Prediction and Simulation of Exponential Self-Exciting Processes

[1]

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1. Hawkes Processes

1.1. The Sum-of-Exponentials Self-Exciting Process of Arbitrary Order.

Let

$$\theta(t) = \begin{cases} 0 & t \leqslant 0 \\ 1 & t > 0 \end{cases} \tag{1}$$

be the Heaviside step function and $\{T_i : T_i < T_{i+1}\} \in \mathbb{R}$ be the time of occurance of the i-th event of a process. The counting-function N_t of a self-exciting process is a uni-variate (linear) self-exciting counting process

$$N_t = \sum_{i: T_i < t} \theta(t - T_i) \tag{2}$$

whose conditional intensity (event rate) is given by

$$\begin{aligned} \lambda(t) &= \lambda_0(t) + \int_{-\infty}^{t} \nu(t-s) \mathrm{d}N_s \\ &= \lambda_0(t) + \sum_{T_k < t} \nu(t - T_k) \end{aligned} \tag{3}$$

where $\lambda_0(t)$ is a deterministic function which will be regarded as a constant $\lambda_0(t) = \lambda_0 = E[\lambda_0(t)]$, [9][4][8][3] Here, $\nu: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a kernel function which expresses the positive influence of past

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events T_i on the current value of the intensity process. The self-exciting process of order P is a defined by the sum-of-exponentials kernel

$$\nu(t) = \sum_{j=1}^{P} \alpha_j e^{-\beta_j t} \tag{4}$$

The intensity is then written as

$$\begin{aligned}
\lambda(t) &= \lambda_0(t) + \int_0^t \sum_{j=1}^{P} \alpha_j e^{-\beta_j(t-s)} \mathrm{d}N_s \\
&= \lambda_0(t) + \sum_{j=1}^{P} \sum_{k=0}^{\check{N}_t} \alpha_j e^{-\beta_j(t-t_k)} \\
&= \lambda_0(t) + \sum_{j=1}^{P} \alpha_j \sum_{k=0}^{\check{N}_t} e^{-\beta_j(t-t_k)} \\
&= \lambda_0(t) + \sum_{j=1}^{P} \alpha_j B_j(N_t)
\end{aligned} \tag{5}$$

where $B_j(i)$ is given recursively by

$$\begin{aligned}
B_j(i) &= \sum_{k=1}^{i-1} e^{-\beta_j(t_i-t_k)} \\
&= e^{-\beta_j(t_i-t_{i-1})} \sum_{k=1}^{i-1} e^{-\beta_j(t_{i-1}-t_k)} \\
&= e^{-\beta_j(t_i-t_{i-1})} \left(1 + \sum_{k=1}^{i-2} e^{-\beta_j(t_{i-1}-t_k)}\right) \\
&= e^{-\beta_j(t_i-t_{i-1})} (1 + B_j(i-1))
\end{aligned} \tag{6}$$

since $e^{-\beta_j(t_{i-1}-t_{i-1})} = e^{-\beta_j 0} = e^{-0} = 1$. A uni-variate self-exciting process is stationary if the branching ratio ρ is less than one.

$$\rho = \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} < 1 \tag{7}$$

If the process is stationary then the stationary unconditional intensity is

$$\begin{aligned}
\lambda = E[\lambda(t)] &= \frac{\lambda_0}{1 - E[\nu(t)]} \\
&= \frac{\lambda_0}{1 - \int_0^\infty \sum_{j=1}^{P} \alpha_j e^{-\beta_j t} \mathrm{d}t} \\
&= \frac{\lambda_0}{1 - \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}} \\
&= \frac{\lambda_0}{1-\rho}
\end{aligned} \tag{8}$$

where $E(\cdot)$ is the Lebesgue integral over the positive real numbers. For consecutive events, the dual-predictable projection, aka compensator, is expressed

$$\begin{aligned}
\Lambda(t_{i-1},t_i) &= \int_{t_{i-1}}^{t_i} \lambda(t)\mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \left(\lambda_0(t) + \sum_{j=1}^{P} \alpha_j B_j(N_t) \right) \mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s + \int_{t_{i-1}}^{t_i} \sum_{j=1}^{P} \alpha_j \sum_{k=0}^{i-1} e^{-\beta_j(t-t_k)}\mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s + \sum_{j=1}^{P} \alpha_j \sum_{k=0}^{i-1} \int_{t_{i-1}}^{t_i} e^{-\beta_j(t-t_k)}\mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s + \sum_{k=0}^{i-1} \int_{t_{i-1}}^{t_i} \nu(t-t_k)\mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s + \sum_{k=0}^{i-1} \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}(e^{-\beta_j(t_{i-1}-t_k)} - e^{-\beta_j(t_i-t_k)}) \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s + \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}(1-e^{-\beta_j\Delta t_i})A_j(i-1)
\end{aligned} \tag{9}$$

where $\Lambda_0(i) = \int_{t_{i-1}}^{t_i} \lambda_0(s)\mathrm{d}s = \lambda_0 \Delta t_i$ if $\lambda_0(t) = \lambda_0$ and

$$\begin{aligned}
A_j(i) &= \sum_{t_k \leqslant t_i} e^{-\beta_j(t_i-t_k)} \\
&= \sum_{k=0}^{i-1} e^{-\beta_j(t_i-t_k)} \\
&= 1 + e^{-\beta_j\Delta t_i} A_j(i-1)
\end{aligned} \tag{10}$$

with $A_j(0) = 0$ since the integral of $\nu(t)$ is

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} \nu(t)\mathrm{d}t &= \int_{t_{i-1}}^{t_i} \sum_{j=1}^{P} \alpha_j e^{-\beta_j(t-t_k)}\mathrm{d}t \\
&= \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}(e^{-\beta_j t_i} - e^{-\beta_j t_{i-1}})
\end{aligned} \tag{11}$$

If $\lambda_0(t)$ does not vary with time, that is, $\lambda_0(t) = \lambda_0$ then (9) simplifies to

$$\begin{aligned}
\Lambda(t_{i-1},t_i) &= \Lambda_0(i) + \sum_{k=0}^{i-1} \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}(e^{-\beta_j(t_{i-1}-t_k)} - e^{-\beta_j(t_i-t_k)}) \\
&= \Lambda_0(i) + \sum_{k=0}^{i-1} \int_{t_{i-1}-t_k}^{t_i-t_k} \nu(t)\mathrm{d}t \\
&= \Lambda_0(i) + \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}(1-e^{-\beta_j(t_i-t_{i-1})})A_j(i-1)
\end{aligned} \tag{12}$$

1.1.1. Maximum Likelihood Estimation.

The log-likelihood of a simple point process is written as

$$\begin{aligned}
\ln\mathcal{L}(N(t)_{t\in[0,T]}) &= \int_0^T (1-\lambda(s))\mathrm{d}s + \int_0^T \ln\lambda(s)\mathrm{d}N_s \\
&= T - \int_0^T \lambda(s)\mathrm{d}s + \int_0^T \ln\lambda(s)\mathrm{d}N_s
\end{aligned} \tag{13}$$

which in the case of the sums-of-exponentials self-exciting process of order P can be explicitly written [6] as

$$\begin{aligned}\ln\mathcal{L}(\{t_i\}_{i=1\ldots n}) &= T-\Lambda(0,T)+\sum_{i=1}^{n}\ln\lambda(t_i)\\ &= T+\sum_{i=1}^{n}\left(\ln\lambda(t_i)-\Lambda(t_{i-1},t_i)\right)\\ &= T-\Lambda(0,T)+\sum_{i=1}^{n}\ln\lambda(t_i)\\ &= T-\Lambda(0,T)+\sum_{i=1}^{n}\ln\left(\lambda_0(t_i)+\sum_{j=1}^{P}\sum_{k=1}^{i-1}\alpha_j e^{-\beta_j(t_i-t_k)}\right)\\ &= T-\Lambda(0,T)+\sum_{i=1}^{n}\ln\left(\lambda_0(t_i)+\sum_{j=1}^{P}\alpha_j B_j(i)\right)\\ &= T-\int_0^T\lambda_0(s)\mathrm{d}s-\sum_{i=1}^{n}\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}(1-e^{-\beta_j(t_n-t_i)})\\ &+\sum_{i=1}^{n}\ln\left(\lambda_0(t_i)+\sum_{j=1}^{P}\alpha_j B_j(i)\right)\end{aligned} \tag{14}$$

where $T=t_n$ and $B_j(i)$ [5] is defined by (6) If the baseline intensity is constant $\lambda_0(t)=\lambda_0$ then the log-likelihood can be written

$$\begin{aligned}\ln\mathcal{L}(\{t_1,\ldots,t_n\}) &= T-\lambda_0 T-\sum_{i=1}^{n}\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}(1-e^{-\beta_j(T-t_i)})\\ &+\sum_{i=1}^{n}\ln\left(\lambda_0+\sum_{j=1}^{P}\alpha_j B_j(i)\right)\end{aligned} \tag{15}$$

Note that it was necessary to shift each t_i by t_1 so that $t_1=0$ and $T=t_n$. Also note that T is just an additive constant which does not vary with the parameters so for the purposes of estimation can be removed from the equation.

1.2. The Hazard Function, Cumulative Hazard Function, and its Inverse.

The hazard function is a conditional density that gives the probability of an event occuring within 1 unit of time given that an event has not yet occurred prior to time t

$$h(t)=\frac{\nu(t)}{s(t)} \tag{16}$$

where $s(t)$ is the survivor function defined by

$$s(t)=1-F(t) \tag{17}$$

and $F(t)$ is the integrated kernel function

$$F(t)=\int_0^t\nu(s)\mathrm{d}s=\frac{\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}(1-e^{-\beta_j t})}{\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}} \tag{18}$$

Let

$$\begin{aligned}ih(t) &= \int_0^t h(t)\mathrm{d}t\\ &= -\ln(1-F(t))\\ &= -\ln\left(1-\frac{\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}(1-e^{-\beta_j t})}{\sum_{j=1}^{P}\frac{\alpha_j}{\beta_j}}\right)\end{aligned} \tag{19}$$

be the integrated, or cumulative, hazard function and define

$$\varphi(t,r)=\left(\sum_{k=1}^{P}(e^{r-\beta_k t}-1)\prod_{j=1}^{P}\begin{cases}\alpha_j & j=k\\ \beta_j & j\neq k\end{cases}\right) \tag{20}$$

then the root of $\varphi(t,r)$ for a given value of r is such that

$$ih^{-1}(r)=\{t\colon ih(t)=r\}=\{t\colon \varphi(t,r)=0\} \tag{21}$$

is a solution to

$$ih(ih^{-1}(r))=r \tag{22}$$

The derivative of $\varphi(t,r)$ is

$$\frac{\partial}{\partial t}\varphi(t,r)=-\left(\sum_{k=1}^{P}\beta_k e^{r-\beta_k t}\prod_{j=1}^{P}\begin{cases}\alpha_j & j=k\\ \beta_j & j\neq k\end{cases}\right) \tag{23}$$

so that the Newton iteration is expressed

$$t-\frac{\varphi(t,r)}{\frac{\partial}{\partial t}\varphi(t,r)}=t+\frac{\sum_{k=1}^{P}\gamma(k)(e^{r-\beta_k t}-1)}{\sum_{k=1}^{P}\gamma(k)\beta_k e^{r-\beta_k t}} \tag{24}$$

where

$$\gamma(k)=\prod_{j=1}^{P}\begin{cases}\alpha_j & j=k\\ \beta_j & j\neq k\end{cases} \tag{25}$$

is the amplitude of j-th joint decay rate.

1.2.1. The case when P=1.

1.3. An Expression for the Density of the Duration Until the Next Event.

The simplest case occurs when the deterministic intensity $\lambda_0(t)=\lambda_0$ is constant and $P=1$ where we have

$$\lambda(\{t_i\})=\lambda_0+\sum_{t_i<t}\sum_{j=1}^{1}\alpha_j e^{-\beta_j(t-t_i)}=\lambda_0+\sum_{t_i<t}\alpha e^{-\beta(t-t_i)} \tag{26}$$

and

$$\lambda=E[\lambda(t)]=\frac{\kappa}{1-\frac{\alpha}{\beta}} \tag{27}$$

is the expected value of the unconditional mean intensity.

$$a_n=\sum_{k=0}^{n}e^{\beta t_k} \tag{28}$$

$$b_n=\sum_{k=0}^{n}e^{\beta(t_k-t_n)} \tag{29}$$

$$c_n=\sum_{k=0}^{n}\sum_{l=0}^{n}e^{\beta(t_k+t_l-t_n)} \tag{30}$$

The expected time until the next point can be obtained by integrating over the unit exponentially distributed parameter ε appearing in the inverse of the compensator

$$\Lambda^{-1}(\varepsilon,\alpha,\beta)=e^{-\beta T}\left(\frac{Ta_n+\frac{a_n}{\beta}W\left(\frac{\alpha}{\lambda_0}A_1(n)\cdot e^{\frac{\alpha b n-\beta\varepsilon}{\lambda_0}}\right)+\frac{e^{-\beta T}}{\lambda_0}\left(a_n\varepsilon-\frac{\alpha}{\beta}c_n\right)}{A_1(n)}\right) \tag{31}$$

where $A_j(i)=\sum_{k=0}^{i-1}e^{-\beta_j(t_i-t_k)}$ is defined recursively in Equation (10) so that

$$\mathrm{E}_\varepsilon[\Lambda^{-1}|\mathcal{F}_t,\alpha,\beta]=\int_0^\infty e^{-\varepsilon}\Lambda^{-1}(\varepsilon,\alpha,\beta)\mathrm{d}\varepsilon \tag{32}$$

$$\begin{aligned} a_n &= a_{n-1} e^{-\beta \Delta t_n} + 1 \\ b_n &= b_{n-1} e^{-\beta \Delta t_n} + 1 \\ c_n &= c_{n-1} e^{-\beta \Delta t_n} + e^{\beta t_n} + 2 a_{n-1} \end{aligned} \tag{33}$$

It would be nice to have expressions like this involving the Lambert W function for $P > 1$ but neither Maple nor Mathematica were able to find any solutions in terms of "known" functions for $P > 1$. It is noted that Equation (31) has the form

$$\int_0^\infty (p + qW(re^{-sx+t}) + ux) e^{-x} \mathrm{d}x \tag{34}$$

which is a function of 6 variables, $\{p, q, r, s, t, u\}$, for which it would be a very nice thing to have a closed form expression, in order to avoid a recourse to numerical or Monte Carlo integration. It seems that such an expression is very likely to exist because if we drop the variable s from Equation (34) we get a closed-form expression of the form

$$\int_0^\infty (p + qW(re^{-x+t}) + ux) e^{-x} \mathrm{d}x = qW(re^t) + \frac{q}{W(re^t)} - q + u + p - \frac{q}{re^t} \tag{35}$$

We can break this problem down into a more manageable one by calculating some more integrals to see if we can find a pattern. Let us begin with the integral

$$\int_0^\infty W(e^{-sx}) e^{-x} \mathrm{d}x = W(1) + \left(-\frac{1}{s}\right)^{-\frac{1}{s}} \left(\Gamma\left(\frac{1}{s}\right) - s\Gamma\left(1 + \frac{1}{s}, -\frac{W(1)}{s}\right)\right) \tag{36}$$

whose closed-form expression was found by Vladimir Reshetnikov.

1.3.1. The Case of Any Order $P = n$.

Let $S = t_n$ where n is the number of points that have occured so far and

$$\begin{aligned} \varphi(P, \varepsilon, x, \alpha, \beta, S) = & \left(\prod_{k=1}^P \beta_k\right)(\lambda_0 x - (\varepsilon + \lambda_0 S)) e^{\sum_{k=1}^P \beta_k (x+S)} + \\ & \sum_{m=1}^P \left(\prod_{k=1}^P \begin{cases} \alpha_k & m = k \\ \beta_k & m \neq k \end{cases}\right) \sum_{k=0}^n e^{\sum_{j=1}^P \beta_j \left(x + \begin{cases} S & j \neq m \\ t_k & j = m \end{cases}\right)} - e^{\sum_{j=1}^P \beta_j \left(S + \begin{cases} x & j \neq m \\ t_k & j = m \end{cases}\right)} \\ = & \tau(x, \varepsilon) + \sum_{j=1}^P \phi_j \sum_{k=0}^{N_T} (\sigma_{j,k}(x, x) - \sigma_{j,k}(x, S)) \end{aligned} \tag{37}$$

then the time of occurance of the next point of the process is equal to the value of x which solves $\varphi(P, \varepsilon, x, \alpha, \beta, S) = 0$ for a given value of ε which is not known ahead-of-time. In simulation, ε is a randomly chosen value drawn from a unit exponential distribution. In operation, ε is the integral of the conditional intensity accumulated over the interval spanning the last point and the arrival of the next point. Let $x(\varepsilon)$ be the implicity defined function

$$x(\varepsilon) = \{x \colon \varphi(P, \varepsilon, x, \alpha, \beta, S) = 0\} \tag{38}$$

The expected value $E_\varepsilon[x(\varepsilon)]$ of the solution over all possible values of ε is equal to

$$x(\varepsilon) = \int_0^\infty e^{-\varepsilon} x(\varepsilon) \mathrm{d}\varepsilon \tag{39}$$

where $x(\varepsilon) = x(\varepsilon; P, x, \alpha, \beta, S)$ is shortened notation to indicate that x is a univariate function of ε only and the variables λ_0, α, β and P are constants. Note that the product of piece-wise functions can be written as

$$\begin{aligned} \prod_{k=1}^P \begin{cases} \alpha_k & m = k \\ \beta_k & m \neq k \end{cases} &= \alpha_m \left(\prod_{k=1}^{m-1} \beta_k\right)\left(\prod_{k=m+1}^P \beta_k\right) \\ &= \alpha_m \prod_{\substack{k=1 \\ k \neq m}}^P \beta_k \end{aligned} \tag{40}$$

where

$$\begin{aligned}
\sigma_{m,k}(x,x) &= \sum_{j=1}^{P} \beta_j\left(x + \begin{cases} T & j \neq m \\ t_k & j = m \end{cases} \right) \\
&= \beta_m(x+t_k) + \sum_{j=1}^{m-1} \beta_j(x+T) + \sum_{j=m+1}^{P} \beta_j(x+T) \\
&= \beta_m(x+t_k) + \sum_{\substack{j=1 \\ j\neq m}}^{P} \beta_j(x+T)
\end{aligned} \tag{41}$$

$$\begin{aligned}
\sigma_{m,k}(x,T) = & \sum_{j=1}^{P} \beta_j\left(T + \begin{cases} x & j \neq m \\ t_k & j = m \end{cases} \right) \\
&= \beta_m(T+t_k) + \sum_{j=1}^{m-1} \beta_j(x+T) + \sum_{j=m+1}^{P} \beta_j(x+T) \\
&= \beta_m(T+t_k) + \sum_{\substack{j=1 \\ j\neq m}}^{P} \beta_j(x+T)
\end{aligned} \tag{42}$$

$$\tau(x,\varepsilon) = ((x-T)\kappa - \varepsilon)\upsilon\eta(x) \tag{43}$$

$$\eta(x) = e^{(x+T)\sum_{k=1}^{P}\beta_k} \tag{44}$$

$$\upsilon = \prod_{k=1}^{P} \beta_k \tag{45}$$

$$\bar{\upsilon}_m = \sum_{\substack{k=1 \\ k\neq m}}^{P} \beta_k \tag{46}$$

so that (37) can be rewritten as

$$\varphi_P(x(\varepsilon)) = \tau(x,\varepsilon) + \sum_{j=1}^{P} \phi_j \sum_{k=0}^{N_T} (\sigma_{j,k}(x,x) - \sigma_{j,k}(x,T)) \tag{47}$$

The derivative

$$\varphi'_P(x(\varepsilon)) = \upsilon(\lambda_0\eta(x) + \tau(x,\varepsilon)) + \sum_{m=1}^{P} \phi_m \sum_{k=0}^{n} (\upsilon\sigma_{m,k}(x) - \bar{\upsilon}_m \sigma_{m,k}(T)) \tag{48}$$

is needed so that the Newton sequence can be expressed as

$$\begin{aligned}
x_{i+1} &= x_i - \frac{\varphi_P(x_i)}{\varphi'_P(x_i)} \\
&= x_i - \frac{\tau(x_i,\varepsilon) + \sum_{m=1}^{P}\phi_m\sum_{k=0}^{n}(\sigma_{m,k}(x_i,x_i) - \sigma_{m,k}(x_i,T))}{\upsilon(\kappa\eta(x_i) + \tau(x_i,\varepsilon)) + \sum_{m=1}^{P}\phi_m\sum_{k=0}^{n}(\mu\,\sigma_{m,k}(x_i) - \mu_m\,\sigma_{m,k}(T))}
\end{aligned} \tag{49}$$

and simplified a bit(at least notationally) if we let

$$\rho(x,d) = \sum_{m=1}^{P} \phi_m \sum_{k=0}^{n} \left(\sigma_{m,k}(x) \begin{cases} 1 & d=0 \\ \upsilon & d=1 \end{cases} - \sigma_{m,k}(T) \begin{cases} 1 & d=0 \\ \bar{\upsilon}_m & d=1 \end{cases} \right) \tag{50}$$

then

$$\begin{aligned}
x_{i+1}(\varepsilon) &= x_i(\varepsilon) - \frac{\varphi_P(x_i(\varepsilon))}{\varphi'_P(x_i(\varepsilon))} \\
&= x_i - \frac{\tau(x_i(\varepsilon),\varepsilon) + \rho(x_i(\varepsilon),0)}{\upsilon(\kappa\eta(x_i(\varepsilon)) + \tau(x_i(\varepsilon),\varepsilon)) + \rho(x_i(\varepsilon),1)}
\end{aligned} \tag{51}$$

so that

$$\Lambda_P^{-1}(\varepsilon; t_0...T) = \lim_{m\to\infty} x_m(\varepsilon) \tag{52}$$

which leads to the expression for the expected arrival time of the next point

$$\int_0^\infty \Lambda_P^{-1}(\varepsilon; t_0...T) e^{-\varepsilon} \mathrm{d}\varepsilon = \int_0^\infty \lim_{m\to\infty} x_m(\varepsilon) e^{-\varepsilon} \mathrm{d}\varepsilon \tag{53}$$

Fatou's lemma[7] can probably be invoked so that the order of the limit and the integral in Equation (53) can be exchanged, with perhaps the introduction of another function, which of course would greatly reduce the computational complexity of the equation. The sequence of functions is known as a Newton sequence [2, 3.3p118] There is also the limit

$$\begin{aligned}\lim_{x\to\infty} \frac{\varphi_P(x_i(\varepsilon))}{\varphi_P'(x_i(\varepsilon))} &= \lim_{x\to\infty} \frac{\tau(x_i(\varepsilon),\varepsilon) + \rho(x_i(\varepsilon),0)}{\upsilon(\kappa\eta(x_i(\varepsilon)) + \tau(x_i(\varepsilon),\varepsilon)) + \rho(x_i(\varepsilon),1)} \\ &= \frac{1}{\lambda}\end{aligned} \tag{54}$$

which converges to the inverse of the stationary rate.

1.4. Filtering, Prediction, Estimation, etc.

The next occurrence time of a point process, given the most recent time of occurrence of a point of a process, can be predicted by solving for the unknown time t_{n+1} when $\{t_n\}$ is a sequence of event times. Let

$$\Lambda_{\text{next}}(t_n, \delta) = \{t_{n+1} \colon \Lambda(t_n, t_{n+1}) = \delta\} \tag{55}$$

where

$$\Lambda(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} \lambda(s; \mathfrak{F}_s) \mathrm{d}s \tag{56}$$

and $\mathfrak{F}_s$ is the σ-algebra filtration up to and including time s and the parameters of λ are fixed. The multivariate case is covered in Section (1.5.1). The idea is to integrate over the solution of Equation (55) with all possible values of ε, distributed according to the unit exponential distribution. The reason for the plural form, time(s), rather than the singular form, time, is that Equation (55) actually has a single real solution and N number of complex solutions, where N is the number of points that have occurred in the process up until the time of prediction. This set of complex expected future event arrival times is deemed the *constellation* of the process, which becomes more detailed with the occurance of each event(the increasing σ-algebra filtration). We shall ignore the constellation for now, and single out the sole real valued element as the expected real time until the next event. After all, does it even make sense to say "something will probably happen around $9.8 + i7.2$ seconds from now?" where i is the imaginary unit, $i = \sqrt{-1}$. The recursive equations for the resemble the heta functions of number theory if you one extends from real valued $\beta \in \mathbb{R}$ to a complex $\beta = i$.

1.5. Calculation of the Expected Number of Events Any Given Time From Now.

The expected number of events given any time from now whatsoever can be calculated by integrating out ε since the process which is adapted to the compensator will be closer to being a unit rate Poisson process the closer the parameters are to being correct and the model actually being a good model of the phemenona it is being applied to. Let F_t be all points up until now, let

$$E(t_{n+1}) = \int_0^\infty \Lambda^{-1}(\varepsilon; \alpha, \beta, F_{t_n}) e^{-\varepsilon} \mathrm{d}\varepsilon$$

then iterate the process, by proceeding as if the next point of the process will occur at the predicted time, simply append the expectation to the current state vector, and project the next point, repeating the process as fast ast the computer will go until some sufficient stopping criteria is met. This equation seems very similiar to the infinite horizon discounted regulator of optimal control; see [1, 1.1].

1.5.1. Prediction.

The next event arrival time of the m-th dimension of a multivariate Hawkes process having the usual exponential kernel can be predicted in the same way as the uni-variate process in Section (1.4), by solving for the unknown t_{n+1} in the equation

$$\left\{ t_{n+1}^m : \varepsilon = \Lambda^m(t_n^m, t_{n+1}^m) = \int_{t_n^m}^{t_{n+1}^m} \lambda^m(s; \mathfrak{F}_s) \mathrm{d}s \right\} \tag{57}$$

where $\Lambda^m(t_n^m,\, t_{n+1}^m)$ is the compensator from Equation (?) and $\mathfrak{F}_s$ is the filtration up to time s and the parameters of λ^m are fixed. As is the case for the uni-variate Hawkes process, the idea is to average over all possible realizations of ε (of which there are an uncountable infinity) weighted according to an exponential unit distribution. Another idea for more accurate prediction is to model the deviation of the generalized residuals from a true exponential distribution and then include the predicted error when calculating this expectation.

Let the most recent arrival time of the pooled and m-th processes respectively be given by

$$T = \max\left(T_m : m = 1 \ldots M\right) \tag{58}$$

$$T_m = \max\left(t_n^m : n = 0 \ldots N^m - 1\right) = t_{N^m - 1}^m \tag{59}$$

and

$$\breve{N}_{T_m}^n = \sum_{k=0}^{\breve{N}^n} \begin{cases} 1 & t_k^n < T_m \\ 0 & \end{cases} \tag{60}$$

count the number of points occurring in the n-th dimension before the most recent point of the m-th dimension and

$$\breve{N}(t_j^m < t_k^n) \tag{61}$$

then the next arrival time for a given value of the exponential random variable ε of the m-th dimension of a multivariate Hawkes process having the standard exponential kernel is found by solving for the real root of

$$\varphi_m(x(\varepsilon); \mathcal{F}_T) = \tau_m(x, \varepsilon) + \sum_{l=1}^{P} \sum_{i=1}^{M} \phi_{m,i,l} \sum_{k=0}^{\breve{N}_{T_m}^i} \left(\sigma_{m,i,l,k}(x, x) - \sigma_{m,i,l,k}(x, T_m)\right) \tag{62}$$

which is similar to the uni-variate case

$$\varphi_P(x(\varepsilon)) = \tau(x, \varepsilon) + \sum_{j=1}^{P} \phi_j \sum_{k=0}^{\breve{N}_T} \left(\sigma_{j,k}(x, x) - \sigma_{j,k}(x, T)\right) \tag{63}$$

where

$$\mathcal{F}_T = \{\kappa_{\ldots}, \alpha_{\ldots}, \beta_{\ldots}, t_0^1 \ldots t_{N^1}^1 \leqslant T, \ldots, t_0^m \ldots t_{N^m}^m \leqslant T, \ldots, t_0^M \ldots t_{N^M}^M \leqslant T\} \tag{64}$$

is the filtration up to time T, to be interpreted as the set of available information, here denoting fitted parameters and observed arrival times of all dimensions, and where

$$\tau_m(x, \varepsilon) = ((x - T_m)\kappa_m - \varepsilon)\upsilon_m \eta_m(x) \tag{65}$$

$$\eta_m(x) = e^{(x + T_m)\sum_{j=1}^{P} \sum_{n=1}^{M} \beta_{m,n,j}} \tag{66}$$

can be seen to be similar to the uni-variate equations $\tau(x,\, \varepsilon) = ((x - T)\kappa - \varepsilon)\upsilon\eta(x)$ and $\eta(x) = e^{(x+T)\sum_{k=1}^{P} \beta_k}$ and

$$\upsilon_m = \prod_{j=1}^{P} \prod_{n=1}^{M} \beta_{m,n,j} \tag{67}$$

$$\phi_{m,p,k} = \prod_{j=1}^{P} \prod_{n=1}^{M} \begin{cases} \alpha_{m,n,j} & n = p \,\text{and}\, j = k \\ \beta_{m,n,j} & n \neq p \,\text{or}\, j \neq k \end{cases} \tag{68}$$

$$\sigma_{m,i,l,k}(x, a) = e^{\sum_{j=1}^{P} \sum_{n=1}^{M} \beta_{m,n,j} \begin{cases} a + t_k^n & n = i \,\text{and}\, j = l \\ x + T_n & n \neq i \,\text{or} j \neq l \end{cases}} \tag{69}$$

For comparison, the uni-variate case is Equation (47) where

$$\sigma_{m,k}(x,a) = e^{(a+t_k)\beta_m + (x+T)\sum_{\substack{j=1\\ j\neq m}}^{P}\beta_j} = e^{\sum_{j=1}^{P}\beta_j \begin{cases} a+t_k & j=m \\ x+T & j\neq m \end{cases}} \tag{70}$$

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