The Zeros of the Hardy Z Function are Simple

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Abstract

It is proved that the non-trivial roots of the Hardy Z function are simple having multiplicity 1 by showing that the fixed-points $N_Z(\alpha)=\alpha$ of the Newton map $N_Z(t)=t-\frac{Z(t)}{\dot{Z}(t)}$ must have a multiplier $\lambda_{N_Z}(\alpha)=|\dot{N_Z}(\alpha)|=\left|\frac{Z(\alpha)\ddot{Z}(\alpha)}{\dot{Z}(\alpha)}\right|=0$ and therefore a multiplicity $m_Z(\alpha)=\frac{1}{1-\lambda_{N_Z}(\alpha)}=\frac{1}{1-0}=1$.

1 Preliminary Outline

1.1 Definitions

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned}\zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s)>1\\ &= (1-2^{1-s})\sum_{n=1}^{\infty} n^{-s}(-1)^{n-1} && \forall \mathrm{Re}(s)>0\end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function

$$\vartheta(t)=-\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{i t}{2}\right)-\ln\Gamma\left(\frac{1}{4}-\frac{i t}{2}\right)\right)-\frac{\ln(\pi)\,t}{2} \tag{2}$$

so that the Hardy Z function[1] can be defined by

$$Z(t) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+i t\right) \tag{3}$$

which is real-valued when t is real and satisfies the identity

$$\zeta(t)=e^{-i\vartheta\left(\frac{i}{2}-i t\right)}Z\left(\frac{i}{2}-i t\right) \tag{4}$$

where $\ln\Gamma(z)$ is the principal branch of the logarithm of the Γ function defined by

$$\ln\Gamma(z)=\ln(\Gamma(z))=(z-1)!=\prod_{k=1}^{z-1} k \forall z\in\mathbb{R}>0 \tag{5}$$

which is analytically continued from the positive real axis when $z\in\mathbb{C}$ is complex. Each of the points $z\in\mathbb{Z}=\{0,-1,-2,\ldots\}$ is a singularity and a branch point so that the union of the branch cuts is the negative real axis. On the branch cuts, the values of $\ln\Gamma(z)$ are determined by continuity from above.

Conjecture 1. *(The Riemann hypothesis) All solutions t of the equation*

$$\zeta(t) = 0 \tag{6}$$

besides the trivial solutions $t = -2n$ with $n \in \mathbb{Z}^+$ have real-part $\frac{1}{2}$, that is, $\mathrm{Re}(t) = \frac{1}{2}$ when $\zeta(t) = 0$ and $t \neq -2n$.

1.2 Iterated Function Systems

Definition 2. *A `fixed-point` α of a function $f(x)$ is a value α such that*

$$f(\alpha) = \alpha \tag{7}$$

[4, 3.]

Definition 3. *The `multiplier` of a fixed point α of a map $f(x)$ where $f(\alpha) = \alpha$ is equal to the absolute value of the derivative of the map evaluated at the point α.*

$$\lambda_f(\alpha) = |\dot{f}(\alpha)| \tag{8}$$

If $\lambda_f(\alpha) < 1$ then α is a said to be an ***attractive fixed-point*** *of the map $f(x)$. If $\lambda_f(\alpha) = 1$ then α is an* ***indifferent fixed point****, and if $\lambda_f(\alpha) > 1$ then α is a* ***repelling fixed-point****. When $\lambda_f(\alpha) = 0$ the fixed-point α is said to be* `super-attractive`*[4, 3.]*

Definition 4. *The Newton map of a function $g(t)$ is given by*

$$N_g(t) = t - \frac{g(t)}{\dot{g}(t)} \tag{9}$$

and the multiplier of its fixed-points $g(\alpha) = 0$ where $N_g(\alpha) = \alpha$ is equal to

$$\lambda_{N_f(\alpha)} = \left| \frac{f(\alpha)\ddot{f}(\alpha)}{\dot{f}(\alpha)} \right| \tag{10}$$

Definition 5. *The multiplicity $m_{N_f}(\alpha)$ of a root $f(\alpha) = 0$ where $N_f(\alpha) = \alpha$ is defined by its Taylor expansion about the point α having the form*

$$f(t) = c(t - \alpha)^{m_f(\alpha)} + (\text{higher order terms...}) \tag{11}$$

where $c \neq 0$ and $m \geqslant 1$. The multiplicity of a root is related to the multipler $\lambda_f(\alpha)$ through the formula

$$\begin{aligned} m_f(\alpha) &= \frac{1}{1 - \lambda_{N_f}(\alpha)} \\ &= \frac{1}{1 - |\dot{N}_f(\alpha)|} \\ &= \frac{1}{1 - \left|\frac{f(\alpha)\ddot{f}(\alpha)}{\dot{f}(\alpha)}\right|} \end{aligned} \tag{12}$$

when the fixed-point $f(\alpha) = \alpha$ is not indifferent, that is, when $\lambda_{N_f}(\alpha) \neq 1$. When $\lambda_{N_f}(\alpha) = 1$ then $m_f(\alpha)$ is not defined thru this formula since $m_f(t) = \frac{1}{1 - \lambda_{N_f}(t)} = \frac{1}{1-1} = \frac{1}{0}$.

See [3, p.52][2]

Definition 6. *Let*

$$Y(t) = N_Z(t) = t - \frac{Z(t)}{\dot{Z}(t)}$$

be the Newton map of $Z(t)$ which has fixed-points $Y(\alpha) = \alpha$ at the zeros $Z(\alpha) = 0$.

Proposition 7. *Y(t) has indifferent fixed-points, $\left\{\lambda_{N_Z}(t) = 1: t = -\frac{i}{2}(-4n-1) \forall n \in \mathbb{Z}^+\right\}$ at each trival zero of α, $Z\left(-\frac{i}{2}(-4n-1)\right) = 0$.*

Definition 8. *A* `removable singularity` *β of a holomorphic function $f(t)$ is a point $f(\beta)$ at which the function is undefined yet it remains possible to redefine the function in such a way that the resulting function is regular in a neighborhood of β.*

Theorem 9. *$Y(t)$ has super-attractive fixed-points at each non-trivial root of Z, $\{\alpha: Z(\alpha) = 0, \mathrm{Re}(\alpha) \neq 0\}$. That is, $\{\lambda_Y(\alpha) = 0: Z(\alpha) = 0, \mathrm{Re}(\alpha) \neq 0\}$.*

Proof. The numerator of the argument of the absolute value function in the multiplier $\lambda_Y(\alpha) = \left|\frac{Z(\alpha)\ddot{Z}(\alpha)}{\dot{Z}(\alpha)}\right|$ is $Z(\alpha)\ddot{Z}(\alpha)$. If α was a geometrically attracting fixed-point $\lambda_Y(\alpha) = l$ where $0 < l < 1$ instead of superattracting then its multiplicity would be $\frac{1}{1-l} = m$ and since m must be an integer, the multiplier l would be equal to $\frac{m-1}{m}$. If $m \geqslant 2$ then $\dot{Z}(\alpha) = 0$ which would require α to be a removable singularity of either $Z(t)$ or $\ddot{Z}(t)$ so the singularity at $\frac{1}{\dot{Z}(t)}$ as $t \to \alpha$ would be canceled by a removable singularity of $Z(t)$ or $\ddot{Z}(t)$ as $t \to \alpha$ so that the limit would exist, but since both $Z(t)$ and $\ddot{Z}(t)$ are well-defined except at their essential singularities then $\ddot{Z}(\alpha)$ does not have any removable singularities. The only essential singularity of $Z(t)$ is at $-\frac{i}{2}$ which corresponds to the pole of $\zeta(t)$ at $t = 1$. The function $\ddot{Z}(t)$ has essential singularities at $-\frac{i}{2}$, $\bigcup_{n=1}^{\infty} -\frac{i}{2}(4n-3)$ and $\bigcup_{n=1}^{\infty} +\frac{i}{2}(4n-3)$. □

Theorem 10. *The roots of $Z(t)$ are simple, that is, have multiplicity 1.*

Proof. Let $\alpha \notin \bigcup_{n=1}^{\infty} -\frac{i}{2}(-4n-1)$ be a non-trivial root where $Z(\alpha) = 0$ then the multiplier of its Newton map is $\lambda_{N_Z}(\alpha) = \left|\frac{Z(\alpha)\ddot{Z}(\alpha)}{\dot{Z}(\alpha)}\right| = 0$ therefore its multiplicity is $m_Z(t) = \frac{1}{1-\lambda_{N_Z}(t)} = \frac{1}{1-0} = \frac{1}{1} = 1$. Every simple root of a function $f(t)$ is a super-attracting fixed-point of $N_f(t)$. See [3, p.52] □

Bibliography

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