Chapter 1

Discrete Random Variable

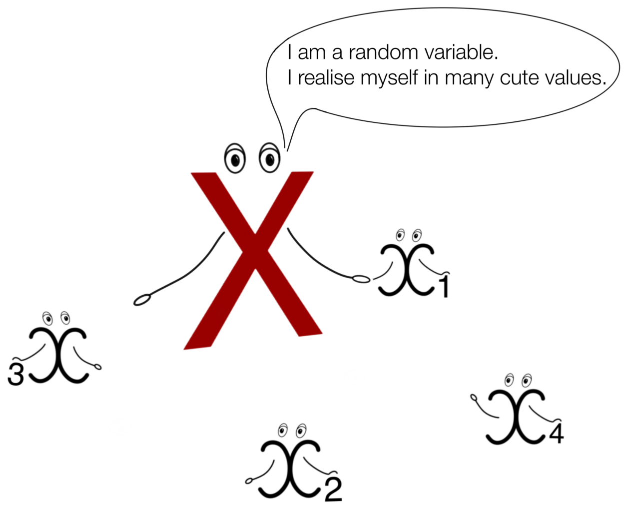
One man's constant is another man's variable

Alan Perlis

Random variable

Random variable is a quantitative characteristic of outcome of a random experiment.

Random variable is conventionally denoted by a capital letter. Contrary, particular values a variable takes are written as small case letters. Therefore, it is common to see expression of the form $P(X = x)$, which refers to the probability that a random variable X takes on a particular value x. Let X denote the age of an ICEF student. $x = 17$means that a particular student is 17 years old. The fact that 25% of students are 17 years old, can be written as $P(X = 17) = 0.25$.



The set of all possible values of a random variable is called its population. Then, population of X is the set of all possible values age of student can be equal to.

Well, we say that X is a random variable. Is it possible that age is random? For example, Masha is 17 years old and value $x = 17$ is not random, but is explained by the date she was born. That's because X denotes the age of *a randomly chosen* ICEF student, and not a particular person. It addresses to all possible outcomes of the random experiment involving choosing an ICEF student randomly and recording his/her age. Before a student is chosen value of the age is random.

Analogically, when you are trying to forecast the tomorrow's weather (say, atmospheric temperature in the area near university at 9 am) you are working with a random variable. One day later, though, this value will not be random any more, it will be a constant equal to the "yesterdays'" atmospheric temperature.

So, when talking about random variable we address to the outcome of an event that did not happen yet or characteristic of an object or individual which is not chosen yet.

Discrete and Continuous Random Variables

Discrete random variable is a random variable with *countable number* of outcomes. Note that "countable" is not the same as "finite". A discrete random variable can have infinite overall number of outcomes (values), but within each closed interval number of possible values is finite. For example if Masha tosses a coin until it comes up tails the number of trials X is potentially infinite. Trials are independent, therefore at each next trial coin can come up heads with a chance of $1/2$. Theoretically the experiment can last forever. Minimal value of X is 1, maximal is infinity, then, the number of possible outcomes is infinite. However, for any closed interval, number of possible values of X inside it is finite, e.g. there are 8 possible values of X between 3 and 10. In this example X is a discrete random variable.

Random variable is **continuous** if it can take any numerical value within an interval. Thus, it has *infinite* number of possible values even on finite interval. For example height of an ICEF boy is continuous random variable. Note, that we usually express height as an integer number of centimeters, which makes it seem to be a discrete number. This is because we operate with approximate height, which rounds off the exact value of the variable. In reality there is infinite number of possible values of height of an ICEF boy even between 182 and 185 cm. We will come back to the discussion of continuous variables in Chapter 4.

Probability Distribution

Probability distribution of a discrete random variable is the way the total probability of 1 is distributed across all values the variable takes. Put simply, it is the set of all possible values x with their corresponding probabilities $P(X = x)$. It can be provided in different forms: as a graph, a formula or a table. Consider an example. Masha wants to get a driving license. Let X be the number of attempts she'll need to pass the driving exam. The table below provides probability distribution of X. The first row represents population of X (all possible values), the second row provides probabilities for these values.

x	1	2	3	4	5	6	7	8
$P(X = x)$	0.05	0.1	0.3	0.25	0.15	0.07	0.05	0.03

Check that the *sum of all probabilities is 1.*

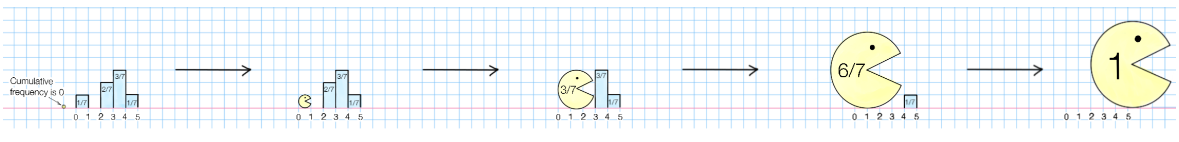
Cumulative probability distribution

Cumulative probability distribution provides the probability $P(X \leqslant x)$ that random variable X does not exceed a particular value x (for each possible x). Obviously, $P(X \leqslant x)$ can be calculated as the sum of probability of x and probability of all values below x.

The lowest row of a table below provides cumulative probability of X for "driving license" example.

x	1	2	3	4	5	6	7	8
P(X=x)	0.05	0.1	0.3	0.25	0.15	0.07	0.05	0.03
P($X \leqslant x$)	0.05	0.15	0.45	0.7	0.85	0.92	0.97	1

As you can check, for each value x_k it is the sum $\sum_{i=1}^{k} P(X = x_i)$. If you are interested in how cumulative distribution looks like, you will find its graph on page ...of Chapter 6.



Expectation of a Discrete Random Variable

Expectation or expected value of a random variable measures the center of its values and answers the question "If having no prior knowledge about X, what value of it should one expect to observe?".

Expectation is the probability-weighted average of all values of random variable. If x_1, x_2, ... , x_N are possible values of the random variable X with corresponding probabilities $p_1, p_2, \ldots, p_N$, then, $E(X) = x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_N \cdot p_N$. This can be written as:

$$E(X) = \sum_{i=1}^{k} x_i \cdot P(X = x_i)$$

You can check that expectation of X in the "getting driving license" example is $E(X) = \sum_{i=1}^{8} x_i \cdot P(X = x_i) = 1 \cdot 0.05 + 2 \cdot 0.1 + \ldots + 8 \cdot 0.03 = 3.91$.

- $E(X)$ is the same as the **population mean** denoted by μ, pronounced as "mu": $E(X) = \mu_X$
- $E(X)$ is a *constant*. It can be equal to any value, even not belonging to the sample space of possible values of X. For example, if Masha tosses a die the expectation of a number of dots X which a die will show up is $E(X) = \sum_{i=1}^{6} x_i \cdot P(X = x_i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$. It means that if the experiment is repeated many times the average result will be close to 3.5, although you may never see 3.5 dots facing up.
- Note that the concept of probability-weighted average can be extended to functions of a random variable. For example, $E(X^2) = x_1^2 \cdot p_1 + x_2^2 \cdot p_2 + \ldots + x_N^2 \cdot p_N$

Properties of Expectation

1. Expectation of a constant is equal to that constant

$$E(c) = c$$

This can be easily proved if you regard a constant c as being variable which takes value c with probability of 1. $E(c) = c \cdot 1 = c$

2. If you multiply a random variable by a constant you multiply its expectation by the same constant. In other words constant can be taken out of the expectation operator.

$$E(cX) = c \cdot E(X)$$

This follows from the rule that a constant can be taken out of a summation operator.

3. Expectation of the sum of variables is equal to the sum of their expectations.

$$E(X + Y) = E(X) + E(Y) \text{ and } E(X - Y) = E(X) - E(Y))$$

For simplicity let's assume that both X *and* Y *take only 2 values* x_1, x_2 *and* y_1, y_2, *correspondingly (in general case proof is the same). To shorten the notation let's denote* $P(X = x_i)$ *by* $P(x_i)$ *and* $P(Y = y_i)$ *by* $P(y_i)$. $(X + Y)$ *is a new random variable which can take four possible values:* $x_1 + y_1$, $x_1 + y_2$, $x_2 + y_1$, $x_2 + y_2$ *with probabilities* $P(x_1 \bigcap y_1)$, $P(x_1 \cap y_2)$, $P(x_2 \cap y_1)$,

$P(x_2 \cap y_2)$, correspondingly. Thus, we can find its expectation as:

$$\begin{aligned} E(X+Y) = (x_1+y_1)\cdot P(x_1\cap y_1) + (x_1+y_2)\cdot P(x_1\cap y_2) + \\ + (x_2+y_1)\cdot P(x_2\cap y_1) + (x_2+y_2)\cdot P(x_2\cap y_2) = \\ = x_1\cdot(P(x_1\cap y_1)+P(x_1\cap y_2)) + x_2\cdot(P(x_2\cap y_1)+P(x_2\cap y_2)) + \\ + y_1\cdot(P(x_1\cap y_1)+P(x_2\cap y_1)) + y_2\cdot P(x_1\cap y_2)+P(x_2\cap y_2)) = \\ = x_1\cdot P(x_1) + x2\cdot P(x_2) + y_1\cdot P(y_1) + y_2\cdot P(y_2) = E(X)+E(Y) \end{aligned}$$

Variance of a discrete random variable

Variance of a random variable X measures its dispersion or variability. It answers the question "How far from μ does X typically fall?" and gives answer in the form of square of the typical deviation.

Variance is the expectation of the squared deviation of X from its population mean μ

$$Var(X) = E(X-\mu)^2 = \sum_{i=1}^{N}(x_i-\mu)^2\cdot p_i$$

Let's calculate variance for the number of Masha's attempts to get the driving license.

$$\begin{aligned} Var(X) = E(X-\mu)^2 = \sum_{i=1}^{8}(x_i-3.91)^2\cdot p_i = \\ = (1-3.91)^2\cdot 0.05 + (2-3.91)^2\cdot 0.1 + \ldots + (8-3.91)^2\cdot 0.03 = 2.5019 \end{aligned}$$

Variance can also be calculated by the formula:

$$Var(X) = E(X^2) - E^2(X) = E(X^2) - \mu^2$$

Proof. $Var(X) = E(X-\mu)^2 = E(X^2-2\mu X+\mu^2) = E(X^2) - E(2\mu X) + E(\mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - 2\mu 2 + \mu^2 = E(X^2) - \mu^2$ Check that: $E(X^2) = \sum_{i=1}^{8} x_i\cdot p_i = 1^2\cdot 0.05 + 2^2\cdot 0.1 + \ldots + 8^2\cdot 0.03 = 17.79$.

Thus, $Var(X) = E(X^2) - \mu^2 = 17.79 - (3.91)^2 = 2.5019$, which equals to value of $Var(X)$ derived above.

Since variance gives values in squared units of measurement, it is inconvenient for direct interpretation, e.g. it is difficult to make sense of 2.5 squared exam attempts. That's why it is more common to measure dispersion by **standard deviation** which is the square root of variance. In economics it is usually used as a measure of risk.

$$\sigma = \sqrt{Var(X)}$$

$Var(X)$ is the same as population variance σ^2 (σ is pronounced as "sigma").
Both $Var(X)$ and σ are non-negative *constants.*
Be careful when writing the "sigma" sign. It is written as σ and not as δ! The latter is another Greek letter called "delta". The rule of thumb is that sigma looks like "6" and not like a Russian small case "Б" letter.

You might be interested why variance uses squared deviations of X from μ. There are 2 reasons for that. Let's illustrate them on the "getting driving license" example.

x	1	2	3	4	5	6	7	8
$P(X=x)$	0.05	0.1	0.3	0.25	0.15	0.07	0.05	0.03

$$E(X) = \mu = 3.91$$

Imagine that you are trying to measure variability without squaring deviations, simply as $\sum_{i=1}^{8}(x_i - 3.91) \cdot p_i$

First, expectation of deviation $(X-\mu)$ would simply produce 0. Values above 3.91 will produce positive deviations, and values below it will produce negative deviations. The resulting sum will reduce to 0. The second argument is that squaring discriminates between small and big deviations. Deviation of value 4 is $(4 - 3.91) = 0.09$. It contributes only $0.09^2 = 0.0081$ to the value of variance. Deviation of value 8 is 4.09, which contributes $4.09^2 \approx 16.73$ to the variance. Thus, $Var(X)$ is very sensitive to large deviations, while allowing small ones to make only a tiny influence. This has a very important practical implication. In practice we don't care about small mistakes, which do not represent big risk for accuracy of overall economic evaluation. However, large deviations may completely change the result and should be taken into account to avoid wrong economic decisions.

Properties of Variance

1. Variance of constant is zero: $Var(c) = 0$
   $Var(c) = E(c^2) - E^2(c) = c^2 - c^2 = 0$. *This result is quite intuitive. Variability of the value which does not change is null.* The reverse is true: if $Var(X) = 0$ then, X takes only one value.

2. If you multiply a random variable by a constant you multiply its variance by the square of this constant: $Var(cX) = c^2 \cdot Var(X)$
   $Var(cX) = E(cX - E(cX))^2 = E(cX - cE(X))^2 = E(cX - c\mu)^2 = E(c^2(X - \mu)^2) = c^2E(X-\mu)^2 = c^2Var(X)$

3. Variance of a sum of random variables X, Y can be calculated as: $Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X,Y)$ *A bit of magic here. For proof and explanation see Chapter 3.*

Why do we need Variance and Expectation?

Let's consider an example. Masha is choosing between the three assets with equal price. The probable returns are estimated as follows:

ASSET 1 — A profit of \$10,000 with probability 0.15 and a loss of \$1,000 with probability 0.85

ASSET 2 — A profit of \$1,000 with probability 0.50, a profit of \$500 with probability 0.30 and a loss of \$500 with probability 0.20 ASSET 3 — A certain profit of \$400

Which asset has the highest expected profit? Would you necessarily advise Masha to choose this asset?

Solution. We have three discrete random variables X_1, X_2, X_3 which are the profits produced by the 1st, 2nd and the 3rd assets, correspondingly. We are given probability distributions for these variables: we know probability for each possible value. To calculate the expected value we use the formula $E(X) = \sum_{i=1}^{N} x_i \cdot p_i$. Below are mean profits for each strategy:

$E(X_1) = 10000 \cdot 0.15 + (-1000) \cdot 0.85 = 650(\$)$

$E(X_2) = 1000 \cdot 0.5 + 500 \cdot 0.3 + (-500) \cdot 0.2 = 550(\$)$

$E(X_3) = 400 \cdot 1 = 400(\$)$

As you can see, the 1st asset has the highest expected profit.

Does it mean that Masha should necessarily choose the first asset? Note that despite of the high average profit this asset brings the loss of 1000\$ in 85% of times! And only in 15% of cases it brings positive profit. That means the asset is very risky contrary to the 3rd one, which does not involve any risk.

Thus, in order to rationally choose between the three options we also need to evaluate and compare another property of assets — their riskiness. Let's calculate standard deviations of the assets:

$$\sigma_1 = \sqrt{Var(X_1)} = \sqrt{E(X_1 - E(X_1))^2} =$$
$$= \sqrt{(10000 - 650)^2 \cdot 0.15 + (-1000 - 650)^2 \cdot 0.85} = 3928(\$)$$

$\sigma_2 = \sqrt{(10000 - 550)^2 \cdot 0.5 + (500 - 550)^2 \cdot 0.3 + (-550 - 550)^2 \cdot 0.2} = 567.9\ (\$)$

$\sigma_3 = \sqrt{(400 - 400)^2 \cdot 1} = 0\ (\$)$

$\sigma_1 > \sigma_2 > \sigma_3$

As you can see, the 1st asset is the most risky, while the 3rd does not involve any risk. Thus, risk increases with the growth of mean profit, which is absolutely typical for real-life assets.

Masha's final choice between the assets will depend on her preferences upon risk (risk-averse, risk-neutral or risk-lover) and the time horizon to use the asset. If she has opportunity to "play the game" many times she should prefer the 1st asset as it will bring the highest profit in the long run. However, if there is only one round to get the return then it's reasonable for Masha to choose the 3rd asset, at least not to lose her money.

Binomial random variable

Imagine a simple experiment. It has only two possible outcomes, denoted for convenience as success and failure. Probability of success equals p. Then, probability of failure equals $(1-p)$. The result of such experiment is called a **Bernoulli random variable**. Let's denote it by A. It takes value 1 in case of success and 0 in case of failure:

A	1	0
$P(A)$	p	$1-p$

As you can see, this distribution is characterized by only one parameter, which is the probability of success p. The conventional way to describe this distribution is $A \sim \text{Bernoulli}(p)$.

Bernoulli choice

Now imagine that this experiment is repeated n times, and their results are independent. The overall number of successes in n trials is the sum of independent Bernoulli variables $X = \sum_{i=1}^{n} A_i$. It can vary *from 0 to n* and is called a **Binomial random variable.**

Binomial random variable X is the number of successes in **n** independent trials with equal probability of success **p** in each trial. It is denoted by **X∼Binom(n,p)**

Remember that Binomial random variable is characterized by three basic properties:

1. There are only 2 outcomes in each trial.
2. Probability of success is the same in each trial.
3. An experiment consists of n *independent* trials (a fixed number).

Consider an example. Masha is late for the seminar taught by a young handsome teacher Alexander. He asks her three multiple choice questions with 5 possible answers in each. Alexander will allow her to join the class if Masha answers at least 2 questions correctly. She knows nothing on the topic concerned and relies on pure guessing. Assuming that her answers on the three questions are independent from each other, the number of right answers given is a binomial random variable with $n = 3$, $p = \frac{1}{5}$. Let's denote this number by X, $X \sim \text{Binom}(3, \frac{1}{5})$

Now let's calculate her probability of attending the seminar, that is $P(X \geqslant 2) = P(X = 2) + P(X = 3)$.

Probability $P(X = 2)$ of giving two right answers out of three means that Masha guesses two times (2 "successes") and gives one wrong answer (1 "failure"). Probability of 2 successes and 1 failure in a row is $(\frac{1}{5})^2 \cdot (\frac{4}{5})^(3-2)$. However, this particular probability corresponds to the "1st correct – 2nd correct – 3rd wrong" event, in this particular order. Since we are searching for the probability of 2 successes and 1 failure in any order, not bothering which 2 questions should be answered correctly, the probability should be multiplied by the number of possible combinations of 2 cases out of 3, that is by C_2^3.
Thus, $P(X = 2) = \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{(3-2)} \cdot C_2^3 = 0.096$. Analogically, $P(X = 3) = \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^{(3-3)} \cdot C_3^3 = 0.008$. Thus, probability that Alexander will allow Masha to attend his seminar is not too high:

$$P(X \geqslant 2) = 0.096 + 0.008 = 0.104$$

In general case probability that a binomial random variable equals k is:

$$P(X = k) = p^k (1 - p)^{n-k} C_n^k$$

Remember that number of successes X is a non-negative whole number between 0 (no successes happened) and n (all trials were successful)

Mean and Variance of a binomial random variable

1. Expectation of a binomial random variable can be calculated as:

$$E(X) = np$$

As we've shown $X = \sum_{i=1}^{n} A_i$, where $A_i \sim$ Bernoulli(p), and A_i, A_j are independent for all i,j.
Then, $E(X) = E(\sum_{i=1}^{n} A_i) = E(A_1 + A_2 + \ldots + A_n) = E(A_1) + E(A_2) + \ldots + E(A_n) = p + p \ldots + p = np$ This result is quite intuitive. Imagine that you tossing a coin 10 times. On average, how many heads do you expect to occur? The immediate answer is 5. You've calculated this as $10 \cdot \frac{1}{2}$, where $\frac{1}{2}$ is the probability of success in each of the 10 independent trials. This is exactly what is suggested by the $n \cdot$ formula for expectation of binomial random variable.

2. Variance of a binomial random variable can be calculated as:

$$Var(X) = np(1-p)$$

$Var(A_i) = E(A_i^2) - E^2(A_i) = \sum_{i=1}^{N} A_i^2 \cdot_i - p^2 = 0^2 \cdot (1-p) + 1^2 \cdot p - p^2 = 0 + p - p^2 = p(1-p)$ Then, $Var(X) = Var(\sum_{i=1}^{n} A_i) = Var(A_1 + A_2 + \ldots + A_n)$. A_i and A_j are independent. As you'll see in Chapter 3, for independent variables variance of their sum is the sum of their variances: $Var(\sum_{i=1}^{n} A_i) = Var(A_1) + Var(A_2) + \ldots + Var(A_n) = p(1-p) + p(1-p) + \ldots + p(1-p) = n \cdot p \cdot (1-p)$.

In practice, we may consider a random variable to be binomial when, in fact, the independence of trials condition is not quite satisfied. This occurs when the probability of occurrence of a given trial is affected only slightly by prior trials. For example, suppose that the probability of a defect in a manufacturing process is 0.005=0.5%. It means there is only 1 defect in 200 items on average. Suppose we check a sample of 60 items out of 10,000 for defects. Initially, we would expect 50 out of 10,000 to be defective (0.5% of it). When we've checked the first item, the proportion of defects changes slightly for the remaining 9,999 items in the sample. For example, if the first one chosen is not defective, probability of the next one to be defective has changed to from 0.5% to 50/9999≈0.5005% (there are 50 defective among 9,999 remaining items). It's a small change but it means that, strictly speaking, the trials are not independent. If we continue and check 59 items and they are not defective, probability of the 60th to be defective is 50/9941≈0.50297%. So, the initial probability of "success" has changed by 0.00297%, which is quite negligible and we can work with it as if probability is the same to approximate the true answer. However, if we were to take a sample of size 9,000, the corresponding probability of success on 9000th trial would be 50/1001≈4.995%, which is absolutely different from the initial 0.5%.

So, this approximations works with small samples (60 out of 10,000), but is inappropriate with large samples (9,000 out of 10,000). How to discriminate between these cases? A common rule of thumb is that in such situations *we may consider a distribution to be binomial if the population size is at least 10 times the sample size.* So, binomial distribution can be applied in the first $(10,000 > 60 \cdot 10)$ but not in the second case $(10,000 < 9,000 \cdot 10)$.

Example of this kind of problem is given in the "Sample AP problems" section

(AP 2010 №4).

Binom-IA-l distribution. While pronouncing the name of the Binomial distribution remind yourself a romantic and melancholic character of the story about "Winnie the Pooh" — Eeyore. In Russian his name is pronounced as "ИА". Recall and ask him to help you correctly address the probability distribution of X. Note, that it is called *binom-IA-l* and not binom-ina-l distribution as students sometimes mistakenly pronounce.

Geometric distribution

Consider a distribution, which is very similar to binomial. Again, each trial has only two possible outcomes, one of which is coded as "success", there is equal probability of success p in each trial. However, experiment does not have fixed number of trials. Contrary, trials continue until the first success happens. In this case number of trials is said to follow geometric distribution.

For example, an ICEF student Lyonya Lenivetz may attempt an AP exam in Statistics again and again until he copes to get a pass score. If we assume that the probability to pass is the same on each attempt and independent of all previous experience, then, the overall number of trials X is geometrically distributed. Such variable takes values from 1 to infinity.

Let's assume that $p = 0.4$. Then, probability to pass the exam during the first year of studies is simply the probability of success p: $P(X = 1) = 0.4$. The chance to succeed in the second attempt is the chance to fail and then to pass, thus, $P(X = 2) = 0.6 \cdot 0.4 = 0.24$. Success on the third attempt means to fail two times and then to succeed: $P(X = 3) = 0.6^2 \cdot 0.4 \approx 0.14$. As you can see, each next probability is a product of the previous one and probability of failure. The general formula for probability of geometric random variable is: $P(X = k) = (1 - p)^{k-1} \cdot p$ Below is the approximate probability distribution of X for Lyonya. As you can see the last column aggregates probability of all values above 10, as X is potentially infinite:

x	1	2	3	4	5	6	7	8	9	10+
$P(X = x)$	0.40	0.24	0.14	0.09	0.05	0.03	0.02	0.01	0.01	0.01

A more realistic example of geometric random variable is the number of trials until a fair die faces up "6".

Full score strategy

Step 1. “Let it be” song — introduce your variable (if it’s not given in conditions).
Step 2. Indicate that the distribution is binomial, indicate its parameters n and p.
Step 3. Indicate what you are asked to find.
Step 4. Write down the relevant formula.
Step 5. Do the calculation, provide the answer.

You’ll see examples of solution following these steps in problems 3, 4, 5 of the “Sample AP problems”section of this chapter (AP 2011 FormB №3, 2010 №4, 2010 FormB №3).

You MUST BE ABLE TO REPRODUCE even being half-awake:

- $E(X) = \sum_{i=1}^{N} x_i \cdot p_i$
  - $E(c) = c$
  - $E(cX) = cE(X)$
  - $E(XY) = E(X) \pm E(Y)$
- $Var(X) = E(X^2) - [E(X)]^2$
  - $Var(c) = 0$
  - $Var(cX) = c^2 Var(X)$
- For a Binomial random variable $X \sim \text{Binom}(n, p)$
  - $P(X = k) = p^k (1-p)^{n-k} C_n^k$
  - $E(X) = np$
  - $Var(X) = np(1-p)$

Calculator BOX

To find $E(X)$, σ, $E(X^2)$:

1. Put values of X into `List 1`, the corresponding probabilities into `List 2`
2. `CALC` $\rightarrow$ `SET`
   `1 Var XList List1`
   `1 Var Freq List2`
   $\rightarrow$ `EXIT`
3. `1VAR`

Now you have results: $E(X)$

Sample AP problems with solutions

Problem 1. AP 2016 №4

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

A company manufactures model rockets that require igniters to launch. Once an igniter is used to launch a rocket, the igniter cannot be reused. Sometimes an igniter fails to operate correctly and the rocket does not launch. The company estimates that the overall failure rate, defined as the percent of all igniters that fail to operate correctly, is 15 percent.

A company engineer develops a new igniter, called the super igniter, with the intent of lowering the failure rate. To test the performance of super igniters, the engineer uses the following process.

- Step 1: One super igniter is selected at random and used in a rocket.
- Step 2: If the rocket launches, another super igniter is selected at random and used in a rocket.

Step 2 is repeated until the process stops. The process stops when a super igniter fails to operate correctly or 32 super igniters have successfully launched rockets, whichever comes first. Assume that super igniter failures are independent.

(a) If the failure rate of the super igniter is 15 percent, what is the probability that the first 30 super igniters selected using the testing process successfully launch rockets?

(b) Given that the first 30 super igniters successfully launch rockets, what is the probability that the first failure occurs on the thirty-first or the thirty-second super igniter tested if the failure rate of the super igniters is 15 percent?

(c) Given that the first 30 super igniters successfully launch rockets, is it reasonable to believe that the failure rate of the super igniters is less than 15 percent? Explain.

Solution

(a) Probability of the event that one super igniter will succeed in launching a rocket is 1-0.15=0.85. For 30 super igniters to succeed this event should independently repeat 30 times. Thus, the sought for probability is $0.85^{30} \approx$ **0.0076**.

(b) Since failures are independent, the information that first 30 super igniters launched rockets successfully does not change the probability of next launches to be successful. Thus, probability of super igniters to fail on the thirty-first or the thirty-second trial is the same as probability to fail on the first or on the second trial. Let's denote the number of trials (starting from 31) on which a super igniter will fail to launch a rocket by X. It is geometrically distributed with $p = 0.15$. This event consists of the 2 elementary outcomes:

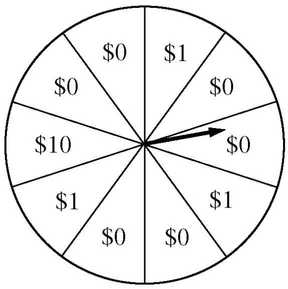
$X = 1$ which means to fail on the first trial and $X = 2$, meaning to fail on the second trial (to succeed first and then to fail). $P(X = 1) + P(X = 2) = 0.15 + 0.85 \cdot 0.15 = \mathbf{0.2775}$.

(c) The chance to observe 30 successful launches given the probability of failure of 0.15 is 0.0076 (found in (a)), which is smaller than 1 percent. Such a small chance provides reasons to doubt that the true probability is so high and advocates that **it is likely to be smaller than 15%**.

Problem 2. AP 2012 №2

You'll have about 10 minutes to solve this problem. It will bring you 11% of score for Free Response section.

A charity fundraiser has a Spin the Pointer game that uses a spinner like the one illustrated in the figure below.



A donation of \$2 is required to play the game. For each \$2 donation, a player spins the pointer once and receives the amount of money indicated in the sector where the pointer lands on the wheel. The spinner has an equal probability of landing in each of the 10 sectors.

(a) Let X represent the net contribution to the charity when one person plays the game once. Complete the table for the probability distribution of X.

x	\$2	\$1	-\$8
$P(X)$			

(b) What is the expected value of the net contribution to the charity for one play of the game?

(c) The charity would like to receive a net contribution of \$500 from this game. What is the fewest number of times the game must be played for the expected value of the net contribution to be at least \$500?

(d) See Chapter 8

Solution

a) Since each game costs \$2 of donation, contribution to the charity equals \$2 when the spinner stops at \$0. That happens with probability $\frac{6}{10}$=0.6. When spinner stops at \$1, net contribution is \$2-\$1=\$1. Probability of that is $\frac{3}{10}$=0.3. Analogically, net contribution of \$2-\$10=-\$8 happens with probability of $\frac{1}{10}$.

x	\$2	\$1	-\$8
$P(X)$	**0.6**	**0.3**	**0.1**

b) $E(X) = \sum_{i=1}^{3} x_i \cdot p_i = 2 \cdot 0.6 + 1 \cdot 0.3 + (-8) \cdot 0.1 = \mathbf{0.7(\$)}$

c) Let Y be the net contribution after the n games. $Y = \sum_{i=1}^{n} X_i$. We need to find n such that $E(Y) \geqslant 500$.

$$E(Y) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = n \cdot E(X) = 0.7n \geqslant 500$$

$$n \geqslant 714.29$$

Thus, the game should be played **at least 715 times**.

Problem 3. AP 2011 Form B №3

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

An airline claims that there is a 0.10 probability that a coach-class ticket holder who flies frequently will be upgraded to first class on any flight. This outcome is independent from flight to flight. Sam is a frequent flier who always purchases coach-class tickets.

(a) What is the probability that Sam's first upgrade will occur after the third flight?

(b) What is the probability that Sam will be upgraded exactly 2 times in his next 20 flights?

(c) Sam will take 104 flights next year. Would you be surprised if Sam receives more than 20 upgrades to first class during the year? Justify your answer.

Solution

(a) Let X denote the number of flight on which the upgrade will occur. X has geometric distribution with p=0.1.

$$P(X > 4) = 1 - P(X4) =$$
$$= 1 - P(X = 1) - P(X = 2) - P(X = 3) - P(X = 4) =$$
$$= 1 - 0.1 - 0.9 \cdot 0.1 - 0.9^2 \cdot 0.1 - 0.9^3 \cdot 0.1 = \mathbf{0.729}$$

(b) (Step 1). Let Y denote the number of upgrades in next 20 flights. (Step 2). $Y \sim$ Binom$(20, 0.1)$ (Steps 3-5). $P(X = 2) = 0.1^2 \cdot 0.9^{(20-2)} \cdot C_{20}^2 = \mathbf{0.2852}$

(c) (Step 1). Now let Y denote number of upgrades in next year.
(Step 2). Then, $Y \sim B(104, 0.1)$
(Step 3). $P(Y > 20) = 1 - P(Y \leqslant 20) = 1 - (P(Y = 0) + P(Y = 1) + \ldots + P(Y = 20))$
(Step 4). Using formula $p^k(1-p)^{n-k}C_n^k$ we get:
(Step 5). $P(Y \leqslant 20) = 0.9986$
$P(Y > 20) = 1 - 0.9986 = 0.0014$
Since the probability of that event is very low, **I would be surprised** if there'll be more than 20 upgrades.

Problem 4. AP 2010 №4

You'll have about 9 minutes to solve this problem. It will bring you 10% of score for Free Response section.

An automobile company wants to learn about customer satisfaction among the owners of five specific car models. Large sales volumes have been recorded for three of the models, but the other two models were recently introduced so their sales volumes are smaller. The number of new cars sold in the last six months for each of the models is shown in the table below.

Car model	A	B	C	D	E	Total
Number of new cars sold in the last six months	112.338	96.174	83.241	3.278	2.323	297.354

The company can obtain a list of all individuals who purchased new cars in the last six months for each of the five models shown in the table. The company wants to sample 2,000 of these owners.

(a) For simple random samples of 2,000 new car owners, what is the expected number of owners of model E and the standard deviation of the number of owners of model E?

(b) When selecting a simple random sample of 2,000 new car owners, how likely is it that fewer than 12 owners of model E would be included in the sample? Justify your answer.

(c) See Chapter 7

Solution

(a) (Step 1) Let X be the number of cars of model E to enter the sample of 2,000. Probability to randomly choose an owner of model E is $\frac{2323}{297354} \sim 0.0078$. Since a sample of 2,000 is more than 10 times smaller than the population size of

297,354 probability of choosing model E owner is almost equal in each of the 2000 “trails”. (Step 2) Thus, approximately $X \sim \mathrm{B}(2000, 0.0078)$. Therefore:
(Steps 3-5) $E(X) = 2000 \cdot 0,0078 = \mathbf{15.6}$
$\sigma_x = \sqrt{V(X)} = \sqrt{2000 \cdot 0.0078 \cdot (1 - 0.0078)} \approx \mathbf{3.934}$

(b) (Step 3) $P(X < 12) = P(X \leqslant 11) = P(X = 0) + P(X = 1) + \ldots + P(X = 11)$
(Step 4) Using formula $p^k(1-p)^{7-k}C_7^k$ we get:
(Step 5) $P(X < 12) \leqslant \mathbf{0.147}$

Problem 5. AP 2010 Form B №3

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

A test consisting of 25 multiple-choice questions with 5 answer choices for each question is administered. For each question, there is only 1 correct answer.

(a) Let X be the number of correct answers if a student guesses randomly from the 5 choices for each of the 25 questions. What is the probability distribution of X? This test, like many multiple-choice tests, is scored using a penalty for guessing. The test score is determined by awarding 1 point for each question answered correctly, deducting 0.25 point for each question answered incorrectly, and ignoring any question that is omitted. That is, the test score is calculated using the following formula.

$$\text{Score} = (1 \times \text{number of correct answers}) - (0.25 \times \text{number of incorrect answers}) + \\ + (0 \times \text{number of omits})$$

For example, the score for a student who answers 17 questions correctly, answers 3 questions incorrectly, and omits 5 questions is Score $= (1 \times 17) - (0.25 \times 3) + (0 \times 5) = 16.25$.

(b) Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. Show that the expected value of the student's score is 18 when using the scoring formula above.

(c) A score of at least 20 is needed to pass the test. Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. What is the probability that the student will pass the test?

Solution.

(a) X has binomial distribution because it represents the number of successes in

n trials with equal probability of success in each trial. $\boldsymbol{X \sim \mathbf{Binom}(25, \frac{1}{5})}$.

(b) (Step 1) Let Y be the number of questions out of 7 the student will randomly answer correctly. Total number of correct answers will be $18 + Y$, and $7 - Y$ will be answered incorrectly. Score $= 1 \cdot (18 + Y) - 0.25 \cdot (7 - Y) = 18 + Y - 1.75 + 0.25Y = 1.25Y + 16.25$,
(Step 2) $Y \sim \text{Binom}(7, \frac{1}{5})$. As for binomial variable: $E(Y) = n \cdot p = \frac{7}{5} = 1.4$
$E(\text{Score}) = E(1.25Y + 16.25) = 1.25E(Y) + 16.25 = 1.25 \cdot 1.4 + 16.25 = 18$.

(c) (Step 3) $P(\text{"pass the test"}) = P(\text{Score} \geqslant 20) = P(1.25Y + 16.25 \geqslant 20) = P(Y \geqslant 3) = P(Y = 3) + P(Y = 4) + P(Y = 5) + P(Y = 6) + P(Y = 7)$.
(Step 4) Using formula $p^k(1-p)^{7-k}C_7^k$ we get:
(Step 5) $P(\text{Score} \geqslant 20) = \mathbf{0.148}$

Problem 6. AP 2005 Form B №2

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

For an upcoming concert, each customer may purchase up to 3 child tickets and 3 adult tickets. Let C be the number of child tickets purchased by a single customer. The probability distribution of the number of child tickets purchased by a single customer is given in the table below.

C	0	1	2	3
$P(C)$	0.4	0.3	0.2	0.1

(a) Compute the mean and the standard deviation of C.

(b) Suppose the mean and the standard deviation for the number of adult tickets purchased by a single customer are 2 and 1.2 respectively. Assume that the numbers of child tickets and adult tickets purchased are independent random variables. Compute the mean and the standard deviation of the total number of adult and child tickets purchased by a single customer.

(c) Suppose each child ticket costs \$15 and each adult ticket costs \$25. Compute the mean and the standard deviation of the total amount spent per purchase.

Solution

(a) $E(C) = \sum_{i=1}^{4} = 0 \cdot 0.4 + 1 \cdot 0.3 + 2 \cdot 0.2 + 3 \cdot 0.1 = \mathbf{1}$
$\sigma = \sqrt{Var(C)} = \sqrt{E(C^2) - E^2(C)}$
$E(C^2) = \sum_{i=1}^{4} c_i^2 \cdot p_i = 0^2 \cdot 0.4 + 1^2 \cdot 0.3 + 2^2 \cdot 0.2 + 3^2 \cdot 0.1 = 2$
$\sigma = \sqrt{2 - 1^2} = \mathbf{1}$

(b) Let A be the number adult tickets purchased by a single customer. We are given that $E(A) = 2$, $\sigma_A = 1.2$

Let T be the total number of adult and child tickets purchased, $T = C + A$
$E(T) = E(C + A) = E(C) + E(A) = 1 + 2 = \mathbf{3}$
Since C and A are independent[a]: $Var(T) = Var(C+A) = Var(C)+Var(A)$
$\sigma_X = \sqrt{Var(15C + 25A)} = \sqrt{15^2 Var(C) + 25^2 Var(A)} = \sqrt{225\sigma_C^2 + 625\sigma_A^2} = \sqrt{225 \cdot 1^2 + 625 \cdot 1.2^2} \approx \mathbf{33.541(\$)}$

[a]For explanation see Chapter 3

Problem 7. AP 2004 №4

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

Two antibiotics are available as treatments for a common ear infection in children. Antibiotic A is known to effectively cure the infection 60 percent of the time. Treatment with antibiotic A costs $50. Antibiotic B is known to effectively cure the infection 90 percent of the time. Treatment with antibiotic B costs $80.

The antibiotics work independently of one another. Both antibiotics can be safely administered to children. A health insurance company intends to recommend one of the following two plans of treatment for children with the ear infection. Plan I: Treat with antibiotic A first. If it is not effective, then treat with antibiotic B. Plan II: Treat with antibiotic B first. If it is not effective, then treat with antibiotic A. If a doctor treats a child with an ear infection using plan I, what is the probability that the child will be cured?

If a doctor treats a child with an ear infection using plan II, what is the probability that the child will be cured?

(a) Compute the expected cost per child when plan I is used for treatment.

(b) Compute the expected cost per child when plan II is used for treatment.

Based on the results in parts (a) and (b), which plan would you recommend?

Explain your recommendation.

Solution Let C_1 and C_2 denote the events that the child will be cured with Plan I and Plan II correspondingly. Let A = "antibiotic A is effective", B = "antibiotic B is effective" The event $C1$ can be viewed as a collection of two elementary outcomes: A and $A \cap B$. That is a child will be cured using plan I either if antibiotic A applied first is effective or when it is ineffective, antibiotic B applied next is effective. From the properties of probability we know that probability of an event can be calculated as the sum of probabilities of all its basic outcomes. Thus, $P(C_1) = P(A) + P(\bar{A} \cap B)$. Since A and B are independent, probability of intersection $\bar{A}$ and B is the product of $P(\bar{A})$ and $P(B)$ $P(C_1) = P(A) + P(\bar{A}) \cdot P(B) = 0.6 + 0.4 \cdot 0.9 = \mathbf{0.96}$ $P(C_2) = P(B) + P(\bar{B} \cap A) = P(B) + P(\bar{B}) \cdot P(A) = 0.9 + 0.1 \cdot 0.6 = \mathbf{0.96}$

Let X_1 and X_2 be costs per child of applying Plan I and Plan II correspondingly. Plan I. If A is effective (probability of 0.6) treatment will require $50. Otherwise both antibiotics will be applied and treatment will cost $50+$80=$130. Below is probability distribution of X_1:

X_1	50	130
$P(X_1)$	0.6	0.4

Plan II. If B is effective (probability 0.9) treatment will cost \$80, otherwise \$80+\$50=\$130. Probability distribution of X_2:

X_2	80	130
$P(X_2)$	0.9	0.1

Thus, $E(X_1) = 50 \cdot 0.6 + 130 \cdot 0.4 = \mathbf{\$82}$, $E(X_2) = 80 \cdot 0.9 + 130 \cdot 0.1 = \mathbf{\$85}$. Probability that the child will be cured is the same for both plans. At the same time expected costs are lower for Plan I. Therefore, from the financial point of view **Plan I should be preferred**.

Problem 8. AP 1999 №5

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

Die A has four 9's and two 0's on its faces. Die B has four 3's and two 11's on its faces. When either of these dice is rolled each face has an equal chance of landing on top. Two players are going to play a game. The first player selects a die and rolls it. The second player rolls the remaining die. The winner is the player whose die has the higher number on top.

(a) Suppose you are the first player and you want to win the game. Which die would you select? Justify your answer.

(b) Suppose the player using die A receives 45 tokens each time he or she wins the game. How many tokens must the player using die B receive each time he or she wins in order for this to be a fair game? (A fair game is one in which the player using die A and the player using die B both end up with the same number of tokens in the long run).

Solution

(a) Let A and B be the number of dots facing up on die A and die B, correspondingly. Below are probability distributions of A and B:

A	9	9
$P(A)$	2/6	4/6

B	3	11
$P(B)$	4/6	2/6

A and B are independent.

Let's calculate the probabilities that the first player wins when he chooses A and chooses B.

$$P(\text{“A wins”}) = P(A > B) = P(A = 9 \cap B = 3) =$$
$$= P(A = 9) \cdot P(B = 3) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

$$P(\text{“B wins”}) = P(A < B) = P(A = 0) + P(A = 9 \cap B = 11) =$$
$$= P(A = 0) + P(A = 9) \cdot P(B = 11) = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{5}{9}$$

Since there is more chance to win with the **die B** I would choose it.

(b) Let C be the number of games won by A if the game is played n times, then, number of times B wins the game is $(n-C)$. $C \sim \text{Binom}(n, \frac{4}{9})$. For the game to be fair expected revenues from playing with A and B should be equal. Revenue of playing with A is 45C. Let k be the number tokens given to the player with die B when he wins. His revenue is $k(n - C)$. For the game to be fair it should be that $E(45C) = E(k \cdot (n - C))$. $E(45C) = 45E(C) = 45 \cdot n \cdot \frac{4}{9} = 20n$, $E(k \cdot (n-C)) = E(kn) - kE(C) = kn - kn\frac{4}{9} = \frac{5kn}{9}$ $20n = \frac{5kn}{9}$ $k = \frac{20 \cdot 9}{5} = 36$ Thus, the player using die B should receive **36 tokens** each time he wins.

Practice AP problems

Problem 1. AP 2008 №3

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

A local arcade is hosting a tournament in which contestants play an arcade game with possible scores ranging from 0 to 20. The arcade has set up multiple game tables so that all contestants can play the game at the same time; thus contestant scores are independent. Each contestant's score will be recorded as he or she finishes, and the contestant with the highest score is the winner.

After practicing the game many times, Josephine, one of the contestants, has established the probability distribution of her scores, shown in the table below.

Josephine's Distribution				
Score	16	17	18	19
Probability	0.10	0.30	0.40	0.20

Crystal, another contestant, has also practiced many times. The probability distribution for her scores is shown in the table below.

Crystal's Distribution			
Score	17	18	19
Probability	0.45	0.40	0.15

(a) Calculate the expected score for each player.

(b) Suppose that Josephine scores 16 and Crystal scores 17. The difference (Josephine minus Crystal) of their scores is -1. List all combinations of possible scores for Josephine and Crystal that will produce a difference (Josephine minus Crystal) of -1, and calculate the probability for each combination.

(c) Find the probability that the difference (Josephine minus Crystal) in their scores is -1.

(d) The table below lists all the possible differences in the scores between Josephine and Crystal and some associated probabilities.

Distribution (Josephine minus Crystal)						
Difference	-3	-2	-1	0	1	2
Probability	0.0015			0.325	0.260	0.090

Complete the table and calculate the probability that Crystal's score will be higher than Josephine's score.

Problem 2. AP 2006 Form B №6

You'll have about 4 minutes to solve this problem. It will bring you 4% of score for Free Response section.

Sunshine Farms wants to know whether there is a difference in consumer preference for the new juice products — Citrus Fresh and Tropical Taste. In an initial blind taste test, 8 randomly selected consumers were given unmarked samples of the two juices. The product that each consumer tasted first was randomly decided by the flip of a coin. After tasting the two juices each consumer was asked to choose which juice he or she preferred, and the results were recorded.

(a), (b), (d), (e), (f) — See Chapter 11.

(d) Let X represent the number of consumers in the sample who prefer Citrus Fresh. Let p represent the population proportion of such consumers. Assuming there is no difference in consumer preference, find the probability for each possible value of X. Record the x-values and the corresponding probabilities in the table below.

x	$p(x)$

Problem 3. AP 2004 №3

You'll have about 10 minutes to solve this problem. It will bring you 11% of score for Free Response section.

At an archaeological site that was an ancient swamp the bones from 20 brontosaur skeletons have been unearthed. The bones do not show any sign of disease or malformation. It is thought that these animals wandered into a deep area of the swamp and became trapped into the swamp bottom. The 20 left femur bones (thigh bones) were located and 4 of these left femurs are to be randomly selected without replacement for DNA testing to determine gender.

(a) Let X be the number out of the 4 selected left femurs that are from males. Based on how these bones where sampled explain why the probability distribution of X is not binomial.

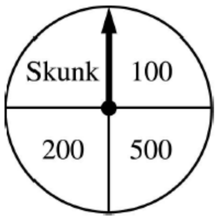
(b) Suppose that the group of 20 brontosaurs whose remains were found in the swamp had been made up of 10 males and 10 females. What is the probability that all 4 in the sample to be tested are male?

(c) The DNA testing revealed that all 4 femurs tested where from males. Based on this result and your answer from part (b), do you think that males and females where equally represented in the group of 20 brontosaurs stuck in the swamp? Explain.

(d) See Chapter 7.

Problem 5. AP 2003 Form B №5

You'll have about 9 minutes to solve this problem. It will bring you 10% of score for Free Response section.



Contestants on a game show spin a wheel like the one shown in the figure below. Each of the four outcomes on this wheel is equally likely and outcomes are independent from one spin to the next.

- The contestant spins the wheel.
- If the result is a skunk, no money is won and the contestant's turn is finished.
- If the result is a number, the corresponding amount in dollars is won. The contestant can then stop with these winnings or can choose the spin again, and his or her turn continues.
- If the contestant spins again and the result is a skunk, all of the money earned on that turn is lost and the turn ends.
- The contestant may continue adding to his or her winnings until he or she chooses to stop or until a spin results in a skunk.

Problem 6. AP 2002 Form B №2

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

Airlines routinely overbook flights because they expect a certain number of no-shows. An airline runs a 5 P.M. commuter flight from Washington D.C. to New York City on a plane that holds 38 passengers. Past experience has shown that if 41 tickets are sold for the flight than the probability distribution for the number who actually show up for the flight is as shown in the table below.

Number who actually show up	36	37	38	39	40	41
Probability	0.46	0.30	0.16	0.05	0.02	0.01

Assume that 41 tickets are sold for each flight.

(a) There are 38 passenger seats in the flight. What is the probability that all passengers who show up for this flight will get a seat?

(b) What is the expected number of no-shows for this flight?

(c) Given that not all passenger seats are filled on a flight what is the probability that only 36 passengers showed up for the flight?

Problem 7. AP 2001 №2

You'll have about 13 minutes to solve this problem. It will bring you 15% of score for Free Response section.

A department supervisor is considering purchasing one of two comparable photocopy machines, A or B. Machine A costs \$10,000 and machine B costs \$10,500. This department replaces photocopy machined every three years. The repair contract for machine A costs \$50 per month and covers an unlimited number of repairs. The repair contract for machine B costs \$200 per repair. Based on past performance, the distribution of the number of repairs needed over any one-year period for machine B is shown below.

X	1	2	3	4	5	6	7	8
$P(X = x)$	0.4	0.24	0.14	0.09	0.05	0.03	0.02	0.01

You are asked to give a recommendation based on overall cost as to which machine, A or B, along with its repair contract, should be purchased. What would your recommendation be? Give a statistical justification to support your recommendation.

You will find more questions on this topic in problems: AP 2009 №2 (b), AP 2007 Form B №2 (b) They are provided in the "CatDog" section of this book.

Answers to the practice problems

Problem 1.

(a) 17.7 for each player

(b)

Josephine	Crystal	probability
16	17	0.045
17	18	0.12
18	19	0.06

(c) 0.225

(d) $P(\text{difference} = -1) = 0.225$, $P(\text{difference} = -2) = 0.085$,
$P(\text{"Crystal's score will be higher than Josephine's score"}) = 0.325$

Problem 2.

X	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.00391	0.03125	0.10937	0.21875	0.27344	0.21875	0.10937	0.03125	0.00391

Problem 3. (a) the trials are not independent ; (b) 0.043; (c) No

Problem 4. 0.1139

Problem 5. (a) 0.4219; (b) 800

Problem 6. (a) 0.92; (b) 4.1; (c) 0.605

Problem 7. Choose B since it has lower expected cost ($ 11,010) compared to A ($ 11,800).