to $V(\mathbf{x}, t)$ without changing the derived force.) To define the potential energy, we must have,

1. $\mathbf{F}$ must be conservative in a simply-connected region, Σ.
2. The potential energy function must be given some arbitrary value at one point in the region which we designate here as point a.
3. The potential energy at all other points in the region (of which b is one such point) is then defined

from

$$V_b = V_a - \int_a^b \mathbf{F} \cdot d\mathbf{s}.$$

Thus we have,

$$W_{ab} = -V_b + V_a.$$

5.2.6 Theorem

If $\mathbf{F}(\mathbf{x})$ is a function of position only and conservative in Σ, then energy is a constant and is said to be *conserved*,

$$E \equiv T + V.$$

Proof: Under the given conditions, we have both that,

$$W_{ab} = T_b - T_a,$$

and,

$$W_{ab} = -V_b + V_a.$$

If we subtract the second from the first, we conclude,

$$E \equiv T_b + V_b = T_a + V_a.$$

QED.

5.2.7 Theorem

If $\mathbf{F}$ is conservative in a simply connected region, then $\mathbf{F} = -\nabla V$ where V is the potential energy function.

Proof: Consider two points which are a differential distance, $d\mathbf{s}$, apart in Σ. From

$$V_b = V_a - \int_a^b \mathbf{F} \cdot d\mathbf{s}$$

we have that

$$\mathbf{F} \cdot d\mathbf{s} = V_a - V_b = -dV.$$

From the definition of the direction derivative, we have

$$dV = \nabla V \cdot d\mathbf{s}.$$

Thus,

$$-\mathbf{F} \cdot d\mathbf{s} = \nabla V \cdot d\mathbf{s},$$

from which we conclude,

$$\mathbf{F} = -\nabla V.$$

QED.

5.2.8 Definition

The *Lorentz Force* is the force exerted on a charged particle (of charge q) by a combination of electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

5.2.9 Theorem

The magnetic field does no work on the particle.

Proof: Let the Lorentz force be the net force on a particle. Since $dW = \mathbf{F} \cdot d\mathbf{s}$,

$$\frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = m\mathbf{a} \cdot \mathbf{v}.$$

Thus, we have,

$$\begin{aligned}\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(v^2) = \frac{1}{2}m\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) &= \frac{1}{2}(\mathbf{a} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{a}) \\ &= m\mathbf{a} \cdot \mathbf{v} \\ &= \frac{dW}{dt}.\end{aligned}$$

So,

$$\frac{dW}{dt} = \frac{dT}{dt} = m\mathbf{a} \cdot \mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$$

which does not depend on the magnetic field so that the magnetic field does no work on the particle. QED.

5.2.10 Theorem

If the electric and magnetic fields are time independent, $T + q\psi$ is conserved.

Proof: If the electric field is time-independent, it may be derived from an electric potential function, $\mathbf{E} = -\nabla\psi$. Using a result from the proof of the previous theorem,

$$\frac{dT}{dt} = \frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v} = -q\nabla\psi \cdot \mathbf{v}.$$

But, $d\psi = \nabla\psi \cdot d\mathbf{s}$ (chain rule.) Thus,

$$\frac{d\psi}{dt} = \nabla\psi \cdot \frac{d\mathbf{s}}{dt} = \nabla\psi \cdot \mathbf{v}.$$

Therefore,

$$\frac{d}{dt}(T + q\psi) = 0,$$

or, $T + q\psi$ is a conserved. QED.

The potential energy, V, of a charged particle in an electric field is $q\psi$, so this is equivalent to saying that the energy, $E = T+V$, is conserved. However, it is not true that the Lagrangian is given by $L = T-V$ in this case, since the Lagrangian must produce both the forces exerted by the electric as well as the magnetic fields. While it goes beyond the scope of the present discussion, we note here for completeness that the correct Lagrangian is,

$$L = T - q\psi + q(\mathbf{A} \cdot \mathbf{v}),$$

where $\mathbf{A}$ is the vector potential which yields the magnetic field according to $\mathbf{B} = \nabla \times \mathbf{A}$.

5.2.11 Definition

Angular momentum,

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{p}.$$

5.2.12 Definition

Torque,

$$\mathbf{N} \equiv \mathbf{r} \times \mathbf{F}.$$

5.2.13 Theorem

If the torque on a particle vanishes, the angular momentum of the particle is conserved.

Proof: We may differentiate the definition of angular momentum to obtain,

$$\frac{d\boldsymbol{\ell}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$
$$= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \mathbf{F} = m\mathbf{v} \times \mathbf{v} + \mathbf{N}.$$

Since $\mathbf{v} \times \mathbf{v} = 0$ and since the torque, $\mathbf{N}$, is assumed to vanish, we conclude that

$$\frac{d\boldsymbol{\ell}}{dt} = 0,$$

i.e. we conclude that the angular momentum is a constant and is conserved. QED.

5.2.14 Theorem

In a system with only conservative forces, the explicit absence of a coordinate q_k in the Lagrangian implies that $\partial L/\partial \dot{q}_k$ is a conserved quantity of the motion. Observe that the explicit absence of a coordinate in the Lagrangian results from a spatial symmetry of the problem. Thus, there is a direct and immediate connection between symmetry and a conserved quantity of the motion.

(Note: In this and subsequent sections, we will attach subscripts to the generalized coordinates rather than superscripts. The use of superscripts is not particularly useful here since we no longer are concerned with maintaining a distinction between covariant and contravariant components.)

Proof: Lagrange's equations apply to a system with only conservative forces,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0.$$

If the Lagrangian has no dependence on a particular coordinate, q_k, the second term of equation k vanishes and we have,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0.$$

We conclude, therefore, that $\partial L/\partial \dot{q}_k$ is a constant with respect to time and is conserved. QED.

5.2.15 Definition

Consider the Lagrangian to be a function of generalized coordinates, q_i, and the corresponding velocities, $\dot{q}_i$, and of time, t. Hence, $L = L(q_i, \dot{q}_i, t)$. The *generalized momenta* are then defined to be,

$$p_i \equiv \partial L/\partial \dot{q}_k.$$

5.2.16 Definition

The *Hamiltonian*, H, is defined to be,

$$H \equiv \sum_i p_i \dot{q}_i - L \equiv p_i \dot{q}_i - L.$$

To complete the definition, however, the $\dot{q}_i$ must be replaced with expressions involving q_i and p_i so that $H = H(p_i, q_i, t)$. The definitions of the generalized momenta are used to make this replacement.

5.2.17 Theorem

The differential equations of motion of the system may be written,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Proof:

1. By definition,
$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}.$$

2. From Lagrange's equations,
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
i.e.,
$$\frac{d}{dt}(p_i) = \dot{p}_i = \frac{\partial L}{\partial q_i}.$$
Observe how the dot moves back and forth in the expressions for p_i and $\dot{p}_i$.

3. By definition,
$$H \equiv p_i\dot{q}_i - L(\dot{q}_i, q_i, t)$$
where a sum is implied by the repeated index in the first term. Hence,
$$dH = p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt$$
$$= (p_i - \frac{\partial L}{\partial \dot{q}_i})d\dot{q}_i + \dot{q}_i dp_i - (\dot{p}_i)dq_i - \frac{\partial L}{\partial t}dt.$$

The coefficient of $d\dot{q}_i$ vanishes by the definition of generalized momenta. We also used the above result from Lagrange's equations to replace $\partial L/\partial q_i$ with $\dot{p}_i$. Thus, on the one hand,

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt.$$

4. On the other hand, consider the Hamiltonian as a function of generalized coordinates and generalized momenta, $H = H(p_i, q_i, t)$. Then

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt.$$

5. Comparing the two expressions for dH, we conclude,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

These equations are called *Hamilton's Equations.* Observe that they are first order rather than second order (as obtained from the Lagrangian). Observe also that p_i and q_i play very symmetrical roles that blur the distinction between coordinates and momenta. This is particularly so for generalized coordinates and momenta that may not even have the units that we ordinarily associate with Cartesian coordinates and momenta. In a system with n degrees of freedom (requiring n coordinates for its description), the phase space of p_i, q_i has $2n$ dimensions.

5.2.18 Theorem

If $\partial L/\partial t = 0$, then the Hamiltonian is a constant of the motion.

Proof: In general $H = H(p_i, q_i, t)$. The chain rule yields,

$$\frac{dH}{dt} = \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial t}$$

$$= (\dot{q}_i)(\dot{p}_i) + (-\dot{p}_i)(\dot{q}_i) + (-\frac{\partial L}{\partial t}).$$

The first two terms add out and the third vanishes by assumption. Thus,

$$\frac{dH}{dt} = 0$$

and H is a constant of the motion. QED.

5.2.19 Theorem

If $\partial L/\partial t = 0$ and the system is conservative, the Hamiltonian can be interpreted as the total energy of the system,

$$H = T + V.$$

Proof: We may write the kinetic energy of the particle as

$$T = \frac{1}{2} m g_{ij} \dot{q}_i \dot{q}_j.$$

From the definition of generalized momenta and given that the system is conservative so that the potential energy is a function only of the generalized coordinates and not the velocities,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = m g_{ij} \dot{q}_j.$$

If we now form the Hamiltonian,

$$H = p_i \dot{q}_i - L = m g_{ij} \dot{q}_j \dot{q}_i - L = 2T - (T - V) = T + V.$$

Thus, $H = T + V$. QED.

5.2.20 Example

Consider the Lagrangian for a two-dimensional harmonic oscillator,

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2).$$

The generalized momenta are, by definition,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}.$$

The Hamiltonian is

$$H = (m\dot{x})(\dot{x}) + (m\dot{y})(\dot{y}) - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2).$$

Thus,

$$H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2).$$

However, we must express the Hamiltonian in terms of generalized coordinates and generalized momenta, i.e.,

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2).$$

Hamilton's equations for this system then become,

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx,$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -ky,$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m},$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}.$$

In this form, these first-order differential equations would lend themselves to direct numerical solution by computer. However, we can combine them to put them into more familiar second-order form. Differentiate the latter two with respect to time and substitute into the former two, eliminating $\dot{p}_i$ in the process. We then obtain the second-order, two-dimensional harmonic oscillator equations that you would obtain from the Lagrangian.

$$m\ddot{x} + kx = 0,$$

$$m\ddot{y} + ky = 0.$$

Although Hamilton's equations can be used readily in place of Lagrange's equations to obtain the differential equations of motion, they are more useful in supplying fundamental postulates in quantum mechanics, statistical mechanics and celestial mechanics. Although quantum mechanics cannot be derived from classical mechanics, the Hamiltonian nevertheless provides a bridge between the two. For example, a certain well defined substitution turns the Hamiltonian of classical mechanics into the Schrödinger equation of quantum mechanics.

5.3 Problems

1. If you shoot a cannon uphill, how should you aim to maximize your range on the hill? Imagine that the angle β characterizes the slope and θ is the angle of elevation relative to the slope. Find the angle γ relative to the horizontal, i.e. $\gamma = \theta + \beta$ that maximizes the range up the hill. Is your answer reasonable if $\beta = 0$? What is the range up the hill? Are your answers reasonable if $\beta = \pi/2$?

   Partial answer: $R_m = (v_0^2/g)/(1 + \sin\beta)$.

2. A gun at the origin of coordinates has a muzzle velocity of 1600 ft/sec. An airplane travels on a course given by $x + z = 3.5 \times 10^4$ ft, and at a constant altitude $y = 10^4$ ft. Over what portion of its course is it in danger? (T. C. Bradbury, *Theoretical Mechanics*, John Wiley & Sons, 1968).

3. Consider a tether ball held at chest height with rope extended. Find the equations of motion, the energy equation and define an "effective potential" for the motion. How fast (feet per second) do you have to propel it horizontally (with the tether fully extended) to keep it from falling below your knees? As the ball revolves around the pivot (assuming the tether keeps the same length) it will oscillate between its highest point and its lowest point. Estimate the frequency of this latter part of its motion. (Let's take chest height to be five feet and knee height to be two feet. Let the length of the tether be 8 feet and let the ball just reach the ground when it is at rest. Let's assume the ball weighs about a pound and the weight of the tether can be neglected.)

   The purpose of the problem is to demonstrate how an essentially two-dimensional problem can be reduced to a one-dimensional problem using conserved quantities, one of which can be identified by symmetry with respect to a coordinate in the Lagrangian. The solution to the problem will require a Taylor's expansion about the minimum of the "effective potential." To find the position of the minimum of the potential for purposes of the expansion, you will be forced to a numerical rather than an analytical solution. When you have finished the problem, you ought to review it to identify the techniques you have used that have more general applicability than this specific problem. The frequency asked for in the problem is found by reducing the problem by approximation to the simple harmonic oscillator.

Chapter 6

Accelerated Frames of Reference

We have already derived the following results:

$$\frac{d\hat{\mathbf{e}}_i'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}_i' = \delta_{ijk}\hat{\mathbf{e}}_j'\omega_k'.$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_o.$$

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0.$$

Therefore, Newton's Second Law can be written,

$$\mathbf{F} = m\mathbf{a} = m(\mathbf{a}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0)$$

Written this way, the second, third and fourth terms inside the parentheses on the right-hand-side are known as the Coriolus acceleration, the transverse acceleration and the centripetal acceleration respectively. If, on the other hand, one would like to preserve the form of Newton's Second Law for observers working in a non-inertial frame, $\mathbf{F} = m\mathbf{a}'$, then one may rewrite the above equation in the equivalent form

$$\mathbf{F} - (2m\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + m\mathbf{A}_0) = m\mathbf{a}'.$$

In this form, we refer to a Coriolus *force*, transverse *force*, and a *centrifugal force*. The extra forces now on the left-hand-side are fictitious forces arising from the choice of a noninertial frame in which to work. In introductory courses in mechanics, instructors often go to great lengths to wean students from the notion of a centrifugal force, but as we see here, there is a perspective in which it is useful.

6.1 Theorems

6.1.1 Theorem: Lagrangian for Motion in a Noninertial Frame

The general Lagrangian for motion in a noninertial frame is,

$$L = T - V + \frac{1}{2}mr'^2\omega^2 - \frac{1}{2}m(\mathbf{r}' \cdot \boldsymbol{\omega})^2 - m(\mathbf{r}' \cdot \mathbf{a}_0) + m\mathbf{v}' \cdot (\boldsymbol{\omega} \times \mathbf{r}')$$

Proof: We shall obtain this result by trial-and-error. We observe that the terms in the Lagrangian are scalars or pseudoscalars. We simply wish to add additional terms to the Lagrangian so that Lagrange's equations produce the non-inertial forces which arise in a non-inertial frame. The non-inertial forces are,

$$\mathbf{F}_{NI} = 2m(\mathbf{v}' \times \boldsymbol{\omega}) + m(\mathbf{r}' \times \dot{\boldsymbol{\omega}}) + m(\boldsymbol{\omega} \times (\mathbf{r}' \times \boldsymbol{\omega})) - m\mathbf{a}_0.$$

We shall identify these terms as follows:

$$\mathbf{I} = 2m(\mathbf{v}' \times \boldsymbol{\omega})$$

$$\mathbf{II} = m(\mathbf{r}' \times \dot{\boldsymbol{\omega}})$$

$$\mathbf{III} = m(\boldsymbol{\omega} \times (\mathbf{r}' \times \boldsymbol{\omega})) - m\mathbf{a}_0.$$

The term **III** depends only on position and is thus similar in nature to a conservative force. Thus, we seek a scalar term ψ_{III} that can be subtracted from the Lagrangian in much the same way that V is subtracted from T. Thus, we seek a function ψ_{III} such that,

$$III_i = -\frac{\partial \psi}{\partial x'_i}.$$

Using the so-called $\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ rule, we can write (taking the i-th component after applying the rule),

$$III_i = x'_i\omega^2 - \omega_i x'_j \omega_j - a_{0i} = x'_j \delta_{ji} \omega^2 - \omega_i x'_j \omega_j - a_{0i}$$

$$= x'_j(\delta_{ij}\omega^2 - \omega_i\omega_j) - a_{0i}.$$

It is straightforward to show that a suitable scalar function is,

$$\psi_{III} = -\frac{1}{2}\delta_{ij}x'_i x'_j \omega^2 + \frac{1}{2}(x'_j\omega_j)(x'_i\omega_i) + x'_j a_{0j} = -\frac{1}{2}r'^2\omega^2 + \frac{1}{2}(\mathbf{r}' \cdot \boldsymbol{\omega})^2 + \mathbf{r}' \cdot \mathbf{a}_0.$$

The Coriolus force is a little harder. Now we seek a scalar that involves both the velocity of the particle and the angular velocity of the frame, but which somehow also yields the cross-product of the Coriolus force. About the simplest thing that one could try is,

$$\psi_I = m\mathbf{v}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') = m\dot{x}'_i \delta_{ijk}\omega_j x'_k.$$

Then, if we apply the Lagrangian operator,

$$\frac{d}{dt}\frac{\partial \psi_I}{\partial \dot{x}'_l} - \frac{\partial \psi_I}{\partial x'_l} = m\delta_{ljk}\dot{\omega}_j x'_k + m\delta_{ljk}\omega_j \dot{x}'_k - m\dot{x}'_i \delta_{ijl}\omega_j,$$

we have three terms. The first is a pleasant bonus. It is just negative of the l-th component of the transverse force which we need anyway. (Observe that Lagrange's equations are in the form $mQ_l - F_l = 0$, so that the operator is supposed to generate the negative components of the non-inertial forces.) The second looks like the negative of the l-th component of the Coriolus force but lacks a factor of two and a minus sign to be correct. The third term, except for some manipulations of the indices, is also a Coriolus force term which combines with the second term to provide the missing factor of two. Thus, we have,

$$L = T - V - \psi_{III} + \psi_I = T - V + \frac{1}{2}mr'^2\omega^2 - \frac{1}{2}m(\mathbf{r}' \cdot \boldsymbol{\omega})^2 - m(\mathbf{r}' \cdot \mathbf{a}_0) + m\mathbf{v}' \cdot (\boldsymbol{\omega} \times \mathbf{r}').$$

QED.

6.1.2 Theorem: Lagrangian for Motion in a Noninertial Frame

In problems describing particle motion in a rotating (noninertial) frame where ω is small (terms proportional to ω^2 can be ignored) and $\mathbf{a}_0 = 0$, the Lagrangian becomes:

$$L = T - V + m(\boldsymbol{\omega} \times \mathbf{r}') \cdot \mathbf{v}'$$

where $\boldsymbol{\omega}$ is the angular velocity of the rotating frame, $\mathbf{r}'$ is the particle position in the frame, and $\mathbf{v}'$ is the particle velocity with respect to the noninertial frame.

6.2 Problems

1. A bicycle travels with constant speed v around a track of radius ρ. From the point of view of the inertial frame,

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_{Coriolus} + \mathbf{a}_{centripetal} + \mathbf{a}_{transverse} + \mathbf{a}_0.$$

   Evaluate each of these terms for a point on the rim at the top of a vertically-upright wheel of radius b with body axes attached to its center and the z' axis always vertical (thus *not* rigidly attached to the wheel). The purpose of the problem is to give you experience in evaluating the different terms in the general expression for acceleration used in noninertial frames.

   Ans:

$$\mathbf{a} = \frac{3v^2}{\rho}\hat{\mathbf{x}}' - \frac{v^2}{b}\hat{\mathbf{z}}'.$$

2. A bug of mass m crawls outward, with constant speed v, along the spoke of a wheel which is rotating with constant angular speed ω about a vertical axis. Find all the forces (real and fictitious) as seen from the point of view of the bug and show them on a diagram. The purpose of the problem is to allow you to demonstrate that you understand how "fictitious" forces arise in noninertial frames and how they are interpreted.

   Ans:

   If the z' axis is vertical and the bug walks outward along the x' axis, then

$$F = 2m\omega v\hat{\mathbf{y}}' - m\omega^2 x\hat{\mathbf{x}}'.$$

3. A projectile is fired due south from a gun located at 45° latitude. The velocity of projection is 100 ft/sec and the angle of elevation of the gun above the horizon plane is 45°. Find the correction to the point of impact due to the Coriolus force.

   The problem is intended to demonstrate the Coriolus effect, but also to demonstrate how differential equations may sometimes be solved using "perturbation theory." In this problem ω is "small." We therefore expect the Coriolus force to result in a small correction (which we call a "first-order" correction) to the motion we would otherwise expect in the absence of ω. We use the presence of ω in the various terms of our equation to decide what to keep and what to ignore. If ω is small, then any terms involving ω^2 (or higher powers) are much smaller and can safely be ignored in the approximation. To apply perturbation theory, write down Newton's Law for the problem in a noninertial, Cartesian frame on the surface of the earth. Include any terms involving one power of ω, but discard any with higher powers of ω. This should leave you with the Coriolus term, but should allow you to discard the centrifugal term. Now assume that $x(t) = x_0(t) + \omega x_1(t) + \ldots$ (and similar expressions for $y(t), z(t)$). Substitute these forms into the equations of motion and separately equate terms that have no ω in them and then separately terms that have a single power of ω. Discard any terms that have higher orders of ω. Now systematically solve first the zeroth-order equations and use these solutions to solve for the first order corrections. When you have completed the problem, review what you have done and try to identify techniques that have more general applicability than this particular example.

4. A ten-ton freight car is headed due north at 45° latitude on the earth's surface at a velocity of 100 ft/sec. What is the sidewise force on the railroad tracks due to the Coriolus force? (Note: This is a very simple illustration of the magnitude of the Coriolus force. You will not need perturbation theory to solve the problem.)

5. The acceleration due to gravity at the equator of the planet Uranus, exclusive of the centrifugal acceleration, is about 950cm/sec^2, a value which is close to that on earth. The period of rotation of the planet is quite short, being about 10.7 hour. The mean radius of Uranus is 23.8×10^3 km or about 3.73 earth radii. What would be the actual measured value of the acceleration of a falling object at the equator (including Coriolus and transverse accelerations but not centrifugal which is second order small for small ω)? A rocket is to be fired vertically from the surface of Uranus at the equator. What is the minimum velocity for escape measured relative to the planet? What would be the initial direction of the rocket as seen by an observer in an inertial frame? (T. C. Bradbury, *Theoretical Mechanics*, John Wiley & Sons, 1968). (Note: While the centrifugal term is proportional to ω^2 and can be neglected in this problem, the term we associate with the symbol a_0 is also pro-

portional to ω^2 but cannot be neglected for a noninertial frame fixed to the surface of the planet. Do you see the difference? The problem also illustrates the concept of "escape velocity," i.e. the firing velocity such that the rocket would just come to rest at infinite distance. It is the velocity which makes the total energy of the rocket be zero.)

Ans:

$$g = 887 \text{cm/sec}^2$$
$$\theta_{\text{initialdirection}} = 10.5°.$$

6. Find the Lagrangian for a satellite launched into an arbitrary orbit around the earth from the point of view of observers on the earth. Use spherical coordinates with origin at the earth's center and fixed with respect to the earth. Is there a missing coordinate in the Lagrangian? If so, find a constant of the motion. What does the orbit look like for the special initial conditions, $\dot{\phi} = -\omega$, i.e. how does the Lagrangian simplify in this special case and what kind of simplified motion does the Lagrangian now describe?

   Ans:
$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta(\dot{\phi} + \omega)^2) + \frac{GmM}{r}$$

7. Experimental: Measure the period of the Foucault pendulum in the foyer of the Eyring Science Center and use it to estimate the length of the wire (in meters) that suspends the pendulum bob. Thus, estimate how high the building rises above the foyer since the top of the pendulum wire is fastened to a point just under the telescope dome at the top of the building. Then, by further observing that the pendulum's motion has a slow progression of 9.65 degrees per hour, estimate what the latitude of Provo is.

8. Experimental: Experience the rotating room (also called the Vomit Vortex). The Vomit Vortex is a large metal cylinder that looks sort of like two livestock watering troughs stacked together to make a round room. The room contains a bench, a table, a pendulum attached to a magnet that sticks to the ceiling, and a box of balls of various sizes and densities. Get a group of 3-6 people together, find either your instructor or a designated alternate so that the room can be located, unlocked, and plugged in. Do the following experiments and write a short report describing what you did.

- Have someone stand outside and measure ω, magnitude and direction, for the room.
- Measure the deviation of the hanging pendulum from the vertical and check to see if your measurement agrees with what you would expect from the centrifugal term in the acceleration expression for non-inertial frames.
- Calculate the direction of the Coriolus force and verify, by playing catch with each other in the room, that your calculated direction is correct.
- Think up something creative to do in the room and see if you can explain what happens.

9. Computer Project 11 (See Appendix)

Chapter 7

Systems of Interacting Particles

We may sum over particles to extend definitions to systems of interacting particles.

7.1 Definitions and Theorems

7.1.1 Definitions

$$\mathbf{F} = \sum \mathbf{F}_\alpha = \sum \mathbf{F}_\alpha^{internal} + \sum \mathbf{F}_\alpha^{external}$$

$$\mathbf{p} = \sum \mathbf{p}_\alpha$$

$$\boldsymbol{\ell} = \sum \boldsymbol{\ell}_\alpha = \sum \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

$$\mathbf{N} = \sum \mathbf{N}_\alpha = \sum \mathbf{r}_\alpha \times \mathbf{F}_\alpha$$

7.1.2 Theorem

$$\mathbf{F} = \dot{\mathbf{p}}$$

Proof: For each particle, we have $\mathbf{F}_\alpha = \dot{\mathbf{p}}_\alpha$. Thus, by definition,

$$\mathbf{F} = \sum \mathbf{F}_\alpha = \sum \dot{\mathbf{p}}_\alpha = \frac{d}{dt} \sum \mathbf{p}_\alpha = \frac{d}{dt}\mathbf{p} = \dot{\mathbf{p}}.$$

QED.

7.1.3 Theorem

$$\mathbf{N} = \dot{\boldsymbol{\ell}}$$

Proof: For each particle, we have

$$\mathbf{N}_\alpha = \dot{\boldsymbol{\ell}}_\alpha.$$

Thus,

$$\begin{aligned}\mathbf{N} = \sum \mathbf{N}_\alpha = \sum \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \sum \mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha \\ = \frac{d}{dt} \sum (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) - \sum \dot{\mathbf{r}}_\alpha \times \mathbf{p}_\alpha.\end{aligned}$$

But,

$$\dot{\mathbf{r}}_\alpha \times \mathbf{p}_\alpha = m_\alpha \dot{\mathbf{r}}_\alpha \times \dot{\mathbf{r}}_\alpha = 0.$$

So,

$$\mathbf{N} = \frac{d}{dt} \sum (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \frac{d}{dt} \sum \boldsymbol{\ell}_\alpha = \frac{d}{dt} \boldsymbol{\ell} = \dot{\boldsymbol{\ell}}.$$

QED.

7.1.4 Definition

An *isolated system* is defined to be a system on which there are no external forces of any kind. For such a system, the total linear and angular momentum is assumed to be constant.

7.1.5 Theorem

For an isolated system,

$$\sum \mathbf{F}_\alpha^{internal} = 0$$
$$\sum \mathbf{N}_\alpha^{internal} = 0$$

Proof: If the total linear momentum of the system is assumed to be constant, we have,

$$\dot{\mathbf{p}} = 0 = \mathbf{F} = \sum \mathbf{F}_\alpha = \sum \mathbf{F}_\alpha^{internal} + \sum \mathbf{F}_\alpha^{external}.$$

But, the sum of external forces vanishes for an isolated system. Hence,

$$\sum \mathbf{F}_\alpha^{internal} = 0.$$

QED.

Similarly,

$$\mathbf{N}_\alpha = \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \mathbf{r}_\alpha \times (\mathbf{F}_\alpha^{int} + \mathbf{F}_\alpha^{ext}) = \mathbf{N}_\alpha^{int} + \mathbf{N}_\alpha^{ext}.$$

If the total angular momentum of the system is assumed to be constant, we have,

$$\dot{\boldsymbol{\ell}} = 0 = \mathbf{N} = \sum \mathbf{N}_\alpha^{int} + \sum \mathbf{N}_\alpha^{ext}.$$

If there are no external forces of any kind on an isolated system, the external torques must vanish because the external forces vanish. Thus,

$$\sum \mathbf{N}_\alpha^{int} = 0.$$

QED.

7.1.6 Theorem

If the net force on a system is zero and if the net torque about any one point is known to be zero, then the net torque about any other point is necessarily zero.

Proof: We assume that the net torque vanishes relative to some point, O. Thus, relative to this point, $\sum \mathbf{r}_\alpha \times \mathbf{F}_\alpha = 0$. Let a point O' have a position $\mathbf{R}$ relative to O, so that $\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$. We then have,

$$0 = \sum(\mathbf{R} + \mathbf{r}'_\alpha) \times \mathbf{F}_\alpha = \mathbf{R} \times \sum \mathbf{F}_\alpha + \sum \mathbf{r}'_\alpha \times \mathbf{F}_\alpha.$$

But the sum of the forces vanishes, so,

$$\sum \mathbf{r}'_\alpha \times \mathbf{F}_\alpha = 0$$

i.e. the net torque about O' vanishes. QED.

7.1.7 Theorem

The work done on a system of particles by the net force (including external and internal forces) equals the change in kinetic energy of the system.

Proof: Let us assume that a general displacement of a system is made so that each particle is displaced by an amount ds_α. We assume that each particle has a net force, $\mathbf{F}_\alpha$, acting on it (which includes both internal and external forces). The work done is,

$$dW = \sum \mathbf{F}_\alpha \cdot d\mathbf{s}_\alpha = \sum m_\alpha \frac{d\mathbf{v}_\alpha}{dt} \cdot \mathbf{v}_\alpha dt = \frac{d}{dt}(\sum \frac{1}{2} m_\alpha v_\alpha^2) dt$$

$$= \sum d(\frac{1}{2} m_\alpha v_\alpha^2)$$

$$= dT.$$

QED. Observe that $\sum \mathbf{F}_\alpha^{int} = 0$ does not imply that $\sum \mathbf{F}_\alpha^{int} \cdot d\mathbf{s}_\alpha = 0$!

7.1.8 Theorem

If the forces on a system are conservative, the work done on a system of particles by the net force is equal to the change in potential energy of the system. In such systems, energy is conserved, i.e., $E = T + V$.

Proof: If the forces on the system are all conservative, we assume that the forces on each particle may be written $\mathbf{F}_\alpha = -\nabla_\alpha V$, where V is a function of the $3N$ coordinates of the N particles,

$$V = V(x_1, x_2, x_3, x_4, ..., x_{3N})$$

and ∇_α is the gradient with respect to the coordinates of a given particle. Using the fact that $dV_\alpha = \nabla_\alpha \cdot d\mathbf{s}_\alpha$, we have,

$$dW = \sum \mathbf{F}_\alpha \cdot d\mathbf{s}_\alpha = -\sum \nabla_\alpha V \cdot d\mathbf{s}_\alpha = -\sum dV_\alpha = -dV.$$

Thus, $dW = -dV$. QED.

For conservative systems, we have, therefore, $dW = dT$ and $dW = -dV$. Subtracting the two expressions, we have $d(T + V) = 0$, from which we conclude that $T + V$ is a constant to which we give the symbol E. Thus, $E = T + V$. QED.

7.1.9 Definition

The position of the *center of mass* $\mathbf{R}_c$ is defined by

$$M = \sum m_\alpha$$

$$\mathbf{R}_c = \frac{1}{M} \sum m_\alpha \mathbf{r}_\alpha.$$

7.1.10 Theorem

$$\mathbf{p} = M\mathbf{V}_c,$$

where,

$$\mathbf{V}_c = \frac{d\mathbf{R}_c}{dt}.$$

Proof: By definition,

$$\mathbf{R}_c = \frac{1}{M}\sum m_\alpha \mathbf{r}_\alpha.$$

If we differentiate this expression with respect to time,

$$\frac{d\mathbf{R}_c}{dt} = \mathbf{V}_c = \frac{1}{M}\sum m_\alpha \dot{\mathbf{r}}_\alpha = \frac{1}{M}\mathbf{p}.$$

Hence,

$$\mathbf{p} = M\mathbf{V}_c.$$

QED.

7.1.11 Theorem

$$\mathbf{F} = M\mathbf{a}_c,$$

where,

$$\mathbf{a}_c = \frac{d\mathbf{V}_c}{dt}.$$

Proof: If we differentiate $\mathbf{p} = M\mathbf{V}_c$ with respect to time,

$$\dot{\mathbf{p}} = \mathbf{F} = M\frac{d\mathbf{V}_c}{dt} = M\mathbf{a}_c.$$

QED.

7.1.12 Theorem

If $\mathbf{r}'_\alpha$ are the position vectors of particles relative to the center of mass, then,

$$\sum m_\alpha \mathbf{r}'_\alpha = 0.$$

Proof: Relative to the center-of-mass, the position vector of the center-of-mass vanishes. Thus, if $\mathbf{r}'_\alpha$ are measured relative to the center of mass,

$$\mathbf{R}_c = \frac{1}{M}\sum m_\alpha \mathbf{r}'_\alpha = 0,$$

which implies that $\sum m_\alpha \mathbf{r}'_\alpha = 0$. QED.

If $\mathbf{v}_\alpha = \mathbf{V}_c + \mathbf{v}'_\alpha$ where $\mathbf{v}'_\alpha = \dot{\mathbf{r}}'_\alpha$, then,

$$\sum m_\alpha \mathbf{v}'_\alpha = 0.$$

Proof: Differentiating $\sum m_\alpha \mathbf{r}'_\alpha = 0$ with respect to time, we have $\sum m_\alpha \mathbf{v}'_\alpha = 0$. QED.

7.1.13 Theorem

The total angular momentum of a system with respect to a fixed point is given by:

$$\boldsymbol{\ell} = \mathbf{R}_c \times M\mathbf{V}_c + \sum(\mathbf{r}'_\alpha \times m_\alpha \mathbf{v}'_\alpha),$$

i.e.,

$$\boldsymbol{\ell} = \boldsymbol{\ell}_{cm} + \boldsymbol{\ell}'.$$

Proof: Let $\mathbf{r}_\alpha = \mathbf{R}_c + \mathbf{r}'_\alpha$. Then, by definition,

$$\boldsymbol{\ell} = \sum \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha = \sum(\mathbf{R}_c + \mathbf{r}'_\alpha) \times m_\alpha(\mathbf{V}_c + \mathbf{v}'_\alpha)$$

$$= (\mathbf{R}_c \times \mathbf{V}_c \sum m_\alpha + \mathbf{R}_c \times (\sum m_\alpha \mathbf{v}'_\alpha) + (\sum m_\alpha \mathbf{r}'_\alpha) \times \mathbf{V}_c + \sum \mathbf{r}'_\alpha \times m_\alpha \mathbf{v}'_\alpha.$$

However, since $\sum m_\alpha \mathbf{v}'_\alpha = 0$ and $\sum m_\alpha \mathbf{r}'_\alpha = 0$, we have,

$$\boldsymbol{\ell} = \mathbf{R}_c \times M\mathbf{V}_c + \sum \mathbf{r}'_\alpha \times m_\alpha \mathbf{v}'_\alpha = \boldsymbol{\ell}_{cm} + \boldsymbol{\ell}'.$$

QED.

7.1.14 Theorem

The kinetic energy of a system can be decomposed,

$$T = \frac{1}{2}MV_c^2 + \sum \frac{1}{2}m_\alpha v_\alpha'^2.$$

Proof:

$$\begin{aligned} T &= \sum \frac{1}{2}m_\alpha v_\alpha^2 = \sum \frac{1}{2}m_\alpha(\mathbf{V}_c + \mathbf{v}_\alpha') \cdot (\mathbf{V}_c + \mathbf{v}_\alpha') \\ &= \frac{1}{2}\sum(m_\alpha)V_c^2 + \mathbf{V}_c \cdot \sum m_\alpha \mathbf{v}_\alpha' + \frac{1}{2}\sum m_\alpha v_\alpha'^2. \end{aligned}$$

But, we have $\sum m_\alpha \mathbf{v}_\alpha' = 0$, so,

$$T = \frac{1}{2}MV_c^2 + \sum \frac{1}{2}m_\alpha v_\alpha'^2.$$

QED.

Chapter 8

Systems of Interacting Particles: Two or Three Particles

8.1 Two-Body Central Force

8.1.1 Definition

A force that depends only on the distance between particles and is directed along the line joining the particles is a *central force*. Central forces are conservative.

8.1.2 Reduced Mass

In this section we will consider two particles that interact via a central force. As a type, we will use force that satisfies an inverse square law, i.e., the force on one particle exerted by the other depends inversely on the square of the distance separating them. However, in treating this specific choice of force law, we will use techniques that can be used with the more general family of force laws, k/r^n. We will show how a "reduced mass" and constants of the motion can be used to reduce the two dimensional problem to a one-dimensional differential equation for motion of a particle in an "effective potential." We will also show how the time-dependent differential equation can be converted to a differential equation for the orbit, i.e., an equation relating r and θ. As you see these ideas introduced, remember that they are more general than the specific (but very important) problem of the inverse square law of force.

Locate a set of axes at the center of mass which will be some point on the line connecting the two particles. Then, relative to the center of mass,

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$$

$$\boldsymbol{\ell} = \mathbf{r}_1 \times m_1\mathbf{v}_1 + \mathbf{r}_2 \times m_2\mathbf{v}_2$$

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2.$$

Define,

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

Thus defined, $\mathbf{r}$ is the vector position of m_1 relative to m_2. We can solve the two equations,

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

to give,

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

and, differentiating with respect to time,

$$\mathbf{v}_1 = \frac{m_2}{m_1 + m_2}\mathbf{v}$$

$$\mathbf{v}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{v}.$$

We can then rewrite the angular momentum and kinetic energy in terms of $\mathbf{r}$ and $\mathbf{v}$,

$$\boldsymbol{\ell} = \mathbf{r} \times \frac{m_1m_2}{m_1 + m_2}\mathbf{v}$$

$$T = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v^2.$$

If we define the *reduced mass*

$$m = \frac{m_1m_2}{m_1 + m_2},$$

we then have the the angular momentum and kinetic energy written in a form suggestive of single particle motion,

$$\boldsymbol{\ell} = \mathbf{r} \times m\mathbf{v}$$

$$T = \frac{1}{2}mv^2.$$

Note that if $m_2 >> m_1$, then $m \approx m_1$, $\mathbf{r}_2$ is small, and $\mathbf{r}_1 \approx \mathbf{r}$. Such is the case, for example, for the earth revolving about the sun and an electron revolving about a nucleus.

8.1.3 Constants of the Motion and Effective Potential

If we adopt plane polar coordinates, r and θ for our system, we may write,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Because we are assuming a central force law, the problem has symmetry in θ. Therefore θ does not appear explicitly in the Lagrangian. Thus, we have a constant of the motion which turns out to be the magnitude of the angular momentum,

$$\ell = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

We may use the fact that

$$\dot{\theta} = \frac{\ell}{mr^2}$$

to eliminate $\dot{\theta}$ from the equations describing the system. Lagrange's equation for r is

$$m\ddot{r} - mr\dot{\theta}^2 - \frac{\partial V}{\partial r} = 0.$$

Eliminating $\dot{\theta}$, we have

$$m\ddot{r} - \frac{\ell^2}{mr^3} - \frac{\partial V}{\partial r} = 0$$

$$E = \frac{1}{2}m\dot{r}^2 + (\frac{\ell^2}{2mr^2} + V).$$

The energy equation now looks very much like the equation for the one-dimensional motion of a single particle with coordinate r. Of course, we have grouped a piece of the true kinetic energy inside the parentheses with $V(r)$ to make the problem look one-dimensional. The true motion is two-dimensional with r and θ both varying. But, given that we have made it look like one-dimensional motion, it is natural to define an *effective potential* V_{eff},

$$V_{eff} = \frac{\ell^2}{2mr^2} + V(r).$$

The term from the kinetic energy that we have moved over into the effective potential gets very large for small r. For a fixed value of E, the presence of this term keeps the system away from sufficiently small values of r and prevents the two particles from coalescing. Hence we refer to this term as a *centrifugal barrier*. The use of an effective potential is a general technique that can be used whenever there is enough symmetry to reduce a multi-dimensional problem to a quasi one-dimensional problem.

If the effective potential is shaped (as a function of r) such that it has a minimum and if E is greater than the minimum of V but $E < 0$, then the particle is said to be in a *potential well* and the motion will be periodic increasing to a maximum r_{max} and decreasing to a minimum r_{min}. The minimum and maximum values of r are those values when $\dot{r}$ vanishes and all of the energy is represented by the effective potential. If E is exactly equal to the minimum of the effective potential, then the minimum and maximum radii merge into one and the periodic orbit is circular.

8.1.4 Orbit Equation

There is a second path that we can follow. Consider,

$$\ell = mr^2\dot{\theta}.$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$m\ddot{r} - \frac{\ell^2}{mr^3} = -\frac{\partial V}{\partial r} = F(r).$$

Define $x = 1/r$. Then,

$$\frac{dx}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -x^2\frac{\dot{r}}{\dot{\theta}}.$$

Thus,

$$\dot{r} = -\frac{\ell}{m}\frac{dx}{d\theta}$$

and,

$$\ddot{r} = -\frac{\ell^2}{m^2}x^2\frac{d^2x}{d\theta^2}. \quad = -\frac{\ell^2}{m^2}x^2 x''$$

Our equations then become,

$x^2 - 2bx - c = 0$

$x = b \pm \sqrt{b^2+c}$

$$\dot{\theta} = \frac{\ell}{m}x^2$$

$-kx^2$

eqn of motion $$\frac{d^2x}{d\theta^2} + x = -\frac{m}{\ell^2}\frac{F(x)}{x^2}$$ $x'' + x = \frac{mk}{\ell^2}$

* energy eq. $$E = \frac{1}{2}\frac{\ell^2}{m}(\frac{dx}{d\theta})^2 + \frac{\ell^2x^2}{2m} + V(x).$$ $x'^2 + x^2 = 2x\underbrace{\frac{mk}{\ell^2}}_{b=1/p} + \underbrace{\frac{2mE}{\ell^2}}_{c}$

$-kx$

In this latter form, the solution of the equations yields the *orbit equation*, i.e., r as a function of θ.

$x'^2 = 0: \; x_m = b \pm \sqrt{b^2+c}$

$= 1/p(1 \pm \sqrt{1 + c/p^2})$

8.1.5 Inverse Square Law: Attraction

$\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0$

We will now take as our force law,

$r^2 + 2br + \omega_0^2 = 0$

$$F_r = -\frac{k}{r^2}.$$

$r = -b \pm \sqrt{b^2 - \omega_0^2}$

$= -b \pm b_1, \; b_1 = i\omega_1$

If $k > 0$, the force is attractive, such as would be the case for the gravitational force ($k = Gm_1m_2$) or the electrostatic attraction of two opposite charges, ($k = |Q_1||Q_2|$). If $k < 0$, the force is repulsive. The inverse square law force is conservative and derivable from a potential,

$$V(r) = -\frac{k}{r}.$$

For the inverse square law the orbit equation becomes,

$$\frac{d^2x}{d\theta^2} + x = \frac{mk}{\ell^2}.$$

This equation has ~~a~~ the general transient solution $x = A\cos(\theta - \theta_0)$ and a particular solution $x_p = mk/\ell^2$. Thus,

$$x = \frac{1}{r} = \frac{mk}{\ell^2} + A\cos(\theta - \theta_0)$$

where A and θ_0 are determined by initial conditions. If we let $\theta_0 = 0$, then r will be at its minimum value r_{min} (also called the perigee or perihelion) when $\theta = 0$. Henceforth, we will assume this initial condition. If there is a maximum at all, r_{max} will occur when $\theta = \pi$.

If we let

$$e = \frac{A\ell^2}{mk}$$

and,

$$p = \frac{\ell^2}{mk},$$

we can put our solution in the form,

$$\frac{p}{r} = 1 + e\cos\theta.$$

$x_m = \frac{1 \pm e}{p} = \frac{mk}{\ell^2} \pm A$ $=$

$r_m = \frac{p}{1 \pm e} = a(1 \mp e)$

$p = a(1-e^2)$

This is a standard form for a conic section (circle, ellipse, parabola, hyperbola) with origin at the focus. In this form e is identified as the *eccentricity* of the orbit. The respective cases are:

$$e = 0 : \textit{circle}$$

$$e < 1 : \textit{ellipse}$$

$$e = 1 : \textit{parabola}$$

$$e > 1 : \textit{hyperbola}.$$

The equation of the conic section can also be written in terms of E and ℓ instead of r_{min} and e. We may solve for r_{min} (and r_{max} if its exists) in terms of E and ℓ from the energy equation.* These turning points occur when $\dot{r} = 0$. Then,

$$\frac{\ell^2}{2mr_m^2} - \frac{k}{r_m} = E.$$

Solving,

$$r_m = -\frac{k}{2E} \pm \sqrt{\frac{k^2}{4E^2} + \frac{\ell^2}{2mE}}.$$

An alternative is to solve for $x_m = 1/r_m$ from

$$\frac{\ell^2 x_m^2}{2m} - kx_m = E$$

yielding,

$$x_{max} = \frac{1}{r_{min}} = \overset{1/p}{\frac{mk}{\ell^2}} \pm \overset{1/p}{\frac{mk}{\ell^2}} \overset{e}{\sqrt{1 + \underbrace{\frac{2\ell^2 E}{mk^2}}_{e/p^2}}}.$$

If this latter form is compared to

$$\frac{1}{r_{min}} = \overset{1/p}{\frac{mk}{\ell^2}} + \overset{e/p}{A}$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

we can identify,

$$A = \frac{mk}{\ell^2}\sqrt{1 + \frac{2\ell^2 E}{mk^2}}$$

$$\frac{\ell^2}{mk} = \frac{(m v r)^2}{m G M m} = \frac{v_\perp^2 r^2}{GM} = r_0$$

and,

$$\frac{GM}{r^2} = \frac{v^2}{r} \quad T = -2V$$

$$e = \sqrt{1 + \frac{2\ell^2 E}{mk^2}}$$

$$x'' \sim \ddot{r} = \frac{v_\perp^2}{r_0} + \frac{GM}{r_0^2} = 0$$

$$\text{at } r_0 = \frac{GM}{v_\perp^2}$$

so that we can write the orbit equation in the alternative form,

$$r = \frac{\ell^2/mk}{1 + \sqrt{1 + (2\ell^2 E)/(mk^2)}\cos\theta}.$$

$$\frac{2mE}{\ell^2} = \frac{m^2 v^2}{(m v_\perp r)^2} = \frac{v_\parallel^2 + v_\perp^2}{v_\perp^2 r_0^2}$$

The respective cases for the conic sections are then,

$$E = -\frac{mk^2}{2\ell^2} \rightarrow e = 0 : circle$$

$$E < 0 \rightarrow e < 1 : ellipse$$

$$E = 0 \rightarrow e = 1 : parabola$$

$$E > 0 \rightarrow e > 1 : hyperbola.$$

If $E < 0$ (ellipse) we can also write the orbit equation in terms of the semi-major axis a and the semi-minor axis b. For an ellipse useful relations are,

$$r_{max} = a(1 + e)$$

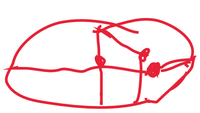
$$r_{min} = a(1 - e).$$

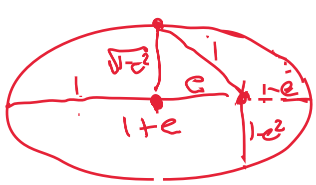
$$e^2 = 1 - (\frac{b}{a})^2.$$

$$a\,\frac{1-e^2}{r} = 1 + e\cos\theta$$

Then,

$$r = \frac{a(1-e^2)}{1 + e\cos\theta} = \frac{(b^2/a)}{1 + \sqrt{1 - (b/a)^2}\cos\theta.} = \frac{b^2}{a + \sqrt{a^2 - b^2}\cos\theta}.$$





$b^2 = a^2 - f^2$ $= ap = r_+ r_-$ $r_\pm = a \pm f$	$a \sim 1$ $b \sim \sqrt{1-e^2}$ $p \sim 1-e^2$ $f \sim e$ $r_\pm \sim 1 \pm e$

Comparing to the earlier form in terms of E and ℓ, we can identify,

$$\frac{b^2}{a} = \frac{\ell^2}{mk}$$

$$(\frac{b}{a})^2 = -\frac{2\ell^2 E}{mk^2}$$

from which we can easily show,

$$a = -\frac{k}{2E}$$

$$b = \frac{\ell}{\sqrt{-2mE}}.$$

We are now position to prove *Kepler's Laws*:

8.1.6 Theorems: Kepler's Laws

1. Each planet moves in an ellipse with the sun as a focus.
2. The radius vector from the sun to the earth sweeps out equal areas in equal times.
3. The square of the period of revolution about the sun is proportional to the cube of the major axis of the orbit.

Proof: We have demonstrated above that the orbits in a central, inverse-square law force field satisfy the standard form for an ellipse,

$$\frac{p}{r} = 1 + e\cos\theta$$

in the case that $e < 1$. This constitutes the proof of Kepler's first law. QED.

We have also demonstrated that the two-body system has a constant of the motion (angular momentum), $\ell = mr^2\dot{\theta}$. If we write this as

$$dt = \frac{m}{\ell}r^2 d\theta = \frac{2m}{\ell}\frac{1}{2}(r)(rd\theta),$$

We observe that $1/2(r)(rd\theta)$ represents a small element of area swept out by the radius vector of the position of the planet. (Think of a very long, narrow triangle with altitude, r, and base, $rd\theta$.) Thus, the radius vector sweeps out equal areas in equal times. This is Kepler's Second Law. QED.

Kepler deduced the first two laws empirically from detailed analysis of data gathered earlier by Tycho Brahe. They were first published in 1609. The third law was published later in 1618. Observe that Kepler's Second Law is a consequence only of the central nature of the force and not of its inverse-square dependence.

If we integrate Kepler's Second Law over one period and use the formula for the area of an ellipse, we have,

$$P = \frac{2m}{\ell}A = \frac{2m}{\ell}(\pi ab).$$

But,

$$\frac{b^2}{a} = \frac{\ell^2}{mk},$$

so we have,

$$P = \frac{2m}{\ell}\pi a(\frac{\ell}{\sqrt{mk}}a^{\frac{1}{2}}).$$

Since $k = Gm_1m_2$ and $m = (m_1m_2)/(m_1 + m_2)$, we have,

$$P^2 = \frac{4\pi^2a^3}{G(m_1 + m_2)}.$$

QED.

8.1.7 Inverse Square Law: Repulsion

If the inverse-square law force is repulsive rather than attractive, then the force constant $k < 0$. We must then modify our derivation of the orbits slightly by returning to

$$\frac{d^2x}{d\theta^2} + x = \frac{mk}{\ell^2}$$

and replacing k with $-|k|$. The explicit minus sign then results in our solution being written as,

$$x = -\frac{m|k|}{\ell^2} + A\cos(\theta - \alpha).$$

Consequently, we have

$$\frac{p}{r} = -1 + e\cos\theta,$$

where,

$$p = \frac{\ell^2}{m|k|},$$

and,

$$e = \sqrt{1 + \frac{2\ell^2 E}{mk^2}}.$$

In this case of repulsion we always have $E > 0$ and $e > 1$; the orbit is always a hyperbola.

8.2 Cross Sections

We now turn to another use of two-body interactions that serves as a type for a much more general technique of physics. When physicists try to understand the atomic and subatomic structure of matter, they are faced with the problem that atoms are very tiny things indeed. Since the time of Rutherford, people have turned to scattering experiments to probe the structure and physics of interaction of atoms and elementary particles. In these experiments a beam of particles is projected onto a target and detectors observe the directions, energies, momenta, etc. of the particles that come out.

What goes in is a beam, a stream J, of some number of particles per unit area per unit time. This is a directly measurable quantity. What comes out is a number of particles dN per unit time. This also is a measurable quantity. The ratio of what comes out to what goes in has the dimensions of [*area*] and is called a *differential cross section.*,

$$d\sigma = \frac{dN}{J}.$$

The cross section $d\sigma$ may depend on the energy of the impinging particles, on the physical size of the projectile and target particles, on the energy, momentum, charge, spin, angle of scatter and type of the emerging particles, etc. In short, much of the physics of what went on in the interaction is buried in $d\sigma$. Theorists calculate cross sections and experimentalists measure them for comparison so that the theories can be sorted for validity. It would be hard to overemphasize the importance of the concept of a cross section in modern physics.

Cross sections are defined for processes. If you project a beam of particles and something happens as a result, you can define a cross section for that process. We are going to concentrate on a particular process, but the ideas we use are much more general than the specific example. The process we choose is the scattering of alpha particles from gold nuclei, the process that allowed Rutherford to postulate the very existence of a nucleus. We will use reduced mass coordinates. Since the gold nucleus is much more massive than the alpha particle, the description is very nearly the scattering of alpha particles from a fixed center of force.

Consider a beam of charged particles particles J impinging on a single charged target particle such as a nucleus in a very thin foil of gold. The particles emerge from the target deflected by an angle ϕ from their original direction. If we draw a line parallel to the beam running through the center of the gold nucleus, the beam will have particles whose line of approach is at varying distances from this line. The perpendicular distance from the center-line to the line of approach of each particle is its *impact parameter* s. Particles with very large impact parameters do not get very close to the nucleus and are scattered very little. Particles with very small impact parameters come very close to the nucleus and are scattered through a large angle. Hence, we conclude that there is a functional relationship between s and ϕ. Particles with impact parameters between s and $s + ds$ are scattered into a cone with angles between ϕ and $\phi - d\phi$. The cone has a solid angle $d\Omega = 2\pi \sin\phi d\phi$.

The process that we are trying to characterize is the scattering of alpha particles into the cone at angle ϕ. All of the beam particles that pass through an annulus (while still very distant from the target) defined by

$$d\sigma = 2\pi s ds$$

will be scattered into the cone. The ratio $d\sigma/d\Omega$ is the differential cross section per unit solid angle,

$$\frac{d\sigma}{d\Omega} = -\frac{2\pi s ds}{2\pi \sin\phi d\phi} = -\frac{s}{\sin\phi}\frac{ds}{d\phi}.$$

(The negative sign is added because increasing s results in decreasing ϕ.) If we can establish a functional relationship between s and ϕ, we can calculate $ds/d\phi$ and have a theoretical prediction for the number of particles expected per second per unit solid angle per target nucleus.

The force law between the alpha particles and the gold nucleus is an inverse-square law. In this case, the force is repulsive. The total energy is the kinetic energy of the incoming particle while it is still at great distance of separation and must therefore be positive. The alpha particle follows a hyperbola. While there is a point of closest approach r_{min}, there is no r_{max}. The alpha particles approach from $-\infty$ and recede to ∞.

We have already seen the equation for the hyperbola,

$$r = \frac{(\ell^2/mk)}{1 + \sqrt{1 + (2\ell^2 E)/(mk^2)} \cos\theta},$$

where θ is measured from the point of closest approach, r_{min}. If v_0 is the speed of the beam particles while they are far removed from the target, then

$$E = \frac{1}{2} m v_0^2$$

and,

$$\boldsymbol{\ell} = \mathbf{r} \times m\mathbf{v}_0.$$

The magnitude of the angular momentum is given by $\ell = mv_0 s$. Since θ is measured from the point of closest approach, let the angle after scattering when the beam particle has moved far away from the target be α. Since the approach and the recession of the beam particle are symmetric with respect to the point of closest approach, we have

$$2\alpha + \phi = \pi,$$

i.e.,

$$\cos\alpha = \sin(\phi/2).$$

After scattering, r recedes to ∞, θ becomes α, and the denominator of the right-hand-side of the expression for the hyperbola must approach zero. We then have,

$$1 + \sqrt{1 + (4s^2E^2)/(k^2)} \sin(\phi/2) = 0$$

which is our desired functional relationship between s and ϕ. After differentiation and use of some trigonometric identities, we have,

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16E^2} \frac{1}{\sin^4(\phi/2)}.$$

This is the differential cross section per unit solid angle for Rutherford scattering. Observe that it is not purely a geometric area. The expression depends on the energy of the beam particles. Beam particles of high energy are not scattered as much as beam particles of low energy. There is also a very sharp dependence on angle. For very small scattering angle (corresponding to minimal deflection) the cross section is very large, meaning that most beam particles pass through the target virtually undeflected.

The *total cross section* is defined by,

$$\sigma = \int d\sigma.$$

In the present case we would compute this as,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int \frac{d\sigma}{d\Omega} 2\pi \sin\phi d\phi.$$

For Rutherford scattering, the total cross section is infinite. Since the inverse square law is a long-range force, all beam particles are scattered to some degree and the target nucleus acts like it reaches out to infinite size. For short range forces, the total cross section is well defined.

Total cross sections are used in a variety of ways. If a molecule moves through a gas of number density n, the mean free path λ between collisions is related to a cross section,

$$\lambda = \frac{1}{n\sigma}$$

and the collision frequency ν is,

$$\nu = v/\lambda = vn\sigma.$$

If a beam passes through a gas of number density n which removes particles by some process characterized by a cross section, the attenuation as a function of the distance through the gas x is given by,

$$I = I_0 e^{-n\sigma x}.$$

8.3 Problems

1. A particle in a central force field moves in a spiral orbit $r = c\theta^2$. Determine the form of the force function as a function of r and how the angle varies with time. (Partial answer:

$$\theta \propto t^{1/5}$$

The problem is really two separate problems. The first part is intended to illustrate the general method by which the differential equation that relates r and t is transformed to one relating r and θ using a constant of the motion and the substitution $r = 1/x$. This is necessary in this case because you are given a relationship between r and θ from which you are to infer a form for the central force. The second part can be worked directly from the constant of the motion identified from the missing coordinate in the Lagrangian. After completing the problem, review the solution and identify techniques that have wider applicability than this particular example.

2. Show that for a circular orbit,

$$v_c^2 = \frac{C}{mr_{min}}.$$

If v_0 is the speed of a particle at $\theta = 0$, show that one may write the orbit equation for inverse square law motion as,

$$r = r_{min}\frac{(v_0/v_c)^2}{1 + [(v_0/v_c)^2 - 1]\cos\theta}.$$

The purpose of the problem is to demonstrate how the orbit equation, $p/r = 1 + e\cos\theta$, can be transformed to another useful form. The force here is assumed to be of the inverse square type, $F = -C/r^2$.

3. A moon rocket is initially travelling in a circular orbit near the earth. It is desired to place the rocket into a new orbit for which the apogee is equal to the radius of the moon's orbit around the earth (240,000 miles). Calculate the ratio of the new speed to the speed in the circular orbit needed to accomplish this. Assume that the radius of the original circular orbit is 4000 miles. Draw a sketch to illustrate the maneuver. Calculate the new apogee if the speed ratio is 0.99 of the value calculated above. The result should illustrate the extreme accuracy needed to achieve a circumlunar orbit. (Note: This problem is meant to be an application of the result of the previous problem.)

4. A comet is observed to have a speed v_0 when it is a distance r_0 from the sun (of mass M) and when its direction of motion makes an angle

ϕ with the radius vector from the sun. Show that the eccentricity of the orbit can be written:

$$e = \sqrt{1 + \left(v_0^2 - \frac{2GM}{r_0}\right) \frac{r_0^2 v_0^2 \sin^2 \phi}{G^2 M^2}}.$$

5. The earth's orbit is nearly circular e=0.017, aphelion=95,000,000 miles, perihelion = 91,000,000 miles). Assuming the earth's orbit to be circular, show that

$$GM = r_e v_e^2$$

$$e = \left(1 + (\frac{v_0^2}{v_e^2} - \frac{2r_e}{r_0})\frac{r_0^2 v_0^2}{r_e^2 v_e^2} \sin^2 \phi\right)^{1/2}.$$

If a comet is first observed to be at a distance of 1/3 astronomical unit (1 a.u.=93,000,000 miles) from the sun and is observed to be traveling at a speed of twice the earth's speed, is the comet's orbit elliptic, parabolic or hyperbolic? (Grant R. Fowles, *Analytical Mechanics*, Holt Rinehart Winston, 1970.)

6. A projectile is fired at an angle of 60 degrees above the horizontal and at a velocity of 5×10^5 cm/sec. Neglecting air resistance, what would be the range of the projectile measured on the earth's surface. To what height above the earth's surface does the projectile rise? Neglect any effect of the earth's rotation. The purpose of the problem is to be an example of orbital equations applied to the motion of an intercontinental ballistic missile.

7. Suppose that it is possible to measure the location of the center of mass and the period of rotation of a double star system. The orbits, both individually as well as relative to one another, may be considered to be approximately circular. r_1 and r_2 are measured from the center-of-mass and are also assumed known. Find formulas for the masses of the stars in terms of these data. The purpose of the problem is to illustrate an application of Kepler's Third Law and center-of-mass coordinates. Partial answer:

$$m_2 = \frac{4\pi^2}{GP^2}(r_1 + r_2)^2 r_1.$$

8. Computer Project 9 (See Appendix)

9. Computer Project 10 (See Appendix)

10. Computer Project 8 (See Appendix)

11. Hard spheres of radius b are elastically scattered by hard spheres of radius a. Find the relation between the scattering angle and the impact parameter in the center of mass coordinate system. Find the differential scattering cross section per unit solid angle, $d\sigma/d\Omega$, in the center-of-mass frame. Find the total cross section. The purpose of the problem is to illustrate the general defining relationship of the differential cross section,
$$\frac{d\sigma}{d\Omega} = -\frac{s}{\sin\phi}\frac{ds}{d\phi}.$$
The impact parameter is s and the scattering angle is ϕ. An analysis of the geometry of the situation is used to relate s and ϕ.

12.
    - A particle of energy E crosses a boundary where the potential changes abruptly from 0 to $-V_1$. As a consequence, the direction of its motion changes from θ_1 (relative to the normal to the interface) to θ_2. If the "index of refraction" is defined as
$$n = \sin\theta_1/\sin\theta_2$$
show that,
$$n = \sqrt{1 + (V_1/E)}.$$
    - Frequently used in nuclear physics is the "square well" to give an approximate description of the interaction between nucleons. It is defined by $V(r) = -V_0$ for $0 \le r \le a$ and $V(r) = 0$ for $r > 0$. Sketch the effective potential for such an interaction between two particles. Can bound orbits exist? Find $d\sigma/d\Omega$ and $d\sigma(\phi)$ for scattering by such a potential. The results are conveniently expressed in terms of the index of refraction defined above.

    Solution: Convince yourself that:

    (a)
$$n = \frac{\sin\theta_1}{\sin\theta_2}$$

(b)

$$\alpha = \theta_1 - \theta_2 = \phi/2$$

(c)

$$s = a \sin\theta_1.$$

Here, s is the impact parameter.

Then show,

$$s^2 = \frac{a^2 n^2 \sin^2(\phi/2)}{1 + n^2 - 2n\cos(\phi/2)}.$$

Eventually show,

$$\frac{d\sigma}{d\Omega} = \frac{a^2 n^2}{4\cos(\phi/2)} \frac{(n\cos(\phi/2) - 1)(n - \cos(\phi/2))}{(1 + n^2 - 2n\cos(\phi/2))^2}.$$

(T. C. Bradbury, *Theoretical Mechanics*, John Wiley & Sons, 1968)

Chapter 9

Systems of Interacting Particles: Rigid Bodies

9.1 Definitions and Theorems

9.1.1 Definition

A *rigid body* is a system of many particles constrained in such a way that the particles making up the body are held fixed relative to one another.

9.1.2 Definition

With respect to an axis, the *moment of inertia* I of a mass distribution is defined,

$$I = \sum (m_\alpha r_\alpha^2) \rightarrow \int r^2 dm.$$

where r_α is the perpendicular distance from a mass particle to the axis.

9.1.3 Theorem: Law of Parallel Axes

If a rigid body consists of a thin lamina, i.e., all the particles lie essentially in a plane, and if $\mathbf{r}'_\alpha$ is the position vector of a particle relative to an axis perpendicular to the plane and through the center of mass and if $\mathbf{R}$ is a vector from an arbitrary axis (also perpendicular to the plane) to the center of mass, then

$$I = I_{cm} + MR^2,$$

where I is the moment of inertia about the arbitrary axis, I_{cm} is the moment of inertia about the center of mass and M is the total mass of the system.

Proof: Let two parallel axes be defined which are perpendicular to the lamina and thus parallel to one another. Let O be the intersection point of the first axis with the lamina and let $\mathbf{r}_\alpha$ mark positions relative to this axis. The second axis passes through the center-of-mass at O'. $\mathbf{R}$ is the position of O' relative to O. We then have $\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$. Hence,

$$I = \sum m_\alpha r_\alpha^2 = \sum m_\alpha (\mathbf{r}'_\alpha + \mathbf{R}) \cdot (\mathbf{r}'_\alpha + \mathbf{R})$$

$$= \sum m_\alpha \mathbf{r}'^2_\alpha + 2\mathbf{R} \cdot \sum m_\alpha \mathbf{r}'_\alpha + R^2 \sum m_\alpha.$$

But, if O' is the center-of-mass, $\sum m_\alpha \mathbf{r}'_\alpha = 0$. Hence,

$$I = I_{cm} + MR^2.$$

QED.

9.1.4 Theorem: Law of Perpendicular Axes

The moment of inertia of any thin lamina about an axis normal to the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina, $I_z = I_x + I_y$.

Proof: Let the x and y axes lie in the lamina with the z axis perpendicular to the lamina. Then,

$$I_z = \sum m_\alpha r_\alpha^2 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = I_x + I_y.$$

QED.

9.1.5 Moments of Inertia of Uniform Objects

1. Thin rod of length a
   - Normal to the rod at its center: $\frac{1}{12}ma^2$
   - Normal to the rod at one end : $\frac{1}{3}ma^2$
2. Thin rectangular lamina with sides a and b
   - Through the center, parallel to b: $\frac{1}{12}ma^2$

- Through the center, perpendicular to the lamina: $\frac{1}{12}m(a^2+b^2)$

3. Thin circular disk of radius a
   - Through the center in the plane of the disk: $\frac{1}{4}ma^2$
   - Through the center normal to the plane of the disk: $\frac{1}{2}ma^2$
4. Thin hoop or annulus of radius a
   - Through the center in the plane of the disk: $\frac{1}{2}ma^2$
   - Through the center normal to the plane of the disk: ma^2
5. Thin cylindrical shell or radius a and length b
   - Central longitudinal axis: ma^2
6. Solid right circular cylinder or radius a and length b
   - Central longitudinal axis: $\frac{1}{2}ma^2$
   - Through the center, normal to the central axis: $m(\frac{a^2}{4}+\frac{b^2}{12})$
7. Thin spherical shell of radius a
   - About any diameter: $\frac{2}{3}ma^2$
8. Solid sphere
   - About any diameter: $\frac{2}{5}ma^2$
9. Solid rectangular parallelepiped with sides a, b, and c
   - Through the center normal to the plane of a and b and parallel to c: $\frac{1}{12}(a^2+b^2)$

9.2 Motion of Laminar Bodies about a Fixed Axis

9.2.1 Theorem

If a rigid body consists of a thin lamina and if the body rotates about a fixed axis which is normal to the plane of the body, then the governing equations of motion are:

$$\boldsymbol{\ell} = I\dot{\theta}\hat{\mathbf{n}},$$

$$\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt} = I\ddot{\theta}\hat{\mathbf{n}},$$

and,

$$T = \frac{1}{2}I\dot{\theta}^2.$$

Proof: Let $\hat{\mathbf{n}}$ be along the axis of rotation, i.e. perpendicular to the lamina with O located at the point of intersection of axis and lamina. Then,

$$\boldsymbol{\ell} = \sum \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha.$$

Because the origin is at the axis of rotation, $\mathbf{r}_\alpha$ is perpendicular to $\mathbf{v}_\alpha$ so that,

$$\begin{aligned}\boldsymbol{\ell} = \hat{\mathbf{n}}\sum r_\alpha m_\alpha v_\alpha = \hat{\mathbf{n}}\sum m_\alpha r_\alpha(r_\alpha\dot{\theta}) = (\hat{\mathbf{n}}\dot{\theta})\sum m_\alpha r_\alpha^2 \\ = I\dot{\theta}\hat{\mathbf{n}}.\end{aligned}$$

QED.

Then, since I and $\hat{\mathbf{n}}$ are constant in time, we may differentiate with respect to time to obtain,

$$\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt} = I\ddot{\theta}\hat{\mathbf{n}}.$$

QED.

Finally,

$$\begin{aligned}T = \sum\frac{1}{2}m_\alpha v_\alpha^2 = \sum\frac{1}{2}m_\alpha(r_\alpha\dot{\theta})^2 = \frac{1}{2}\dot{\theta}^2\sum\frac{1}{2}m_\alpha r_\alpha^2 \\ = \frac{1}{2}I\dot{\theta}^2.\end{aligned}$$

QED.

9.3 Laminar Motion

9.3.1 Definition

If a rigid body is constrained to move parallel to a plane (such as a wheel rolling on a flat surface or a block sliding down an inclined plane), the motion is said to be *laminar motion*.

9.3.2 Theorem

The governing equations of laminar motion are,

$$\mathbf{F} = M\mathbf{a}_{cm},$$

$$\mathbf{N}_{cm} = I_{cm}\ddot{\theta}\hat{\mathbf{n}},$$

and,

$$T = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2.$$

Proof: We have already proven for any system of particles: $\mathbf{F} = M\mathbf{a}_{cm}$, i.e. that the system of total mass M taken as a whole obeys a form of Newton's Second Law for the motion of the center-of-mass. $\mathbf{F}$ is the net force on the system. If P is taken as an arbitrary fixed point, we have also shown that relative to this point,

$$\mathbf{N}_P = \frac{d\boldsymbol{\ell}_P}{dt}.$$

These two results apply to the more specific case of laminar motion as well. It is also true generally as well as in the specific case of laminar motion, that the angular momentum and kinetic energy can be decomposed into two terms,

$$\boldsymbol{\ell}_P = \mathbf{R}_{cm} \times M\mathbf{V}_{cm} + \sum(\mathbf{r}'_\alpha \times m_\alpha \mathbf{v}'_\alpha) = \boldsymbol{\ell}_{cm} + \boldsymbol{\ell}'$$

$$T = \frac{1}{2}MV_{cm}^2 + \sum \frac{1}{2}m_\alpha v_\alpha'^2.$$

Thus, the latter two equations each decompose into two parts which describe a "particle" of total mass, M, located at the center-of-mass and which describe motion of particles relative to the center-of-mass. We may use the decomposition of the angular momentum to show,

$$\mathbf{N}_P = \frac{d\boldsymbol{\ell}_P}{dt} = \frac{d\mathbf{R}_c}{dt} \times MV_{cm} + \mathbf{R}_{cm} \times \frac{d\mathbf{V}_{cm}}{dt} + \frac{d\boldsymbol{\ell}_{cm}}{dt}$$

$$= \mathbf{V}_{cm} \times M\mathbf{V}_{cm} + \mathbf{R}_{cm} \times M\mathbf{a}_{cm} + \frac{d\boldsymbol{\ell}_{cm}}{dt}.$$

The first term on the right vanishes. Using $\mathbf{F} = M\mathbf{a}_{cm}$, we have,

$$\mathbf{N}_P = \frac{d\boldsymbol{\ell}_{cm}}{dt} + \mathbf{R}_{cm} \times \mathbf{F} = \mathbf{N}_{cm} + \mathbf{R}_{cm} \times \mathbf{F}.$$

We see once again the decomposition into a term representing the center-of-mass "particle" and action relative to the center-of-mass.

Therefore, we may divide our problem into two parts. One part is the motion of the center of mass, which obeys the equation, $\mathbf{F} = M\mathbf{a}_{cm}$. We may then solve, as a second part, the problem of motion relative to an axis through the center-of-mass perpendicular to the lamina and thus reduce it to the problem of laminar motion relative to a fixed axis. The unit vector $\hat{\mathbf{n}}$ lies along the axis. Thus, the governing equations for the system are,

$$\mathbf{F} = M\mathbf{a}_{cm},$$

$$\mathbf{N}_{cm} = I_{cm}\ddot{\theta}\hat{\mathbf{n}},$$

and,

$$T = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2.$$

QED.

9.4 Rotational Motion with respect to a Fixed Point or the Center of Mass

9.4.1 Definition

An inertial set of axes relative to which a rigid body moves is called a *space set of axes.* A set of axes fixed to a rigid body is called a *body set of axes.*

9.4.2 Euler's Equations

Consider a body set of axes fixed to a rigid body at the center of mass. The governing equations are then,

$$\mathbf{F} = M\frac{d\mathbf{V}_c}{dt}$$

$$\boldsymbol{\ell} = \mathbf{R}_c \times M\mathbf{V}_c + \sum \mathbf{r}'_\alpha \times m_\alpha \mathbf{v}'_\alpha,$$

and,

$$\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt}.$$

Since the translational motion of the center of mass decouples from the motion relative to the center of mass, we may focus on the rotational motion relative to the center of mass taking the motion of the center of mass itself to be a separate problem. The problem of the rotation relative to the center of mass is completely equivalent to motion of a rigid body with one constrained fixed point (not necessarily a fixed axis), although in the latter case the point need not be the center of mass. In either case, the general result,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0,$$

reduces to,

$$\mathbf{v}'_\alpha = \boldsymbol{\omega} \times \mathbf{r}'_\alpha.$$

Then, relative to the center of mass or to the single fixed point,

$$\boldsymbol{\ell} = \sum m_\alpha \mathbf{r}'_\alpha \times \mathbf{v}'_\alpha = \sum m_\alpha \mathbf{r}'_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}'_\alpha).$$

We may reduce the triple cross product to yield,

$$\begin{aligned}\boldsymbol{\ell} &= \sum m_\alpha [\boldsymbol{\omega} r'^2_\alpha - \mathbf{r}'_\alpha(\mathbf{r}'_\alpha \cdot \boldsymbol{\omega})] \\ &= \sum m_\alpha [(\omega'_i \hat{\mathbf{e}}'_i) r'^2_\alpha - (x'_{\alpha i}\hat{\mathbf{e}}'_i)(x'_{\alpha j}\omega_j)'] \\ &= \sum m_\alpha [\delta_{ij} r'^2_\alpha - x'_{\alpha i} x'_{\alpha j}]\omega'_j \hat{\mathbf{e}}'_i \\ &\equiv I'_{ij}\omega'_j \hat{\mathbf{e}}'_i \\ &= \mathbf{I} \cdot \boldsymbol{\omega}.\end{aligned}$$

The matrix I_{ij} has *moments of inertia* I'_{11},I'_{22} and I'_{33} on the diagonal. These are the moments of inertia about the three body axes respectively. Because the body set of axes is fixed to the object, the moments of inertia are constants, i.e., they do not change with time. The off-diagonal elements of I are called the *products of inertia.* They also are constants in the body set of axes. If the body axes are chosen carefully, the products of inertia can be made to vanish, thus simplifying most problems.

A body set of axes in which the products of inertia vanish is called the *principal axes* and the corresponding moments of inertia are the *principal* *moments of inertia*. If you consider a typical moment of inertia tensor with off-diagonal elements as a matrix, you can find the principle moments of inertia by finding the eigenvalues of the matrix. The eigenvalues of a matrix then become the diagonal elements of its diagonal form (the principal moments) and the corresponding eigenvectors are the along the principal axes. However, in most simple cases, one can find the principal axes by inspection using symmetry arguments. For example, you may be able to exploit symmetry to make the product of inertia,

$$I_{xy} = \sum m_\alpha x_\alpha y_\alpha$$

vanish if an axis can be oriented such that for every x-value, there is both a positive and a negative y-value for equal masses so that in the sum a positive term cancels a negative term. For example, for a square plate, the principal axes could be located at the center of the plate with x and y axes parallel to the sides of the plate and the z axis perpendicular to the plate. For a circular disk with origin at the center, any two perpendicular diagonals would serve as principal axes, with a third perpendicular to the plane of the disk.

If we take the time derivative of $\boldsymbol{\ell}$,

$$\begin{aligned}\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt} &= I'_{ij}\dot{\omega}'_j\hat{\mathbf{e}}'_i + I'_{ij}\omega'_j\frac{d\hat{\mathbf{e}}'_i}{dt} \\ &= I'_{ij}\dot{\omega}_j\hat{\mathbf{e}}'_i + I'_{ij}\omega'_j(\delta_{ikl}\hat{\mathbf{e}}'_k\omega'_l) \\ &= (I'_{ij}\dot{\omega}'_j + \delta_{ijk}\omega'_j I'_{kl}\omega'_l)\hat{\mathbf{e}}'_i.\end{aligned}$$

Thus, we have *Euler's Equations*,

$$N'_i = I'_{ij}\dot{\omega}'_j + \delta_{ijk}\omega'_j I'_{kl}\omega'_l.$$

If the axis set is a principal set of axes, the Euler equations have the simpler (and more common) form:

$$\begin{aligned}N'_1 &= I'_{11}\dot{\omega}'_1 + \omega'_2\omega'_3(I'_{33} - I'_{22}),\\ N'_2 &= I'_{22}\dot{\omega}'_2 + \omega'_1\omega'_3(I'_{11} - I'_{33}),\\ N'_3 &= I'_{33}\dot{\omega}'_3 + \omega'_1\omega'_2(I'_{22} - I'_{11}).\end{aligned}$$

9.4.3 Theorem

$$T = \frac{1}{2} I'_{ij}\omega'_i\omega'_j = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}\cdot\ell.$$

Proof: If we exclude any translational kinetic energy and thus restrict ourselves to the rotational kinetic energy relative to a fixed point or to the center-of-mass, we have,

$$T = \frac{1}{2}\sum m_\alpha v'^2_\alpha = \frac{1}{2}\sum m_\alpha(\boldsymbol{\omega}\times\mathbf{r}'_\alpha)\cdot(\boldsymbol{\omega}\times\mathbf{r}'_\alpha)$$

$$= \frac{1}{2}\sum m_\alpha(\delta_{ijk}\omega'_j x'_{\alpha k})(\delta_{ilm}\omega'_l x'_{\alpha m})$$

$$= \frac{1}{2}\sum m_\alpha(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk})\omega'_j\omega'_l x'_{\alpha k}x'_{\alpha m}$$

$$= \frac{1}{2}\omega'_j\omega'_l\sum m_\alpha(\delta_{jl}r'^2_\alpha - x'_{\alpha j}x'_{\alpha l})$$

$$= \frac{1}{2}\omega'_j I'_{jl}\omega'_l.$$

QED.

9.4.4 Theorem

$$\frac{dT}{dt} = N'_i\omega'_i.$$

Proof: Multiply Euler's equations by ω'_i to form the sum,

$$N'_i\omega'_i = I'_{ij}\omega'_i\dot{\omega}'_j + \omega'_i\delta_{ijk}\omega'_j I'_{kl}\omega'_l.$$

Since, the double sum $\omega'_i\delta_{ijk}\omega'_j = 0$ (because even and odd permutations of δ_{ijk} have opposite signs), we have,

$$N'_i\omega'_i = I'_{ij}\dot{\omega}'_j\omega'_i = \frac{dT}{dt}.$$

QED. Thus, the rate at which the net torque does work on the rigid body is equal to its time rate of change of kinetic energy.

9.4.5 Euler Angles

The solution to Euler's equations yields angular velocity, not position. To specify the arbitrary position of a rigid body (ignoring translational motion) requires three generalized coordinates since the arbitrary orientation in space has three degrees of freedom. If we freeze the body axes to the rigid body, we can equivalently specify the orientation of the body axes relative to the space axes. A common way to do this is in terms of three angles (called Euler angles) defined in the following way.

Imagine a space set of axes and a body set of axes. Imagine originally that the two are identical, with the x_3 and x_3' axes coincidental and oriented vertically. Imagine the x_1 and x_1' axes coming out of the page and the x_2 and x_2' axes lying horizontally to the right in the page to complete the triad. Now imagine rotating the primed axis set around the common x_3, x_3' axis through an angle ϕ. This moves the x_1' axis toward the right and the x_2' axis back into the page. The angle ϕ is the first of three simple rotations which define the Euler angles. If the angle ϕ increased in a time-dependent way, the corresponding angular velocity would be $\dot{\phi}\hat{\mathbf{e}}_3$.

The new x_1' axis is called the "line of nodes." Define a unit vector, $\hat{\mathbf{n}}$, along the line of nodes. Now, imagine that the line of nodes is held fixed and that the primed axis system is rotated through an angle θ about it. This rotation separates the x_3 and x_3' axes by the angle θ. The angle θ is the second of the Euler angles and a corresponding angular velocity about the line of nodes would be $\dot{\theta}\hat{\mathbf{n}}$.

Finally, hold the new x_3' axis fixed and rotate the primed system through an angle ψ about the x_3' axis. This is the third of the Euler angles and the corresponding angular velocity is $\dot{\psi}\hat{\mathbf{e}}_3'$.

The three angles ϕ, θ, ψ are the Euler angles and they specify an arbitrary orientation of the body (primed) set of axes relative to the space set of axes. An arbitrary angular velocity is given by,

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{e}}_3 + \dot{\theta}\hat{\mathbf{n}} + \dot{\psi}\hat{\mathbf{e}}_3',$$

where,

$$\hat{\mathbf{n}} = \cos\psi\hat{\mathbf{e}}_1' - \sin\psi\hat{\mathbf{e}}_2'.$$

The utilitarian value of the Euler angles is greatly increased if we can relate the unit vectors in the two sets of axes (body and space) to one another via the Euler angles. This is done in Table 9.1.

Table 9.1: Unit vectors of the body axes as combinations of unit vectors of the space axes (read horizontally) or unit vectors of the space axes as combinations of unit vectors of the body axes (read vertically).

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\mathbf{e}}'_1$	$\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi$	$\sin\phi\cos\psi + \cos\theta\cos\phi\cos\psi$	$\sin\theta\sin\psi$
$\hat{\mathbf{e}}'_2$	$-\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi$	$-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi$	$\sin\theta\cos\psi$
$\hat{\mathbf{e}}'_3$	$\sin\theta\sin\phi$	$-\sin\theta\cos\phi$	$\cos\theta$

For example, using Table 9.1,

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{e}}_3 + \dot{\theta}\hat{\mathbf{n}} + \dot{\psi}\hat{\mathbf{e}}'_3$$

$$= \dot{\phi}(\sin\theta\sin\psi\hat{\mathbf{e}}'_1 + \sin\theta\cos\psi\hat{\mathbf{e}}'_2 + \cos\theta\hat{\mathbf{e}}'_3) + \dot{\theta}(\cos\psi\hat{\mathbf{e}}'_1 - \sin\psi\hat{\mathbf{e}}'_2) + \dot{\psi}\hat{\mathbf{e}}'_3.$$

By collecting the coefficients of the unit vectors, we then have,

$$\omega'_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi,$$

$$\omega'_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi,$$

$$\omega'_3 = \dot{\psi} + \dot{\phi}\cos\theta.$$

These latter expressions are useful for expressing the kinetic energy in terms of Euler angles and their time derivatives via,

$$T = \frac{1}{2}\omega'_i I'_{ij}\omega'_j.$$

9.5 Problems

1. A sphere of radius a rolls off a hemisphere of radius b placed convex-side-up on a table. Does the sphere leave the hemisphere before it hits the table? At what angle? How does this compare with a particle that slides from a frictionless hemisphere?

2. Find the inertia tensor for a square plate of side a and mass m in a coordinate system $O'x'y'z'$ centered at one corner with the x' and y' axes along the two edges. Find the angular momentum about O' of the above plate when it is rotating with angular frequency ω about the diagonal of the plate that passes through O'. Find the kinetic energy of the above rotating plate.

   Partial Ans:
   $$I_{xx} = \frac{1}{3}ma^2$$
   $$I_{xy} = -\frac{1}{4}ma^2$$
   $$\ell_x = \frac{1}{12\sqrt{2}}ma^2\omega.$$

3. For the plate above, find the principal moments of inertia, i.e. moments of inertia in the axis system in which the products of inertia vanish. Find the directions of the principal axes. Principal axes are axes chosen so that the products of inertia (the off-diagonal terms in the inertial tensor) vanish.

4. Find the moments and products of inertia of a uniform rectangular block of sides a, b, c for a coordinate system with origin at one corner and with axes along the edges of the block. If the block spins about a *long* diagonal through O', find the angular momentum about the fixed origin. Does your answer reduce to the result above for a thin plate?

5. Consider a thin square plate with one corner fixed (at which we center body axes) and rotating about the x' axis (oriented along one side)with constant angular speed ω. Find the torque necessary to maintain the angular velocity along this axis. If you orient the x' axis along the diagonal, what torque is now needed to maintain the angular velocity along this new direction? What obvious advantage is there to orienting the angular velocity along the principal axes of the plate?

6. Find the solution to Euler's Equations in the absence of torque for principal axes in which $I_{11} = I_{22} = I$. Assume $I < I_{33}$ and $\omega_3 > 0$. You might try defining
$$\gamma = \frac{I_{33} - I}{I}\omega_3$$
and using complex numbers to uncouple the equations.

7. Find the solutions of Euler's equations for principal axes when $I_{11} = I_{22} > I_{33}$ and first component of torque N_1' as measured in the body coordinates is not zero but is constant. Assume $\omega_3 > 0$. (Note: Observe that in this problem $I_{22} > I_{33}$ but that it is just opposite in the previous problem.)

   Ans:
$$\gamma = \frac{I_{11} - I_{33}}{I_{11}}\omega_3$$
$$\omega_3 = constant$$
$$\omega_1 = a\cos\gamma(t - t_0)$$
$$\omega_2 = -a\sin\gamma(t - t_0) - \frac{N_1}{\omega_3(I_{11} - I_{33})}.$$

8. An artificial satellite which is approximately cylindrical in shape has principal moments of inertia $I_{11} = I_{22} = 10^4$slug ft^2 and $I_{33} = 0.4 \times 10^4$slug ft^2. Initially, the satellite is not rotating. There are two rockets mounted directly opposite one another parallel and antiparallel to the x_1 body axis such that they cause a torque in the positive x_3-direction. These rockets have a thrust of 10 lb and are 4 ft from the center of mass. The are fired for 100 sec and then shut off. Now rockets, which are at opposite ends of the satellite mounted parallel and antiparallel to the body x_2-axis such that they produce a torque in the positive x_1-direction, are fired for 2.62 sec. These rockets each produce 300 lb of thrust and are 10 ft from the center of mass. Assume that the amount of material ejected by the rockets is small enough so that the moments of inertia are not affected. Find the components of angular velocity in the body set of axes for the final motion. (T. C. Bradbury, *Theoretical Mechanics*, John Wiley & Sons, 1968).

   Solution: This is a straightforward application of Euler's equations and is meant to illustrate the results of the two previous problems.

The initial condition is $\omega_1 = \omega_2 = 0$ at $t = 0$ and should allow you to evaluate arbitrary constants. At the time all motors are shut off you should be able to show that $\omega_1 = 0$, $\omega_2 = -1.0/sec$, and $\omega_3 = 2.0/sec$. By reestablishing your time origin at the time the motors shut off, you should be able to show that thereafter,

$$\omega_1 = -\sin\gamma t$$

$$\omega_2 = -\cos\gamma t$$

$$\omega_3 = 2.0/sec.$$

9. For the previous problem, what is the angle between ω and ℓ? What is the angle between ω and the body x_3'-axis? At what angular rate does ω precess about the body x_3'-axis? At what angular rate does ω precess about ℓ.

   Solution:

   - Draw carefully a diagram and read most of the answers from it.
   - Since ℓ is a constant of the motion, it will have a constant magnitude and a fixed direction in space. It might be helpful to draw your diagram at the time when the precessing ω has a zero component along the x_1'-axis and only ω_2 and ω_3 are non-zero in the body axes.
   - To find the angular rate of precession of ω about ℓ, whisper softly to yourself the magic words: "The polhode cone rolls on the herpolhode cone without slipping." With ℓ and ω sharing a common origin, the herpolhode (or space) cone is the cone that ω traces out as it precesses about ℓ where ℓ is fixed in space. The polhode (or body) cone is the cone that ω traces out as it precesses about the symmetry axis (x_3') of the object (body). Can you visualize the motion of the object being represented by the rolling of the body cone on the space cone without slipping? (Ans: 1.28/sec).

10. Experimental: Go to the walk-in laboratory area designated by your instructor and find the air-suspension top apparatus. Do the following and write a simple report about what you did.

    - Demonstrate the effects of precession, retrograde motion, nutation and the so-called "sleeping top."

- Find the torque on the ball when the rod is attached, measure the precession frequency, and calculate the rapid (and therefore hard-to-measure) rotation frequency of the ball. Make sure the ball is rotating really fast. Minimize the nutation so that the motion is pure spin and precession.

11. Computer Project 12 (see Appendix)

Chapter 10

The Special Theory of Relativity

The Special Theory of Relativity can seem to be a fantastic, almost unbelievable, subject. In introductory courses, it is often illustrated with thought experiments involving rocket ships or trains moving near the speed of light. The disclaimer is usually made that the effects of relativity are very, very small in ordinary experience, thus giving the impression of an esoteric subject of little practical consequence.

But for many areas of physics, Special Relativity is very important and very practical. Relativistic effects are important in nuclear physics, elementary particle physics, high-temperature plasma physics, beam physics, accelerator technology, astrophysics, space physics, cosmic-ray physics and even sometimes in high-precision atomic physics. But even if it were not so, it would still be true that the education of a physicist is not complete until he or she understands this great revolutionary subject of the twentieth century. The four-vector formulation of Special Relativity can bring one to a deeper understanding and reveal an elegance and structure the one may otherwise not recognize.

10.1 Galilean Transformation

In developing the Newtonian mechanics of mass particles, we have made some assumptions. To be sure, we are reassured by experiment that these assumptions are true within the realm where the experiments have actually been performed. For example, we have taken the masses of the particles to be constants that are independent of the motion of the particle. The mass is a measure of inertia, i.e. a measure of the resistance to acceleration by a given force. Chemistry of the nineteenth century concluded that mass was a strictly conserved quantity and that any motion of the particles did not change this conclusion.

We have also made some assumptions about time and space. Consider a coordinate system with origin O that is fixed in space (the "laboratory frame") and consider another frame with origin O' moving with constant velocity v along the x-axis of the laboratory frame. For simplicity assume that the x'-axis is parallel to the x-axis. Assume that a clock hangs on the wall of the laboratory and that observers at rest in the laboratory or moving with the primed frame can read time intervals from this clock. This is a second assumption on our part, i.e. that the single clock on the wall of the laboratory serves stationary or moving observers equally as well. Assume that there is a particle of mass m that has position $\mathbf{r}'$ in O' such that its position in O is $\mathbf{r} = \mathbf{r}' + vt$. The particle is not necessarily at rest in O', but we are assuming that if it moves, it moves along the common x and x' direction.

Now, if the particle moves in some small time interval dt that is read from the clock on the wall, we have,

$$dx = dx' + vdt$$

$$dy = dy'$$

$$dz = dz'$$

and, since time is read from the single clock on the wall,

$$dt = dt'.$$

We call this connection between the unprimed and primed coordinates of the particle a *Galilean transformation*. If we divide the first equation by dt, we get a familiar relationship between the speeds, $u_x = dx/dt$ and $u_{x'} = dx'/dt'$, along the x-axis,

$$u_x = u_{x'} + v.$$

When we search for the laws of physics, we seek generalities that serve the observer in the laboratory as well as the observer who moves with the primed frame. Our own experience, perhaps in an airliner moving in a straight line at constant speed, tells us that uniform motion is not observable. By this we mean that we would observe no violation of the laws of motion that we deduced from experiments on the ground when applied to experiments in the airliner. This is very desirable, since one would not want to dignify with the label "law" a relationship that is only valid in a particular frame of reference.

If we differentiate our velocity formula, remembering that v is constant,

$$a_x = a_{x'}.$$

For Newton's Second Law, this means that if $\mathbf{F} = \mathbf{F}'$, then

$$\mathbf{F} = m\mathbf{a}$$

and,

$$\mathbf{F}' = m\mathbf{a}'.$$

Thus, Newton's Second Law takes the same form in each of the two frames of reference, and this is made possible, in part, by the assumption that the Galilean transformation is correct as well as the assumption that the forces themselves are equal as seen from the two frames.

To illustrate, imagine a platform on a flatcar on a railroad track that is moving at constant speed on a straight section of track. If a cannon is mounted to the platform in such a way that it fires a projectile exactly vertically, the projectile shares the horizontal speed of the flatcar when it is fired. This horizontal speed of both the platform and the projectile is unchanged (in the absence of air currents) during the motion of the projectile. Hence the projectile falls back on the platform exactly where it was first fired. Newton's Second Law applied to the motion, either from the frame of reference of the platform or from the stationary laboratory frame at the side of the track, predicts this same outcome. The same law is applicable and works in both frames to describe the projectiles's motion, although the motions will appear different to observers in the two frames.

But, consider what happens if our particle is replaced by a photon. If the photon moves along the x-axis, the Galilean transformation tells us that the speed of the photon should be the sum of its speed in the primed system plus the speed of the primed system relative to the unprimed system. It is not so. In a paper published in 1965, researchers at CERN in Geneva produced two photons from the decay of a neutral pion. The pion (which here defines the origin of the primed frame) was moving at 0.999 times the speed of light. The time of flight of these photons was measured over a distance of 80 meters and to within experimental error, the speed of both (moving in opposite directions along the x axis) was equal to the speed of light in vacuum. The motion of the pion made no difference in the speeds of the photons in the laboratory to within a precision of about 130 parts per million. This is a very remarkable result and it means that for photons, at least, something is very wrong with the Galilean transformation, something that was first suspected in 1887 when the Michelson-Morley experiment was first performed.

If you try to fix the Galilean transformation with the simplest linear generalization of the Galilean transformation, one might try,

$$dx' = \alpha dx + \beta dt$$

$$dy' = dy$$

$$dz' = dz$$

and,

$$dt' = \gamma dt + \delta dx.$$

The coefficients $\alpha, \beta, \gamma, \delta$ would presumedly be chosen in such as way as to somehow account for the unexpected outcome of the pion experiment. In doing so, the Galilean transformation would be replaced by a more general one, the relationship between velocities would also be changed, and quite possibly, we would conclude that Newton's Second Law itself as we have known it would have to be replaced with another equation in order to achieve form invariance (so-called *covariance*). Thus, the consequence of the pion experiment is far-reaching.

10.2 Spacetime and the Lorentz Transformation

One of the consequences of our proposed new transformation is that our observers can no longer share the common clock on the wall. You can see this particularly in the fourth equation of the set where it is proposed to make dt' equal to a linear combination of dt and dx. So, from henceforth, we will provide observers with their own rulers and clocks which they carry with themselves to make measurements of space and time.

Because of this mixing of space and time, it is helpful to relax the distinction between space and time that is natural to us by beginning to think of a four-dimensional world. The "points" in this world are called *events* and have four coordinates. Three of these are the regular (x, y, z) coordinates of position relative to some suitably chosen origin and the fourth is a time coordinate measured by a clock that belongs to the frame of reference from which the event is observed. The events are "things that happen" at a certain place and at a certain time. Each event can be specified uniquely by a quartet of coordinates, (x, y, z, t) or (x', y', z', t') depending on the frame of reference and origin chosen.

Separations between events can also be specified by a quartet, (dx, dy, dz, dt) and it is these numbers that are related by the new linear transformation that we are seeking. As long as the separations are very small, we can at least assume that the transformation is approximated by a linear transformation and it is for this reason that we concentrate on the transformation for the separations. If we think of the coordinates in the primed system as being some functions of the coordinates in the unprimed system,

$$x'_i = x'_i(x_j, t)$$
$$t' = t'(x_j, t),$$

then, for small displacements and a reasonably behaved transformation, the chain rule of differentiation yields,

$$dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j + \frac{\partial x'_i}{\partial t} dt$$

$$dt' = \frac{\partial t'}{\partial x_j} dx_j + \frac{\partial t'}{\partial t} dt.$$

This is exactly the form of the transformation we seek. By "reasonably behaved transformation," we mean one that is differentiable and has a unique inverse.

Now think of the experiment that we used to illustrate Galilean invariance, i.e. the cannon fired vertically from a horizontally moving platform. Only this time we will use a photon that reflects from a mirror that is overhead and moving with the platform. The primed frame is attached to the platform which we take to be moving from left to right along the direction chosen for the x and x' axes. In the primed frame, the platform is at rest and the photon moves vertically, strikes the mirror at a distance $d\ell$ away, and returns to its point of origin along the same line as the line of its ascent. The line is a series of events which is called a *world line*.

We want to focus on two of the events of this world line of the photon. Event 1 is the event of the photon being produced. Event 2 is the event of the photon striking the platform after being reflected from the mirror. These two events occur at exactly the same place in the primed system, but the time between them is the time it takes for the photon to travel a distance $2d\ell$ at the speed of light, c. Thus, in the primed system,

$$dx' = 0$$

$$dy' = 0$$

$$dz' = 0$$

$$dt' = \frac{2d\ell}{c}.$$

Viewed in the unprimed (laboratory) frame, the photon moves upward and to the right along a straight line, strikes the mirror, then moves downward and to the right before hitting the moving platform. In the unprimed system, the platform has moved a distance vdt. Thus, in the unprimed (laboratory) system,

$$dx = vdt$$

$$dy = 0$$

$$dz = 0$$

$$dt = \frac{2\sqrt{(\frac{vdt}{2})^2 + (d\ell)^2}}{c}.$$

The expression for dt is a straightforward application of Pythagoras' theorem for right triangles.

Exercise: Draw a diagram showing Event 1 and Event 2 and show that the above expressions for dx, dx', dt, dt' are correct.

By assuming that the photon has the same speed in both frames of reference, we are forcing the transformation to describe a world in which the speed of light does not depend on the motion of its source, i.e., the experimental outcome of the pion experiment.

If we substitute these relationships into our proposed transformation, we have,

$$0 = \alpha dx + \beta dt$$

which implies, since $dx = vdt$, that

$$\alpha v + \beta = 0.$$

The fact that $dt' = \frac{2d\ell}{c}$ can be combined with,

$$dt = \frac{2\sqrt{(\frac{vdt}{2})^2 + (d\ell)^2}}{c},$$

to show that,

$$dt' = \sqrt{1 - \frac{v^2}{c^2}} dt.$$

Finally, we can combine this last result with $dt' = \gamma dt + \delta dx$, to show that,

$$\gamma + \delta v = \sqrt{1 - \frac{v^2}{c^2}}.$$

Unfortunately, this gives us just two equations for the four unknown coefficients, $\alpha, \beta, \gamma, \delta$. So let's consider a second experiment to get another set of two equations. This time, hold the mirror stationary and let the photon move exactly vertically up and back in the unprimed system so that for Events 3 and 4 (which are not to be confused with Events 1 and 2 in the quite different first experiment),

$$dx = 0$$

$$dy = 0$$

$$dz = 0$$

$$dt = \frac{2d\ell}{c}$$

and,

$$dx' = -vdt'$$

$$dy' = 0$$

$$dz' = 0$$

$$dt' = \frac{2\sqrt{(\frac{vdt'}{2})^2 + (d\ell)^2}}{c}.$$

The minus sign in the expression for dx' arises because the platform moves from left to right. The reflected photon strikes a point behind the point of origin in the primed system.

Exercise: Draw a diagram showing Event 3 and Event 4 and show that the above expressions for dx, dx', dt, dt' are correct.

Because $dx = 0$, the transformation is simplified and we use the dx' and dt' equations to show, respectively,

$$\beta = \frac{-v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Combining with the results from the first experiment,

$$\alpha = \frac{-\beta}{v} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

$$\delta = \frac{\sqrt{1 - \frac{v^2}{c^2}} - \gamma}{v} = \frac{-v}{c^2\sqrt{1 - \frac{v^2}{c^2}}}.$$

We may now write down our transformation as

$$dx' = \gamma(dx - vdt)$$

$$dy' = 0$$

$$dz' = 0$$

$$dt' = \gamma(dt - \frac{vdx}{c^2}).$$

We call this transformation the *special or restricted Lorentz transformation.* It is "restricted" because we have constrained all of our motions to be along the x, x' axes. Later we will relax this restriction and derive the full Lorentz transformation in three dimensions.

The special Lorentz transformation can be inverted simply by taking the view of an observer in the primed system. From her perspective, the primed system is at rest and the unprimed system moves from right to left with speed $-v$. Thus,

$$dx = \gamma(dx' + vdt')$$
$$dy = 0$$
$$dz = 0$$
$$dt = \gamma(dt' + \frac{vdx'}{c^2}).$$

Inertial frames are frames in which there is no physical consequence of assuming that the frame is at rest or in uniform motion and this assumption is now built into our Lorentz transformation. Observe also that in the limit as $v/c \to 0$, we have $\gamma \to 1$ and the Lorentz transformation becomes the Galilean transformation. This is as it should be because the Galilean transformation serves very well for physics when speeds are much less than the speed of light. The speeds in the pion experiment were all at or near the speed of light. We have also assumed throughout that the speed of light is the same for observers in either the primed or unprimed system. The Lorentz transformation thus embodies two essential postulates which can be thought of as the foundation of the Special Theory of Relativity,

1. **The Postulate of Special Relativity**: Every "law of physics" that holds in any reference frame must equally well hold in any reference frame that moves in a straight line at constant speed relative to the first.

2. **Postulate of the Speed of Light**: The speed of light in vacuum is the same for observers at rest or moving relative to one another in a straight line at constant speed. It is an absolute invariant.

The Lorentz transformation has a singularity at $v \to c$. But, what would be the consequences of velocities that exceed the speed of light? Specifically, we focus on the transfer of information at superluminary speeds. Imagine two events, A and B, such that A precedes B. We may think of A being a "cause" of B and that information generated at the event A must propagate to B before B can happen. For example, event A might be the creation of a muon and event B might be its decay. The muon itself carries the information from event A to event B. For these two events $dt_{AB} > 0$. The

Lorentz transformation tells us that in a primed system moving relative to the first with speed v,

$$dt'_{AB} = \gamma(dt_{AB} - \frac{vdx_{AB}}{c^2}).$$

Since $dx_{AB} = udt_{AB}$, we may write,

$$dt'_{AB} = \gamma dt_{AB}(1 - \frac{vu}{c^2}).$$

If there is no restriction on the magnitude of u at which the information is carried from event A to event B, i.e. $u > c$, the sign of dt'_{AB} may be made negative. Then, in the primed frame, the effect precedes its cause. We therefore add a third postulate to insure causality,

3. **Postulate of Causality**: Information is not propagated at greater than the speed of light.

10.3 Consequences of the Lorentz Transformation

There are several consequences that immediately follow from the Lorentz transformation.

1. **Non-Simultaneity of Events**: The first consequence has to do with what we mean when we say that two events happen "at the same time." Consider a frame of reference in which two events occur. We make the following operational definition: Two events which generate a light signal are said to be *simultaneous* if an observer located midway between the spatial positions of the two events receives the light signals coincidentally. (An "operational definition" is one made in terms of an experiment. Thus experiment and not semantics can be invoked to decide whether the events are simultaneous.)

   Now, if two events that occur at different places separated by dx in the unprimed (laboratory) frame are simultaneous, then $dt = 0$. Thus, the Lorentz transformation tells us,

$$dt' = \gamma(dt - \frac{vdx}{c^2}) = -\frac{\gamma vdx}{c^2} \neq 0.$$

Two events that are simultaneous in one frame are not simultaneous in a frame moving relative to the first with speed v.

Think of an unprimed system at rest and a primed stystem moving from left to right. Think of two event at points A and B (to the right of A) that emit light signals just as two points A' and B' are lined up side-by-side with A and B. These events are such that an observer halfway between A and B at point C receives a light signal from each coincidentally. We have purposely created two events that are simultaneous in the unprimed system according to the operational definition. But the observer at C' (halfway between A' and B' in the primed system and moving with it) moves forward after the light signals are emitted so that the light has less of a distance to travel to meet her. The speed of light is unaffected by the motion of the unprimed system and the signal from B' arrives first followed by the signal from A'. The observer in the primed system judges that the events are not simultaneous, just as the Lorentz transformation predicted.

2. Lorentz Contraction: A second immediate consequence of the Lorentz transformation is called the Lorentz contraction. Let us imagine a separation dx' that marks the length of a stick that lies along the x' axis and is at rest in the primed system. As the stick moves by, an observer in the unprimed system wants to measure the length of the stick with his own ruler. One way to accomplish this is for the unprimed observer to mark the positions of the two ends of the moving stick on his stationary ruler as the stick flies by. The markings of the two ends are two events which the unprimed observer must make simultaneous for his measurement to make sense. Certainly he does not want to mark the front end first, then allow some time to elapse before marking the position of the trailing end. So, in the unprimed system, $dt = 0$. The Lorentz transformation then tells us,

$$dx' = \gamma(dx - vdt) = \gamma dx$$

i.e.,

$$dx = dx'\sqrt{1 - \frac{v^2}{c^2}}.$$

The measured length, dx, is always shorter than dx'. This we call the *Lorentz Contraction*. The length, dx', measured in the rest system of the stick (primed) is called the *proper length* of the stick.

To the primed observer, the Lorentz transformation yields,

$$dt' = -\frac{\gamma v dx}{c^2} = -\frac{\gamma v}{c^2}(\frac{dx'}{\gamma}) = -\frac{v dx'}{c^2} \neq 0.$$

The primed observer judges that the primed observer's measurement of the length of the stick is "incorrect" because, from the primed observer's point of view, the unprimed observer did not mark the ends of the stick simultaneously. Indeed, the negative sign for dt' means that from the primed point of view, the trailing end of the stick (A') was measured after the forward edge (B') so that $t_{B'} - t_{A'} < 0$, i.e. $t_{B'} < t_{A'}$.

3. **Time Dilation**: Finally, the Lorentz transformation gives us a result called *time dilation*. The tick of a clock is an event. If the clock is at rest in the primed system, two ticks of the clock occur at the same place and $dx' = 0$. The Lorentz transformation then tells us,

$$dt = \gamma(dt' + \frac{v dx'}{c^2}) = \gamma dt'$$

$$dt = \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The time interval measured in the rest frame of the clock is called the *proper time*. The temporal interval dt is always greater than dt' if the clock moves relative to the unprimed frame so that the moving clock is judged by the unprimed observer to be running slow.

Time dilation is very commonly observed for radioactively unstable particles moving at high speeds. Muons are unstable and decay to an electron, a neutrino and an antineutrino. The proper lifetime of a muon is 2.2 microseconds. Muons moving near the speed of light have a dilated mean lifetime in the laboratory. For muons in cosmic-ray showers, the dilation factor, γ, may be 10^3 or more.

4. **Addition Law for Velocities**: Having replaced the Galilean transformation with a more general Lorentz transformation, we must ask what relationship replaces the Galilean rule for adding velocities, $u'_x = u_x + v$. Assume that the primed and unprimed frames are oriented so that the special Lorentz transformation holds. Then,

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}.$$

Thus, we have the new addition formulas for velocities,

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

and, similarly,

$$u_y = \frac{1}{\gamma}\frac{u'_y}{1 + \frac{u'_x v}{c^2}}$$

$$u_z = \frac{1}{\gamma}\frac{u'_z}{1 + \frac{u'_x v}{c^2}}.$$

Observe that as u'_x and v both approach the speed of light, u_x nevertheless remains less than the speed of light. For $u'_x << c$ and $v << c$, the formulas are essentially the Galilean formulas.

Exercise: Demonstrate that the formulas for u_y and u_z are correct.

Notice again the emphasis in each of these examples on the concept of an "event." Once the Lorentz transformation is defined, the consequences of Special Relativity follow by systematically identifying events and their spatial and temporal separations. Notice also that one must carefully differentiate between what the primed and the unprimed observer measures for these separations. If one is not careful in making the distinctions between the viewpoints of the primed and unprimed frames, Special Relativity can sometimes seem paradoxical. Many of these apparent paradoxes can be resolved by focussing on the spatial and temporal separation of events as was done above.

Exercise: Consider the following. Perhaps you have seen a pole vaulter carrying a pole as he runs down the ramp to approach the crossbar. The pole is held horizonal. Imagine an unprimed (laboratory) observer watching such a pole vaulter approach a small shed equipped with garage doors, both open, at the front and back. The pole vaulter is approaching the front of the shed. Let us imagine that the proper length of the pole and of the shed is 10 meters, but that the vaulter can run very, very fast. To the unprimed observer, the moving pole is Lorentz contracted to half its length, but the shed is at rest and is not contracted. So, the five-meter contracted pole fits nicely inside the shed and the garage doors could be closed quickly, trapping the vaulter and pole inside. However, from the vaulter's (primed) point of view, the pole is at rest and has its proper length of 10 meters. The shed is moving relative to the vaulter and has a Lorentz-contracted length of 5 meters. The 10-meter pole can never fit inside the 5-meter shed. How then can relativity be correct? What two events might you wish to focus on? Are these events simultaneous? How is the paradox resolved?

10.4 Three-Vectors

The results that we have derived thus far from the Lorentz transformation are, for the most part, treated in every introduction to Special Relativity in about the way that we have done here. We wish now to develop a very elegant and powerful formalism that will allow us to understand Special Relativity much more deeply and to derive additional important consequences of the theory. To see what we are doing and why, it is probably worthwhile to revisit some ideas and formalism associated with ordinary vectors.

The concept of a vector is often first introduced to represent physical quantities that have a magnitude and a direction. We sometimes come to think of a vector as being defined by the characteristics of magnitude and direction. We often represent a vector by an arrow with the direction of the arrow giving the direction of the vector and the length of the arrow scaled to represent the magnitude. We will now define a vector in a more general and abstract way, which, nevertheless incorporates the characteristics of magnitude and direction.

On a piece of paper, draw a vector $\mathbf{a}$ which reaches upward and to the right. After drawing $\mathbf{a}$, draw an x_1-axis horizontally and an x_2-axis vertically. Observe that the vector existed independently before the coordinate system was drawn, but once you draw the axes, the vector acquires components in the coordinate system. In a three-dimensional coordinate system, these might be represented as (a_1, a_2, a_3).

Now add a second coordinate system. Imagine x'_1 originally along x_1, and x'_2 along x_2. Then rotate the primed system through an angle θ counterclockwise around a common x_3, x'_3-axis imagined extending perpendicularly out of the page. The vector $\mathbf{a}$ has components in the primed system, (a'_1, a'_2, a'_3) as well, but they are not the same as the components in the unprimed system. Indeed,

$$a'_1 = a_1 \cos\theta + a_2 \sin\theta$$

$$a'_2 = -a_1 \sin\theta + a_2 \cos\theta$$

$$a'_3 = a_3.$$

Since this is a very special rotation, we shall refer to it as a *limited or special rotation.*

Exercise: Draw a diagram and derive the special rotation transformation above.

The triplet of numbers that represents the vector depends on your choice of orientation of the coordinate system. We might write the relationship among the components in matrix form,

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

The matrix,

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is called the *rotation transformation matrix* for the special rotation. It connects the unprimed and primed components of a vector when the coordinate systems is rotated in the special way described above. In a completely general rotation of the primed systemstem relative to the unprimed system, we can represent the matrix by,

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

We can also represent the transformation in the much more economical form,

$$a'_i = \sum_{j=1}^{3} R_{ij} a_j \equiv R_{ij} a_j.$$

In what follows, a repeated index, such as j in this example, will imply a sum on that index. Once the sum has been carried out, j disappears altogether from the expression and is called a *dummy index.* The index could, of course, be replaced by any other symbol, i.e., $R_{ij}a_j \equiv R_{ik}a_k$. The matrix R for a general rotation has a number of important generic characteristics which we can only summarize here:

1. The transformation R is an *orthogonal transformation.* This means that the transpose of the matrix R is the inverse of R, i.e.,

$$R^{-1} = R^t$$

$$R^t R = R R^t = I.$$

   These inverse relations may also be expressed,

$$R_{ij} R_{ik} = \delta_{jk}$$

$$R_{ji} R_{ki} = \delta_{jk},$$

   where δ_{jk} is the Kronecker delta function.

2. The determinant of $R = +1$. This feature is characteristic of a *proper* orthogonal transformation and includes rotations. If a single coordinate axis is inverted, the determinant is -1 and the transformation, although orthogonal, is said to be *improper.*

A *scalar* under rotations is a single function of position in space. Its value at a particular point in space is independent of the coordinate systems chosen, although the value of the scalar may be some function of the coordinates and may be a different function in different coordinate systems. Temperature is a classic example of a scalar function under rotations.

We can define a *3-vector* to be a triplet of components that transforms under rotation of the coordinate system according to the specific pattern:

$$a'_i = R_{ij} a_j.$$

This definition includes such familiar vectors as displacement, velocity and acceleration.

We can define a *3-tensor* to be a nonet of components that transforms under rotation of the coordinate system according to the specific pattern:

$$A'_{ij} = R_{ik} R_{jl} A_{kl}.$$

Some physical quantities, such as the dielectric tensor, are tensor quantities. Tensors of higher order with more than two subscripts can be defined in an analogous fashion. In this way, scalars, vectors and tensors are defined by the patterns or their transformation under rotations.

A scalar which has the same functional form in different coordinate systems is said to be an *invariant.* The scalar dot product of two vectors is an example of an invariant,

$$\mathbf{a}\cdot\mathbf{b} = a'_i b'_i = R_{ij}a_j R_{ik}b_k = R_{ij}R_{ik}a_j b_k = \delta_{jk}a_j b_k = a_k b_k.$$

Thus, the dot product has the same form and value in either coordinate system ($a'_i b'_i$ or $a_i b_i$). It is an invariant. Since the dot product of a vector with itself is the square of its magnitude, the invariance property says that the length of a vector does not change when one simply rotates the coordinate system.

10.5 Lorentz Transformations

The quartet (dx, dy, dz, cdt) are said to be components of a *four-vector.* Any quartet of variables that transforms under a special Lorentz transformation in the same way as this quartet transforms is likewise a four-vector under Lorentz transformations. We may write the transformation as,

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ dy' &= 0 \\ dz' &= 0 \\ dt' &= \gamma(dt - \frac{vdx}{c^2}). \end{aligned}$$

However, to develop the formalism, it helps to make some notational changes. Let us define,

$$\begin{pmatrix} dx \\ dy \\ dz \\ cdt \end{pmatrix} \equiv \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{pmatrix}.$$

We can adapt our summation convention on repeated indices to the Lorentz transformation if Greek letters are used when the range of the indices is four rather than three. We may then write our special Lorentz transformation as,

$$\begin{pmatrix} dx'^1 \\ dx'^2 \\ dx'^3 \\ dx'^4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{pmatrix}.$$

We might also write this as,

$$dx^{\lambda'} = L_{\mu}^{\lambda'} dx^{\mu}.$$

This is the pattern that defines a 4-vector under Lorentz transformations.

The inverse Lorentz transformation can be read from

$$dx = \gamma(dx' + vdt')$$

$$dy = 0$$

$$dz = 0$$

$$dt = \gamma(dt' + \frac{vdx'}{c^2}),$$

namely,

$$L^{-1} = \begin{pmatrix} \gamma & 0 & 0 & +\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

It follows that,

$$LL^{-1} = L^{-1}L = I.$$

All of this is done in terms of the special Lorentz transformation. However, we can include our discussion of 3-vectors into the 4-vector formalism by simply expanding R by one dimension, i.e., the rotation transformation becomes a special case of the Lorentz transformation. For example, the matrix for the special rotation transformation is generalized to,

$$R \rightarrow \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can generate more general Lorentz transformations by building them from an ordered sequence of special Lorentz transformations (including special rotations). For example, consider a primed frame whose origin does not move along the x-axis, but rather moves along a line in the $x - y$ plane. The line is inclined by an angle θ to the x-axis. Thus, unlike the special Lorentz transformation, the velocity of the primed system has two components, $v_x = v\cos\theta$ and $v_y = v\sin\theta$. However, assume that the x'-axis is still parallel to the x-axis and that the y'-axis is still parallel to the y-axis.

Begin with the primed and unprimed systems both at rest with respective axes parallel. First, rotate the primed system by an angle θ. Then give the primed system a *boost* to give it a speed v relative to the unprimed system. Finally, rotate the x'-axis back through an angle $-\theta$ to get it parallel to the x-axis again. The overall Lorentz transformation is a combination of these three,

$$L = R_{-\theta} L_v R_\theta,$$

or, $L =$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+(\gamma-1)\frac{v_x^2}{c^2} & (\gamma-1)\frac{v_x v_y}{c^2} & 0 & -\frac{v_x}{c}\gamma \\ (\gamma-1)\frac{v_x v_y}{c^2} & 1+(\gamma-1)\frac{v_y^2}{c^2} & 0 & -\frac{v_y}{c}\gamma \\ 0 & 0 & 1 & 0 \\ -\frac{v_x}{c}\gamma & -\frac{v_y}{c}\gamma & 0 & \gamma \end{pmatrix}$$

In a similar fashion or simply by extending the patterns that are evident from our example, one can generate the *general Lorentz transformation without rotation*,

$$L = \begin{pmatrix} 1+(\gamma-1)\frac{v_x^2}{c^2} & (\gamma-1)\frac{v_x v_y}{c^2} & (\gamma-1)\frac{v_x v_z}{c^2} & -\frac{v_x}{c}\gamma \\ (\gamma-1)\frac{v_x v_y}{c^2} & 1+(\gamma-1)\frac{v_y^2}{c^2} & (\gamma-1)\frac{v_y v_z}{c^2} & -\frac{v_y}{c}\gamma \\ (\gamma-1)\frac{v_x v_z}{c^2} & (\gamma-1)\frac{v_y v_z}{c^2} & 1+(\gamma-1)\frac{v_z^2}{c^2} & -\frac{v_z}{c}\gamma \\ -\frac{v_x}{c}\gamma & -\frac{v_y}{c}\gamma & -\frac{v_z}{c}\gamma & \gamma \end{pmatrix}.$$

This may also be written in 3-vector notation as,

$$d\mathbf{r}' = d\mathbf{r} + \mathbf{v}[(\gamma-1)\frac{d\mathbf{r}\cdot\mathbf{v}}{v^2} - \gamma dt]$$

$$dt' = \gamma[dt - \frac{d\mathbf{r}\cdot\mathbf{v}}{c^2}].$$

Exercise: Use the general Lorentz transformation without rotation to derive the expression for $d\mathbf{r}'$ in 3-vector notation and the expression for dt'.

Exercise: How do ordinary velocities transform under the general Lorentz transformation without rotation?

The subsuming of rotation transformations into the larger class of Lorentz transformations can be made more natural in the following way. Define $\tanh\theta = v/c$. The function θ is called the *rapidity*. Since, $-1 < \tanh\theta < 1$, this definition is consistent with all possible frame speeds less than the speed of light. Using,

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta},$$

it follows that,

$$\cosh\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\sinh\theta = \gamma\frac{v}{c}.$$

We may then write the special Lorentz transformation as,

$$dx' = dx\cosh\theta - cdt\sinh\theta$$

$$cdt' = -dx\sinh\theta + cdt\cosh\theta,$$

or,

$$\begin{pmatrix} dx'^1 \\ dx'^4 \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^4 \end{pmatrix}.$$

For special Lorentz transformations, the transformation matrix has the appearance of a rotation transformation matrix with hyperbolic functions replacing their trigonometric counterparts.

The rapidities have an important characteristic that makes them very useful. Consider a primed frame, a double primed frame and an unprimed frame that are originally identical. Then, boost the primed frame and double primed frame together to a velocity v' with respect to the first. Finally, boost the double primed frame to a velocity v'' with respect to the primed frame. The overall Lorentz transformation that relates the double primed frame to the unprimed frame is obtained,

$$L = L''L' = \begin{pmatrix} \cosh\theta'' & -\sinh\theta'' \\ -\sinh\theta'' & \cosh\theta'' \end{pmatrix} \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\theta''\cosh\theta' + \sinh\theta''\sinh\theta' & -\cosh\theta''\sinh\theta' - \cosh\theta'\sinh\theta'' \\ -\cosh\theta''\sinh\theta' - \cosh\theta'\sinh\theta'' & \cosh\theta''\cosh\theta' + \sinh\theta''\sinh\theta' \end{pmatrix}.$$

Thus,

$$L = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} = \begin{pmatrix} \cosh(\theta' + \theta'') & -\sinh(\theta' + \theta'') \\ -\sinh(\theta' + \theta'') & \cosh(\theta' + \theta'') \end{pmatrix}.$$

Thus, rapidities have a much simpler addition property than relativistic velocities. Rapidities add like Galilean velocities!

10.6 A Brief Tensorial Tutorial

We noted in passing that the special Lorentz transformation,

$$dx' = \gamma(dx - vdt)$$

$$dy' = dy$$

$$dz' = dz$$

$$dt' = \gamma(dt - \frac{vdx}{c^2}),$$

can be compared to the chain rule result,

$$dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j + \frac{\partial x'_i}{\partial t} dt$$

$$dt' = \frac{\partial t'}{\partial x_j} dx_j + \frac{\partial t'}{\partial t} dt.$$

By comparison of the two sets of expressions and defining $dx_1 = dx^1, dx_2 = dx^2, dx_3 = dx^3, cdt = dx^4$ we may write the Lorentz transformation as, $dx^{\mu'} = L^{\mu'}_{\nu} dx^{\nu}$. There is an implied sum on the repeated index, ν. A four-vector is defined to be a quartet that transforms in this way. In fact, this pattern must be distinguished from a closely related pattern, so we will call it the *contravariant* transformation pattern. We will distinguish quartets that transform according to this pattern by using superscripts and we will refer to the quartet as the *contravariant components* of the four-vector. The positioning of the primes tells us that the elements of the matrix L are given by,

$$L^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}.$$

Now, we could have done the transformation the other way,

$$dx = \gamma(dx' + vdt')$$
$$dy = dy'$$
$$dz = dz'$$
$$dt = \gamma(dt' + \frac{vdx'}{c^2}),$$

and, the chain rule,

$$dx_i = \frac{\partial x_i}{\partial x'_j} dx'_j + \frac{\partial x_i}{\partial t'} dt'$$
$$dt = \frac{\partial t}{\partial x'_j} dx'_j + \frac{\partial t}{\partial t'} dt'.$$

If we compare these expressions, we obtain the elements of the inverse Lorentz transformation, L^{-1}

$$(L^{-1})^{\mu}_{\nu'} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}.$$

We shall usually let the position of the primes alone indicate that this is the inverse matrix and write, $LL^{-1} = I$ and $L^{-1}L = I$ as

$$L^{\mu'}_{\nu} L^{\nu}_{\lambda'} = \delta^{\mu}_{\lambda},$$
$$L^{\mu}_{\nu'} L^{\nu'}_{\lambda} = \delta^{\mu}_{\lambda}.$$

The symbol, δ^{μ}_{λ}, is the Kronecker delta and it represents the elements of the identity matrix. Note again, that a repeated index indicates a sum. In virtually all cases in expressions that follow, the repeated index will appear once as a subscript and once as a superscript as it does here. This is part of the pattern that gives elegance to the formalism.

The chain rule yields another very important result. Think of a function, $f = f(x_i, t)$. Then,

$$\frac{\partial f}{\partial x'_i} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x'_i} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x'_i} = \left[\frac{\partial x_j}{\partial x'_i}\frac{\partial}{\partial x_j} + \frac{\partial t}{\partial x'_i}\frac{\partial}{\partial t}\right] f.$$

We may think of this relationship defining a transformation property for the derivative operation, namely,

$$\frac{\partial}{\partial x'^{\mu}} = L^{\lambda}_{\mu'} \frac{\partial}{\partial x^{\lambda}}.$$

Observe that this is not the contravariant pattern (the primes on L are in the wrong place.) Instead, we will write this expression as,

$$\partial_{\mu'} = L^{\lambda}_{\mu'} \partial_{\lambda}.$$

This transformation pattern is called the *covariant* transformation pattern. Quartets which transform in this way are said to be the *covariant components* of a four-vector and they are labelled with subscripts rather than superscripts.

Similarly, 4-tensors have contravariant and covariant components as well. They transform according to the patterns:

$$A'^{\alpha\beta} = L^{\alpha'}_{\mu} L^{\beta'}_{\nu} A^{\mu\nu}$$

$$A'_{\alpha\beta} = L^{\mu}_{\alpha'} L^{\nu}_{\beta'} A_{\mu\nu}.$$

Exercise: Demonstrate explicitly that

$$F^{\mu\nu} U_{\nu} = A^{\mu}$$

is a form-invariant equation under Lorentz transformations.

10.7 Relativistic Four-Vectors

Four-vectors play a very important role in relativity for the following reasons:

1. All four-vectors share the same Lorentz transformation pattern from one frame to another. If you know a thing is a four-vector, you immediately know how it transforms from frame to frame.

2. The generalized scalar dot product of two four-vectors is an invariant. If you know the magnitude in one frame, you know it immediately in all frames. This property is useful if the invariant is very simply evaluated in one frame, but much more complicated in another.

3. If we wish our "laws of physics" to have the same form (covariance) in all inertial frames, we must look for laws that express relationships among four-vectors. The Lorentz transformation will transform such relationships so that they have the same form in all admissable frames.

At this point in our treatment, however, we have encountered only one four-vector, (dx^1, dx^2, dx^3, cdt). We shall see below how to create other useful four-vectors beginning with this one.

If we define $d\mathbf{s}$ to be a separation between events, we can generalize a result from differential geometry, namely,

$$ds^2 = g_{ij}dq^i dq^j \to ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$

This pattern to create $d\mathbf{s}\cdot d\mathbf{s}$ serves as a generalization of a scalar dot product. Scalar dot products are scalar invariants under rotations. In four-space, the thing which is invariant under the special Lorentz transformation is,

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = dx'^2 + dy'^2 + dz'^2 - c^2dt'^2.$$

You can verify that this is true by direct substitution of the special Lorentz transformation.

Exercise: Demonstrate the invariance of ds^2 under the special Lorentz transformation.

Therefore, to make the formalism work, we must define our *metric tensor* (in Cartesian coordinates),

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then we can write,

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

in analogy to the form taken from differential geometry. Indeed, we can use curvilinear coordinates rather than Cartesian coordinates. For spherical coordinates, the metric tensor would be,

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

However, one familiar characteristic of ds^2 that we give up by this generalization is that, unlike the case in three-dimensional differential geometry, we may (and usually do) have $ds^2 < 0$! The function ds^2 is an invariant. It has the same form, $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, and same value in all admissable frames.

Observe that the inner product of two four-vectors can be written,

$$g_{\mu\nu}A^\mu B^\nu = A^\mu B_\mu = A_\nu B^\nu.$$

The practical consequence of the metric tensor for Cartesian four-vectors is to associate a negative sign with the fourth-component term in the sum,

$$g_{\mu\nu}A^\mu B^\nu = A^1B^1 + A^2B^2 + A^3B^3 - A^4B^4 = A_1B_1 + A_2B_2 + A_3B_3 - A_4B_4.$$

In the rest frame of a moving clock,

$$d\mathbf{s} = (0, 0, 0, cdt') \equiv (0, 0, 0, cd\tau).$$

Thus, $ds^2 = -c^2d\tau^2$ is the invariant "squared length" of $d\mathbf{s}$. The squared proper time, $d\tau^2$, being equal to the invariant ds^2 divided by a constant of nature, c, is also an invariant. If we attach a frame of reference to a moving particle, its velocity in the unprimed system and the velocity of the primed frame are the same, $u = v$. In the laboratory frame, $d\mathbf{s} = (dx, 0, 0, cdt)$, so that

$$ds^2 = dx^2 - c^2dt^2 = -c^2dt^2(1 - \frac{1}{c^2}(\frac{dx}{dt})^2) = -\frac{c^2dt^2}{\gamma^2}.$$

By the invariance property, the value in the laboratory frame is the value in the rest frame, so,

$$-c^2d\tau^2 = -\frac{c^2dt^2}{\gamma^2},$$

or,

$$d\tau = \frac{dt}{\gamma}.$$

We have thus recovered the result called "time dilation."

A four-vector divided by a scalar invariant is a four-vector because it has the transformation pattern of a four-vector. We can create a new four-vector by dividing $d\mathbf{s}$ by $d\tau$. The resulting four-vector is called the *four-velocity* and has components,

$$U^\mu = \frac{dx^\mu}{d\tau}.$$

We then have,

$$\frac{dx^1}{d\tau} = \frac{dx^1}{dt/\gamma} = \gamma\frac{dx^1}{dt} = \gamma u_x,$$

$$\frac{dx^4}{d\tau} = \frac{cdt}{dt/\gamma} = \gamma c.$$

We might also write the four-velocity as,

$$U^\mu = (\gamma u_x, \gamma u_y, \gamma u_z, \gamma c) = (\gamma \mathbf{u}, \gamma c) = (\hat{\mathbf{n}} c \sinh\theta, c\cosh\theta),$$

where $\hat{\mathbf{n}}$ is a unit vector tangent to the particle motion in three-space.

Because U is a four-vector, we immediately know that it transforms from frame to frame according to the pattern, $U^{\mu'} = L^{\mu'}_{\nu} U^\nu$. We also know that its "squared length," $g_{\mu\nu}U^\mu U^\nu$, is an invariant. Its value is most easily obtained in the rest system of the particle where $u = 0$ and $\gamma = 1$. Then, $U = (0, 0, 0, c)$ and $g_{\mu\nu}U^\mu U^\nu = -c^2$.

Like proper time, the proper length and the proper mass are invariants. We may multiply U by the proper mass, m_0, to create yet another four-vector, the *four-momentum*,$P^\mu = m_0 U^\mu = (m_0\gamma\mathbf{u}, m_0\gamma c)$. We immediately know that this four-vector transforms according to the usual pattern and we know that it has an invariant squared length,

$$g_{\mu\nu}P^\mu P^\nu = m_0^2(g_{\mu\nu}U^\mu U^\nu) = -m_0^2 c^2.$$

It is customary to combine the factor of γ and the proper mass to create a *relativistic mass*, $m \equiv \gamma m_0$ and a *relativistic momentum*, $\mathbf{p} \equiv \gamma m_0 \mathbf{u} \equiv m\mathbf{u}$. We may also define a *relativistic energy*, $E \equiv \gamma m_0 c^2 \equiv mc^2$. Then, the four-momentum in an arbitrary frame can be written, $P^\mu = (\mathbf{p}, E/c)$. Using the invariance property of the squared length, we obtain the useful relationship,

$$p^2 - \frac{E^2}{c^2} = -m_0^2 c^2,$$

or,

$$E^2 = p^2c^2 + m_0^2 c^4.$$

Exercise: Derive this latter result, taking care to show that the algebraic signs are handled correctly.

We may create a *four-acceleration* by differentiating the four-velocity with respect to proper time, $A^\mu = \dot{U}^\mu$. To maintain form invariance (covariance) under Lorentz transformations, the laws of physics must be expressed as relationships among four-vectors and, possibly, four-tensors. With $m_0 A^\mu$, we have a four-vector with which we can build a covariant form of Newton's Second Law. But, first, we need to develop four-vector forces.

We can obtain another important four-vector from a different line of reasoning. Consider a plane electromagnetic wave propagating in space. We may represent the fields of the electromagnetic wave as,

$$\mathbf{A} = \mathbf{A}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

$\mathbf{A}$ represents either the electric field, $\mathbf{E}$, or the magnetic field, $\mathbf{B}$, associated with the electromagnetic wave. The vector $\mathbf{k}$ is the *wave number* defined by

$$\mathbf{k} = \frac{2\pi}{\lambda}\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of propagation and λ is the wavelength of the wave. The magnitude of $\mathbf{k}$ is the number of waves in 2π units of length. Since it depends on a measured length, the Lorentz contraction implies that $\mathbf{k}$ is not a relativistic invariant. Similarly, the frequency, ω, of the wave is a certain number of radians in a measured period of time and because of time dilation is also not a relativistic invariant.

The quantity $\mathbf{k}\cdot\mathbf{x}-\omega t$ is called the *phase* of the wave. It is a dimensionless quantity and is unaffected by either Lorentz contraction or time dilation. The phase is a function of a position and a time. We might think of an observer making an observation of the wave at $(\mathbf{x}, t)$ thus defining an event in space time. The primed coordinates of this same event are $(\mathbf{x}', t')$. The phase of the wave determines whether the observers are observing the peak, the trough, or some other part of the cycle of the wave. But, since they will both agree that they are observing the same part of the wave cycle, the phase must be the same for both. The phase is a relativistic invariant, which means that,

$$\mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t.$$

For simplicity, consider a special Lorentz transformation and an electromagnetic wave propagating along the common x, x'-axes. Then,

$$k_x' dx' - \omega' dt' = k_x'[\gamma(dx - vdt)] - \omega'[\gamma(dt - \frac{vdx}{c^2})]$$

$$= \gamma(k_x' + \frac{v\omega'}{c^2})dx - \gamma(\omega' + k_x' v)dt$$

$$= k_x dx - \omega dt.$$

If the phase is an invariant, we must have,

$$k_x = \gamma(k'_x + \frac{v\omega'}{c^2})$$

$$\frac{\omega}{c} = \gamma(\frac{\omega'}{c} + \frac{k'_x v}{c}).$$

But, this is just the pattern for the special Lorentz transformation of a four-vector, $k^\mu = (\mathbf{k}, \omega/c)$. We will call this four-vector the *four-wavenumber*.

Two important results follow immediately from the Lorentz transformation of this four-vector. Imagine electromagnetic waves that are observed approaching an observer at an angle, α, in the x, y-plane. Electromagnetic waves have an especially simple *dispersion relation*, that relates the wave number and the frequency, namely $k = \omega/c$. Because the four-wavenumber is a four-vector, we know immediately how it transforms,

$$\begin{pmatrix} k' \cos\alpha' \\ k' \sin\alpha' \\ 0 \\ \omega'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} k \cos\alpha \\ k \sin\alpha \\ 0 \\ \omega/c \end{pmatrix}.$$

In particular,

$$k' \cos\alpha' = \gamma k \cos\alpha - \frac{v\gamma}{c}\frac{\omega}{c}$$

$$\frac{\omega'}{c} = -\frac{v}{c}\gamma k \cos\alpha + \gamma\frac{\omega}{c}.$$

Using the dispersion relationship for electromagnetic waves, the second of these can be written,

$$\omega' = \gamma\omega(1 - \frac{v}{c}\cos\alpha).$$

This relationship is the *relativistic Doppler shift*. It tells what the frequency of electromagnetic waves is in a primed frame if those same waves are observed with frequency, ω, in the laboratory and arriving at angle, α.

The second important relationship comes from dividing the first of the transformation equations by the second and using the fact that $k' = \omega'/c$ and $k = \omega/c$. We have then,

$$\cos\alpha' = \frac{\cos\alpha - v/c}{1 - (v/c)\cos\alpha}.$$

This is the *relativistic aberation formula* and gives the angle of approach, α', in the primed system in terms of the angle in the unprimed system.

10.8 The Four-Vector Formulation of Maxwell's Equations

To develop a Lorentz form-invariant replacement for Newton's Second Law, we need a four-vector expression for force. Of the four fundamental forces, the strong and weak forces are quantum mechanical and do not fall in the domain of phenomena described by Newton's Second Law, although the quantum field theories that treat them are usually expressed in relativistically covariant form. Gravity, on the other hand, is treated in an extension of the Special Theory of Relativity called *General Relativity*. The laws of General Relativity are expressed in covariant form, but the treatment goes beyond what we can do here.

We are left with the electromagnetic force. It is the force exerted on charged particles by electric and magnetic fields. In nonrelativistic mechanics, it is expressed as a 3-vector, called the *Lorentz Force*,

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}).$$

Since the force depends on a combination of electric and magnetic fields, as well as the charge, q, and velocity, $\mathbf{u}$, of a particle, it is a somewhat more difficult task than we have yet encountered to express the electromagnetic force in four-vector form.

Maxwell's equations express the relationship between electromagnetic fields and the sources (charge and current) that generate them. For our purposes, it is not necessary to understand many of the applications of Maxwell's equations to electromagentic phenomena. What we shall show is that Maxwell's equations are already fully consistent with Special Relativity. We do this by showing how Maxwell's equations can be written down in terms of four-vectors and four-tensors, thus ensuring that the equations have exactly the same form in all coordinate systems of Special Relativity. We shall also show how the Lorentz Force is expressed as part of a four-vector. With this force as an example, we will then show how Newton's Second Law and Newtonian dynamics are incorporated into Special Relativity. Finally, we will conclude with some examples.

The sources of the electromagnetic field are stationary or moving charges. Here we must add another postulate to our theory of Special Relativity:

4. **Postulate of Charge Invariance**: The electric charge on an elementary particle is an absolute invariant.

The charge density, on the other hand, is the amount of charge per unit volume,

$$\rho = \frac{Nq}{Adx}$$

where, N is the total number of charged particles, q is the charge on each particle, $A = dydz$ is an area that is invariant under the special Lorentz transformation, and dx is a differential of distance along the common x, x'-axes. For a special Lorentz transformation, $dy = dy'$ and $dz = dz'$ so that $A = A'$. However, $dx = \gamma(dx' + vdt')$ so that charge density is not relativistically invariant. Neverthless, like proper length, proper time and proper mass, the proper density, ρ_0, is a relativistic scalar and can be used to create a new four-vector (called the *four-current*),

$$J^\mu = \rho_0 U^\mu = (\rho_0 \gamma \mathbf{u}, \rho_0 \gamma c).$$

As we have done to create relativistic mass, it is customary to define the *relativistic charge density* as $\rho \equiv \gamma\rho_0$ so that $J^\mu = (\rho\mathbf{u}, \rho c) = (\mathbf{j}, \rho c)$. Thus, we have defined the *relativistic current*, $\mathbf{j}$, with units of charge per unit area per unit time. In this way we have created a four-vector that describes the sources of the electromagnetic field.

The electromagnetic fields, $\mathbf{E}$ and $\mathbf{B}$, do not individually extend naturally to become parts of four-vectors. Indeed, it is one of the surprising insights of Special Relativity that the six components of these 3-vectors are elements of a single electromagnetic field tensor. In a sense, electric and magnetic fields are not different things, but rather equivalent parts of the same thing. What one observer reckons to be a pure electric field, is reckoned by another observer in a different frame to be a combination of electric and magnetic fields!

This time we are going to do it backwards. We are not going to start with electromagnetic experiments and develop equations to describe them, which, after much work are to be shown to be relativistically consistent. Rather, we are going to do a very remarkable thing. We are going to assume as a basic premise that the laws of physics must be expressed in relativistically form-invariant equations and let that assumption tell us essentially what the laws of electrodynamics must be. We will have to make a few "judicious choices" along the way so that the laws of physics come out in their familiar form, but these are more convention than anything else.

We begin by assuming that there exists a four-vector, $A^\mu = (\mathbf{A}, \phi/c)$. The factor $1/c$ is not essential, but including it here makes it easier to connect our results with the familiar equations of electrodynamics. The four-vector A is generated in some way by the sources that we have described by the four-current, J. The four-vector A has only four components, so it alone cannot incorporate the six components of $\mathbf{E}$ and $\mathbf{B}$. Therefore, we must use one of the methods we have identified for generating four-tensors from four-vectors.

We can generate a four-tensor by differentiating a four-vector. For example, $T_{\mu\nu} = \partial_\mu A_\nu$, but the result is a tensor that has sixteen components. We only need six slots for our electromagnetic fields. However, we could add two tensors and create another tensor. Consider, $T_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu$. This tensor is symmetric because $T_{\mu\nu} = T_{\nu\mu}$. Symmetric tensors have ten independent components which you can see if you write down a 4×4 symmetric matrix of the elements. But then we have ten slots for our six components of electromagnetic field. Finally, consider, $T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This tensor is anti-symmetric because $T_{\mu\nu} = -T_{\nu\mu}$. This means that the four diagonal elements of its matrix must be zero. The anti-symmetric tensor has six independent elements and that is just what we need! So, we define the covariant components of an antisymmetric electromagnetic field tensor,

$$T_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}.$$

Arranging the electromagnetic fields in just this way is one of the "judicious choices" mentioned above. We have done it so that the four-vector equation, $T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is equivalent to the two vector equations,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

These relationships are familiar in electrodynamics. Ordinarily, they are used to define the *vector potential*, $\mathbf{A}$, and scalar potential, ϕ, by defining their relationships to the more familiar electrical and magnetic fields. In our approach here, we introduced the four-vector $A^\mu = (\mathbf{A}, \phi/c)$ first. By arranging the components of the electric and magnetic fields in the field tensor as we did, we identify that our assumed $\mathbf{A}$ is the vector potential of conventional electrodynamics and that ϕ is the scalar potential of conventional electrodynamics.

We can also "raise the indices" to create the contravariant components of the electromagnetic field tensor, $T^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta}$. Then,

$$T^{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix}.$$

Exercise: Derive the expression for $T^{\mu\nu}$ by raising the indices of $T_{\alpha\beta}$.

With a little patience you can show that the following two form-invariant equations,

$$\partial_\mu T^{\nu\mu} = \frac{4\pi}{c} \rho_0 U^\nu$$
$$\partial_\mu T_{\nu\lambda} + \partial_\lambda T_{\mu\nu} + \partial_\nu T_{\lambda\mu} = 0$$

are completely equivalent to Maxwell's Equations in Gaussian units,

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.$$

The first two of Maxwell's equations have sources on the right as does the first of the two relativistic equations. The second two of Maxwell's equations and the second of the relativistic equations are equal to zero on the right.

Exercise: Demonstrate how $\nabla \cdot \mathbf{B} = 0$ follows from the four-vector form of Maxwell's equations.

It follows that the Lorentz force, being a function of fields and velocity, can be written as some combination of the field tensor and the four-velocity. It can be readily verified that the expression,

$$K^\mu = \frac{q}{c} T^{\mu\nu} U_\nu = (\gamma \mathbf{F}, \frac{1}{c}\gamma \mathbf{F} \cdot \mathbf{u}),$$

is the four-vector that contains the Lorentz force, $\mathbf{F}$. The four-force, K^μ, is sometimes called the *Minkowski force*. Therefore, the relativistic version of Newton's Second Law in those instances where the force is electromagnetic must be,

$$\frac{q}{c} T^{\mu\nu} U_\nu = m_0 \frac{dU^\mu}{d\tau}.$$

Exercise: Verify the relationship between the Minkowski force and the Lorentz force by expanding $T^{\mu\nu}U_\nu$.

10.9 Relativistic Particle Mechanics

Special Relativity deals with frames of reference that move relative to one another at constant velocity. Occasionally, the newcomer to the subject will have the mistaken idea that Special Relativity only deals with constant velocities, while General Relativity is the extension of Special Relativity to deal with accelerated motions. This is a misconception. General Relativity is an extension of Special Relativity to deal with the force of gravity. The only restriction of Special Relativity is that motion, including accelerated motion, be described from a frame which is at rest or in uniform motion. Many of the applications of Special Relativity are to the acceleration of charged particles in electromagnetic fields, either in accelerators or elsewhere in the cosmos.

Consider a charged particle moving with velocity $\mathbf{u}$ in the laboratory frame. We may write the four-vector law of motion as,

$$K^\mu = (\gamma\mathbf{F}, \frac{1}{c}\gamma\mathbf{F}\cdot\mathbf{u}) = m_0\frac{dU^\mu}{d\tau}.$$

Since $d\tau = dt/\gamma$, we may write this as,

$$\left(\gamma\mathbf{F}, \frac{1}{c}\gamma\mathbf{F}\cdot\mathbf{u}\right) = m_0\gamma\left(\frac{d}{dt}(\gamma\mathbf{u}), \frac{d}{dt}(\gamma c)\right).$$

Thus,

$$\left(\mathbf{F}, \frac{1}{c}\mathbf{F}\cdot\mathbf{u}\right) = \left(\frac{d}{dt}(\gamma m_0\mathbf{u}), \frac{d}{dt}(\gamma m_0 c)\right) = \left(\frac{d\mathbf{p}}{dt}, \frac{1}{c}\frac{dE}{dt}\right),$$

where, $\mathbf{p}$ is the relativistic momentum and E is the relativistic energy of the particle. We may separate the single four-equation into two,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F}\cdot\mathbf{u} = \frac{dE}{dt}.$$

The first is a relativistic statement of Newton's Second Law, except that $\mathbf{p}$ is the relativistic momentum rather than the Newtonian momentum. The second of these two equations is a statement that the power, $\mathbf{F}\cdot\mathbf{u}$, delivered to the particle by the force, $\mathbf{F}$, equals the rate of increase of the total relativistic energy of the particle. The total relativistic energy, $E = \gamma m_0 c^2$, is the sum of the kinetic energy of the particle plus its rest mass energy, $m_0 c^2$.

There is a peculiar consequence of the relativistic Second Law.

Exercise: Show that

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} u \frac{du}{dt}.$$

Exercise: Show that

$$\gamma + \gamma^3 \frac{u^2}{c^2} = \gamma^3.$$

We have then that,

$$\frac{d}{dt}(\gamma \mathbf{u}) = \gamma \frac{d\mathbf{u}}{dt} + \mathbf{u}\frac{d\gamma}{dt} = \gamma \mathbf{a} + \mathbf{u}\frac{\gamma^3}{c^2} u \frac{du}{dt} = \gamma(\mathbf{a}_\perp + \mathbf{a}_\parallel) + \mathbf{u}\frac{\gamma^3}{c^2} u \frac{du}{dt},$$

where $\mathbf{a}_\perp$ is the component of acceleration perpendicular to the direction of motion and $\mathbf{a}_\parallel$ is the component of acceleration parallel to the direction of motion.

Now, $\mathbf{u}\cdot\mathbf{u} = u^2$. If we differentiate,

$$\mathbf{u}\cdot\mathbf{a} = u\frac{du}{dt} = u a_\parallel.$$

Because $\mathbf{u}$ and $\mathbf{a}_\parallel$ are in the same direction, we may use that fact that $\mathbf{u} a_\parallel = u \mathbf{a}_\parallel$ to write,

$$\frac{d}{dt}(\gamma \mathbf{u}) = \gamma \mathbf{a}_\perp + (\gamma + \frac{\gamma^3 u^2}{c^2})\mathbf{a}_\parallel = \gamma \mathbf{a}_\perp + \gamma^3 \mathbf{a}_\parallel.$$

Thus,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m_0 \gamma \mathbf{a}_\perp + m_0 \gamma^3 \mathbf{a}_\parallel \equiv m_\perp \mathbf{a}_\perp + m_\parallel \mathbf{a}_\parallel.$$

The particle behaves as if it has two different masses, one for parallel acceleration and one for perpendicular acceleration! It is more difficult to accelerate a particle by an electromagnetic force along its direction of motion than it is to accelerate it perpendicular to its motion.

10.10 The Twin Paradox

Imagine twins, Albert and Henri, who part one day for separate, extended voyages in spacetime. Let Albert stay behind and remain at the origin of his stationary, unprimed reference frame, but let Henri move away in a rocket ship which defines the origin of his primed frame. After some time, the two meet again and compare their ages (clocks). Let us assume that they meet where they first start with the origins of the two systems once again coincident. The parting and the reuniting define two events in space time for which $dx = dx' = 0$. Thus, the invariant $ds^2 = -c^2 d\tau^2$. But which clock has actually measured the proper time? From Albert's point of view, his clock has remained stationary and has measured the proper time, while Henri's clock has moved and measured a dilated time. But from Henri's point of view, it is Henri's clock that has remained stationary, while Albert and his clock have moved and have measured a dilated time. Do their clocks agree or not? How can they agree if each thinks the other has measured a dilated time relative to his own? How can they disagree if there is symmetry in their points of view? We have here an apparent paradox, often called the *twin paradox*.

The paradox is resolved in the following way. First, the viewpoints of the twins are not necessarily symmetric. If the two are to be reunited, at least one of the pair must turn around, i.e., be accelerated during at least part of the journey. Special Relativity is a description of motion from a frame of reference that is at rest or in uniform motion. Thus, the unprimed frame of Albert is an admissable frame for using the relationships of Special Relativity, but Henri's accelerated frame is not during the period of acceleration. Thus the symmetry is broken. It is true that $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g'_{\alpha\beta}dx'^\alpha dx'^\beta$, but the metric tensor in the primed frame is no longer a function of constant values of γ and v. Indeed, without some thought, it is not obvious what it should actually be.

Nevertheless, we can calculate a comparison of Albert's elapsed time relative to the proper time measured on Henri's clock if we use Special Relativity in Albert's frame. Let us assume that Henri's rocket ship is accelerated by some kind of constant force, $\mathbf{F} = m_0\mathbf{g}$. Newton's relativistic Second Law is then,

$$\mathbf{F} = m_0\mathbf{g} = \frac{d}{dt}(\gamma m_0 \mathbf{u}).$$

If we assume that the acceleration is one-dimensional along the common

x, x'-axes, we have

$$g = \frac{d}{dt}(\gamma u).$$

In this instance we can make good use of the rapidity variable by using,

$$\sinh\theta = \gamma\frac{u}{c}$$

$$\cosh\theta = \gamma$$

$$\tanh\theta = \frac{u}{c}.$$

Then,

$$g = \frac{d}{dt}(\gamma u) = \frac{1}{\gamma}\frac{d}{d\tau}(\gamma u) = \frac{c}{\cosh\theta}\frac{d}{d\tau}(\cosh\theta\tanh\theta) = c\frac{d\theta}{d\tau}.$$

Assuming a boundary condition of $\theta = 0$ at $\tau = 0$, we can integrate to obtain,

$$\theta = \frac{g\tau}{c}.$$

The four-velocity yields,

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx}{d\tau}, 0, 0, c\frac{dt}{d\tau}\right) = (c\sinh\theta, 0, 0, c\cosh\theta),$$

i.e.,

$$\frac{dx}{d\tau} = c\sinh(\frac{g\tau}{c})$$

$$\frac{dt}{d\tau} = \cosh(\frac{g\tau}{c}).$$

Each can be integrated using appropriate boundary conditions to yield,

$$x_a = \frac{c^2}{g}\left(\cosh(\frac{g\tau}{c}) - 1\right)$$

$$t_a = \frac{c}{g}\sinh(\frac{g\tau}{c}).$$

Observe that the function for t_a is an even function of g, so that it serves as well for acceleration and deceleration.

If Henri accelerates to a terminal coasting speed, u_c, we have during the coasting period,

$$\frac{d}{dt}(\gamma u) = 0$$

from which $\gamma u = \text{constant} = \gamma_c u_c$. Then,

$$\sinh\theta_c = \frac{\gamma_c u_c}{c}$$

$$\cosh\theta_c = \gamma_c.$$

Again,

$$U^\mu = \left(\frac{dx}{d\tau}, 0, 0, c\frac{dt}{d\tau}\right) = (c\sinh\theta_c, 0, 0, c\cosh\theta_c),$$

from which,

$$x_c = c\tau\sinh(\frac{\gamma_c u_c}{c})$$

$$t_c = \tau\cosh(\frac{\gamma_c u_c}{c}).$$

By combining periods of acceleration, coasting, and deceleration, one can demonstrate that when Henri returns to meet with Albert, his (proper) time and age will be less than that of Albert who stayed behind.

Exercise: If g is chosen for comfort to be the local acceleration of gravity (which in truly unusual units is 1.03 light years/year/year), how far would Henri travel in 22 years of his own time? How much time will elapse on earth if Henri makes a round trip by accelerating for 6 years (proper time), then decelerating for six years and then reversing the process to come back? How much time will he add to the elapsed time on earth if he coasts for a year (proper time) on the way out and again for a year on the way back?

10.11 Conservation Laws

The force law,

$$K^\mu = m_0\frac{dU^\mu}{d\tau}$$

tells us that for a particle,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F}\cdot\mathbf{u} = \frac{dE}{dt}.$$

In the absence of force, the four-velocity is a constant of the motion, i.e., $\mathbf{p}$ and E are constants of the motion. If u is modest in magnitude relative to c, we can expand γ,

$$\frac{1}{\sqrt{1-u^2/c^2}} = 1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \ldots.$$

Hence,

$$E = m_0c^2 + \frac{1}{2}m_0u^2 + \frac{3}{8}\frac{m_0u^4}{c^2} + \ldots.$$

The first term exists even when $u = 0$. We call m_0c^2 the *rest energy* of the particle. The remaining terms, beginning with $1/2m_0u^2$, are associated with motion and together form the *relativistic kinetic energy*. Because relativistic kinetic energy is not directly identifiable with a component of a four-vector, it usually plays a less important role in relativistic mechanics than the relativistic momentum and the total relativistic energy.

It is possible within Special Relativity to have a particle for which $m_0 = 0$ if that particle moves at the speed of light. In such a case, both the numerator and the denominator of γm_0 become zero. However, if we let this happen in such a way that the total energy of the particle is finite, we have,

$$\gamma m_0 c^2 \to E$$

$$\gamma m_0 \to E/c^2$$

$$\mathbf{p} = \gamma m_0 u\hat{\mathbf{n}} \to (E/c^2)(c)\hat{\mathbf{n}} = E/c\hat{\mathbf{n}}.$$

The four momentum then becomes, $P^\mu = (E/c\hat{\mathbf{n}}, E/c)$. The photon of quantum mechanics is such a massless particle for which the total energy is $\hbar\omega$. The four-momentum of a photon is

$$P^\mu = (\frac{\hbar\omega}{c}\hat{\mathbf{n}}, \frac{\hbar\omega}{c}).$$

For a system of n particles, we define the total relativistic momentum of the system to be the sum of the relativistic momenta of the individual particles of the system,

$$\mathbf{P} \equiv \sum_{i=1}^{n} \mathbf{p}_i$$

and the total relativistic energy, E_t, to be the sum of the relativistic energies of the individual particles. If Newton's Third Law holds relativistically, this means that $\mathbf{P}$ and E_t are conserved in the absence of external forces.

The center of mass system is defined to be that (primed) system for which $\mathbf{P}' = 0$. The total relativistic three-momentum $\mathbf{P}$ and E_t/c are the elements of a four-vector and transform according to the pattern of four-vectors. This means that,

$$E_t = \gamma(E_t' + \mathbf{v} \cdot \mathbf{P}').$$

If we choose the primed system to be the center-of-mass system, $\mathbf{P}' = 0$, we have $E_t = \gamma E_t'$. If we now think of our system of particles as a body made up of the constituent particles, we can give it a proper mass, M_0,

$$E_t = \gamma M_0 c^2 = \gamma E_t',$$

from which,

$$M_0 = \sum_{i=1}^{n} \left(\frac{m_{0i}}{\sqrt{1 - u_i'^2/c^2}} + \frac{V_i'}{c^2} \right) = \sum_{i=1}^{n} m_{0i} c^2 + \sum_{i=1}^{n} \frac{T_i'}{c^2} + \sum_{i=1}^{n} \frac{V_i'}{c^2}.$$

Here, T_i' is the relativistic kinetic energy of the particles in the center-of-mass frame and V_i' represents the potential energies that the particles might have as a result of their interaction. Clearly, the proper mass of the body is more than the sum of the rest masses of the constituent particles. If the kinetic energy of the particles is increased by heating, for example, the mass M_0 of the body increases! Similarly, if the potential energy of the particles decreases as a result of their interaction, the mass of the body might even be less than the sum of the rest masses by an amount Δm called the *mass defect*. In any case, it is clear that mass and energy are not different things, but manifestations of the same thing, *mass-energy*.

10.12 Uses of Invariants

Conservation laws tell us that certain quantities are the same at different times. Invariant quantities tell us that certain quantities are the same in different frames. Sometimes the two can work together advantageously.

Imagine a proton accelerator that collides protons on stationary targets. Imagine that the intent of the collision is to collide a proton on a proton target to produce a proton-antiproton pair from the kinetic energy of the accelerated proton. We want to know how much energy the initial proton must have if an proton and antiproton exist after collison in addition to the original two protons. In other words, we want to know the *threshold energy* for the process.

In this case we are not concerned with the dynamics of the interaction itself. We are just interested in a relationship between quantities before the collision (beam energy) and conditions after the collision (four masses). This is something about which conservation laws tell us. Experiments of this type are carried out in the laboratory frame, but they are usually most easily analyzed in the center-of-mass frame. Invariants tell us something about relationships between frames.

Both in the laboratory frame and in the center-of-mass frame, we have a "before" four-momentum and an "after" four-momentum, i.e., four different four-momentum vectors. The "before" four-momentum in the laboratory frame consists of the 3-momentum of the single moving proton as well as the total relativistic energies of the two protons,

$$P^{\mu}_{b,lab} = \left(p_{1b}, 0, 0, \frac{E_{p1} + m_p c^2}{c} \right).$$

The "after" four-momentum (for threshold) in the center-of-mass frame consists of four stationary particles clustered together. The four-momentum is,

$$P^{\mu}_{a,cm} = \left(0, 0, 0, \frac{4 m_p c^2}{c} \right).$$

Since there are no external forces on the system, four-momentum is conserved: $P^{\mu}_{b,cm} = P^{\mu}_{a,cm}$. Thus we have,

$$g_{\mu\nu} P^{\mu}_{b,lab} P^{\nu}_{b,lab} = g_{\mu\nu} P^{\mu}_{b,cm} P^{\nu}_{b,cm} = g_{\mu\nu} P^{\mu}_{a,cm} P^{\nu}_{a,cm}.$$

The first equality follows from invariance and the second follows from the conservation law. Skipping the intermediate step,

$$p_{1b}^2 - \frac{(E_{1b} + m_p c^2)^2}{c^2} = -\frac{(4 m_p c^2)^2}{c^2}.$$

However, we may also use the fact that the four-momentum of the original beam particle alone forms an invariant connecting the laboratory and the rest frame of that single particle,

$$p_{1b}^2 - \frac{E_{1b}^2}{c^2} = -\frac{m_p^2 c^4}{c^2}.$$

Combining these two relationships, we have

$$E_{1b} = 7 m_p c^2.$$

If we add the rest energy of the stationary target, we have $8m_pc^2$ in the laboratory frame, of which $4m_pc^2$ is available in the center-of-mass frame to produce particles.

How much energy would we have in the center-of-mass frame if we accelerated both protons to an energy of $7m_pc^2$ and figured out a way to collide them head-on. In this case, the laboratory frame becomes the center-of-mass frame and all $14m_pc^2$ is available to produce particles, i.e. 3.5 times as much as before. In terms of four-vectors,

$$P^\mu_{b,lab} = \left(0, 0, 0, \frac{2E_{1b}}{c}\right)$$

$$P^\mu_{a,cm} = \left(0, 0, 0, \frac{E_{cm}}{c}\right),$$

from which $E_{cm} = 14m_pc^2$. For this reason, many of the modern accelerators are designed as colliders rather than stationary target accelerators, although there are other tradeoffs, including the rate of collision events, that also must be considered.

As another example, consider the decay of a neutron. A free neutron decays into a proton, an electron and an antineutrino with a proper mean lifetime of 14.8 minutes. Suppose we want to know the maximum possible energy of the ejected electron. We shall suppose that in this case the antineutrino carries away negligible energy and momentum. In this special circumstance, it is as if the antineutrino doesn't exist.

Let the neutron be at rest, so that the laboratory frame and the center-of-mass frame are the same. In the absence of external forces, the four-momentum of the system is conserved,

$$(0, 0, 0, m_n c)_{b,cm} = (\mathbf{p}_e + \mathbf{p}_p, (E_e + E_p)/c)_{a,cm} = (p_e - p_p, 0, 0, (E_e + E_p)/c)_{a,cm}$$

and has an invariant scalar product. Thus, we have,

$$p_e = p_p$$

$$m_n c^2 = E_e + E_p$$

$$(p_e - p_p)^2 - \frac{(E_e + E_p)^2}{c^2} = -m_n^2 c^2.$$

The four-momenta of the electron and proton taken separately have invariant scalar products,

$$p_e^2 - (\frac{E_e}{c})^2 = -m_e^2 c^2$$

$$p_p^2 - (\frac{E_p}{c})^2 = -m_p^2 c^2.$$

Combining these conservation and invariance relationships,

$$E_{e,max} = \frac{(m_n^2 - m_p^2 + m_e^2)c^2}{2m_n}.$$

Since the neutron was assumed to be at rest, we can take this result to be a center-of-mass result. If the neutron were moving along the x-axis of the laboratory with velocity v, we could transform the result back to the laboratory frame to obtain a result for the maximum energy of an electron resulting from the decay of a moving neutron.

Let us imagine now that the electron is emitted at an angle θ relative to the x-axis in the laboratory system. We want to know what $E_{e,max}$ is in the laboratory as a function of θ. The value for $E_{e,max}$ already obtained in the center-of-mass frame now becomes $E'_{e,max}$. Since energy is an element of a four-vector, we know how it transforms,

$$\frac{E'_e}{c} = \gamma\left(\frac{E_e}{c} - \frac{v}{c^2} p_e \cos\theta\right)$$

or,

$$\frac{E_e}{c} = \gamma\left(\frac{E'_e}{c} + \frac{v}{c^2} p'_e \cos\theta'\right).$$

Because we want an answer in terms of the laboratory angle, we use the former of the two. We eliminate p_e by using the invariant scalar product relationship for the electron,

$$p_e^2 - (E_e^2/c^2) = -m_e^2 c^2.$$

With p_e eliminated, the transformation becomes a quadratic equation in E_e, for which, after some algebraic simplification,

$$E_{e,max} = \frac{E'_{e,max} + \frac{v}{c}\cos\theta\sqrt{E'^2_{e,max} - \gamma^2 m_e^2 c^4\left(1 - \frac{v^2}{c^2}\cos^2\theta\right)}}{\gamma\left(1 - \frac{v^2}{c^2}\cos^2\theta\right)}.$$

As a final example of this type, let us consider a phenomenon called *Compton scattering*. A photon, originally moving along the x-axis scatters from a stationary electron. As a result, the electron recoils at an angle ϕ relative to the x-axis and the photon, with an altered wavelength and energy, is scattered to an angle θ relative to the x-axis. We are interested in the change in wavelength of the photon as a function of the angle of scatter, θ.

In this instance, nothing is gained by working in the center-of-mass frame and all calculations are done directly in the laboratory frame. We have a four-momentum for the original photon before the scattering,

$$P^\mu_{b,\omega} = \left(\frac{\hbar\omega}{c}, 0, 0, \frac{\hbar\omega}{c}\right),$$

for which $g_{\mu\nu}P^\mu_{b,\omega}P^\nu_{b,\omega} = 0$. There is also a total four-momentum before the scattering,

$$P^\mu_{b,tot} = P^\mu_{b,e} + P^\mu_{b,\omega} = \left(\frac{\hbar\omega}{c}, 0, 0, \frac{\hbar\omega}{c} + m_e c\right).$$

There is a total four-momentum after the scattering, $P^\mu_{a,e} + P^\mu_{a,\omega^*}$,

$$P^\mu_{a,tot} = \left(\frac{\hbar\omega^*}{c}\cos\theta + p_e\cos\phi, \frac{\hbar\omega^*}{c}\sin\theta - p_e\sin\phi, 0, \frac{\hbar\omega^*}{c} + \frac{E_e}{c}\right).$$

The scalar product of each with itself is an invariant. Conservation of four-momentum tells us that $P^\mu_{b,tot} = P^\mu_{a,tot}$, so that,

$$g_{\mu\nu}P^\mu_{b,tot}P^\nu_{b,tot} = \left(\frac{\hbar\omega}{c}\right)^2 - \left(\frac{\hbar\omega}{c} + m_e c\right)^2 = g_{\mu\nu}P^\mu_{a,tot}P^\nu_{a,tot}.$$

We simplify the "after" expression by systematically eliminating four-vectors that contain information about the scattered electron for which we are not presently concerned. Hence,

$$g_{\mu\nu}P^\mu_{a,tot}P^\nu_{a,tot} = g_{\mu\nu}(P^\mu_{a,e} + P^\mu_{a,\omega^*})(P^\nu_{a,e} + P^\nu_{a,\omega^*})$$

$$
\begin{aligned}
&= g_{\mu\nu} P^{\mu}_{a,e} P^{\nu}_{a,e} + g_{\mu\nu} P^{\mu}_{a,\omega^*} P^{\nu}_{a,\omega^*} + 2 g_{\mu\nu} P^{\mu}_{a,e} P^{\nu}_{a,\omega^*} \\
&= -m_e^2 c^2 + 0 + 2 g_{\mu\nu} (P^{\mu}_{a,tot} - P^{\mu}_{a,\omega^*}) P^{\nu}_{a,\omega^*} \\
&= -m_e^2 c^2 + 2 g_{\mu\nu} P^{\mu}_{b,tot} P^{\nu}_{a,\omega^*} - 0 \\
&= -m_e^2 c^2 + 2 \left(\frac{\hbar\omega}{c} \frac{\hbar\omega^*}{c} \cos\theta - (\frac{\hbar\omega}{c} + m_e c)(\frac{\hbar\omega^*}{c}) \right).
\end{aligned}
$$

After some simplifying algebra and using,

$$
\hbar\omega = \frac{2\pi\hbar c}{\lambda},
$$

we obtain the desired Compton scattering result,

$$
\Delta\lambda = \frac{2\pi\hbar}{m_e c}(1 - \cos\theta).
$$

There are other kinds of important invariants. The dynamics of an interaction of colliding particles is characterized by a *cross-section*. Conceptually, the cross-section is an area around a target seen by the approaching particle, such that if the approaching particle passes within this area, some process will take place. For classical Rutherford scattering, alpha particles approach gold nuclei and are scattered by Coulomb repulsion. Surrounding the gold nucleus like a halo with radius s is a strip of area, $d\sigma = 2\pi s ds$ in the plane containing the gold nucleus but perpendicular to the motion of the alpha particle. If the alpha particle passes through this area, it will be scattered into an angle between ϕ and $\phi + d\phi$. The area $d\sigma$ is said to be the *differential cross-section* for this particular process. Similar cross-sections are defined for other nuclear processes.

Since $d\sigma$ represents an area perpendicular to the direction of motion of the approaching particle, taken to be along the x-axis, the cross-section is invariant under the special Lorentz transformation. It provides a simple bridge to connect measurements in the laboratory and analysis in the center-of-mass system.

Variants of the differential cross-section are also used. For Rutherford scattering, the differential cross-section per unit solid angle is defined,

$$
\frac{d\sigma}{d\Omega} = \frac{2\pi s ds}{2\pi \sin\phi d\phi} = \frac{s}{\sin\phi} \frac{ds}{d\phi},
$$

but this cross-section is not Lorentz invariant because the angle ϕ is not Lorentz invariant. For relativistic purposes it is sometimes convenient to note that a four-volume,

$$(dxdydzcdt) = (\frac{dx'}{\gamma}dy'dz'(\gamma cdt')) = dx'dy'dz'cdt'$$

is invariant. Since four-vectors have the same transformation properties, it follows that,

$$dp_x dp_y dp_x \frac{E}{c}$$

is an invariant volume in four-momentum space. Thus the cross-section,

$$\frac{1}{E}\frac{d\sigma}{d\mathbf{p}}$$

is an invariant. It is a cross-section for a process that produces a particle into a certain volume of four-momentum space.

The proper mass of a system of particles is an invariant with another use. Suppose that a proton collides with a proton, producing a shower of pions. One suspects that some of the pions result from the decay of a very short-lived particle that is also produced in the collision. Let us imagine that the short-lived particle is expected to decay into three pions. The detector has sufficient resolution to allow one to reckon the relativistic momenta and energies of the pions, but not enough to even detect the suspected particle.

The four-momentum invariant of the suspected particle in its own frame, $P_M^\mu = (0, 0, 0, M_0 c)$, has a value $-M_0^2 c^2$. If one takes the four-momenta of the pions, three at a time, forms

$$P_t^\mu = ((\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3), (E_1 + E_2 + E_3)/c)$$

and computes $g_{\mu\nu} P_t^\mu P_t^\nu$, and finds that the values cluster around $-M_0^2 c^2$, one may take it as compelling evidence for the short-lived existence of the suspected particle.

Thus, invariants are used in a variety of ways to simplify relativistic calculations.

Computer Exercises

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In what follows, "ODE" stands for whatever Ordinary Differential Equation solver that you have available.

1. **Computer Project: Theory** One of the most commonly occurring differential equations in physics is the one describing the simple harmonic oscillator. In mechanics it occurs when a particle experiences a restoring force toward a point of equilibrium that is proportional to the displacement away from that point, $F = -kx$. The differential equation arising from Newton's Second Law becomes,

$$-kx = m\ddot{x}$$

or,

$$\ddot{x} + \omega_0^2 x = 0 \quad ,$$

with $\omega_0^2 = k/m$.

Consider, for example, a spring of negligible mass hanging vertically from a support. If you fasten a mass to the spring and gently lower it, the spring will extend an additional distance s until the force exerted upward by the spring balances the weight of the attached object,

$$ks = mg.$$

Now lift the spring a distance x in preparation to releasing it and setting the system into oscillatory motion. When lifted, the force exerted by the spring is $k(s - x)$, so that

$$m\ddot{x} = k(s - x) - mg = (ks - mg) - kx = -kx$$

since $ks = mg$.

If we now let

$$\dot{x} = v$$

$$\ddot{x} = \dot{v},$$

we then have the two first order differential equations,

$$\dot{x} = v$$

$$\dot{v} = -\omega_0^2 x,$$

which is the form that numerical differential equation solvers use.

A solution to this set of equations yields $x(t)$ and $v(t)$. You probably know already that the equations can be solved analytically and that

$$x = A\cos(\omega_0 t - \alpha)$$

$$v = -A\omega_0 \sin(\omega_0 t - \alpha).$$

A and α are determined by initial conditions. You can verify that these are solutions by substituting the solutions into the differential equations.

Since we already know the analytical solutions, let us use the harmonic oscillator to illustrate another useful way of looking at motion. Think of x and v as being two "coordinates" and make a plot of x versus v. Put x along the abscissa and v along the ordinate. Such a plot is called a *phase plot*, each pair of coordinates (x, v) is a *phase point*, and the curve defined by the sequence of points is called a *trajectory in phase space*.

It is easy to see what the trajectory should be for a harmonic oscillator. The energy is conserved, so

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

Rewrite this in the form

$$\frac{v^2}{(2E/m)} + \frac{x^2}{(2E/k)} = 1,$$

which is the form of an ellipse,

$$\frac{v^2}{a^2} + \frac{x^2}{b^2} = 1,$$

where a and b are the semimajor and semiminor axes. Thus, the trajectory of the simple harmonic oscillator in phase space is an ellipse. Changing the total energy E changes the semimajor and semiminor axes of the ellipse.

Computer Project

(a) Analytically find the formula for $x(t)$ that solves this differential equation with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. Find the values of x and $\dot{x}$ at the time $t = 10\pi/\omega_0$.

(b) Let $m = 1.0$ and $k = 2.0\pi$ (so that $\omega_0^2 = 2.0\pi$) and let $\dot{x}(0) = 0$. Then let $E = 1.0/4\pi$, followed by $E = 1.0/8\pi$ and $E = 1/16\pi$. Under these conditions, show that $x(0) = (2E/k)^{1/2}$. Plot $v(t)$ versus $x(t)$ for several values of E to see if the expected ellipses are obtained and to compare them for different choices of E. In this case you don't want to erase the screen after each computation because you want to change something and compare a new graph with the old one. You just want to change the initial conditions.

2. **Computer Project: Theory** If an object falls through a medium that exerts a resistive frictional force proportional to some power of the speed, we can imagine that its initial motion will be close to free fall (since v is still very small), but that the frictional force will grow as the speed increases until the frictional force eventually balances the weight. At this point the forces are in balance, acceleration ceases and the object falls thereafter at a constant speed called the *terminal velocity*.

Newton's Second Law for an object of mass m can be written approximately for a short time interval Δt as

$$F = m\frac{\Delta v}{\Delta t}$$

or,

$$F\Delta t = \Delta(mv).$$

We can read this to mean that over a short time interval, the change in momentum, mv, equals the product of the force times the time interval. $F\Delta t$ is called the *impulse*.

Consider a billiard ball falling through the air. The ball has a radius R so that as it falls for a short time Δt it sweeps out a cylinder of cross section πR^2 and length $v\Delta t$. If the density of air is ρ_{air}, the mass of air in the imaginary cylinder is the density of the air times the volume of the cylinder, $\rho_{air}\pi R^2 v\Delta t$. To a crude approximation, this air is set in motion by collision with the ball and changes its downward velocity from zero to the velocity of the ball v. The change of momentum of the air equals the impulse, so

$$m\Delta v = (\rho_{air}\pi R^2 v\Delta t)(v - 0) = F\Delta t.$$

Thus, $F \propto v^2 = bv^2$. This is the force exerted by the ball on the air. By Newton's Third Law, it is the magnitude of the frictional force exerted on the ball by the air. Crudely, then, we conclude that the air might exert a force on the ball that is proportional to the square of its speed.

If this were true, the Newton's Second Law for the falling billiard ball, including gravity, would be,

$$m\ddot{x} = mg - b\dot{x}^2$$

where we have taken the positive direction to be downward. We can write this as

$$\frac{d\dot{x}}{dt} = g - \frac{b}{m}\dot{x}^2.$$

Separating variables,

$$\int_0^v \frac{d\dot{x}}{(mg/b) - \dot{x}^2} = \frac{b}{m}\int_0^t dt.$$

Now, (mg/b) has dimensions of velocity squared, so we will replace it with v_t^2. The integral on the left-hand-side is of the form

$$\int \frac{dx}{p^2 - x^2} = \frac{1}{p}\tanh^{-1}(\frac{x}{p}).$$

So,

$$v = v_t \tanh(gt/v_t).$$

When t is zero, the hyperbolic tangent vanishes and $v_0 = 0$. When t gets very large, the hyperbolic tangent approaches one and $v \to v_t$. Thus, v_t is the terminal velocity.

Here is an only-slightly-crude *measured* formula for the force on a billiard ball falling through still air at speed v:

$$F = mg - a * v \exp\left(4.4|v|^{.232 - \frac{\ln|v|}{63}}\right)$$

In all cases, let $a = 1.0 \times 10^{-5}$. The result is a numerical formula in SI units, meaning that if v has units of m/sec, then F has units of Newtons. The billiard ball has mass $m = 0.220$ kg. Note that the direction of positive displacement x is taken to be down, i.e., gravity is a positive force and air resistance is a negative force.

Computer Project

(a) Use trial and error on your calculator, or use a numerical solver, to find the terminal speed of this ball, i.e., the speed at which the force vanishes.

(b) Use ODE to solve for the motion of the ball if it is dropped from rest, i.e., get $v(t)$ and $x(t)$. (You will have to make v_0 very small but not exactly zero to avoid the singularity at $ln(0)$.) From the plot of v vs t, obtain the terminal velocity.

(c) Make plots of x and v as functions of time and check to see if they are reasonable; describe what you see. Only use a time interval long enough to get pretty close to terminal velocity.

(d) Compare the output from ODE with the prediction of the hyperbolic tangent formula derived above analytically.

3. **Computer Project: Theory** The differential equation for the one-dimensional harmonic oscillator can be modified to include a frictional force. For simplicity, assume that the frictional force is proportional to the speed of the moving object. With this additional drag force, Newton's Second Law becomes

$$m\ddot{x} = -kx - b\dot{x}$$

or,

$$\ddot{x} + (b/m)\dot{x} + \omega_0^2 x = 0,$$

where $\omega_0^2 = k/m$ is the frequency of the undamped oscillator.

We can solve this equation by using a trial function, $e^{\alpha t}$. The parameter α can be used to make the solution fit our equation. If we substitute the trial function into the equation, we get

$$(\alpha^2 + (b/m)\alpha + \omega_0^2)e^{\alpha t} = 0.$$

Since $e^{\alpha t}$ does not vanish, the quantity in parentheses must vanish. The resulting quadratic equation can be solved for α and will have *two* solutions,

$$\alpha = -\frac{b}{2m} \pm \sqrt{(\frac{b}{2m})^2 - \omega_0^2}.$$

If $(b/2m)^2 < \omega_0^2$ the quantity under the radical will be negative and our two solutions will be complex numbers. Define,

$$\omega_1^2 = \omega_0^2 - (b/2m)^2 = \omega_0^2(1 - (1/\omega_0^2)(b/2m)^2).$$

We can then write our most general solution to the differential equation as a linear combination of the two solutions. In doing so, we introduce two constants, c_1 and c_2, that will be determined by the initial conditions. They will be chosen so that at $t = 0$, our solution will have the correct values for $x(0)$ and $\dot{x}(0)$. We have,

$$x(t) = e^{-(bt/2m)}(c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}).$$

This result can be written in two equivalent forms. First we can use

$$e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$$

to write our result as

$$\begin{aligned} x(t) &= e^{-bt/2m}((c_1 + c_2)\cos\omega_1 t + i(c_2 - c_1)\sin\omega_1 t) \\ &= e^{-bt/2m}(a_1 \cos\omega_1 t + a_2 \sin\omega_1 t). \end{aligned}$$

In this form a_1 and a_2 are determined by initial conditions.

Second, we can define $B^2 = a_1^2 + a_2^2$. Think of this as Pythagoras' Theorem for an imaginary right triangle of which B is the hypotenuse and a_1 and a_2 are adjacent sides such that $\tan\beta = a_2/a_1$. Then,

$$x(t) = e^{-bt/2m}B(\frac{a_1}{B}\cos\omega_1 t + \frac{a_2}{B}\sin\omega_1 t)$$

$$= e^{-bt/2m} B(\cos\beta\cos\omega_1 t + \sin\beta\sin\omega_1 t)$$
$$= e^{-bt/2m} B\cos(\omega_1 t - \beta).$$

Now B and β are to be determined by initial conditions.

In each of the three forms, the oscillation frequency is

$$\omega_1 = \omega_0\sqrt{1 - (1/\omega_0^2)(b/2m)^2},$$

i.e., the presence of the damping term (parameterized by b) causes a shift from the undamped frequency ω_0.

We can also use the averaging method to solve our differential equation. Think of the differential equation in the form,

$$\ddot{x} + \omega_0^2 x = f(x, \dot{x}),$$

with the right-hand side taken to be a small perturbation on otherwise harmonic motion. The averaging method tells us to assume a solution of the form,

$$x(t) = A(t)\cos(\omega_0 t + \phi(t))$$
$$\dot{x}(t) = -\omega_0 A(t)\sin(\omega_0 t + \phi(t)),$$

where A and ϕ are slowly varying functions of time. The method consists of substituting this assumed solution into the differential equation and solving for $\dot{A}$ and $\dot{\phi}$,

$$\dot{A} = -\frac{1}{\omega_0} f(x, \dot{x})\sin(\omega_0 t + \phi(t))$$

$$\dot{\phi} = -\frac{1}{\omega_0 A(t)} f(x, \dot{x})\cos(\omega_0 t + \phi(t)).$$

The method then assumes that A and ϕ are varying slowly enough that these equations can be averaged on the right-hand-side over one cycle of the motion taking A and ϕ to be constant. In our case, $f(x, \dot{x}) = -(b/m)\dot{x}$, so,

$$\dot{A} = -(1/\omega_0)\langle(-(b/m)\dot{x})\sin(\omega_0 t + \phi)\rangle$$

$$\dot{\phi} = -(1/\omega_0)\langle((-(b/m)\dot{x})\cos(\omega_0 t + \phi))/A\rangle,$$

assuming that A and ϕ are taken as constants for one cycle of the motion. In this case we have,

$$\dot{A} = -(bA/m)\langle \sin^2(\omega_o + \phi)\rangle$$

$$\dot{\phi} = -(b/m)\langle \sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi)\rangle.$$

Since the average of $\sin^2(\omega_0 + \phi)$ over one cycle is $1/2$, and the average of the product of a sin and a cos over one cycle is zero, we have,

$$\dot{A} = -(bA/2m)$$

$$\dot{\phi} = 0.$$

The first can be solved by separating variables,

$$A = A_0 e^{-bt/2m}$$

and the solution to the second is a constant, ϕ_0, by inspection. Hence,

$$x(t) = A_0 e^{-bt/2m}\cos(\omega_0 t + \phi_0).$$

Computer Project

To verify the quality of the averaging assumption in the method, consider the damped harmonic oscillator:

$$\ddot{x} = -\omega_o^2 x - \frac{\dot{x}}{\tau}.$$

(a) Use ODE to solve the *un-averaged* amplitude and phase equations. Use $\omega_o = 1.0$, $m = 1.0$, $x(0) = 1.0$, $\dot{x}(0) = 0.0$ and use several values of $\tau = m/b$, say $\tau = 1,\ 3,\ 10,\ 100$. The unaveraged amplitude and phase equations are

$$\dot{A} = -(bA/m)\sin^2(\omega_o t + \phi)$$

$$\dot{\phi} = -(b/m)\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi).$$

(b) Describe what the plots of $A(t)$ and $\phi(t)$ look like and verify that

$$x(t) = A(t)\cos(\omega_o t + \phi(t))$$

is indeed very close to the correct solution of the equation for $\tau = 1$. Run a solution to the unaveraged phase phase/amplitude equations with the given initial conditions. Then add to the plot the exact analytical solution, i.e.,

$$x(t) = e^{-bt/2m} B \cos(\omega_1 t - \beta),$$

and another for the result of the averaging assumption,

$$x(t) = e^{-bt/2m} B \cos(\omega_0 t - \beta),$$

But, remember that for these latter two equations, the values of B and β are determined from initial conditions. The values of B and β are not the same in the two equations and must be separately determined (analytically) if the comparison of the graphs is to make any sense.

4. **Computer Project: Theory** What happens if the damped harmonic oscillator is driven by some harmonic force of frequency ω? The differential equation becomes

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos\omega t$$

or,

$$\ddot{x} + (b/m)\dot{x} + \omega_0^2 x = (F_0/m)\cos\omega t.$$

Perhaps the slickest way to solve this differential equation is to consider a second, companion equation,

$$\ddot{y} + (b/m)\dot{y} + \omega_0^2 y = (F_0/m)\sin\omega t.$$

Define the complex number $z = x + iy$, multiply the second equation by i, add the two together and rewrite as,

$$\ddot{z} + (b/m)\dot{z} + \omega_0^2 z = (F_0/m)e^{i\omega t}.$$

If we solve this equation for z, we can reclaim the solution to our original equation because x is the real part of z. This method works for linear differential equations because the operations d^2/dt^2, d/dt and multiplication by a constant are linear operations. When performed on a complex number z, these operations act separately on the real and imaginary parts of z, never mixing the two.

A general solution is a sum of a particular solution and the solution to the homogeneous equation. In the previous problem we obtained a solution to the homogeneous equation. An obvious candidate for the particular solution is something like

$$z = z_0 e^{i\omega t}.$$

Substituting this form into the equation to see if it will work, we obtain

$$(-\omega^2 + i(b/m)\omega + \omega_0^2) z_0 e^{i\omega t} = (F_0/m) e^{i\omega t},$$

from which we see that we must require,

$$z_0 = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i(b/m)\omega}.$$

We can write z_0 in polar form,

$$z_0 = R e^{i\theta} = \frac{(F_0/m) e^{i\theta}}{[(\omega_0^2 - \omega^2)^2 + (b/m)^2\omega^2]^{1/2}},$$

with,

$$\tan\theta = \frac{-b\omega}{m(\omega_0^2 - \omega^2)}.$$

If we let $\alpha = -\theta$, we can write the general solution to our original equation by recovering the real part of $z = z_0 e^{i\omega t}$ and adding it to the previously derived solution to the homogeneous equation,

$$x = \frac{(F_0/m)\cos(\omega t - \alpha)}{[(\omega_0^2 - \omega^2)^2 + (b/m)^2\omega^2]^{1/2}} + e^{-bt/2m} B \cos(\omega_1 t - \beta),$$

where B and β are determine from initial conditions,

$$\omega_1 = \omega_0 \sqrt{1 - (1/\omega_0^2)(b/2m)^2},$$

and,

$$\tan\alpha = \frac{b\omega}{m(\omega_0^2 - \omega^2)}.$$

The second (transient) term dies out exponentially with time leaving the first term as the long-term solution. The first term has an amplitude that depends on ω. It is largest when the denominator is smallest, i.e., when $\omega = \omega_0$. Thus, the system *resonates* if driven at or near the natural frequency, ω_0.

Computer Project

Explore the properties of the driven-damped harmonic oscillator. The differential equation is

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_o^2 x = \frac{F_o}{m}\cos\omega t$$

where ω_o is the natural frequency, $\omega_o^2 = k/m$, and where ω is the driving frequency.

(a) Set $F_o/m = 0$, $x(0) = 2.0$, $v(0) = 0.0$ and use ODE to verify the amplitude decay formula

$$x \propto e^{-t/2\tau}$$

and the damping-modified frequency formula

$$\omega_o' = \sqrt{\omega_o^2 - \frac{1}{4\tau^2}} \quad .$$

I recommend using $\omega_o = 2\pi$ and $\tau = 0.2$ and running ODE for a time interval of 1 or 2 s.

(b) Leave your model set as in part (a), but change the damping term from $\dot{x}/\tau$ to $\dot{x}^3/\omega_o^2\tau$. Run your model with initial values of $x = 2.0$ and $v = 0.0$ for both the model of part (a) and for the model with the modified damping term. Use a time interval of 20 s. Tell how the plots of these two models differ and explain the difference qualitatively. (Why does the nonlinear term result in extended oscillation instead of exponentially dying away as it does in the linear case?) Before continuing with the rest of this problem, change the damping term back to the standard $\dot{x}/\tau$ term.

(c) Let $F_o/m = 1$, set the damping to zero by making τ very large, and set the driving frequency to $\omega = 0.9\omega_o$. Set the initial position to $x = 0.5$ and set the initial velocity to zero. Run ODE for 40 s, or so, and plot $x(t)$. Explain the pattern you see in the plot. (Do you know what "beats" are?) Now set $\tau = 1$ and run ODE for 20 s. Explain the new pattern displayed in the plot of $x(t)$.

(d) Scan the driving frequency, ω, from $\omega = 0.1\omega_o$ to $\omega = 10\omega_o$ and use ODE to verify the expected resonance. Run ODE with $\tau = 1$, use a time interval of 20 s, and initially set $x = 0$ and $v = 0$. It may be easier to get the steady-state amplitude from a data file than from the plots. Sketch your results for amplitude as a function of ω.

(e) To become more familiar with phase space, make phase space plots of the orbits, x versus v. I suggest using a driving frequency of about $\omega = 0.95\omega_0$. Plot at least five phase space plots on the same graph for comparison. Explain why all of them end up doing the same thing in spite of their different starting points. We say that the motions all approach the same *limit cycle*.

5. **Computer Project: Theory** What happens if the driven, damped oscillatory motion occurs with a restoring force more general than $F_x = -kx$? Assume that the restoring force is still derivable from a potential, $F_x = -dV/dx$ that has either a minimum or a maximum V_0 at x_0.

   If we make a Taylor's expansion around the minimum or maximum, we can write,

$$V = V_0 + (dV/dx)_{x_0}(x - x_0) + \frac{1}{2}(d^2V/dx^2)_{x_0}(x - x_0)^2 + \ldots$$

   Since we have chosen to expand around a maximum or minimum, $(dV/dx)_{x_0} = 0$. If we move our axes so that $x_0 = 0$, and let constants stand for the derivatives evaluated at the point of expansion,

$$V = V_0 + \frac{1}{2}kx^2 + \frac{1}{6}k_2x^3 + \frac{1}{24}k_3x^4 + \ldots$$

   If the potential is symmetric ($V(x) = V(-x)$), the odd powers of x must vanish because their coefficients vanish. If we assume that the terms in the series are getting smaller as one goes to higher order and if we then truncate the series with the quartic term, we have

$$F = -\frac{dV}{dx} = -kx - \frac{1}{6}k_3x^3.$$

We will take $k > 0$ as it would be for a harmonic oscillator. Depending on k_3, the cubic term might have a positive or a negative coefficient. If the overall coefficient is positive, the potential looks like a rounded letter "M" centered on the origin. (Or think of a top view of a nudist bending over to pick up his collar button!) It is this case that we want to investigate, so we will write,

$$F = -kx + \gamma x^3,$$

with both k and γ positive. Motion will no longer be strictly harmonic, but will instead be referred to as *anharmonic.*

For our driven, damped anharmonic oscillator, the differential equation becomes,

$$m\ddot{x} = -kx + \gamma x^3 - b\dot{x} + F_0 \cos \omega t,$$

or,

$$\ddot{x} + (b/m)\dot{x} + \omega_0^2 x - (\gamma/m)x^3 = (F_0/m)\cos \omega t.$$

The equation is now nonlinear, but we might expect that motion of sufficiently modest energy to be somewhat like the motion for a damped, driven harmonic oscillator, i.e, after some time the motion settles into periodic motion with the frequency of the driving force, ω. We might also expect that the amplitude of the motion will get large (resonance) if the driving frequence ω is close to the natural frequency, ω_0.

If the motion is periodic, we should be able to represent it with a Fourier series expansion,

$$x(t) = a_0 + \sum (a_n \cos n\omega t + b_n \sin n\omega t),$$

$$\dot{x}(t) = \sum (-n\omega a_n \sin n\omega t + n\omega b_n \cos n\omega t).$$

If we set our origin of time so that $\dot{x} = 0$ at $t = 0$, all of the coefficients b_n of the cos terms in the expression for $\dot{x}$ must vanish. Furthermore, Fourier analysis tells us that

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)d(\omega t).$$

But this is just the average position of the particle and for a symmetric potential such as we have assumed, the average position is expected to be zero. So $a_0 = 0$.

The final argument will take some sketches of $\cos 2\omega t$ and an imagined periodic $x(t)$. If the motion were truly periodic with frequency ω and if we were to start it with initial conditions that $x(0) = x_0$ and $\dot{x}(0) = 0$ in a symmetric potential, then when $\omega t = \pi$, the particle will have moved to $-x_0$ and will then return to x_0 by the time $\omega t = 2\pi$ as if time had simply reversed. (Damping would invalidate this assumption since damping is not time symmetric. The test for time symmetry is whether changing t and dt to $-t$ and $-dt$ leaves the equation of motion unchanged. For the moment we now limit our discussion to undamped motion.) The consequence of time symmetric motion is that,

$$a_2 = \frac{1}{\pi} \int_0^{2\pi} x(t) \cos 2\omega t d(\omega t) = 0.$$

A sketch of the product of $x(t)$ and $\cos 2\omega t$ shows that for every positive piece of the integral, there is a corresponding negative piece that cancels it. The argument can be extended to show that $a_n = 0$ for all even values of n.

Thus, our periodic solution for $x(t)$ must look something like,

$$x(t) = a_1 \cos \omega t + a_3 \cos 3\omega t + \ldots$$

if we choose $\dot{x}(0) = 0$ in a symmetric potential well. If the motion is close to harmonic, the series will converge rapidly and we can keep just the first two terms to get an approximation. We can also assume for rapid convergence that $a_3 << a_1$.

In the absence of damping *and* the absence of a driving force,

$$\ddot{x} + \omega_0^2 x - (\gamma/m)x^3 = 0.$$

We can substitute our trial solution into the differential equation, gather the coefficients of $\cos \omega t$ and $\cos 3\omega t$, separately set them to zero and solve for ω and a_3 in terms of a_1. The troublesome part of this agenda is the handling of

$$x^3 = (a_1 \cos \omega t + a_3 \cos 3\omega t)^3.$$

Terms involving $\cos^2 \omega t$ and $\cos^3 \omega t$ have to be reduced using,

$$\cos^3 \omega t = \frac{1}{4}\cos 3\omega t + \frac{3}{4}\cos \omega t$$

$$\cos^2 \omega t \cos 3\omega t = \frac{1}{4}\cos 5\omega t + \frac{1}{2}\cos 3\omega t + \frac{1}{4}\cos \omega t.$$

Terms with $\cos 5\omega t$ that originate in this process are discarded since we have already thrown away higher harmonics in our trial solution. Further, since we have assumed that $a_3 << a_1$ and have thrown away all other terms, we are not justified in keeping some other small terms that arise in the products obtained by expanding x^3. The coefficient a_1 is the largest and is said to be *first order small.* The coefficient a_3 is said to be *third order small.* A term like a_1^3 is also third order small, $a_1^2 a_3$ is fifth order small. Since we threw away everything smaller than third order small in our trial solution, we are not justified in keeping anything smaller that may appear in the process of expanding x^3.

We then have,

$$[a_1(\omega_0^2 - \omega^2) + \frac{3}{4}(\gamma/m)a_1^3]\cos \omega t$$

$$+[a_3(\omega_0^2 - 9\omega^2) + \frac{1}{4}(\gamma/m)a_1^3]\cos 3\omega t = 0.$$

Since $\cos \omega t$ and $\cos 3\omega t$ are linearly independent, their coefficients must separately equal zero if the equation is to be satisfied.

We use these two equations to solve for ω and a_3 in terms of a_1 and the other parameters of the problem. We chose the initial condition $\dot{x} = 0$ already, leaving the initial condition on x to determine a_1.

We obtain,

$$\omega^2 = \omega_0^2 + \frac{3}{4}(\gamma/m)a_1^2,$$

and,

$$a_3 = \frac{(\gamma/m)a_1^3}{32\omega_0^2[1 + \frac{27}{32\omega_0^2}(\gamma/m)a_1^2]} \approx \frac{(\gamma/m)a_1^3}{32\omega_0^2}.$$

But what happens if the damping and driving forces are added? The result is both remarkable and surprising.

Computer Project

Consider the driven, damped, anharmonic oscillator

$$\ddot{x} = -x + 4x^3 - \alpha\dot{x} + f\cos\omega t$$

where for numerical convenience ω_o has been chosen to be 1.0 and where the size of the nonlinear term has been arbitrarily chosen as well ($4x^3$).

(a) With $\alpha = 0$ and $f = 0$, find the equilibrium points for this system(points of vanishing force) and tell whether each is stable or unstable. Make a phase space sketch of the orbits for this system, then make phase space plots for several different initial conditions to verify your sketch. When you run the model, I suggest that you just choose $x_o = -0.4$, $v_o = 0$ and a time interval of about 4 s. Set the *max* and *min* values of the x-axes to (-0.6, 0.6) and of the v-axes to (-0.5, 0.5). You should be able to generate a wide variety of interesting orbits. Be careful not to select starting points far into the unstable regions or the model will achieve numerical overflow and possibly throw you out of ODE. You will then have to start over.

(b) Add damping by setting $\alpha = 1.5$ and make the same kind of phase-flow picture you made in part (a). Describe how the flow has changed near the stable and unstable equilibrium points.

(c) Let $\alpha = 0.4$, $f = 0.115$, $x_o = 0$, $\dot{x}_o = 0$, and run the model for a time interval of 100 s for the following choices of the driving frequency, ω:

$$\omega = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 1,\ 2,\ 3$$

Use the results of your computer runs to sketch the amplitude as a function of ω^2. Do you see evidence for resonance?

6. **Computer Project: Theory** This project is a continuation of the previous one. There is an interesting effect to be seen here, but it takes a bit of perserverance. The theory for this project is given in the book *Newtonian Mechanics*, Ralph Baierlein (McGraw Hill, 1993, pages 82-87).

Computer Project

Consider the driven, damped, anharmonic oscillator

$$\ddot{x} = -x + 4x^3 - \alpha\dot{x} + f\cos\omega t$$

where for numerical convenience ω_o has been chosen to be 1 and where the size of the nonlinear term has been arbitrarily chosen as well ($4x^3$). Let $\alpha = 0.4$, $f = 0.115$, $x_o = 0$, $\dot{x}_o = 0$.

(a) Try to scan the frequency more carefully past the region near $\omega = 0.60 - 0.70$. It is in this vicinity that the amplitude has unexpected behavior. If you slowly approach this frequency range from below, the amplitude takes a sudden jump, but if you approach from above, the jump occurs at a different place! This is not a computer glitch, it is the way the system actually behaves. We have spent a lot of time trying to make this work, and have only been successful by making very long runs. The key is apparently to scan the frequency in a special way. If you just do the obvious thing and let the frequency change very slowly with time to move along the ω axis, the time-dependent frequency completely changes the dynamics and nothing resembling the figure emerges. The apparent right thing to do is to move the frequency up in steps, with enough time between the steps to allow the oscillator to settle into its limit cycle. This is done by using the *ceil(x)* function (i.e., the integer just greater than the argument) in ODE in the following way:

$$\omega = \omega_s + \frac{(\omega_f - \omega_s)}{20} * \mathrm{ceil}(.0007958 * t - 0.5)$$

where ω_s is the starting value of the frequency and ω_f is the final value of the frequency. This formula is built to make 20 frequency steps between ω_s and ω_f if the time interval you have chosen is 25132.0 s. Note: you will have to type this whole rotten thing into the argument of the cosine function, then multiply it by t to get $\cos(\omega t)$, so we suggest first writing the coded equation out carefully on paper. Check your parentheses carefully! This is a long run. We found that a time step of 0.3 s does a reasonable job. Make two scans, one with $\omega_s = 0.6$ and $\omega_f = 0.8$, and the other with $\omega_s = 0.8$ and $\omega_f = 0.6$. Try to use a plot of x vs t to determine the time at which the system jumped from one branch to another and see if those times were the same or different for the two scans. Explain what you find using the term *hysteresis*. It will help you to understand when the jump occurs if you make the frequency, given in the formula above, a variable and plot it together with x. Then you can see at what frequency the jump occurred. (Do *not* plot the entire x vs t curve. It would take forever! Use the computer screen to observe where the transition occurs and read the data from the screen or a data table and make a sketch for your report.)

7. **Computer Project: Theory** The Lagrangian for a simple pendulum of mass m suspended by a massless string of length ℓ is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta).$$

The differential equation of motion that follows from this Lagrangian is,

$$\ddot{\theta} + (g/\ell)\sin\theta = 0.$$

If the angle θ is small, $\sin\theta \approx \theta$, and the equation becomes that of a simple harmonic oscillator with $\omega_0^2 = g/\ell$.

If we add a damping force proportional to speed, $F_d = -b\dot{\theta}$, we have

$$\ddot{\theta} + \omega_0^2\sin\theta + (b/m)\dot{\theta} = 0.$$

If we add a harmonic driving force,

$$\ddot{\theta} + \omega_0^2\sin\theta + (b/m)\dot{\theta} = (F_0/m)\cos\omega t.$$

For certain choices of parameters, including the driving frequency, this supposedly simple system exhibits a complex and erratic behavior that is called *chaos*.

Computer Project

Study the pendulum with damping and a sinusoidal driving force. Use the equation of motion

$$\ddot{\theta} = -\sin\theta - \alpha\dot{\theta} + f\cos(\omega t)$$

which has a small amplitude oscillation frequency of $\omega_o = 1$.

(a) Study the motion of the pendulum with no damping and no drive ($\alpha = 0$ and $f = 0$) by using several different initial conditions between $\theta_o = 0.1$ radians and $\theta_o = 3.13$ radians ($\dot{\theta}_o = 0$). Explain why your plots of θ as a function of t make physical sense.

(b) Add damping by setting $\alpha = 0.2$ (keep $f = 0$) and use an initial angle of $\theta_o = 3$ and two different initial angular velocities, $\dot{\theta}_o = 2.2$ and $\dot{\theta}_o = 2.3$. Use a time interval of 100 s. Explain why the phase space plots for these two cases are different by explaining what the pendulum is doing in each case.

(c) If the system is driven with the driving frequency set to $\omega = 0.3$ and the drive amplitude set to $f = 1$, you should see the system settle into a limit cycle (use initial conditions $\theta_o = 0$ and $\dot{\theta}_o = 0$ and keep $\alpha = 0.2$). As f is increased you should see a transition to chaos somewhere between $f = 1$ and $f = 1.1$. Study this transition in as much detail as possible. (You may have to select a higher-than usual precision solution option to get as much accuracy as possible – the transition to chaos is a very delicate business.) The best way to observe chaos is to look at a phase space plot. Select an initial point, start it near (0,0), and just let it run for a long time (>30 seconds) to see if it settles into a limit cycle or if it just bounces around.

(d) One of the hallmarks of chaos is that two points started close to each other in phase space very quickly become widely separated. Observe this effect by using $f = 1.1$ and starting two points close to each other and plotting their orbits on the same graph.

8. **Computer Project: Theory** The following treatment illustrates a general method for solving vibration problems in a linear approximation.

   A carbon dioxide molecule is a linear structure with a carbon atom in the center and an oxygen atom on each end. We can model the bonds that attach the oxygen atoms to the carbon as linear springs of constant k. For simplicity let us assume the masses of the three atoms are equal to m. When stretched and released, the system begins to vibrate along the line joining the atoms and it is this motion that we seek to describe.

   If we assign Cartesian coordinates x_1, x_2, x_3 relative to some arbitrary origin, the kinetic energy of the system is given by,

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2.$$

We can define a *mass matrix* M with the masses distributed along the diagonal elements and zeros elsewhere. If the masses are not equal as in this example, $M_{ij} = m_i\delta_{ij}$, where δ_{ij} is the Kronecker delta. Then we can write the kinetic energy as,

$$T = \frac{1}{2}\dot{x}_i M_{ij}\dot{x}_j,$$

where,

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

The potential energy lies in the stretch of the two springs,

$$V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2.$$

We can also write this as,

$$V = \frac{1}{2}x_i K_{ij} x_j$$

if we define a matrix

$$K = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}.$$

The Lagrangian is,

$$L = T - V = \frac{1}{2}\dot{x}_i M_{ij}\dot{x}_j - \frac{1}{2}x_i K_{ij} x_j,$$

from which the differential equations of motion follow,

$$M_{ij}\ddot{x}_j + K_{ij}x_j = 0.$$

(The factor of 1/2 disappears because of the *double* sum in the Lagrangian. If you have difficulty seeing it, you should write out the sum.)

We can write the equations of motion in matrix form,

$$M\ddot{X} + KX = 0,$$

where X is a column vector,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

A trial solution to this set of linear differential equations is, $X = X_0 \cos(\omega t - \alpha)$. When substituted into the equations of motion we get the matrix equation,

$$(-\omega^2 M + K)X_0 = 0.$$

This set of equations has a solution if and only if the determinant of the coefficients vanishes, i.e.,

$$\left| K - \omega^2 M \right| = 0.$$

In our particular case this becomes,

$$\begin{vmatrix} k - m_1\omega^2 & -k & 0 \\ -k & 2k - m_2\omega^2 & -k \\ 0 & -k & k - m_3\omega^2 \end{vmatrix} = 0.$$

We evaluate this equation and solve for ω^2. The resulting values of ω are the only values for which the trial solution is a solution to the differential equations. In our example, (if we set $m_1 = m_2 = m_3 = m$ for simplicity) there are three values of ω^2,

$$\omega_1^2 = 0$$

$$\omega_2^2 = k/m$$

$$\omega_3^2 = 3k/m,$$

and for each there is a corresponding X_0 which arises from solving

$$(K - \omega_i^2 M)X_0 = 0,$$

$$X_{01} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad X_{02} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad X_{03} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

The general solution to our differential equations is a linear combination of these three,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \alpha_1)$$

$$+a_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\omega_2 t - \alpha_2) + a_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cos(\omega_3 t - \alpha_3).$$

The ω_i are called the *eigenfrequencies* and the X_{0i} are the *eigenvectors*. The constants a_i and α_i are determined by initial conditions. In general the motion is a complicated combination of oscillations all happening at once. But, if you choose the initial conditions just right, you can get the system to pick out one of these frequencies to the exclusion of the others. These special motions are *normal modes* of the system.

Computer Project

Set up a ODE model for this system.

(a) Solve the differential equations of motion for several arbitrary choices of initial conditions. Plot $x_1(t)$, $x_2(t)$ and $x_3(t)$. Let $m_1 = m_2 = m_3 = 1$ and $k = 1$. Run the model long enough so that you can see the frequencies of the oscillations.

(b) Solve the equations for the initial conditions $\dot{X}(0) = 0$ and $X_0(0) = X_{02}$ (the second eigenvector.) Again, plot $x_i(t)$ and observe the frequencies of oscillation. Describe the motion of the system qualitatively in your own words.

(c) Solve the equations for the initial conditions $\dot{X}(0) = 0$ and $X_0(0) = X_{03}$ (the third eigenvector.) Again, plot $x_i(t)$ and observe the frequencies of oscillation. Describe the motion of the system qualitatively in your own words.

(d) One of the eigenfrequencies is zero. What does the system do if you start it from rest with this eigenvector of positions?

(e) What differences in parts (a)-(c) do you see if the mass of carbon is taken to be 1.0 and the mass of oxygen is taken to be 1.333 to more realistically simulate a carbon dioxide molecule? Use the same initial conditions that you originally used in parts (b)-(c) even though these are no longer the exact eigenvectors.

9. **Computer Project:** Use ODE to numerically solve for the motion of two gravitating masses in the x-y plane. This will require some care because the phase space for this problem is 8-dimensional. To make things simple, assume that the gravitational potential energy between the two masses is given by

$$U(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

(a) Write down the eight first-order equations of motion for

$$(x_1,\ y_1,\ v_{x1},\ v_{y1},\ x_2,\ y_2,\ v_{x2},\ v_{y2}).$$

(b) Make a ODE model that correctly codes your equations of motion. This will be difficult and tedious because you will have to put the denominator

$$[(x_1 - x_2) * (x_1 - x_2) + (y_1 - y_2) * (y_1 - y_2)]^{3/2}$$

in each of the four equations for the time derivatives of the velocities. Use masses $m_1 = 1$ and $m_2 = 1.5$ and initial conditions $x_1 = 1$, $y_1 = 0$, $v_{x1} = 0.1$, $v_{y1} = 0.7$, $x_2 = -1$, $y_2 = 0$, $v_{x2} = 0.12$, $v_{y2} = -0.55$. Make a plot of x_1 vs. y_1, i.e., look at the orbit of mass 1 in the x-y plane. Explain qualitatively why it looks the way it does. For high speed and reasonable accuracy, use $\Delta t = 0.05$ and run for a time interval of 60 s.

(c) Define a variable containing the total energy of the system and verify that it is constant.

(d) Define variables containing the x and y components of the position of the center of mass. Plot x_{cm} vs. y_{cm} on a plot – does it look like it should?

(e) Define variables containing the x and y components of the difference vector between the two masses, $\mathbf{r}_1 - \mathbf{r}_2$. Plot the x component of this vector vs the y component and verify that the difference vector does indeed trace out an ellipse. (Note: ellipses may look circular unless the axes on the plot are perfectly chosen, so unless you change the axis ranges, you will probably see more or less circular orbits. But they are really ellipses.

10. **Computer Project:** In the previous exercise we used all 8 variables for the motion of two attracting particles in the plane to numerically solve the orbit equations. In this exercise we will go to the center of mass frame and use conservation of angular momentum to reduce the orbit problem to three variables, $(r,\ v_r,\ \theta)$.

   (a) Use the law of angular momentum conservation and the radial equation of motion to write down three first-order differential equations for $(r,\ v_r,\ \theta)$. The purpose of this part is to find ordinary differential equations for ODE to solve.

   (b) The purpose of this part is to find initial conditions for circular motion. (See homework Problem 8.2.) Set $G = 1$, $m_1 = m_2 = 1$, and $\ell = mr^2\dot{\theta} = 1$. Under these conditions, solve for the radius and angular velocity of a circular orbit. Verify your solution by numerically solving the three equations of motion with ODE. Note that to get an interesting orbit to plot, you will need to convert r and θ into x and y in the usual way by defining them as new variables in ODE:

   $$x = r\cos\theta \qquad ; \qquad y = r\sin\theta$$

   Make a phase plot with x and y as the two variables.

   (c) The purpose of this part is to find a v_{r0} to change the orbit from the circle of Part (b) to an ellipse. Changing v_r does not change the angular momentum since it is a change of velocity along the line connecting the two particles. Perturb your circular orbit by adding just enough radial velocity to make an elliptical orbit with the same angular momentum as the circular orbit, but with eccentricity $e = 0.5$. Verify your solution with ODE.

(d) The purpose of this part is to test Kepler's Third Law by finding initial conditions and solving for three ellipses of the same ℓ but different perigees. Plot r as a function of time for $\ell = 0.65$ and read the period of the motion from the graph. We suggest running for about 50 seconds. Estimate the length of the major axis of the ellipse from a parametric plot of y versus x. Try three different values of $r(0)$ and roughly verify Kepler's Third Law that relates the period of the orbit to the major axis.

(e) Now modify the inverse square law force term in the radial equation of motion and describe what happens to the orbit. This will require changing the ordinary differential equations themselves. Some possible suggestions are to replace Gm_1m_2/r^2 by $Gm_1m_2/r^{2\pm\delta}$, where δ is a small number, or to add a small term to the force law, e.g., $F = Gm_1m_2(1/r^2 + \delta/r^4)$.

11. **Computer Project:** You are playing championship basketball on a rotating floor. The game is tied with time running out, and you must sink a 13.1-foot (4-meter) jump shot to win the game. You are on the x-axis, one meter away from floor's center of rotation, which is located directly behind you. The basket is also on the x-axis, 5 meters away from the center of rotation. The floor is rotating in the counterclockwise direction, viewed from above, at $\omega = 0.4\ \mathrm{s}^{-1}$. You arbitrarily choose the vertical speed of your shot to be 7 m/s, giving a total elapsed time between the release of the ball and its arrival at the vertical level of the basket of 1.2676 s. The problem then is to choose a radial velocity and an angular velocity so that when the ball comes down through the vertical level of the basket, it is also at $r = 5$ m and $\theta = 0$. A mere mortal would panic, but you have been selected to the GCAA Academic All-Galaxy team, so as you dribble between frantic defenders, you calmly imagine a blackboard in your mind on which you mentally write down the rotating-frame-Lagrangian in cylindrical coordinates (r, θ). You correctly deduce that gravity plays no role on these coordinates since you have already taken it into account with your choice of a vertical velocity of 7 m/s and your calculation of the hang time of the ball (1.2676 s). Deriving the (r, θ) equations of motion of the ball you discover that there is a conserved angular momentum, and you use it to write down three first order differential equations for (r, $\dot{r}$, and θ). You quickly call up ODE from your upper left frontal lobe and

code the three equations. The initial conditions for r and θ are easy: $r = 1$ and $\theta = 0$. But you also must choose the correct initial values of $\dot{r}$ and $\dot{\theta}$ so that at time $t = 1.2676$ s, the ball will arrive at $r = 5$ m and $\theta = 0$. The equations are nonlinear, so even you can't solve them analytically in the short time remaining in the game, but with your new coprocessor implant, you are able to run ODE repeatedly to find, by trial and error, the correct initial values of $\dot{r}$ and $\dot{\theta}$ and win the game. What are your winning values?

This way of using a differential equation solver is called *shooting*, and it is used all the time to solve differential equations where boundary conditions are specified instead of initial conditions.

(Note: it is also possible to solve this problem by getting out of the rotating frame, using Cartesian coordinates, and shooting at the moving basket. Then you transform back to the rotating frame to get the correct values. If you do it this way, you will foul out of the game and not win the most valuable sentient life-form trophy. You will also not get any points for this problem.)

12. **Computer Project:** Use ODE to solve Euler's equations for a spinning object with principal moments of inertia $I_1 = 1$, $I_2 = 2$, and $I_3 = 3$.

    (a) Show that if the initial conditions are either $\omega_1 = 1$ and the other two ω's are small, say 0.05, or $\omega_3 = 1$ and the other two are small, then the motion is stable (the other two ω's stay small). Also show that if $\omega_2 = 1$ and the other two are small, then the motion is unstable, i.e., the other two ω's don't stay small.

    (b) Now fasten a rubber band around a book to keep it closed and throw it into the air while spinning it about each of its principal axes in succession. Tell how what you see makes it possible to tell which axis is the unstable axis, and check to see if your observations agree with the results you just got from ODE.

    (c) Set ω_1 and ω_3 to be small and set $\omega_2 = 1.0$. Try to decide whether these nonlinear equations have chaotic solutions or regular solutions. Change the initial ω_i's enough times that you are reasonably convinced that you know the answer to this question. Do you see "sensitive dependence on initial conditions?" To be unstable is not the same as being chaotic. Which do you have?

(d) Finally, set all three $\omega_i = 1.0$ and make phase space plots of the motion with all three possible pairings of the three ω's. This should convince you that classical mechanics, though very nearly ancient, is still rich and complex.

Appendix A

What to do before you start

While physicists spend a lot of time solving mathematical problems, they often spend as much time trying to convince themselves that the answers they have written down are correct. There are no "answers at the end of the book" in research. Physicists who are really good are constantly applying tests to their work to convince themselves that what they are doing is correct. What they do is a kind of craftsmanship. Like any craftsmanship, it requires practice and conscious attention and effort. The following is an attempt to list some things that one can do if one is asked to "convince the reader that what you have done is correct."

A.1 What to do *before* you start writing down equations.

Guess. Write it down. John Wheeler is said to have warned, "Never begin to calculate until you know what the answer is." A good physicist tries to develop intuition about physical systems, an intuition that tells him/her what to expect before beginning to calculate. Your guess does not have to be precise or even right. If you are seeking a numerical answer, you may only be looking for an order-of-magnitude estimate. But you can't lose by guessing. If your guess is right, your calculation will confirm your good sense and you will grow in confidence. If your guess is wrong, you will build your intuition by correcting your misconceptions by the calculation. Here are some things to help with your "guess":

- Draw a diagram, sketch, or rough diagram. Label it with things you think are important.

- Ask yourself what the answer could depend upon. For example, what could the frequency of a simple pendulum depend upon? Mass? Length of string? Phase of the moon? Color of the bob? Temperature? Density of air? Size of the bob? Gravitational constant?

- From the above list choose the appropriate parameters and variables that you think govern the answer you are seeking.

- Apply dimensional analysis. Is there a combination of the fundamental parameters of the system that have the dimensions of the answer you seek? Are their "natural" time scales, distance scales, frequencies, etc. that present themselves from the parameters of the problem. For example, if, for the simple pendulum, you identify the length of the string ℓ, the mass m, and the gravitational constant g as things on which you believe the frequency will depend, you may note that ℓ has dimensions of $[length]$, g has the dimensions of acceleration, i.e. $[length]/[time]^2$ and that, therefore, $\sqrt{g/\ell}$ has the dimensions of $[time]^{-1}$, i.e. the dimensions of frequency. It is almost certain, therefore, that the frequency of the pendulum is set by this "natural frequency."

  To be more systematic, reduce the dimensions (*not* the units) of the parameters of your system to products of the fundamental dimensions of distance, time, mass, electric charge, i.e. to the dimensions of the quantities established by standards. This is what we did when we established the dimensions of g to be $[length][time]^{-2}$. To find what combination of m, g, and ℓ have dimensions of $[time]^{-1}$, write,

  $$[m^{\alpha}\ell^{\beta}g^{\gamma}] = [time]^{-1}.$$

  The square brackets [ ] are to be interpreted to mean "dimensions of the things inside.". In this example, α, β, and γ are unknown exponents that are to be chosen to make the dimensions on the left-hand-side equal to the dimensions of the right-hand-side. Since the dimensions of mass are established by a standard, the dimensions of mass can never be expressed as combinations of $[length]$ or $[time]$. We then have,

  $$[mass]^{\alpha}[length]^{\beta}[length]^{\gamma}[time]^{-2\gamma} = [time]^{-1}$$

Thus, since the dimensions of mass do not appear on the right-hand-side, α must be zero. Similarly, to cancel the dimensions of [*length*], we must have that $\beta = -\gamma$. And, finally, to get the power on [*time*] correct, we must have $\gamma = 1/2$.

- Make an order-of-magnitude estimate. In general, the answer you seek to a problem *may* (but not always) be the one given by dimensional analysis but multiplied by some dimensionless number, usually in the range 0.1-10.

- Reason from symmetry.

- Reason from conservation laws.

- Solve a simpler problem first. Ignore air friction, the variation of g with altitude, the mass of a string, etc. Solving the simpler problem may give you a plan for solving the more complicated problem and it gives a limiting case to which the solution to the more complicated problem must reduce in the limit that certain parameters are allowed to go to zero (or 1 or infinity, etc).

A.2 What to do along the way

- Use vectors and vector notation to describe the problem economically.

- Choose coordinates wisely

    1. Choose coordinates that fit the symmetry or the preferred directions of the problem.
    2. If necessary, move to coordinates in which the description is simpler, such as coordinates fixed to a rotating frame or to the center of mass of the system.

- Use constants of the motion such as energy, momentum, angular momentum, etc. Sometimes you can replace combinations of variables with a single constant.

- Check your units. Try to find the natural scales of the problem, i.e. the combinations of parameters with dimensions of length, time, frequency, velocity, etc. and write variables as dimensionless ratios such as x/x_0, t/t_0, ω/ω_0, v/v_0, etc. wherever possible. Remember, the arguments of such functions as sin, exp, log should be dimensionless. If you express things as dimensionless ratios, you will be able to see that this condition is being met.

- If the problem has more than one natural time scale, i.e. something about the problem that changes very quickly superimposed on something about the problem that is changing much more gradually, you may want to separate the problem into two separate parts, solve them separately, and combine the results. Sometimes the fast motion can be averaged and the average behavior used in solving the slow problem.

- If you do complicated algebraic manipulations on an expression and wonder if you have made an algebra mistake, you can sometimes check your work with a computer or calculator by substituting randomly chosen values of variables separately into the right-hand and left-hand-side of the expression and seeing if you get the same number in both cases.

- If you reach an impasse in your ability to solve the problem, try:

    1. Dropping small terms.
    2. Ignoring slow variations.
    3. Linearizing the equations.
    4. Using a series expansion and keeping leading terms (Taylor, binomial, etc.).
    5. Iterating.
    6. Guessing a form for the answer with adjustable parameters that can be used to fit it to your problem.

- Is your answer reasonable?

- Does your answer agree with your original guess? Why not?

- Does your answer give the correct result for certain simple limits?

- Can you compare an analytic result with a numerical computation for some choice of parameters to see if they are the same?

- Does your answer agree with one obtained independently by someone else?