> $with(cl) : with(plots) :$

> $\text{vartheta} := t \rightarrow -\frac{I}{2} \cdot \left(\text{lnGAMMA}\left(\frac{1}{4} + \frac{I \cdot t}{2}\right) - \text{lnGAMMA}\left(\frac{1}{4} - \frac{I \cdot t}{2}\right)\right) - \frac{\ln(\text{Pi})}{2} \cdot t$

$$\vartheta := t \mapsto -\frac{\mathrm{I}\left(\ln\Gamma\left(\frac{1}{4} + \frac{\mathrm{I}\,t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{\mathrm{I}\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2} \tag{1}$$

> $S := to1pf\left(\frac{\text{argument}\left(\text{Zeta}\left(\frac{1}{2} + I \cdot t\right)\right)}{\text{Pi}}\right)$

$$S := t \mapsto \frac{arg\left(\zeta\left(\frac{1}{2} + \mathrm{I}\,t\right)\right)}{\pi} \tag{2}$$

> $varthetaStirling := to1pf\left(\frac{t}{2} \cdot \log\left(\frac{t}{2 \cdot \text{Pi} \cdot \exp(1)}\right) - \frac{\text{Pi}}{8}\right)$

$$varthetaStirling := t \mapsto \frac{t \ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2} - \frac{\pi}{8} \tag{3}$$

>

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>

> $N := to1pf\left(\frac{\text{vartheta}(t)}{\text{Pi}} + S(t) + 1\right);$

 # Backlund zeta-zero counting function, valid over the whole critical strip

$$N := t \mapsto \frac{-\frac{\mathrm{I}\left(\ln\Gamma\left(\frac{1}{4} + \frac{\mathrm{I}\,t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{\mathrm{I}\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{\pi} + \frac{arg\left(\zeta\left(\frac{1}{2} + \mathrm{I}\,t\right)\right)}{\pi} + 1 \tag{4}$$

>

>

> $T := unapply\left(piecewise\left(n = 1, 1, 1 + \text{floor}\left(\frac{t \cdot \ln\left(\frac{t}{2 \cdot \text{Pi} \cdot \exp(1)}\right)}{2 \cdot \text{Pi}} + \frac{7}{8}\right) - n\right), n, t\right);$

 # proposed to be equal to the branch of the argument of zeta at the n-th zero

$$T := (n, t) \mapsto \begin{cases} 1 & n = 1 \\ 1 + \left\lfloor \frac{t \ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2\,\pi} + \frac{7}{8} \right\rfloor - n & otherwise \end{cases} \tag{5}$$

>

>

>

```
> zeros := readdata("c:\\maple\\zeros") :
    # This file can be retrieved from http://www.dtc.umn.edu/~odlyZko/zeta_tables/index.html
```

$$> Sn := unapply\left(\left(\left(\frac{1}{2} - \mathrm{frac}\left(\left(\frac{\mathrm{vartheta}(t)}{\mathrm{Pi}}\right)\right) - T(n, t)\right)\right), t, n\right)$$

$$Sn := (t, n) \mapsto \frac{1}{2} - \mathrm{frac}\left(\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4} + \frac{\mathrm{I}\,t}{2}\right) - ln\Gamma\left(\frac{1}{4} - \frac{\mathrm{I}\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{\pi}\right) - \begin{cases} 1 & n = 1 \\ 1 + \left\lfloor \frac{t \ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2\,\pi} + \frac{7}{8} \right\rfloor - n & otherwise \end{cases} \tag{6}$$

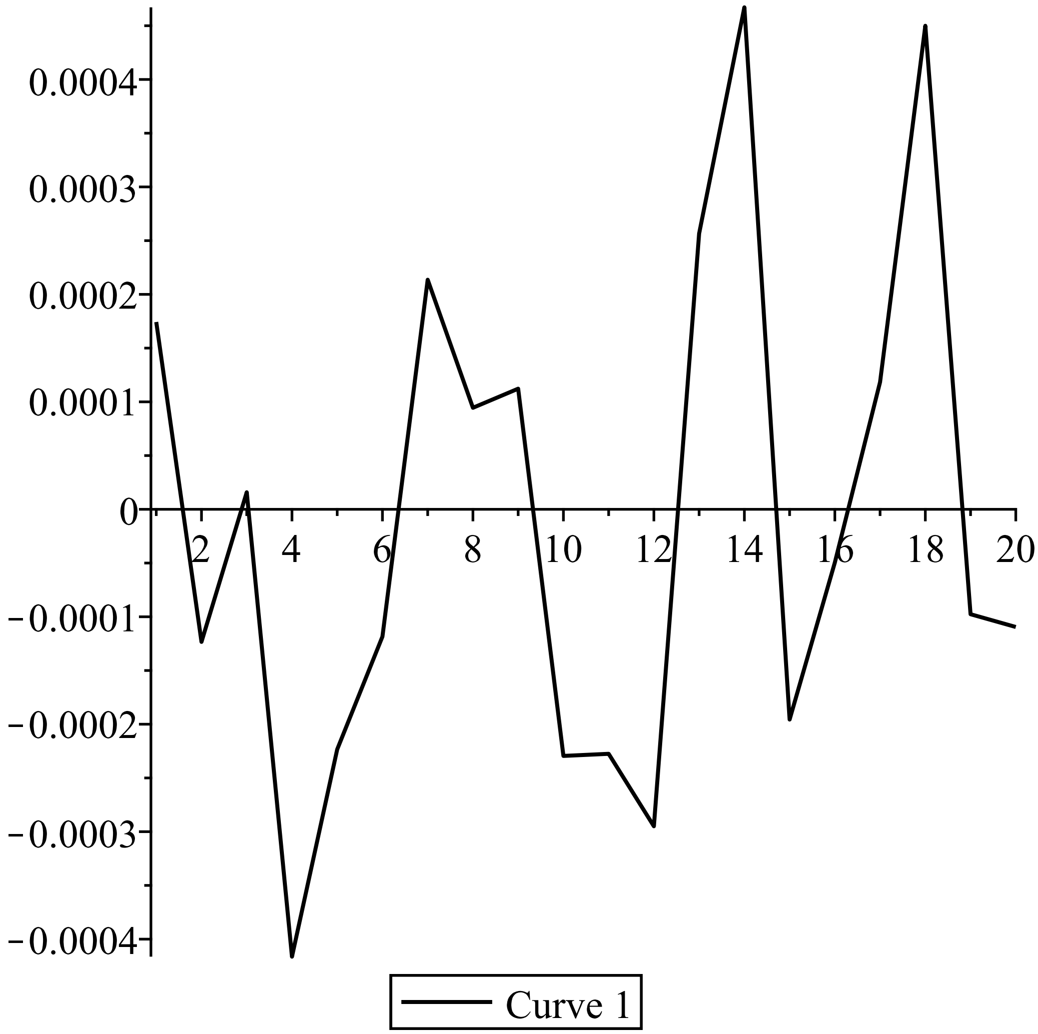
```
>
> # The conjecture is that Sn(t[n],n)=S(t[n]) where t[n] is the imaginary part of the n-th zero on the
    critical line
> # I am not aware of an expression that gives the precise value of the argument at the n-th zero,
    since it is discontinuous there, and is equal to the mean value of the left and right one-sided
    limits
> # therefore, to numerically check the conjecture I evaluate the argument shifted off of the critical
    line by just a very small amount, 0.0001 , and check that the resulting difference is close to 0
> Zd := n→evalf(Sn(zeros[n], n))·Pi - argument(Zeta(0.50001 + I·zeros[n]))
```

$$Zd := n \mapsto evalf\left(Sn\left(zeros_n, n\right)\right)\pi - arg\left(\zeta\left(0.50001 + \mathrm{I}\,zeros_n\right)\right) \tag{7}$$

```
> flist(Zd, 1 ..20)
```

[0.0001742923, −0.0001233752, 0.0000158643, −0.0004163028, −0.0002234084, −0.0001184350, 0.0002136172, 0.0000944674, 0.0001122380, −0.0002294904, −0.0002275425, −0.0002949642, 0.0002565506, 0.000467109, −0.0001956492, −0.00004957018, 0.0001183858, 0.0004499533, −0.0000976363, −0.0001095092] **(8)**

```
>
> listplot((8));
    # If the conjecture is true, all of these numbers should be very close to 0 for any n. If there
    was another way of calculating S(t[n]) at a zero t[n] then the resulting difference should be
    ·exactly· zero
```



> $flist\left(n \rightarrow is\left(evalf\left(Zd(n)\right) < 10^{-4}\right), 1..20\right)$

[*false*, *true*, *true*, *true*, *true*, *true*, *false*, *true*, *false*, *true*, *true*, *true*, *false*, *false*, *true*, *true*, *false*, *false*, *true*, *true*] **(9)**

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> *# below, anotherS is an equation, that I conjecture equals S(t)*

>

>

>

>

> $fsolve(\mathrm{vartheta}(t)=0, t=-18..-16)$

$$-17.84559954 + -0.\,\mathrm{I} \tag{10}$$

>

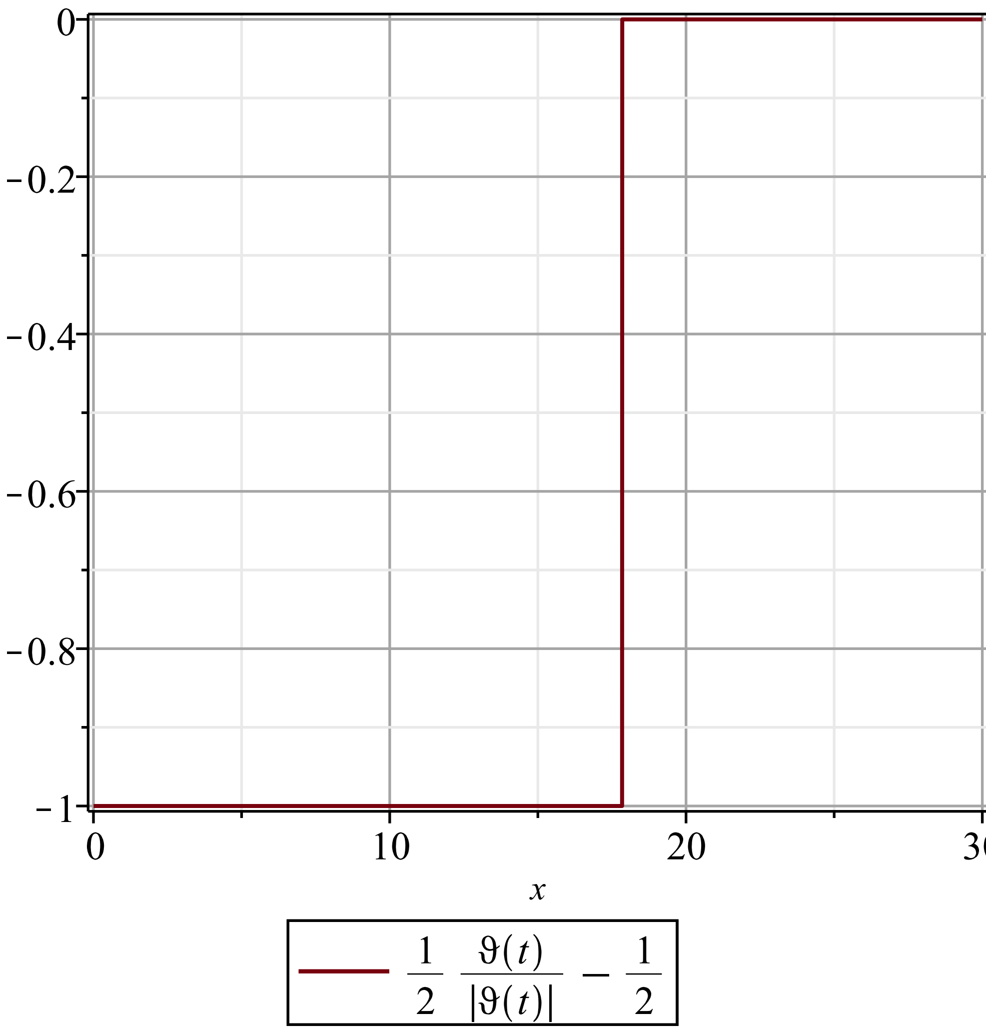
>

> $B := to1pf(N(t)+2)$; # *the zeta zero counting function for the whole critical strip, plus two*

$$B := t \mapsto \frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{\pi}+\frac{arg\left(\zeta\left(\frac{1}{2}+\mathrm{I}\,t\right)\right)}{\pi}+3 \tag{11}$$

>

> $pgl\left(\frac{\frac{\mathrm{vartheta}(t)}{|\mathrm{vartheta}(t)|}-1}{2}, 0..30, legend=\left[\frac{\frac{'\vartheta'(t)}{|'\vartheta'(t)|}-1}{2}\right]\right)$



> # *here, n=B(t) is used to select the function Sn(t,n) which is shifted down by 1.5 and has* $\frac{1}{2}(1/(|$

'ϑ'(t)|) ('ϑ'(t)) -1) *added* **to** *it*

> $anotherS := to1pf\left(Sn(t, B(t)) - \frac{3}{2} + \frac{\frac{\mathrm{vartheta}(t)}{|\mathrm{vartheta}(t)|} - 1}{2}\right)$

$$anotherS := t \mapsto -\frac{3}{2} - \mathrm{frac}\left(\frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{\pi}\right) - \Bigg\{ \quad -2 + \left\lfloor \frac{t\ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2\,\pi}\right.$$

$$+ \frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{2\left|-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}\right|}$$

> *# The definition actually seems recursive, since S(t) can be dropped from both sides , the proposal is that*

> $S(t) = anotherS(t)$

$$\frac{arg\left(\zeta\left(\frac{1}{2}+I\,t\right)\right)}{\pi} = -\frac{3}{2} - \mathrm{frac}\left(\frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{\pi}\right)$$

$$- \Bigg\{ \begin{array}{c} 1 \\ -2 + \left\lfloor \frac{t\ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2\,\pi} + \frac{7}{8}\right\rfloor - \frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{\pi} - \frac{arg\left(\zeta\left(\frac{1}{2}+\right.\right.}{\pi} \end{array}$$

$$+ \frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}}{2\left|-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right) - ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2}\right|}$$

> $eval(\mathbf{(13)}, t = 23.4);\ evalf(\%);$

$$-0.04788923144 = -0.0478892314 + \frac{-834269039 + 585000000\ln(\pi)}{50000000\,\pi}$$

$$+\frac{16.68538078-0.\,\mathrm{I}-11.70000000\ln(\pi)}{\pi}$$

$$-0.04788923144=-0.047889232+-0.\,\mathrm{I} \tag{14}$$

> $shouldbezero := to1pf(anotherS(t)-S(t))$

$$shouldbezero := t\mapsto -\frac{3}{2}-\mathrm{frac}\left(\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{\pi}\right)$$

$$-\left\{\begin{array}{l} 1 \\ \\ -2+\left\lfloor\frac{t\ln\left(\frac{t}{2\pi\mathrm{e}}\right)}{2\pi}+\frac{7}{8}\right\rfloor-\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{\pi}-\frac{arg\left(\zeta\left(\frac{1}{2}+\right.\right.}{\pi}\end{array}\right.$$

$$+\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{2\left|-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}\right|}-\frac{arg\left(\zeta\left(\frac{1}{2}+\mathrm{I}\,t\right)\right)}{\pi}$$

> *# subtracting S(t) from both sides of S(t)=anotherS(t) we get a function which should be identically zero for all real t >0, at t it is discontinuous yet its limit from the right equals 0 whereas its limit from the left equals 1*

>

>

> $limit(shouldbezero(t), t=0, left)$

$$1 \tag{16}$$

> $limit(shouldbezero(t), t=0, right)$

$$0 \tag{17}$$

>

> $is(shouldbezero(14)=0)$ assuming $0<t::real$ *# Maple cannot verify this*

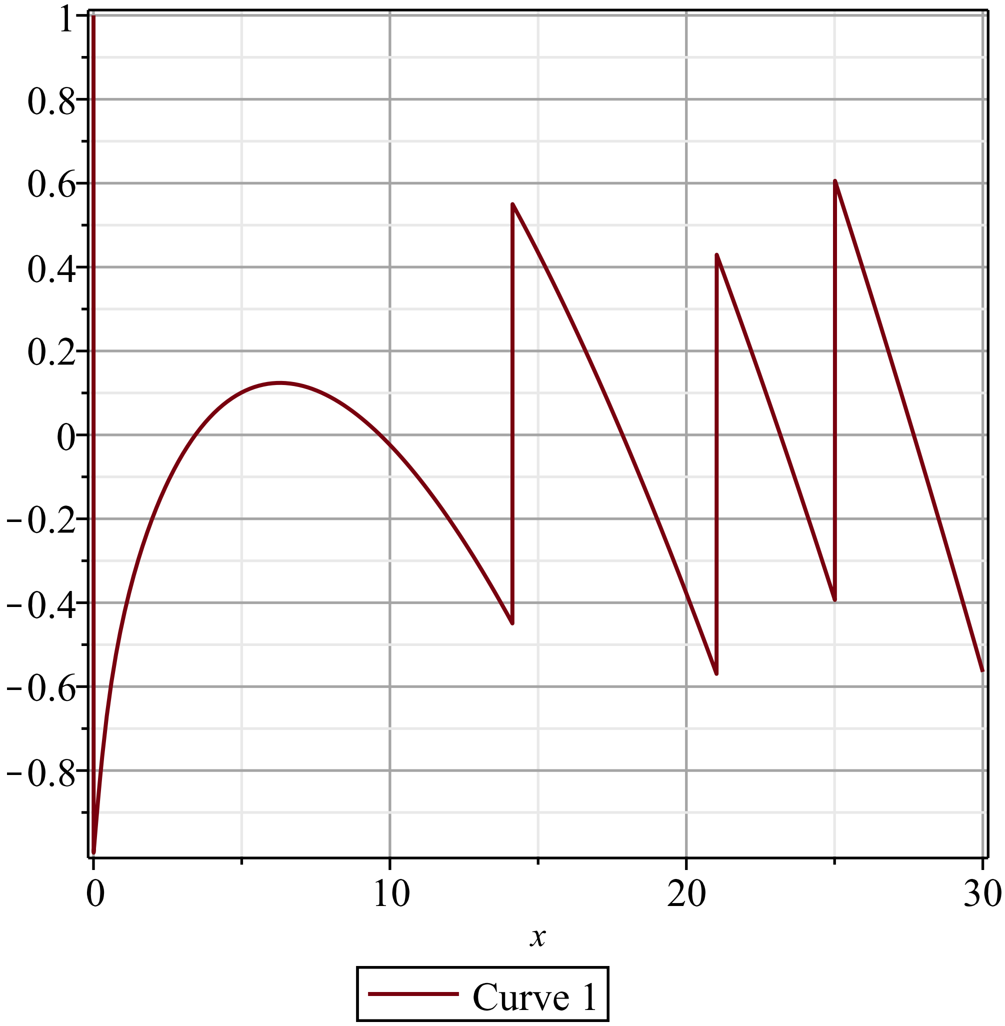
$$FAIL \tag{18}$$

> $evalf(shouldbezero(14));$
 # the error is numerical rounding error introduced by finite precision arithmetic, it equals $2\cdot 10^{-Digits}$

$$3.\,10^{-10}+0.\,\mathrm{I} \tag{19}$$

>

> $pgl(S, 0..30)$



> *shouldbezero* := *to1pf*(*anotherS*(*t*) − *S*(*t*))

$$shouldbezero := t \mapsto -\frac{3}{2} - \mathrm{frac}\left(\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{\pi}\right)$$

$$-\left\{\frac{1}{-2+\left\lfloor\frac{t\ln\left(\frac{t}{2\,\pi\,\mathrm{e}}\right)}{2\,\pi}+\frac{7}{8}\right\rfloor-\frac{-\frac{\mathrm{I}\left(ln\Gamma\left(\frac{1}{4}+\frac{\mathrm{I}\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{\mathrm{I}\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{\pi}-\frac{arg\left(\zeta\left(\frac{1}{2}+\right.\right.}{\pi}}\right.$$

$$+\frac{-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}}{2\left|-\frac{I\left(ln\Gamma\left(\frac{1}{4}+\frac{I\,t}{2}\right)-ln\Gamma\left(\frac{1}{4}-\frac{I\,t}{2}\right)\right)}{2}-\frac{\ln(\pi)\,t}{2}\right|}-\frac{arg\left(\zeta\left(\frac{1}{2}+I\,t\right)\right)}{\pi}$$

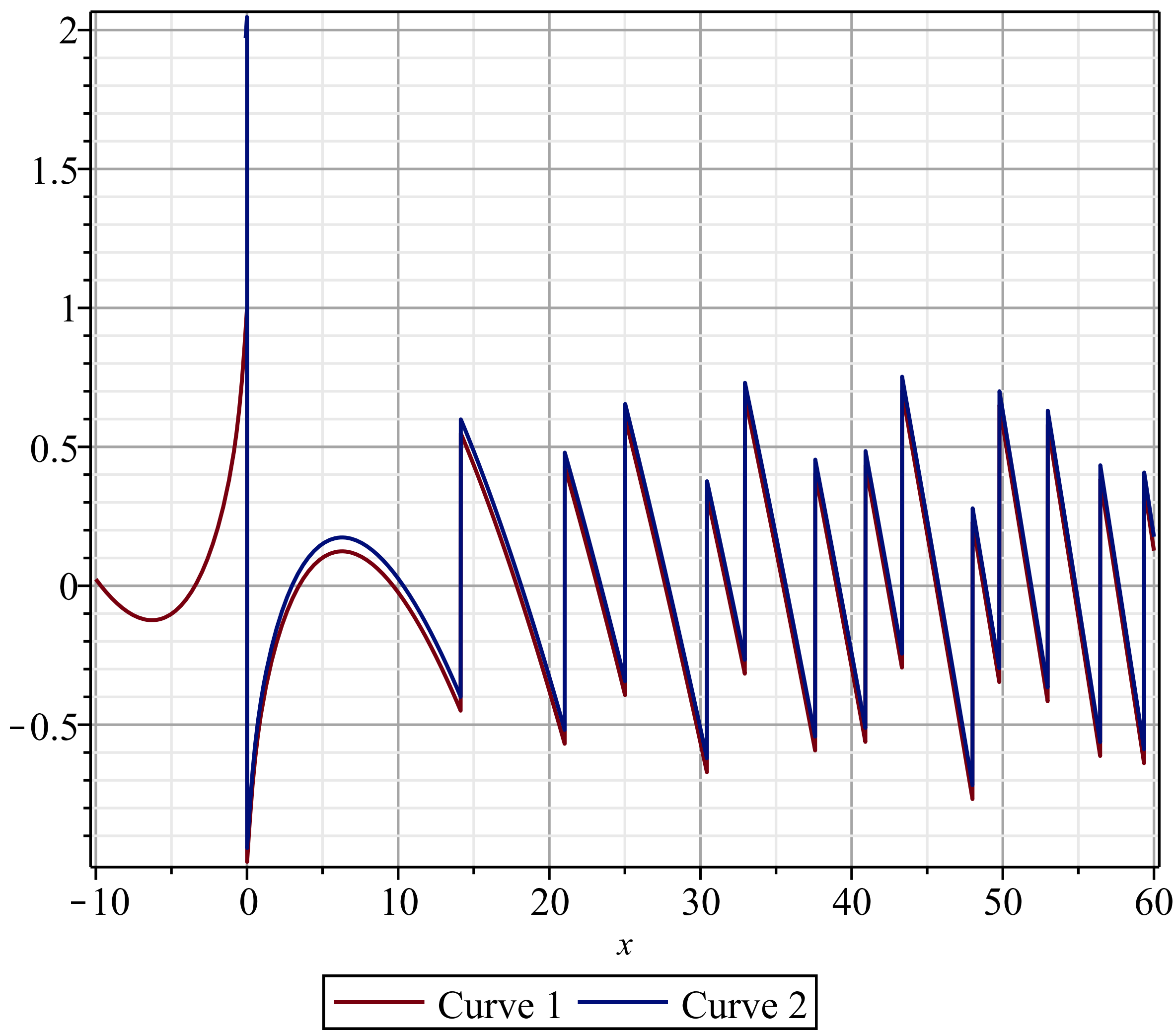
> $Digits := 20$

$$Digits := 20 \tag{21}$$

>

> $pgl\left(\left[S(t), 0.05+Sn(t, B(t))-\frac{3}{2}+\frac{\frac{\mathrm{vartheta}(t)}{|\mathrm{vartheta}(t)|}-1}{2}\right], -10..60\right);$

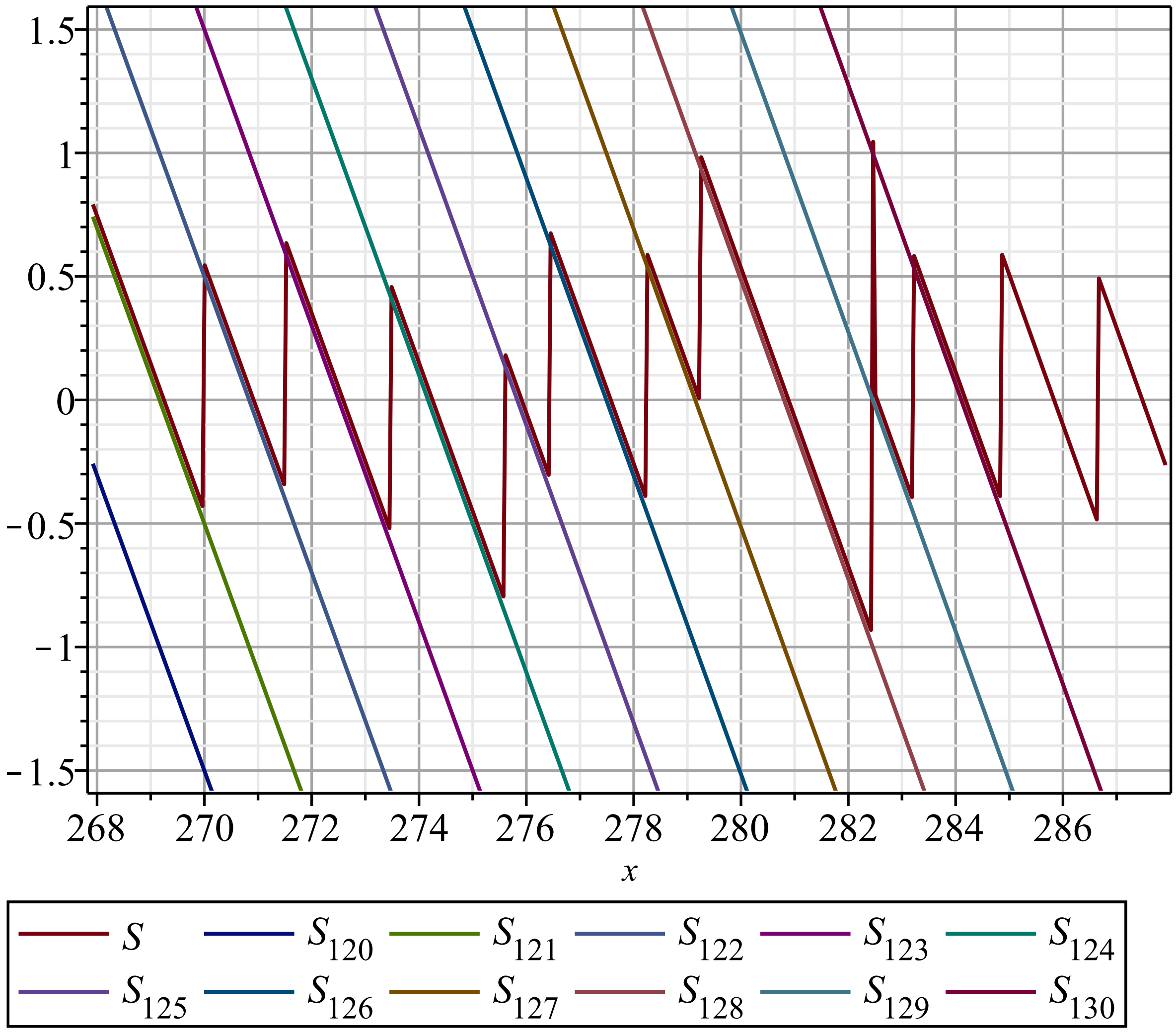
 # Introduce just a small delta, 0.05, so the lines aren't drawn directly on top of one another



>

> $pgl\left(\left[S(t)+0.05, op\left(flist\left(n \to Sn(t,n)-\frac{3}{2}, 120..130\right)\right)\right], zeros[119]..zeros[131], view = \left[zeros[119]..zeros[131], -\frac{\mathrm{Pi}}{2}..\frac{\mathrm{Pi}}{2}\right], legend = ['S', op(flist(n \to 'S'[n], 120..130))],\right.$

$adaptive = false, numpoints = 500\big)$



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