2

Formulation of Two- and Three-dimensional Boundary-value Problems

2.1 INTRODUCTORY REMARKS

It makes no sense to attempt to "solve" a boundary-value problem without a precise knowledge of what the problem is. The truth of this statement seems self-evident. Unfortunately, attempts are often made to solve vaguely defined problems, creating considerable confusion and, sometimes, totally erroneous results. In this chapter we present precise statements of multidimensional boundary-value problems in classical linear heat conduction and elastostatics. The presentation is similar in many respects to that for the one-dimensional model problem of Chapter 1. In particular, we discuss strong and weak forms, their equivalence, corresponding Galerkin formulations, the definitions of element arrays, and pertinent data processing concepts. In multi-dimensions, the data processing ideas necessarily become more involved. The reader is urged to study them carefully as they are necessary in order to understand the computer implementation of finite element techniques.

2.2 PRELIMINARIES

Let $n_{sd}(= 2 \text{ or } 3)$ denote the ***number of space dimensions*** of the problem under consideration. Let $\Omega \subset \mathbb{R}^{n_{sd}}$ be an open set[1] with piecewise smooth boundary Γ. A general point in $\mathbb{R}^{n_{sd}}$ is denoted by $\boldsymbol{x}$. We will identify the point $\boldsymbol{x}$ with its position vector emanating from the origin of $\mathbb{R}^{n_{sd}}$. The ***unit outward normal vector to*** Γ is denoted by $\boldsymbol{n}$.

[1] For our purposes, it is sufficient to think of an open set as one without its boundary.

We shall employ the following alternative representations for $\boldsymbol{x}$ and $\boldsymbol{n}$:

$$(n_{sd} = 2): \quad \boldsymbol{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \boldsymbol{n} = \{n_i\} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \tag{2.2.1}$$

$$(n_{sd} = 3): \quad \boldsymbol{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad \boldsymbol{n} = \{n_i\} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.2.2}$$

where x_i and n_i, $1 \leq i \leq n_{sd}$, are the Cartesian components of $\boldsymbol{x}$ and $\boldsymbol{n}$, respectively; see Figure 2.2.1. Unless otherwise specified we shall work in terms of Cartesian components of vectors and tensors.

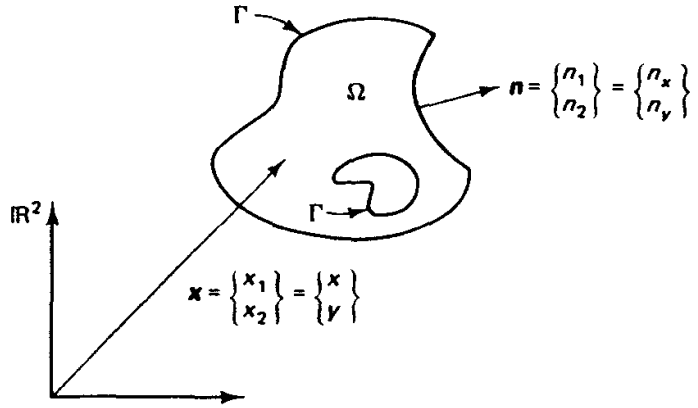


Figure 2.2.1

We assume that Γ admits the decomposition

$$\Gamma = \overline{\Gamma_g \cup \Gamma_h} \tag{2.2.3}$$

where

$$\Gamma_g \cap \Gamma_h = \varnothing \tag{2.2.4}$$

and Γ_g and Γ_h are open sets in Γ. The notations are defined as follows: $\cup$ is the ***set union*** symbol. Thus $\Gamma_g \cup \Gamma_h$ means the set of all points $\boldsymbol{x}$ contained in either Γ_g or Γ_h. Also, $\cap$ is the ***set intersection*** symbol. Thus $\Gamma_g \cap \Gamma_h$ means the set of all points contained in both Γ_g and Γ_h. The ***empty set*** is denoted by $\varnothing$. Thus (2.2.4) means that there is no point $\boldsymbol{x}$ contained in both Γ_g and Γ_h (i.e., Γ_g and Γ_h do not intersect or overlap). A bar above a set means ***set closure***, i.e., the union of the set with its boundary. Thus

$$\overline{\Omega} = \Omega \cup \Gamma \tag{2.2.5}$$

To understand the meaning of $\overline{\Gamma_g \cup \Gamma_h}$, we must define the boundaries of Γ_g and Γ_h in Γ. We shall do this with the aid of an example.

Example 1

Let $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x^2 + y^2 < 1$, i.e., the interior of the unit disc$\}$. The boundary of Ω is $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x^2 + y^2 = 1$, i.e., the unit circle$\}$. Let

$$\Gamma_g = \Gamma \cap \{\boldsymbol{x} \in \mathbb{R}^2 \mid y > 0\} \tag{2.2.6}$$

The "boundary of Γ_g" consists of the endpoints of the upper semicircle, i.e., $\{x \in \mathbb{R}^2 \mid x = -1, y = 0, \text{ and } x = +1, y = 0\}$. Thus

$$\overline{\Gamma_g} = \Gamma \cap \{x \in \mathbb{R}^2 \mid y \geq 0\} \tag{2.2.7}$$

Similarly, let

$$\Gamma_h = \Gamma \cap \{x \in \mathbb{R}^2 \mid y < 0\} \tag{2.2.8}$$

Thus

$$\overline{\Gamma_h} = \Gamma \cap \{x \in \mathbb{R}^2 \mid y \leq 0\} \tag{2.2.9}$$

Clearly

$$\overline{\Gamma_g \cup \Gamma_h} = \overline{\Gamma_g} \cup \overline{\Gamma_h} = \overline{\Gamma} \tag{2.2.10}$$

These sets are depicted in Fig. 2.2.2.

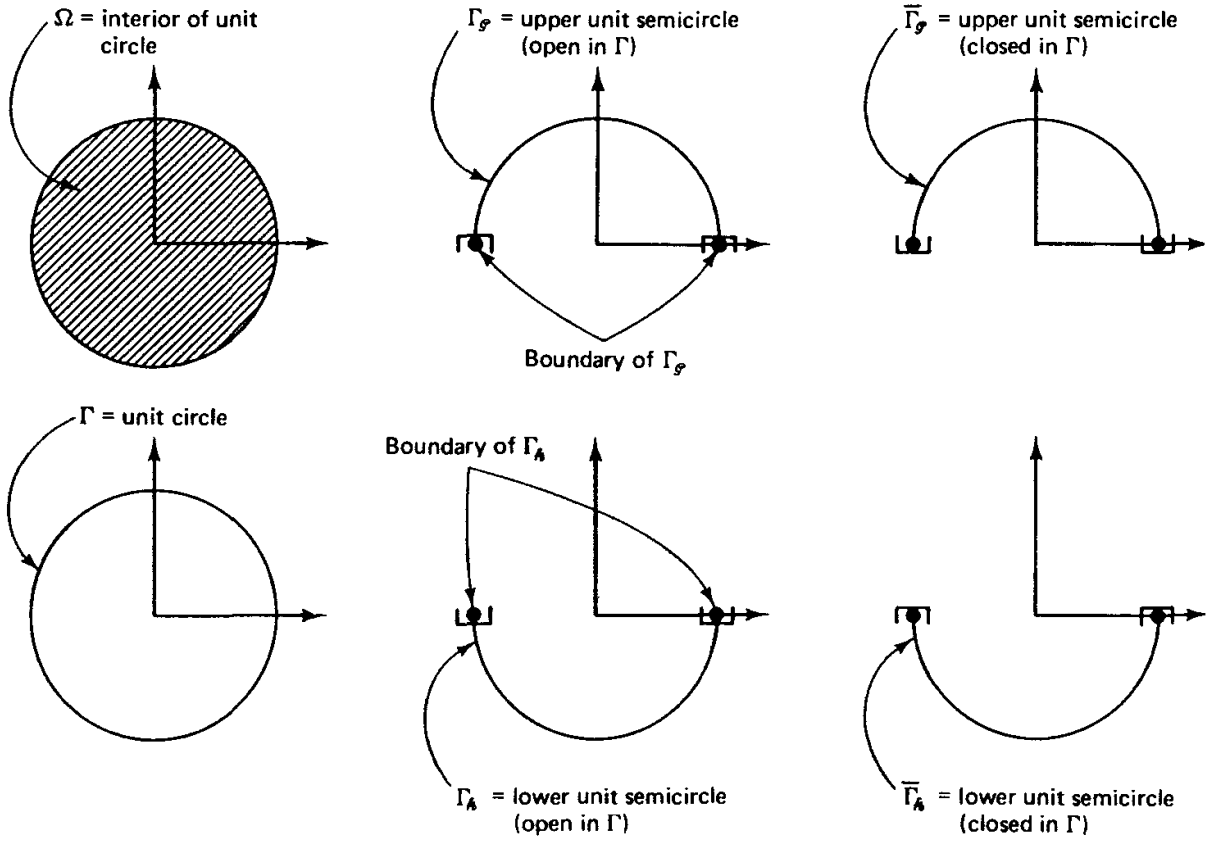


Figure 2.2.2

We shall assume throughout that $\Gamma_g \neq \varnothing$ but allow for the case $\Gamma_h = \varnothing$.

Let the indices i, j, k, l, run over the values $1, \ldots, n_{sd}$. Differentiation is denoted by a comma (e.g., $u_{,i} = u_{,x_i} = \partial u/\partial x_i$) and repeated indices imply summation (e.g., in $\mathbb{R}^3$, $u_{,ii} = u_{,11} + u_{,22} + u_{,33} = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$). The summation convention only applies to the indices i, j, k, and l and only to two repeated indices. If there are three, or more, repeated indices in an expression, then

the summation convention is *not* in effect. (This is in keeping with the usual convention.)

Divergence theorem. Let $f: \overline{\Omega} \to \mathbb{R}$ be C^1. Then

$$\boxed{\int_\Omega f_{,i}\, d\Omega = \int_\Gamma f n_i\, d\Gamma} \tag{2.2.11}$$

The proof may be found in [1].

Integration by parts. Let f be as above and also let $g: \overline{\Omega} \to \mathbb{R}$ be C^1. Then

$$\boxed{\int_\Omega f_{,i} g\, d\Omega = -\int_\Omega f g_{,i}\, d\Omega + \int_\Gamma f g n_i\, d\Gamma} \tag{2.2.12}$$

Proof. We integrate the identity (i.e., "product rules of differentiation," see [1])

$$(fg)_{,i} = f_{,i} g + f g_{,i}$$

to get

$$\int_\Omega (fg)_{,i}\, d\Omega = \int_\Omega f_{,i} g\, d\Omega + \int_\Omega f g_{,i}\, d\Omega$$

and then use the divergence theorem to convert the left-hand side into a boundary integral. ■

2.3 CLASSICAL LINEAR HEAT CONDUCTION: STRONG AND WEAK FORMS; EQUIVALENCE

Let q_i denote (Cartesian components of) the ***heat flux vector***, let u be the ***temperature***, and let ℓ be the ***heat supply per unit volume***. Assume the heat flux vector is defined in terms of the temperature gradient by the ***generalized Fourier law***[2]:

$$q_i = -\kappa_{ij} u_{,j}, \qquad \kappa_{ij} = \kappa_{ji} \quad \text{(symmetry)} \tag{2.3.1}$$

where the ***conductivities***, κ_{ij}'s, are given functions of x. (If the κ_{ij}'s are constant throughout Ω, the body is said to be ***homogeneous***.) The ***conductivity matrix***, $\boldsymbol{\kappa} = [\kappa_{ij}]$, is assumed positive definite (see the definition in Sec. 1.9). The most common situation in practice is the ***isotropic case*** in which $\kappa_{ij}(x) = \kappa(x)\delta_{ij}$, where δ_{ij} is the Kronecker delta.

[2]The generalized fourier law is a ***constitutive equation***, or ***equation of state***, which reflects the heat conduction properties of the body (i.e., Ω) under consideration.

A formal statement[3] of the strong form of the boundary-value problem is as follows:

$$(S)\quad \text{Given } \ell : \Omega \to \mathbb{R},\ g : \Gamma_g \to \mathbb{R} \text{ and } h : \Gamma_h \to \mathbb{R}, \text{ find } u : \overline{\Omega} \to \mathbb{R} \text{ such that}$$

$$q_{i,i} = \ell \quad \text{in } \Omega \qquad (\textit{heat equation}) \tag{2.3.2}$$

$$u = g \quad \text{on } \Gamma_g \tag{2.3.3}$$[4]

$$-q_i n_i = h \quad \text{on } \Gamma_h \tag{2.3.4}$$

where q_i is defined by (2.3.1).

The functions g and h are the ***prescribed boundary temperature and heat flux***, respectively. This problem possesses a unique solution for appropriate restrictions on the given data.

In more mathematical terminology, (2.3.2) is a generalized Poisson equation, (2.3.3) is a Dirichlet boundary condition, and (2.3.4) is a Neumann boundary condition.

We shall now construct a weak formulation of the boundary-value problem analogous to that for the one-dimensional problem of Chapter 1. In particular, (2.3.3) and (2.3.4) will be treated as essential and natural boundary conditions, respectively. As before, let $\mathcal{S}$ denote the trial solution space and $\mathcal{V}$ the variation space. This time $\mathcal{S}$ and $\mathcal{V}$ consist of real-valued functions defined on $\overline{\Omega}$ satisfying certain smoothness requirements, such that all members of $\mathcal{S}$ satisfy (2.3.3), whereas if $w \in \mathcal{V}$, then

$$w = 0 \quad \text{on } \Gamma_g \tag{2.3.5}$$

The weak formulation of the problem goes as follows:

$$(W)\quad \text{Given } \ell : \Omega \to \mathbb{R},\ g : \Gamma_g \to \mathbb{R} \text{ and } h : \Gamma_h \to \mathbb{R}, \text{ find } u \in \mathcal{S} \text{ such that for all } w \in \mathcal{V}$$

$$\boxed{-\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w\ell \, d\Omega + \int_{\Gamma_h} wh \, d\Gamma} \tag{2.3.6}$$

where q_i is defined by (2.3.1).

Theorem. Assume all functions involved are smooth enough to justify the manipulations. Then a solution of (S) is a solution of (W) and vice versa.

Proof. **1.** Assume u is the solution of (S). By virtue of (2.3.3), $u \in \mathcal{S}$. Pick any $w \in \mathcal{V}$ and proceed as follows:

[3] By a formal statement, we mean one in which we do not precisely delineate the spaces to which the functions involved belong.

[4] The statement $u = g$ on Γ_g means $u(x) = g(x)$ for all $x \in \Gamma_g$, and so on.

$$0 = \int_\Omega w(\underbrace{q_{i,i} - \ell}_{0})\, d\Omega \qquad \text{(heat equation, i.e., (2.3.2))}$$

$$= -\int_\Omega w_{,i} q_i \, d\Omega + \int_\Gamma w q_i n_i \, d\Gamma - \int_\Omega w\ell \, d\Omega \qquad \text{(integration by parts)}$$

$$= -\int_\Omega w_{,i} q_i \, d\Omega - \int_{\Gamma_h} wh \, d\Gamma - \int_\Omega w\ell \, d\Omega \qquad \text{(generalized Fourier law, } w = 0 \text{ on } \Gamma_q\text{, and heat flux boundary condition, i.e. (2.3.4))}$$

Therefore (2.3.6) is satisfied and so u is a solution of (W).

2. Assume u is the solution of (W). Then $u = g$ on Γ_g, and for all $w \in \mathcal{V}$

$$0 = \int_\Omega w_{,i} q_i \, d\Omega + \int_\Omega w\ell \, d\Omega + \int_{\Gamma_h} wh \, d\Gamma \qquad \text{(by (2.3.6))}$$

$$= \int_\Omega w(-q_{i,i} + \ell) \, d\Omega + \int_{\Gamma_h} w(n_i q_i + h) \, d\Gamma \qquad (2.3.7)$$

Let

$$\alpha = -q_{i,i} + \ell$$

$$\beta = q_i n_i + h$$

To show (2.3.2) and (2.3.4) are satisfied and thus complete the proof, we must prove that

$$\alpha = 0 \qquad \text{on } \Omega$$

$$\beta = 0 \qquad \text{on } \Gamma_h$$

First pick $w = \alpha\phi$ where

i. $\phi > 0$ on Ω;
ii. $\phi = 0$ on Γ; and
iii. ϕ is smooth.

(These conditions insure that $w \in \mathcal{V}$.) With this choice for w, (2.3.7) becomes

$$0 = \int_\Omega \alpha^2 \phi \, d\Omega$$

which implies $\alpha = 0$ on Ω.

Now pick $w = \beta\psi$, where

i′. $\psi > 0$ on Γ_h;
ii′. $\psi = 0$ on Γ_g; and
iii′. ψ is smooth.

(These conditions insure that $w \in \mathcal{V}$.) With this choice for w, (2.3.7) becomes

(making use of $\alpha = 0$):

$$0 = \int_{\Gamma_k} \beta^2 \psi \, d\Gamma$$

from which it follows that $\beta = 0$ on Γ_k. Thus u is a solution of (S). ■

It is convenient to introduce an abstract version of (2.3.6). Let

$$a(w, u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} \, d\Omega \tag{2.3.8}$$

$$(w, f) = \int_\Omega w f \, d\Omega \tag{2.3.9}$$

$$(w, h)_\Gamma = \int_{\Gamma_h} w h \, d\Gamma \tag{2.3.10}$$

Then (2.3.6) may be written as

$$a(w, u) = (w, f) + (w, h)_\Gamma \tag{2.3.11}$$

Exercise 1. Verify that $a(\cdot\,, \cdot)$, $(\cdot\,, \cdot)$ and $(\cdot\,, \cdot)_\Gamma$, as just defined, are symmetric bilinear forms. (Note that the symmetry of $a(\cdot\,, \cdot)$ follows from the symmetry of the conductivities.)

In manipulating terms in theories involving vector and tensor quantities, the indicial notation used is very explicit and convenient. However, when we come to the Galerkin formulation analogous to (2.3.6), additional indices necessarily appear. The situation becomes very complicated due to the greater number of indices involved and due to the ranges of the various indices being different. When we come to elasticity theory, the situation is even worse as the corresponding terms have an even greater number of indices. For these reasons it is useful at this point to adopt an *index-free* notation for (2.3.6). Aside from stemming the proliferation of indices, we shall find later on that this formulation is conducive to the computer implementation of the element arrays, especially in more complicated situations such as elasticity.

In introducing our index-free notation we shall assume for definiteness that $n_{sd} = 2$. Let ∇ denote the gradient operator; thus

$$\nabla u = \{u_{,i}\} = \begin{Bmatrix} u_{,1} \\ u_{,2} \end{Bmatrix} \tag{2.3.12}$$

$$\nabla w = \{w_{,i}\} = \begin{Bmatrix} w_{,1} \\ w_{,2} \end{Bmatrix} \tag{2.3.13}$$

In the case of two space dimensions, the conductivity matrix may be written as

$$\boldsymbol{\kappa} = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \qquad \text{(symmetric)} \tag{2.3.14}$$

In the isotropic case, (2.3.14) simplifies to

$$\boldsymbol{\kappa} = \kappa[\delta_{ij}] = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.3.15}$$

In terms of the above expressions, the integrand of (2.3.8) may be written in index-free fashion:

$$\boxed{w_{,i}\kappa_{ij}u_{,j} = (\nabla w)^T \boldsymbol{\kappa}(\nabla u)} \tag{2.3.16}$$

Thus in place of (2.3.8) we may write

$$\boxed{a(w,u) = \int_\Omega (\nabla w)^T \boldsymbol{\kappa}(\nabla u)\, d\Omega} \tag{2.3.17}$$

Exercise 2. Verify (2.3.16) for the cases $n_{sd} = 2$ and 3.

2.4 HEAT CONDUCTION: GALERKIN FORMULATION; SYMMETRY AND POSITIVE-DEFINITENESS OF K

Let $\mathcal{S}^h$ and $\mathcal{V}^h$ be finite-dimensional approximations to $\mathcal{S}$ and $\mathcal{V}$, respectively. We assume all members of $\mathcal{V}^h$ vanish, or vanish approximately, on Γ_q and that each member of $\mathcal{S}^h$ admits the representation

$$u^h = v^h + g^h \tag{2.4.1}$$

where $v^h \in \mathcal{V}^h$ and g^h results in satisfaction, or at least approximate satisfaction, of the boundary condition $u = g$ on Γ_g.

The Galerkin formulation is given as follows:

(G) Given ℓ, g, and h [as in (W)], find $u^h = v^h + g^h \in \mathcal{S}^h$ such that for all $w^h \in \mathcal{V}^h$ (cf. Sec. 1.5):

$$\boxed{a(w^h, v^h) = (w^h, \ell) + (w^h, h)_\Gamma - a(w^h, g^h)} \tag{2.4.2}$$

We now view our domain as "discretized" into element domains Ω^e, $1 \leq e \leq n_{el}$. In two dimensions the element domains might be simply triangles and quadrilaterals; see Fig. 2.4.1. Nodal points may exist anywhere on the domain but most frequently appear at the element vertices and interelement boundaries and less often in the interiors.

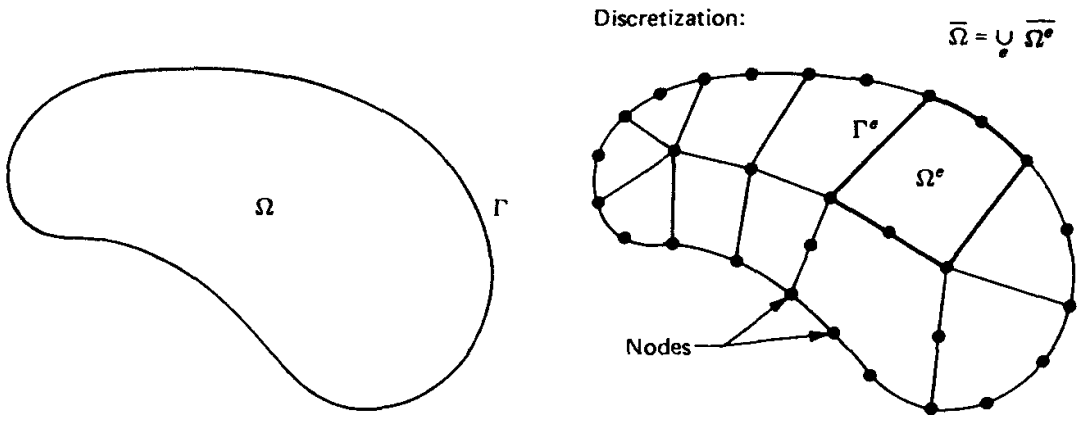


Figure 2.4.1

In Sec. 1.9 the global nodal ordering and ordering of equations in the matrix system coincided. In multidimensional applications this would prove to be an inconvenient restriction with regard to data preparation.

In what follows, a more flexible scheme is described. Let $\eta = \{1, 2, \ldots, n_{np}\}$, the ***set of global node numbers*** where n_{np} is the ***number of nodal points.*** By the terminology q-node we shall mean a node, A, at which it is prescribed that $u^h = q$. Let $\eta_q \subset \eta$ be the set of "q-nodes." The ***complement of*** η_q ***in*** η, denoted by $\eta - \eta_q$, is the set of nodes at which u^h is to be determined. The number of nodes in $\eta - \eta_q$ equals n_{eq}, the ***number of equations.***

A typical member of $\mathcal{V}^h$ is assumed to have the form

$$w^h(x) = \sum_{A \in \eta - \eta_q} N_A(x) c_A \tag{2.4.3}$$

where N_A is the shape function associated with node number A and c_A is a constant. We assume throughout that $w^h = 0$ if and only if $c_A = 0$ for each $A \in \eta - \eta_q$. Likewise

$$v^h(x) = \sum_{A \in \eta - \eta_q} N_A(x) d_A \tag{2.4.4}$$

where d_A is the unknown at node A (i.e., temperature) and

$$q^h(x) = \sum_{A \in \eta_q} N_A(x) q_A, \qquad q_A = q(x_A) \tag{2.4.5}$$

From (2.4.5), we see that q^h has been *defined* to be the nodal interpolate of q by way of the shape functions.[5] Consequently, q^h will be, generally, only an approximation of q. See Fig. 2.4.2. Additional sources of error are (1) the use of approximations f^h and h^h in place of f and h, respectively; and (2) domain approximations in which the element boundaries do not exactly coincide with Γ. Analyses of these approximations are presented in Strang and Fix [2].

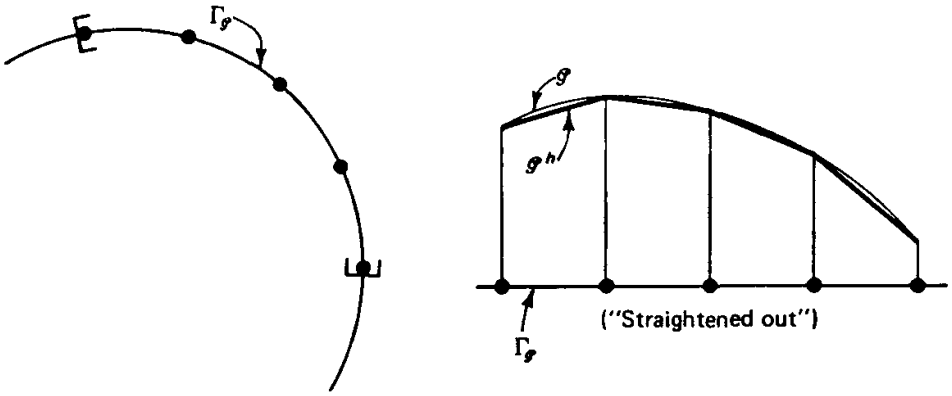


Figure 2.4.2 Piecewise linear approximation of boundary data (schematic).

Substituting (2.4.3)–(2.4.5) into (2.4.2) and arguing as in Sec. 1.6, results in

$$\sum_{B \in \eta - \eta_q} a(N_A, N_B) d_B = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_q} a(N_A, N_B) q_B, \qquad A \in \eta - \eta_q \tag{2.4.6}$$

To define the global stiffness matrix and force vector, we need to first specify the global ordering of equations. For this purpose we introduce the ID array, sometimes called the ***destination array,*** which assigns to node A the corresponding global equation number, viz.,

$$\mathrm{ID}(A) = \begin{cases} P & \text{if } A \in \eta - \eta_q \\ 0 & \text{if } A \in \eta_q \end{cases} \tag{2.4.7}$$

(P: Global equation number)

[5] This is not the only possibility, nor the best from the standpoint of accuracy. However, in practice it is generally the most convenient.

where $1 \leq P \leq n_{eq}$. The dimension of ID is n_{np}. As may be seen from (2.4.7), nodes at which q is prescribed are assigned "equation number" zero. An example of the setup of ID and other important data-processing arrays is presented in Sec. 2.6.

The matrix equivalent of (2.4.6) is given as follows:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \tag{2.4.8}$$

$$\boldsymbol{K} = [K_{PQ}], \qquad \boldsymbol{d} = \{d_Q\}, \qquad \boldsymbol{F} = \{F_P\}, \qquad 1 \leq P, Q \leq n_{eq} \tag{2.4.9}$$

$$K_{PQ} = a(N_A, N_B), \qquad P = \mathrm{ID}(A), \qquad Q = \mathrm{ID}(B) \tag{2.4.10}$$

$$F_P = (N_A, \ell) + (N_A, h)_\Gamma - \sum_{B \in \eta_q} a(N_A, N_B) q_B \tag{2.4.11}$$

The main properties of $\boldsymbol{K}$ are established in the following theorem.

Theorem

1. $\boldsymbol{K}$ is symmetric.
2. $\boldsymbol{K}$ is positive definite.

Proof

1. The symmetry of $\boldsymbol{K}$ follows directly from the symmetry of $a(\cdot\ ,\ \cdot)$, viz.,

$$\begin{aligned} K_{PQ} &= a(N_A, N_B) && \text{(by definition)} \\ &= a(N_B, N_A) && \text{(symmetry of } a(\cdot\ ,\ \cdot)) \\ &= K_{QP} && \text{(by definition)} \end{aligned}$$

2. (Recall that we must show (i) $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} \geq 0$ and (ii) $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$ implies $\boldsymbol{c} = \boldsymbol{0}$.)

To each n_{eq}-vector $\boldsymbol{c} = \{c_P\}$, we may associate a member $w^h \in \mathscr{V}^h$ by the expression $w^h = \Sigma_{A \in \eta - \eta_g} N_A \bar{c}_A$, where $\bar{c}_A = c_P$, $P = \mathrm{ID}(A)$.

i.

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} &= \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q \\ &= \sum_{A,B \in \eta - \eta_g} \bar{c}_A a(N_A, N_B) \bar{c}_B \\ &= a\left(\sum_{A \in \eta - \eta_g} N_A \bar{c}_A, \sum_{B \in \eta - \eta_g} N_B \bar{c}_B \right) && \text{(bilinearity of } a(\cdot\ ,\ \cdot)) \\ &= a(w^h, w^h) && \text{(definition of } w^h) \\ &= \int_\Omega \underbrace{w^h_{,i} \kappa_{ij} w^h_{,j}}_{\geq 0} \, d\Omega && \text{(positive-definiteness of conductivities)} \\ &\geq 0 \end{aligned}$$

ii. Assume $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$. By the proof of part (i),

$$\int_\Omega \underbrace{w^h_{,i}\kappa_{ij}w^h_{,j}}_{\geq 0}\, d\Omega = 0$$

and thus it follows that

$$w^h_{,i}\kappa_{ij}w^h_{,j} = 0$$

By the positive-definiteness hypothesis on the conductivities, this requires $w^h_{,i} = 0$ and so w^h must be constant. However $w^h = 0$ on Γ_q (which is not empty) and so w^h must be zero throughout Ω. By the definition of w^h, it follows that each $c_P = 0$; that is $\boldsymbol{c} = \boldsymbol{0}$, which was to be proved. ■

Remarks

1. Observe that it is the positive-definiteness hypothesis on the constitutive coefficients (i.e., κ_{ij}'s) *and* the boundary condition incorporated in the definition of $\mathscr{V}^h$ which together result in the positive-definiteness of $\boldsymbol{K}$ and thus ensure its invertibility.

2. The explicit structure of the shape functions, which will be delineated in Chapter 3, will also result in $\boldsymbol{K}$ being banded.

Exercise 1. (This exercise is a multidimensional analog of the one contained in Sec. 1.8.) Let

$$\Gamma_{\text{int}} = \left(\bigcup_{e=1}^{n_{el}} \Gamma^e \right) - \Gamma \qquad (\textit{interior element boundaries})$$

One side of Γ_{int} is (arbitrarily) designated to be the "+ side" and the other is the "− side." Let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ be unit normals to Γ_{int} which point in the plus and minus directions, respectively. Clearly $\boldsymbol{n}^+ = -\boldsymbol{n}^-$. Let q_i^+ and q_i^- denote the values of q_i obtained by approaching $\boldsymbol{x} \in \Gamma_{\text{int}}$ from + and − sides, respectively. The "jump" in $q_n = q_i n_i$ at $\boldsymbol{x}$ is defined to be

$$\begin{aligned}[q_n] &= (q_i^+ - q_i^-)n_i^+ \\ &= q_i^+ n_i^+ + q_i^- n_i^-\end{aligned}$$

As may be easily verified, the jump is invariant with respect to reversing the + and − designations.

Consider the weak formulation (i.e., (2.3.6)) and assume w and u are smooth on the element interiors but may experience discontinuities in gradient across element boundaries. (Functions of this type contain the standard C^0 finite element interpolations; see Chapter 3.) Show that

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i} - \ell)\, d\Omega - \int_{\Gamma_h} w(q_n + h)\, d\Gamma + \int_{\Gamma_{\text{int}}} w[q_n]\, d\Gamma$$

from which the Euler-Lagrange conditions may be readily deduced:

i. $q_{i,i} = \ell$ in $\bigcup_{e=1}^{n_{el}} \Omega^e$

ii. $-q_n = h$ on Γ_h

iii. $[q_n] = 0$ on Γ_{int}

As may be seen, (i) is the heat equation on the *element interiors* and (iii) is a continuity condition across element boundaries on the heat flux. Contrast the present results with those obtained assuming w and u are *globally* smooth.

The Galerkin finite element formulation obtains an *approximate* solution to (i) through (iii).

2.5 HEAT CONDUCTION: ELEMENT STIFFNESS MATRIX AND FORCE VECTOR

As before, we can break up the global arrays into sums of elemental contributions:

$$\boldsymbol{K} = \sum_{e=1}^{n_{el}} \boldsymbol{K}^e, \qquad \boldsymbol{K}^e = [K^e_{PQ}] \tag{2.5.1}$$

$$\boldsymbol{F} = \sum_{e=1}^{n_{el}} \boldsymbol{F}^e, \qquad \boldsymbol{F}^e = \{F^e_P\} \tag{2.5.2}$$

where

$$K^e_{PQ} = a(N_A, N_B)^e = \int_{\Omega^e} (\nabla N_A)^T \boldsymbol{\kappa} (\nabla N_B)\, d\Omega \tag{2.5.3}$$

$$\begin{aligned} F^e_P &= (N_A, \ell)^e + (N_A, h)^e_\Gamma - \sum_{B \in \eta_q} a(N_A, N_B)^e q_B \\ &= \int_{\Omega^e} N_A \ell \, d\Omega + \int_{\Gamma^e_h} N_A \, h \, d\Gamma - \sum_{B \in \eta_q} a(N_A, N_B)^e \, q_B \end{aligned} \tag{2.5.4}$$

$$\Gamma^e_h = \Gamma_h \cap \Gamma^e, \qquad P = \text{ID}(A), \qquad Q = \text{ID}(B) \tag{2.5.5}$$

See Fig. 2.5.1 for an illustration of Γ^e_h.

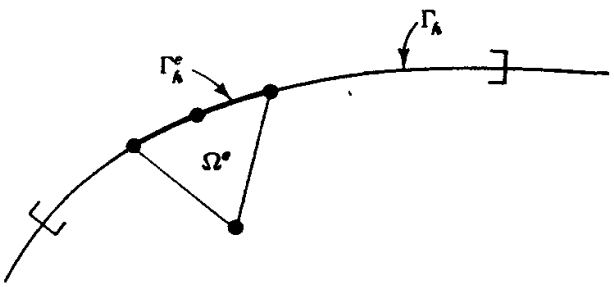


Figure 2.5.1

The element stiffness, $\boldsymbol{k}^e$, and element force vector, $\boldsymbol{f}^e$, may be deduced from these equations:

$$\boldsymbol{k}^e = [k^e_{ab}], \qquad \boldsymbol{f}^e = \{f^e_a\}, \qquad 1 \le a, b \le n_{en} \tag{2.5.6}$$

$$k^e_{ab} = a(N_a, N_b)^e = \int_{\Omega^e} (\nabla N_a)^T \boldsymbol{\kappa} (\nabla N_b)\, d\Omega \tag{2.5.7}$$

$$f^e_a = \int_{\Omega^e} N_a f\, d\Omega + \int_{\Gamma^e_h} N_a h\, d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} q^e_b \tag{2.5.8}$$

where (recall) n_{en} is the number of element nodes, and $q^e_b = q(\boldsymbol{x}^e_b)$ if q is prescribed at node number b and *equals zero otherwise.*[6]

The global arrays, $\boldsymbol{K}$ and $\boldsymbol{F}$ may be formed from the element arrays $\boldsymbol{k}^e$ and $\boldsymbol{f}^e$, respectively, by way of an assembly algorithm as described in Sec. 1.14.

The element stiffness matrix can be written in a standard form convenient for programming:

$$\boldsymbol{k}^e = \int_{\Omega^e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, d\Omega \tag{2.5.9}$$

where, in the present case,

$$\underbrace{\boldsymbol{D}}_{n_{sd} \times n_{sd}} = \boldsymbol{\kappa} \tag{2.5.10}$$

$$\underbrace{\boldsymbol{B}}_{n_{sd} \times n_{en}} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_{n_{en}}] \tag{2.5.11}$$

$$\underbrace{\boldsymbol{B}_a}_{n_{sd} \times 1} = \nabla N_a \tag{2.5.12}$$

The component version of (2.5.9) is

$$k^e_{ab} = \int_{\Omega^e} \boldsymbol{B}^T_a \boldsymbol{D} \boldsymbol{B}_b\, d\Omega \tag{2.5.13}$$

[6] An implicit assumption in localizing the q-term is that if $\boldsymbol{x}_A$ is not a node attached to element e, then $N_A(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \overline{\Omega}^e$. Otherwise, the last term in (2.5.4) may involve q-data of nodes not attached to element e, which is not accounted for in (2.5.8).

Exercise 1. Let

$$\underbrace{\mathbf{d}^e}_{n_{en} \times 1} = \{d_a^e\} = \begin{Bmatrix} d_1^e \\ d_2^e \\ \vdots \\ d_{n_{en}}^e \end{Bmatrix} \tag{2.5.14}$$

where

$$d_a^e = u^h(x_a^e) \tag{2.5.15}$$[7]

$\mathbf{d}^e$ is called the ***element temperature vector.*** Show that the heat flux vector at point $x \in \Omega^e$ can be calculated from the formula

$$\mathbf{q}(x) = -\mathbf{D}(x)\mathbf{B}(x)\mathbf{d}^e = -\mathbf{D}(x) \sum_{a=1}^{n_{en}} \mathbf{B}_a d_a^e \tag{2.5.16}$$

Exercise 2. Consider the strong statement of the boundary-value problem in classical linear heat conduction in which the h-type boundary condition (i.e., eq. (2.3.4)) is replaced by the following expression:

$$\lambda u - q_i n_i = h \qquad \text{on } \Gamma_h \tag{2.5.17}$$

where $\lambda \geq 0$ is a given function of $x \in \Gamma_h$. Generalize the weak formulation to include (2.5.17) as a natural boundary condition. Obtain an expression for the additional contribution to k_{ab}^e (cf. (2.5.13)) arising from (2.5.17). Show that $\mathbf{K}$ is positive-definite.

The boundary condition (2.5.17) is equivalent to what is often called ***Newton's law of heat transfer;*** λ is called the ***coefficient of heat transfer.*** This boundary condition applies to the case in which the heat flux is proportional to the difference of the surface temperatures of the body and surrounding medium, the latter formally represented by h/λ in (2.5.17).

2.6 HEAT CONDUCTION: DATA PROCESSING ARRAYS; ID, IEN, AND LM

The element nodal data is stored in the array IEN, the ***element nodes array,*** which relates local node numbers to global node numbers, viz.,

$$\underbrace{\text{IEN}(a, e)}_{\substack{\nearrow \\ \text{Local node number}}\ \ \substack{\nwarrow \\ \text{Element number}}} = \underbrace{A}_{\substack{\text{Global node number}}} \tag{2.6.1}$$

The relationship between global node numbers and global equation numbers as well as nodal boundary condition information is stored in the ID array (see 2.4.7). In

[7] The g-boundary conditions are accounted for in this definition.

practice, the IEN and ID arrays are set up from input data. The LM array, which was described in the context of the one-dimensional model problem in Sec. 1.14, may then be constructed from the relation

$$\boxed{\mathrm{LM}(a, e) = \mathrm{ID}(\mathrm{IEN}(a, e))} \tag{2.6.2}$$

Because of the previous relationship, we often think of LM as the element "localization" of ID. Strictly speaking the LM array is redundant. However, it is generally convenient in computing to set up LM once and for all, rather than make use of (2.6.2) repeatedly.

Example 1 illustrates the structure of the ID, IEN, and LM arrays.

Example 1

Consider the mesh of four-node, rectangular elements shown in Fig. 2.6.1. We assume that the local node numbering begins at the lower left-hand node of each element and proceeds in counterclockwise fashion. This is illustrated in Fig. 2.6.1 for element 2, which is typical. We also assume that essential boundary conditions (i.e., "g-type") are specified at nodes 1, 4, 7, and 10. Thus there will only be eight equations in the global system $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$. We adopt the usual convention that the global equation numbers run in ascending order with respect to the ascending order of global node numbers. The ID, IEN, and LM arrays are given in Fig. 2.6.2. The reader is urged to verify the details.

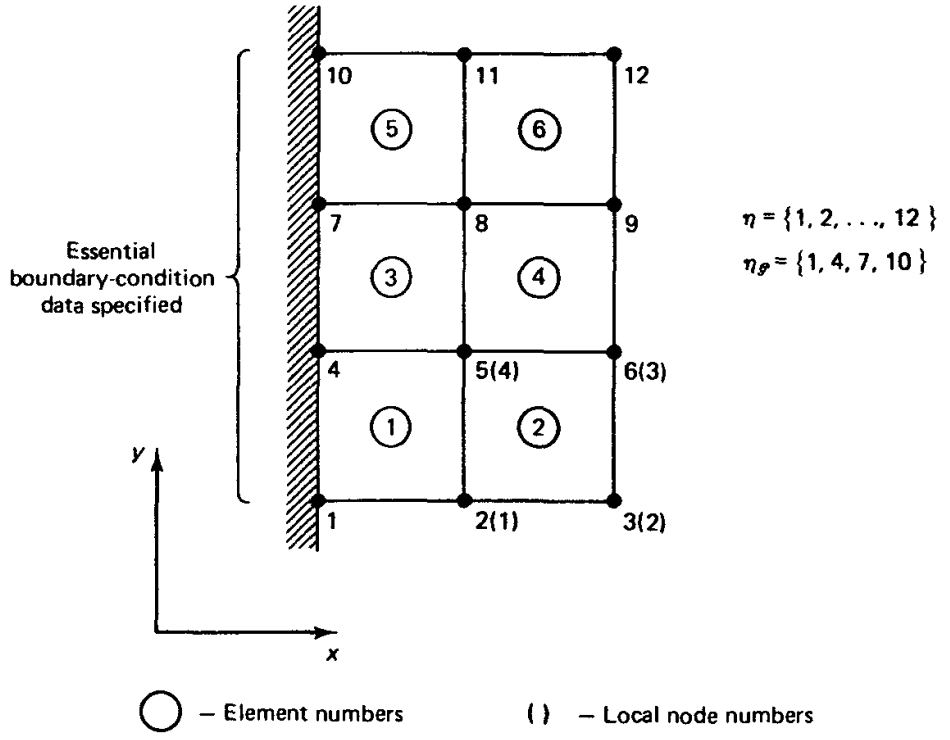


Figure 2.6.1 Mesh of four-node, rectangular elements; global and local node numbers, element numbers, and essential boundary condition nodes.

ID array:

Global node numbers (A)

1	2	3	4	5	6	7	8	9	10	11	12
0*	1	2	0	3	4	0	5	6	0	7	8

($n_{np} = 12$)

($n_{eq} = 8$)

$$P = \mathrm{ID}(A)$$

IEN array:

Element numbers (e)

Local node numbers (a)	1	2	3	4	5	6
1	1	2	4	5	7	8
2	2	3	5	6	8	9
3	5	6	8	9	11	12
4	4	5	7	8	10	11

($n_{el} = 6$)

($n_{en} = 4$)

$$A = \mathrm{IEN}(a, e)$$

LM array:

Element numbers (e)

Local node numbers (a)	1	2	3	4	5	6
1	0*	1	0	3	0	5
2	1	2	3	4	5	6
3	3	4	5	6	7	8
4	0	3	0	5	0	7

($n_{el} = 6$)

($n_{en} = 4$)

$$P = \mathrm{LM}(a, e) = \mathrm{ID}(\mathrm{IEN}(a, e))$$

* Temperature boundary conditions ("q-type") denoted by zeros.

Figure 2.6.2. ID, IEN, and LM arrays for the mesh of Fig. 2.6.1.

In terms of the IEN and LM arrays, a precise definition of the q_a^e's may be given (see (2.5.8)):

$$q_a^e = \begin{cases} 0 & \text{if } \mathrm{LM}(a,e) \neq 0 \\ q_A & \text{if } \mathrm{LM}(a,e) = 0, \quad \text{where } A = \mathrm{IEN}(a,e) \end{cases} \tag{2.6.3}$$

This definition may be easily programmed.

In our final example of this section we shall illustrate the assembly procedure for a typical element subjected to essential boundary conditions.

Example 2

Consider a typical, four-noded element e. Assume values of the LM array, for this element, are given as follows:

$$\left.\begin{aligned} \mathrm{LM}(1,e) &= 5 \\ \mathrm{LM}(2,e) &= 0 \\ \mathrm{LM}(3,e) &= 0 \\ \mathrm{LM}(4,e) &= 9 \end{aligned}\right\} \tag{2.6.4}$$

We deduce from (2.6.4) that the contributions to the global arrays are given as follows:[8]

$$\left.\begin{aligned} K_{55} &\leftarrow K_{55} + k_{11}^e \\ K_{59} &\leftarrow K_{59} + k_{14}^e \\ K_{95} &\leftarrow K_{95} + k_{41}^e \\ K_{99} &\leftarrow K_{99} + k_{44}^e \end{aligned}\right\} \tag{2.6.5}$$

$$\left.\begin{aligned} F_5 &\leftarrow F_5 + f_1^e \\ F_9 &\leftarrow F_9 + f_4^e \end{aligned}\right\} \tag{2.6.6}$$

Note that all terms in the second and third rows and columns of $\boldsymbol{k}^e$ do *not* contribute to $\boldsymbol{K}$. However, they may contribute to $\boldsymbol{F}$ via f_1^e and f_4^e, since, by (2.5.8) we have

$$f_1^e = \cdots - k_{12}^e q_2^e - k_{13}^e q_3^e \tag{2.6.7}$$

$$f_4^e = \cdots - k_{42}^e q_2^e - k_{43}^e q_3^e \tag{2.6.8}$$

in which, for clarity, we have omitted the first two terms of the right-hand side of (2.5.8).

Special subroutines are easily programmed to carry out the operations indicated in (2.6.5)–(2.6.8).

It is instructive to visualize the contributions of the eth element to the global stiffness and force. These contributions are depicted in Fig. 2.6.3.

We note that all necessary element assembly information is provided by the LM array.

[8] Due to symmetry, k_{41}^e is not actually assembled in practice.

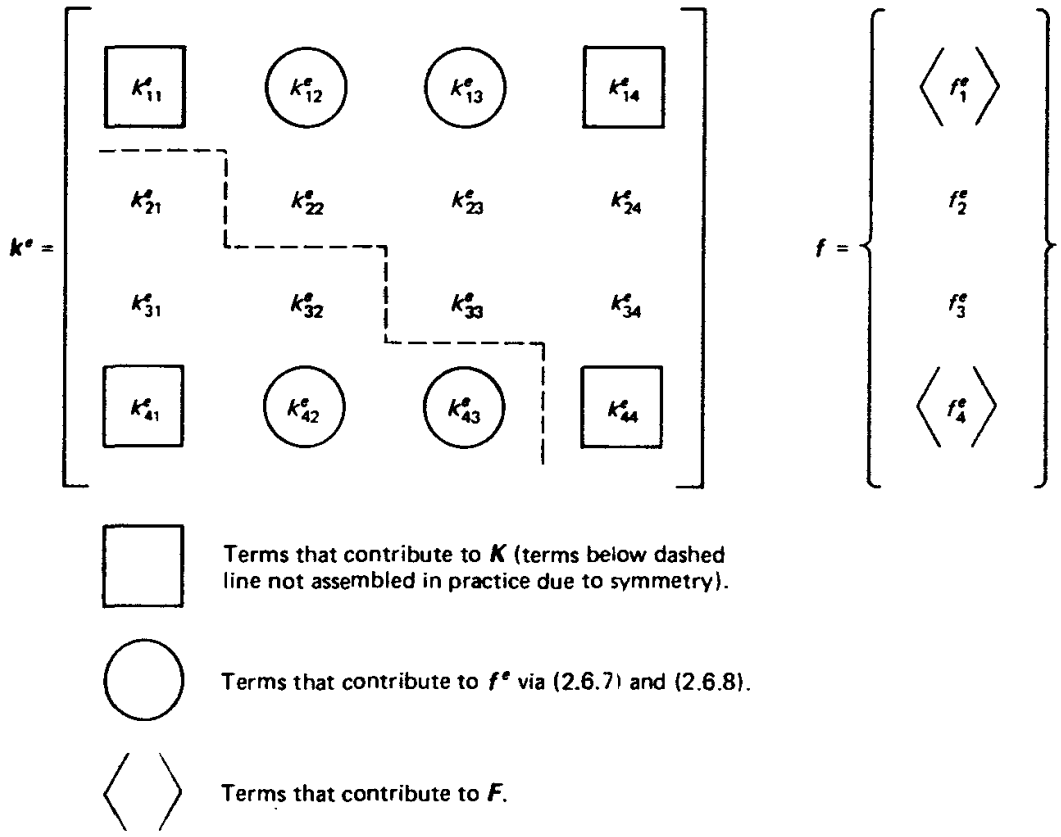
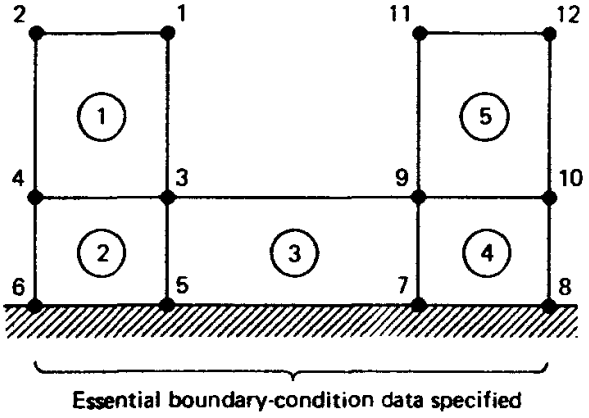


Figure 2.6.3 Contributions of heat conduction element in Example 2 to global arrays.

Exercise 1. Consider the accompanying mesh. Set up the ID, IEN, and LM arrays.



2.7 CLASSICAL LINEAR ELASTOSTATICS: STRONG AND WEAK FORMS; EQUIVALENCE

Classical elastostatics is a rich subject in its own right. See [3–7] for background and references to the literature. These references range from the very physical to the very

mathematical. The most physical book is the one by Timoshenko and Goodier. In ascending order of mathematical content are Sokolnikoff, Gurtin, Duvaut-Lions, and Fichera.

The reader is reminded that indices i, j, k, and l take on values $1, \ldots, n_{sd}$, where n_{sd} is the number of spatial dimensions, and the summation convention applies to repeated indices i, j, k, and l *only*.

Let σ_{ij} denote (Cartesian components of) the (Cauchy) ***stress tensor,*** let u_i denote the ***displacement vector,*** and let f_i be the ***prescribed body force per unit volume.*** The ***(infinitesimal) strain tensor,*** ϵ_{ij}, is defined to be the symmetric part of the displacement gradients, viz.,

$$\epsilon_{ij} = u_{(i,j)} \overset{\text{def.}}{=} \frac{u_{i,j} + u_{j,i}}{2} \qquad (\textit{strain-displacement equations}) \tag{2.7.1}$$

The stress tensor is defined in terms of the strain tensor by the ***generalized Hooke's law:***[9]

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{2.7.2}$$

where the c_{ijkl}'s, the ***elastic coefficients,*** are given functions of $\boldsymbol{x}$. (If the c_{ijkl}'s are constants throughout, the body is called ***homogeneous.***) The elastic coefficients are assumed to satisfy the following properties:

Symmetry

$$c_{ijkl} = c_{klij} \qquad \text{(major symmetry)} \tag{2.7.3}$$

$$\left.\begin{aligned} c_{ijkl} &= c_{jikl} \\ c_{ijkl} &= c_{ijlk} \end{aligned}\right\} \qquad \text{(minor symmetries)} \tag{2.7.4}$$

Positive-definiteness

$$c_{ijkl}(\boldsymbol{x})\psi_{ij}\psi_{kl} \geq 0 \tag{2.7.5}$$

$$c_{ijkl}(\boldsymbol{x})\psi_{ij}\psi_{kl} = 0 \quad \text{implies} \quad \psi_{ij} = 0 \tag{2.7.6}$$

for all $\boldsymbol{x} \in \overline{\Omega}$ and all $\psi_{ij} = \psi_{ji}$.

Note. The positive-definiteness condition is in terms of symmetric arrays, ψ_{ij}.

We shall see in Sec. 2.8 that a consequence of the major symmetry (2.7.3) is that $\boldsymbol{K}$ is symmetric. The first minor symmetry implies the symmetry of the stress tensor (i.e., $\sigma_{ij} = \sigma_{ji}$).[10] The positive-definiteness condition, when combined with appropriate boundary conditions on the displacement, leads to the positive-definiteness of $\boldsymbol{K}$.

[9] This is another ***constitutive equation,*** which reflects the elastic properties of the body under consideration.

[10] From a fundamental continuum mechanics standpoint, the symmetry of the Cauchy stress tensor emanates from the balance of angular momentum.

In the present theory, the unknown is a vector (i.e., the displacement vector). Consequently, a generalization of the boundary conditions considered previously is necessitated. We shall assume that Γ admits decompositions

$$\left.\begin{aligned} \Gamma &= \overline{\Gamma_{g_i} \cup \Gamma_{h_i}} \\ \varnothing &= \Gamma_{g_i} \cap \Gamma_{h_i} \end{aligned}\right\} \qquad i = 1, \ldots, n_{sd} \tag{2.7.7}$$

For example, in two dimensions the situation might appear as in Fig. 2.7.1. As can be seen there can be a different decomposition for each $i = 1, 2, \ldots, n_{sd}$.

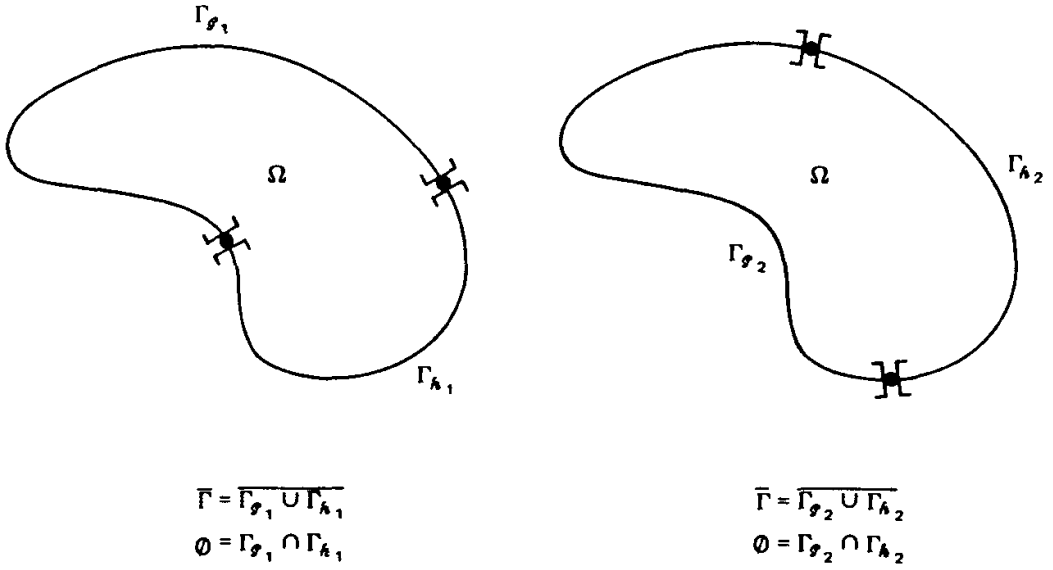


Figure 2.7.1

A formal statement of the strong form of the boundary-value problem goes as follows:

(S) Given $f_i : \Omega \to \mathbb{R}$, $g_i : \Gamma_{g_i} \to \mathbb{R}$, and $h_i : \Gamma_{h_i} \to \mathbb{R}$, find $u_i : \overline{\Omega} \to \mathbb{R}$ such that

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega \qquad (\textit{equilibrium equations}) \tag{2.7.8}$$

$$u_i = g_i \quad \text{on } \Gamma_{g_i} \tag{2.7.9}$$

$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_{h_i} \tag{2.7.10}$$

where σ_{ij} is defined in terms of u_i by (2.7.1) and (2.7.2).

Remarks

1. The functions g_i and h_i are called the ***prescribed boundary displacements*** and ***tractions,*** respectively.

2. (S) is sometimes referred to as the ***mixed boundary-value problem of linear elastostatics.*** Under appropriate hypotheses on the data, (S) possesses a unique solution (see Fichera [4]).

3. The additional complexity of the present theory, when compared with heat conduction, is that the unknown (i.e., $\boldsymbol{u} = \{u_i\}$) is vector-valued rather than scalar-valued.

4. In practice, it is important to be able to deal with somewhat more complicated boundary-condition specification. In order not to encumber the present exposition, this generalization will be considered later on in the form of an exercise (see Exercise 5 in Sec. 2.12).

Let $\mathcal{S}_i$ denote the trial solution space and $\mathcal{V}_i$ the variation space. Each member $u_i \in \mathcal{S}_i$ satisfies $u_i = q_i$ on Γ_{q_i}, whereas each $w_i \in \mathcal{V}_i$ satisfies $w_i = 0$ on Γ_{q_i}. Equation (2.7.10) will be a natural boundary condition.

The weak formulation goes as follows:

(W) Given $\ell_i : \Omega \to \mathbb{R}$, $q_i : \Gamma_{q_i} \to \mathbb{R}$ and $h_i : \Gamma_{h_i} \to \mathbb{R}$, find $u_i \in \mathcal{S}_i$ such that for all $w_i \in \mathcal{V}_i$,

$$\boxed{\int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega = \int_\Omega w_i \ell_i\, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \right)} \qquad (2.7.11)$$

where σ_{ij} is defined in terms of u_i by (2.7.1) and (2.7.2).

Remarks

5. In the solid mechanics literature, (W) is sometimes referred to as the ***principle of virtual work,*** or ***principle of virtual displacements,*** w_i being the ***virtual displacements.***

6. The existence and uniqueness of weak solutions is discussed in Duvaut-Lions [3].

Note. The boundary integral in (2.7.11) takes on the explicit form:

$$\sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \right) = \int_{\Gamma_{h_1}} w_1 h_1\, d\Gamma + \cdots + \int_{\Gamma_{h_{n_{sd}}}} w_{n_{sd}} h_{n_{sd}}\, d\Gamma \qquad (2.7.12)$$

Theorem. Assume all functions are smooth enough to justify the manipulations. Then a solution of (S) is a solution of (W), and vice versa.

Remark

7. The proof of the equivalence theorem requires some preliminary results, which we shall establish in the following lemmas.

Lemma 1. *(Euclidean decomposition of a second-rank tensor.)* Let s_{ij} denote a general nonsymmetric, second-rank tensor. Then $s_{ij} = s_{(ij)} + s_{[ij]}$, where $s_{(ij)}$ is symmetric (i.e., $s_{(ij)} = s_{(ji)}$) and $s_{[ij]}$ is skew-symmetric (i.e., $s_{[ij]} = -s_{[ji]}$).

Proof. Define

$$s_{(ij)} = \frac{s_{ij} + s_{ji}}{2} \qquad (2.7.13)$$

$$s_{[ij]} = \frac{s_{ij} - s_{ji}}{2} \tag{2.7.14}$$

It is easily verified that $s_{(ij)}$ is symmetric and $s_{[ij]}$ is skew-symmetric. ■

Remark

8. $s_{(ij)}$ and $s_{[ij]}$ are called the ***symmetric*** and ***skew-symmetric parts*** of s_{ij}, respectively.

Lemma 2. Let s_{ij} be a nonsymmetric tensor and let t_{ij} be a symmetric tensor. Then

$$s_{ij}t_{ij} = s_{(ij)}t_{ij} \tag{2.7.15}$$

Proof. By Lemma 1, $s_{ij} = s_{(ij)} + s_{[ij]}$. Since

$$s_{ij}t_{ij} = s_{(ij)}t_{ij} + s_{[ij]}t_{ij}$$

the lemma will be established if we can show that $s_{[ij]}t_{ij} = 0$. We proceed as follows:

$$\begin{aligned} s_{[ij]}t_{ij} &= -s_{[ji]}t_{ij} && \text{(skew-symmetry of } s_{[ij]}\text{)} \\ &= -s_{[ji]}t_{ji} && \text{(symmetry of } t_{ij}\text{)} \\ &= -s_{[ij]}t_{ij} && \text{(redefinition of dummy indices)} \end{aligned}$$

from which the result follows. ■

Proof of Theorem

1. Let u_i be a solution of (S). Thus $u_i \in \mathcal{S}_i$. Multiply (2.7.8) by $w_i \in \mathcal{V}_i$ and integrate over Ω, viz.,

$$0 = \int_\Omega w_i(\sigma_{ij,j} + \ell_i)\, d\Omega = -\int_\Omega w_{i,j}\sigma_{ij}\, d\Omega + \int_\Gamma w_i\sigma_{ij}n_j\, d\Gamma + \int_\Omega w_i\ell_i\, d\Omega$$

(integration by parts)

$$= -\int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \right) + \int_\Omega w_i\ell_i\, d\Omega$$

(symmetry of σ_{ij}, Lemma 2, $w_i = 0$ on Γ_{g_i}, and (2.7.10))

Therefore, u_i is a solution of (W).

2. Assume u_i is a solution of (W). Since $u_i \in \mathcal{S}_i$, $u_i = g_i$ on Γ_{g_i}. From (2.7.11)

$$0 = -\int_\Omega \underbrace{w_{(i,j)}\sigma_{ij}}\, d\Omega + \int_\Omega w_i\ell_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma$$

$(= w_{i,j}\sigma_{ij}$ by Lemma 2)

$$= \int_\Omega w_i(\sigma_{ij,j} + \ell_i)\, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i(\sigma_{ij}n_j - h_i)\, d\Gamma \qquad \begin{array}{l}\text{(integration by parts,}\\ w_i = 0 \text{ on } \Gamma_{g_i})\end{array} \tag{2.7.16}$$

Let

$$\alpha_i = \sigma_{ij,j} + \ell_i$$
$$\beta_i = \sigma_{ij}n_j - h_i$$

Thus to complete the proof we must show that

$$\alpha_i = 0 \qquad \text{on } \Omega$$
$$\beta_i = 0 \qquad \text{on } \Gamma_{h_i}$$

These conditions can be proved by the techniques used in Sec. 2.3. Let $w_i = \alpha_i\phi$, where

i. $\phi > 0$ on Ω;
ii. $\phi = 0$ on Γ; and
iii. ϕ is smooth.

(These conditions insure that $w_i \in \mathcal{V}_i$.) Substituting this w_i into (2.7.16) yields

$$0 = \int_\Omega \underbrace{\alpha_i\alpha_i}_{\geq 0} \underbrace{\phi}_{>0}\, d\Omega$$

which implies $\alpha_i = 0$ on Ω.

Now take $w_i = \delta_{i1}\beta_1\psi$, where

i′. $\psi > 0$ on Γ_{h_1};
ii′. $\psi = 0$ on Γ_{g_1}; and
iii′. ψ is smooth.

(Again, $w_i \in \mathcal{V}_i$.) Substituting this w_i into (2.7.16) and making use of $\alpha_i = 0$ results in

$$0 = \int_{\Gamma_{h_1}} \beta_1^2\psi\, d\Gamma$$

from which it follows that $\beta_1 = 0$ on Γ_{h_1}.

We may proceed analogously to show $\beta_2 = 0$, and so on. Consequently, u_i is a solution of (S). ■

The abstract notation for the present case is

$$a(w, u) = \int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega \tag{2.7.17}$$

$$(w, \ell) = \int_\Omega w_i \ell_i\, d\Omega \tag{2.7.18}$$

$$(w, h)_\Gamma = \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \right) \tag{2.7.19}$$

Exercise 1. Verify that $a(\cdot, \cdot)$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_\Gamma$, as just defined, are symmetric, bilinear forms (cf. Sec. 1.4).

Let $\mathcal{S} = \{u \mid u_i \in \mathcal{S}_i\}$ and let $\mathcal{V} = \{w \mid w_i \in \mathcal{V}_i\}$. Then the weak form can be concisely written in terms of (2.7.17–2.7.19) as follows:

(W) Given ℓ, g, and h (in which the components are defined in (W)), find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

$$a(w, u) = (w, \ell) + (w, h)_\Gamma \tag{2.7.20}$$

As discussed in Sec. 2.3, it is desirable to construct index-free counterparts of the expressions on the right-hand sides of (2.7.17)–(2.7.19). For concreteness we shall assume that $n_{sd} = 2$; thus $1 \le i, j, k, l \le 2$.

Let[11]

$$\boldsymbol{\epsilon}(u) = \{\epsilon_I(u)\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \tag{2.7.21}$$

$$\boldsymbol{\epsilon}(w) = \{\epsilon_I(w)\} = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{Bmatrix} \tag{2.7.22}$$

$$\boldsymbol{D} = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{symmetric} & & D_{33} \end{bmatrix} \tag{2.7.23}$$

[11] According to our previous notational conventions, $\boldsymbol{\epsilon} = [\epsilon_{ij}]$, the matrix of strain components. However, we will have no need for this matrix, and consequently we reserve $\boldsymbol{\epsilon}$ for the "strain vector" defined in (2.7.21). A similar notational conflict occurs with respect to the "stress vector," $\boldsymbol{\sigma}$, defined in Exercise 4. Note that factors of one-half have been eliminated from the *shearing components* (i.e., last components) of (2.7.21) and (2.7.22). (Compare (2.7.21) and (2.7.22) with (2.7.1).) This will considerably simplify subsequent writing.

where

$$D_{IJ} = c_{ijkl} \tag{2.7.24}$$

in which the indices are related by the following table:

TABLE 2.7.1

I / J	i / k	j / l
1	1	1
3	1	2
3	2	1
2	2	2

As should be clear, we have "collapsed" pairs of indices (i, j, and k, l) into single indices (I and J, respectively) taking account of the symmetries of c_{ijkl}, $u_{(k,l)}$ and $w_{(i,j)}$. Observe that the indices I and J take on values 1, 2, and 3. In n_{sd} dimensions, the I and J indices will take on values $1, 2, \ldots, n_{sd}(n_{sd} + 1)/2$.

It can be shown that

$$w_{(i,j)}c_{ijkl}u_{(k,l)} = \epsilon(w)^T D\, \epsilon(u) \tag{2.7.25}$$

and so

$$\boxed{a(w, u) = \int_\Omega \epsilon(w)^T D\, \epsilon(u)\, d\Omega} \tag{2.7.26}$$

Exercise 2. Verify (2.7.25) for $n_{sd} = 2$.

Exercise 3. Construct the analog of Table 2.7.1 for the case $n_{sd} = 3$. For definiteness of the ordering, take

$$\epsilon(u) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3} + u_{3,2} \\ u_{1,3} + u_{3,1} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \tag{2.7.27}$$

Exercise 4. Show that

$$\sigma = D\, \epsilon(u) \tag{2.7.28}$$

where

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \qquad n_{sd} = 2 \tag{2.7.29}$$

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}, \qquad n_{sd} = 3 \tag{2.7.30}$$

Exercise 5. If the body in question is ***isotropic***, then

$$c_{ijkl}(x) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(x)\,\delta_{ij}\delta_{kl} \tag{2.7.31}$$

where λ and μ are the ***Lamé parameters***; μ is often referred to as the shear modulus and denoted by G. The relationships of λ and μ to E, ***Young's modulus***, and ν, ***Poisson's ratio***, are given by

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \tag{2.7.32}$$

$$\mu = \frac{E}{2(1 + \nu)} \tag{2.7.33}$$

(See Sokolnikoff [6], p. 71, for further relations with other equivalent moduli.) If (2.7.31) is not satisfied, the body is said to be ***anisotropic***. Using (2.7.31) set up $\boldsymbol{D}$ for $n_{sd} = 2$ and $n_{sd} = 3$. *Hint:* The answer for $n_{sd} = 2$ is

$$\boldsymbol{D} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ & \lambda + 2\mu & 0 \\ \text{Symmetric} & & \mu \end{bmatrix} \tag{2.7.34}$$

This matrix manifests the ***plane strain*** hypothesis. (See Sokolnikoff [6] for elaboration.)

Remark

9. The case of isotropic ***plane stress*** may be determined from (2.7.34) by replacing λ by $\bar{\lambda}$, where

$$\bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu} \tag{2.7.35}$$

(See Sokolnikoff [6] or Timoshenko and Goodier [7] for elaboration on the physical ideas.)

Exercise 6. (See Exercise 1, Sec. 2.4 for background and an analogous result.)
Let the "jump" in $\sigma_{in} = \sigma_{ij}n_j$ be denoted by $[\sigma_{in}]$. Consider the weak formulation (i.e., (2.7.11)) and assume w_i and u_i are smooth on element interiors, but experience gradient discontinuities across element boundaries. Show that

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w_i(\sigma_{ij,j} + \ell_i)\, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i(\sigma_{in} - h_i)\, d\Gamma - \int_{\Gamma_{\text{int}}} w_i[\sigma_{in}]\, d\Gamma$$

from which the Euler-Lagrange conditions may be read:

i. $\sigma_{ij,j} + \ell_i = 0$ in $\bigcup_{e=1}^{n_{el}} \Omega^e$
ii. $\sigma_{in} = h_i$ on Γ_{h_i}
iii. $[\sigma_{in}] = 0$ on Γ_{int}

Here (i) is the equilibrium equation on *element interiors*, and (iii) is a traction continuity condition across element boundaries. Compare these results with those obtained assuming w_i and u_i are *globally* smooth.

2.8 ELASTOSTATICS: GALERKIN FORMULATION, SYMMETRY, AND POSITIVE-DEFINITENESS OF K

Let $\mathcal{S}^h$ and $\mathcal{V}^h$ be finite-dimensional approximations to $\mathcal{S}$ and $\mathcal{V}$, respectively. We assume members $\boldsymbol{w}^h \in \mathcal{V}^h$ result in satisfaction, or approximate satisfaction, of the boundary condition $w_i = 0$ on Γ_{g_i}, and members of $\mathcal{S}^h$ admit the decomposition

$$\boldsymbol{u}^h = \boldsymbol{v}^h + \boldsymbol{g}^h \tag{2.8.1}$$

where $\boldsymbol{v}^h \in \mathcal{V}^h$ and $\boldsymbol{g}^h$ results in satisfaction, or approximate satisfaction, of the boundary condition $u_i = g_i$ on Γ_{g_i}.

The Galerkin formulation of our problem is given as follows:

(G) Given $\boldsymbol{\ell}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ (as in (W)), find $\boldsymbol{u}^h = \boldsymbol{v}^h + \boldsymbol{g}^h \in \mathcal{S}^h$ such that for all $\boldsymbol{w}^h \in \mathcal{V}^h$

$$a(\boldsymbol{w}^h, \boldsymbol{v}^h) = (\boldsymbol{w}^h, \boldsymbol{\ell}) + (\boldsymbol{w}^h, \boldsymbol{h})_\Gamma - a(\boldsymbol{w}^h, \boldsymbol{g}^h) \tag{2.8.2}$$

To define the global stiffness matrix and force vector for elasticity, it is necessary to introduce the ID array. This entails a generalization of the definition given in Sec. 2.6, since in the present case there will be more than 1 degree of freedom per node. For elasticity there are n_{sd} degrees of freedom per node, but in order to include in our

definition cases such as heat conduction, we shall take the fully general situation in which it is assumed that there are n_{dof} ***degrees of freedom per node***.[12] In this case

$$\text{ID}(i, A) = \begin{cases} P & \text{if } A \in \eta - \eta_{q_i} \\ 0 & \text{if } A \in \eta_{q_i} \end{cases} \tag{2.8.3}$$

Global equation number

Degree of freedom number

Global node number

where $1 \leq i \leq n_{\text{dof}}$. Thus ID has dimensions $n_{\text{dof}} \times n_{np}$. If $n_{\text{dof}} = 1$, we reduce to the case considered previously in Sec. 2.6 (i.e., $\text{ID}(i, A) = \text{ID}(A)$).

Recall that $\eta = \{1, 2, \ldots, n_{np}\}$ denotes the set of global node numbers. Let $\eta_{q_i} \subset \eta$ be the set of nodes at which $u_i^h = q_i$ and let $\eta - \eta_{q_i}$ be the complement of η_{q_i}. For each node in $\eta - \eta_{q_i}$, the nodal value of u_i^h is to be determined.

The explicit representations of v_i^h and q_i^h, in terms of the shape functions and nodal values are

$$v_i^h = \sum_{A \in \eta - \eta_{q_i}} N_A d_{iA} \qquad \text{(no sum on } i\text{)} \tag{2.8.4}$$

Degree of freedom number

Global node number

$$q_i^h = \sum_{A \in \eta_{q_i}} N_A q_{iA} \qquad \text{(no sum on } i\text{)} \tag{2.8.5}$$

Let e_i denote the ith Euclidean basis vector for $\mathbb{R}^{n_{sd}}$; e_i has a 1 in slot i and zeros elsewhere. For example

$$(n_{sd} = 2) \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{2.8.6}$$

[12] In general, this is taken to mean the maximum number of degrees of freedom per node in the global model. It is possible in practice to have elements with fewer degrees of freedom per node contributing to the model.

$$(n_{sd} = 3) \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \qquad e_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \qquad e_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{2.8.7}$$

The vector versions of (2.8.4) and (2.8.5) may be defined with the aid of e_i, viz.,

$$\boxed{\begin{aligned} \boldsymbol{v}^h &= v_i^h \boldsymbol{e}_i \qquad (2.8.8) \\ \boldsymbol{q}^h &= q_i^h \boldsymbol{e}_i \qquad (2.8.9) \end{aligned}}$$

Likewise, a typical member $\boldsymbol{w}^h \in \mathcal{V}^h$ has the representation

$$\boxed{\boldsymbol{w}^h = w_i^h \boldsymbol{e}_i, \qquad w_i^h = \sum_{A \in \eta - \eta_{q_i}} N_A c_{iA} \qquad \text{(no sum on } i\text{)}} \tag{2.8.10}$$

Substituting (2.8.4), (2.8.5), and (2.8.8)–(2.8.10) into (2.8.2) and arguing along the lines of Sec. 1.6 results in (verify!)

$$\boxed{\begin{aligned} \sum_{j=1}^{n_{\text{dof}}} \left(\sum_{B \in \eta - \eta_{q_j}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_j) d_{jB} \right) &= (N_A \boldsymbol{e}_i, \boldsymbol{\ell}) + (N_A \boldsymbol{e}_i, \boldsymbol{h})_\Gamma \\ -\sum_{j=1}^{n_{\text{dof}}} \left(\sum_{B \in \eta_{q_j}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_j) q_{jB} \right), \qquad A \in \eta - \eta_{q_i}, \qquad 1 \le i \le n_{sd} \end{aligned}}$$

(2.8.11)[13]

This is equivalent to the matrix equation

[13]For correct interpretation of the meaning of these equations, the sum on j should be taken first. For example, in two dimensions

$$\sum_{j=1}^{2} \left(\sum_{B \in \eta - \eta_{q_j}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_j) d_{jB} \right) = \sum_{B \in \eta - \eta_{q_1}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_1) d_{1B} + \sum_{B \in \eta - \eta_{q_2}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_2) d_{2B}$$

and

$$\sum_{j=1}^{2} \left(\sum_{B \in \eta_{q_j}} (N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_j) q_{jB} \right) = \sum_{B \in \eta_{q_1}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_1) q_{1B} + \sum_{B \in \eta_{q_2}} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_2) q_{2B}$$

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \tag{2.8.12}$$

where

$$\boldsymbol{K} = [K_{PQ}] \tag{2.8.13}$$

$$\boldsymbol{d} = \{d_Q\} \tag{2.8.14}$$

$$\boldsymbol{F} = \{F_P\} \tag{2.8.15}$$

$$K_{PQ} = a(N_A e_i, N_B e_j) \tag{2.8.16}$$

$$F_P = (N_A e_i, \boldsymbol{\ell}) + (N_A e_i, \boldsymbol{h})_\Gamma - \sum_{j=1}^{n_{\text{dof}}} \left(\sum_{B \in \eta_{q_j}} a(N_A e_i, N_B e_j) q_{jB} \right) \tag{2.8.17}$$[13]

in which

$$P = \text{ID}(i, A), \qquad Q = \text{ID}(j, B) \tag{2.8.18}$$

Equation (2.8.16) may be written in more explicit form by using (2.7.26) and noting that (see (2.7.21) and (2.7.22)):

$$\boldsymbol{\epsilon}(N_A e_i) = \boldsymbol{B}_A e_i \tag{2.8.19}$$

where

$$(n_{sd} = 2) \qquad \boldsymbol{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \tag{2.8.20}$$

$$(n_{sd} = 3) \qquad \boldsymbol{B}_A = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix} \tag{2.8.21}$$

Exercise 1. Verify (2.8.19)–(2.8.21).

With these, (2.8.16) becomes

$$\underbrace{K_{PQ}}_{\substack{\text{Global} \\ \text{equation} \\ \text{numbers}}} = \underbrace{e_i^T \int_\Omega \boldsymbol{B}_A^T \boldsymbol{D} \boldsymbol{B}_B \, d\Omega \, e_j}_{\substack{\text{Global node numbers} \\ \text{Degree of freedom numbers}}} \tag{2.8.22}$$

and the indices are related by (2.8.18).

Equation (2.8.17) is also amenable to explication. Note that, by (2.7.18)

$$(N_A e_i, \ell) = \int_\Omega N_A \ell_i \, d\Omega \tag{2.8.23}$$

and likewise by (2.7.19)

$$(N_A e_i, h)_\Gamma = \int_{\Gamma_{h_i}} N_A h_i \, d\Gamma \qquad \text{(no sum)} \tag{2.8.24}$$

Thus (2.8.17) may be written as

$$F_P = \int_\Omega N_A \ell_i \, d\Omega + \int_{\Gamma_{h_i}} N_A h_i \, d\Gamma - \sum_{j=1}^{n_{dof}} \left(\sum_{B \in \eta_{q_j}} a(N_A e_i, N_B e_j) q_{jB} \right) \tag{2.8.25}$$[14]

Now that we have defined $\boldsymbol{K}$, we can establish its fundamental properties. We shall need the following preliminary results.

Let $n_{sd} = 2$ or 3 and let $w : \Omega \to \mathbb{R}^{n_{sd}}$. If $w_{(i,j)} = 0$ ("zero strains"), then w admits the representations:

$$\begin{array}{lll} & \overbrace{}^{\text{Translation}} \quad \overbrace{}^{\text{Rotation}} & \\ (n_{sd} = 2) \qquad w(x) = c + c_3(x_1 e_2 - x_2 e_1) & & (2.8.26) \\ (n_{sd} = 3) \qquad w(x) = \underbrace{c_1}_{\text{Translation}} + \underbrace{c_2 \times x}_{\text{Rotation}} & & (2.8.27) \end{array}$$

where

$$c = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \qquad c_1 = \begin{Bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{Bmatrix} \qquad c_2 = \begin{Bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{Bmatrix} \tag{2.8.28}$$

and c_3 are constants; and $\times$ denotes the vector cross product. Equations (2.8.26) and (2.8.27) define ***infinitesimal rigid-body motions***.

Assumption R

We assume that the homogeneous boundary conditions incorporated into the definition of $\mathcal{V}^h$ *preclude* nontrivial infinitesimal rigid-body motions. In other words, we assume that if $w^h \in \mathcal{V}^h$ is a rigid-body motion, then w^h is identically zero.

Theorem

1. $\boldsymbol{K}$ is symmetric.
2. If Assumption R holds, then $\boldsymbol{K}$ is also positive definite.

Proof of 1. **Symmetry** We may note that symmetry of $\boldsymbol{K}$ follows from (2.8.16) and the symmetry of $a(\cdot\,, \cdot)$. However, we shall provide an alternative proof in terms of (2.8.22).

[14]See footnote 13 of this chapter.

$$K_{PQ} = \boldsymbol{e}_i^T \int_\Omega \boldsymbol{B}_A^T \boldsymbol{D} \boldsymbol{B}_B \, d\Omega \, \boldsymbol{e}_j$$

$$= \boldsymbol{e}_j^T \int_\Omega \boldsymbol{B}_B^T \boldsymbol{D}^T \boldsymbol{B}_A \, d\Omega \, \boldsymbol{e}_i$$

$$= \boldsymbol{e}_j^T \int_\Omega \boldsymbol{B}_B^T \boldsymbol{D} \boldsymbol{B}_A \, d\Omega \, \boldsymbol{e}_i \qquad \text{(symmetry of } \boldsymbol{D}\text{)}$$

$$= K_{QP} \qquad \blacksquare$$

Remark

Note that the symmetry of $\boldsymbol{K}$ followed from the symmetry of $\boldsymbol{D}$, which was a consequence of the major symmetry of the c_{ijkl}'s (see (2.7.3)).

Proof of 2. **Positive definite** (Recall from Sec. 1.9 that we must show (i) $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} \geq 0$ and (ii) $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$ implies $\boldsymbol{c} = \boldsymbol{0}$.)

Let $w_i^h = \Sigma_{A \in \eta - \eta_{q_i}} N_A c_{iA}$ be a member of $\mathcal{V}_i^h$. Then $c_P = c_{iA}$, where $P = \mathrm{ID}(i, A)$, $1 \leq P \leq n_{eq}$, defines the components of an n_{eq}-vector $\boldsymbol{c}$.

i.

$$\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q$$

$$= \sum_{i,j=1}^{n_{\mathrm{dof}}} \left(\sum_{\substack{A \in \eta - \eta_{q_i} \\ B \in \eta - \eta_{q_j}}} c_{iA} a(N_A \boldsymbol{e}_i, N_B \boldsymbol{e}_j) c_{jB} \right) \qquad \text{(definition of } K_{PQ}\text{)}$$

$$= a\left(\sum_{i=1}^{n_{\mathrm{dof}}} \left(\sum_{A \in \eta - \eta_{q_i}} c_{iA} N_A \boldsymbol{e}_i \right), \sum_{j=1}^{n_{\mathrm{dof}}} \left(\sum_{B \in \eta - \eta_{q_j}} c_{jB} N_B \boldsymbol{e}_j \right) \right) \qquad \text{(bilinearity of } a(\cdot\,,\cdot)\text{)}$$

$$= a(w^h, w^h) \qquad \text{(definition of } w^h\text{)}$$

$$= \int_\Omega \underbrace{w^h_{(i,j)} c_{ijkl} w^h_{(k,l)}}_{\geq 0} \, d\Omega \qquad \text{(by (2.7.5) and (2.7.17))}$$

$$\geq 0$$

ii. Assume $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$. By the proof of part (i), we deduce that

$$w^h_{(i,j)} c_{ijkl} w^h_{(k,l)} = 0$$

From (2.7.6), this means that $w^h_{(i,j)} = 0$, and so $\boldsymbol{w}^h$ is an infinitesimal rigid motion. By Assumption R, $\boldsymbol{w}^h = \boldsymbol{0}$, from which it follows that $c_p = 0$; hence $\boldsymbol{c} = \boldsymbol{0}$. $\blacksquare$

Remark

Positive definiteness of $\boldsymbol{K}$ is based upon two requirements: a positive-definiteness condition on the constitutive coefficients and suitable boundary conditions being incorporated into $\mathcal{V}^h$.

2.9 ELASTOSTATICS: ELEMENT STIFFNESS MATRIX AND FORCE VECTOR

As usual, K and F may be decomposed into sums of elemental contributions. These results will be omitted here as the reader should now be familiar with the ideas involved (cf. Sec. 2.5). We will proceed directly to the definitions of k^e and f^e:

$$k^e = [k^e_{pq}], \qquad f^e = \{f^e_p\}, \qquad 1 \le p, q \le n_{ee} = n_{ed} n_{en} \tag{2.9.1}$$

[15]

$$k^e_{pq} = e_i^T \int_{\Omega e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega \, e_j, \qquad p = n_{ed}(a-1) + i, \quad q = n_{ed}(b-1) + j \tag{2.9.2}$$

$$(n_{sd} = 2) \qquad \boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix} \tag{2.9.3}$$

$$(n_{sd} = 3) \qquad \boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix} \tag{2.9.4}$$

and

$$f^e_p = \int_{\Omega^e} N_a b_i \, d\Omega + \int_{\Gamma^e_{h_i}} N_a h_i \, d\Gamma - \sum_{q=1}^{n_{ee}} k_{pq} g^e_q, \qquad \Gamma^e_{h_i} = \Gamma_{h_i} \cap \Gamma^e \quad \text{(no sum on } i\text{)} \tag{2.9.5}$$

where $g^e_q = g^e_{jb} = g_j(\boldsymbol{x}^e_b)$ if g_j is prescribed at node b, and equals zero otherwise.

It is useful for programming purposes to define the ***nodal submatrix***

$$\underbrace{\boldsymbol{k}^e_{ab}}_{n_{ed} \times n_{ed}} = \int_{\Omega^e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega \tag{2.9.6a}$$

From (2.9.2) we see that

[15] n_{ee} stands for the ***number of element equations*** and n_{ed} is the ***number of element degrees of freedom*** (per node). It is possible in practice to have $n_{ed} < n_{dof}$, although they are usually equal.

$$\boxed{k^e_{pq} = e_i^T k^e_{ab} e_j} \tag{2.9.6}$$

This means, "the pq-component of k^e is the ij-component of the submatrix k^e_{ab}."

By (2.9.1) through (2.9.4), we see that k^e may be written as

$$\boxed{k^e = \int_{\Omega^e} B^T D B \, d\Omega} \tag{2.9.}$$

where

$$B = [B_1, B_2, \ldots, B_{n_{en}}] \tag{2.9.}$$

For example, in the case of a two-dimensional (i.e., $n_{sd}=n_{ed}=2$), four-noded elemei k^e takes the form

$$\underbrace{k^e}_{8 \times 8} = \left[\begin{array}{cccc} k^e_{11} & k^e_{12} & k^e_{13} & k^e_{14} \\ \hdashline k^e_{21} & k^e_{22} & k^e_{23} & k^e_{24} \\ k^e_{31} & k^e_{32} & k^e_{33} & k^e_{34} \\ k^e_{41} & k^e_{42} & k^e_{43} & k^e_{44} \end{array}\right] \tag{2.9.}$$

In practice, the submatrices above the dashed line are computed and those below, required, are determined by symmetry.

The global arrays K and F may be formed from the element arrays k^e and f respectively, by way of an assembly algorithm as outlined in Sec. 1.14.

Exercise 1. Let

$$\underbrace{d^e}_{n_{ee} \times 1} = \{d^e_a\} = \begin{Bmatrix} d^e_1 \\ d^e_2 \\ \vdots \\ d^e_{n_{en}} \end{Bmatrix} \tag{2.9.1}$$

$$(n_{ed} = 2) \qquad d^e_a = \begin{Bmatrix} d^e_{1a} \\ d^e_{2a} \end{Bmatrix} \tag{2.9.1}$$

$$(n_{ed} = 3) \qquad d^e_a = \begin{Bmatrix} d^e_{1a} \\ d^e_{2a} \\ d^e_{3a} \end{Bmatrix} \tag{2.9.1}$$

where

$$d_{ia}^e = u_i^h(x_a^e) \tag{2.9.13}$$[16]

d^e is called the ***element displacement vector***. Show that the stress vector (see Exercise 4, Sec. 2.7.) at point $x \in \Omega^e$ can be calculated from the formula

$$\sigma(x) = D(x)B(x)d^e = D(x)\sum_{a=1}^{n_{en}} B_a(x)d_a^e \tag{2.9.14}$$

2.10 ELASTOSTATICS: DATA PROCESSING ARRAYS ID, IEN, AND LM

We have already noted that the definition of the ID array must be generalized for the present case as indicated in Sec. 2.8. We must also generalize our definition of the LM array. However, the IEN array remains the same as before.

In the present and fully general cases, the LM array is three-dimensional, with dimensions $n_{ed} \times n_{en} \times n_{el}$, and is defined by

$$\mathrm{LM}(i, a, e) = \mathrm{ID}(i, \mathrm{IEN}(a, e)) \tag{2.10.1}$$

Degrees of freedom number — Local node number — Element number

Alternatively, it is sometimes convenient to think of LM as two-dimensional, with dimensions $n_{ee} \times n_{el}$, viz.,[17]

$$\mathrm{LM}(p, e) = \mathrm{LM}(i, a, e), \qquad p = n_{ed}(a - 1) + i \tag{2.10.2}$$

Local equation number — Element number

To see how everything works in practice, it is helpful to run through a simple example.

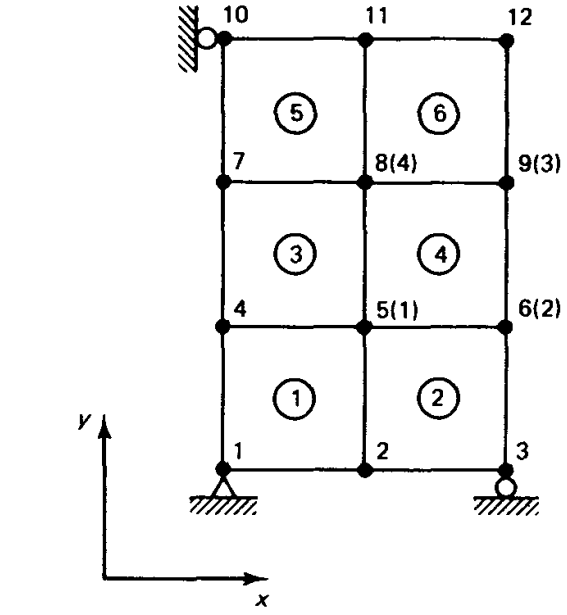
Example 1

Consider the mesh of four-node, rectangular elements illustrated in Fig. 2.10.1. We assume that the local node numbering begins in the lower left-hand corner for each element and proceeds in counterclockwise fashion.

[16] The q-boundary conditions are accounted for in this definition.

[17] The reader knowledgeable in FORTRAN will realize that the internal computer storage of (2.10.1) and (2.10.2) is identical.

This is shown for element 4, which is typical. Four displacement (i.e., "q-typ boundary conditions are specified; namely, the horizontal displacement is specifie nodes 1 and 10, and the vertical displacement is specified at nodes 1 and 3. Si $n_{np} = 12$, $n_{\text{dof}} = n_{ed} = 2$, and 4 displacement degrees of freedom are specified, we h $n_{eq} = 20$. As is usual, we adopt the convention that the global equation numbers rur ascending order with respect to the ascending order of global node numbers.[18] The IEN, and LM arrays are given in Figure 2.10.2. The reader is urged to verify the resu



Element numbers

( ) Local node numbers

Horizontal "roller" (vertical displacement specified*)

Vertical "roller" (horizontal displacement specified*)

"Fixity" (vertical and horizontal displacements specified*)

*Possibly nonzero

Figure 2.10.1 Mesh of four-node, rectangular, elasticity elements; globa and local node numbers, element nun bers, and displacement boundary con tions.

In terms of the IEN and LM arrays, a precise definition of the q_p^e's may be give (see (2.9.5)):

$$q_p^e = q_{ia}^e = \begin{cases} 0, & \text{if LM}(i, a, e) \neq 0 \\ q_{iA}, & \text{where } A = \text{IEN}(a, e), \text{ if LM}(i, a, e) = 0 \end{cases} \tag{2.10.}$$

This definition may be easily programmed.

[18] In practice, equation numbers are often renumbered internally to minimize the bandwidth of and thus decrease storage and solution effort. This is especially important in analyzing large-scale systen involving tens of thousands of equations. An algorithm for reducing bandwidth is presented in [8].

ID array:

Global node numbers (A)

Global degrees of freedom (i)	1	2	3	4	5	6	7	8	9	10	11	12
1	0*	1	3	4	6	8	10	12	14	0	17	19
2	0	2	0	5	7	9	11	13	15	16	18	20

($n_{np} = 12$)

($n_{dof} = 2$) $P = \mathrm{ID}(i, A)$ ($n_{eq} = 20$)

IEN array:

Element numbers (e)

Local node numbers (a)	1	2	3	4	5	6
1	1	2	4	5	7	8
2	2	3	5	6	8	9
3	5	6	8	9	11	12
4	4	5	7	8	10	11

($n_{el} = 6$)

($n_{en} = 4$)

$A = \mathrm{IEN}(a, e)$

LM array:

Element numbers (e)

		1	2	3	4	5	6
1	1	0*	1	4	6	10	12
	2	0	2	5	7	11	13
2	1	1	3	6	8	12	14
	2	2	0	7	9	13	15
3	1	6	8	12	14	17	19
	2	7	9	13	15	18	20
4	1	4	6	10	12	0	17
	2	5	7	11	13	16	18

($n_{el} = 6$)

1, 2, 3, 4, 5, 6, 7, 8 — Local equation numbers ($p = n_{ed}(a - 1) + i$)

($n_{ee} = 8$)

Element degree of freedom numbers (i, where $1 \le i \le n_{ed}$ and here $n_{ed} = 2$)

Local node numbers (a)

$P = \mathrm{LM}(p, e) = \mathrm{LM}(i, a, e) = \mathrm{ID}(i, \mathrm{IEN}(a, e))$

*Displacement boundary conditions are denoted by zeros.

Figure 2.10.2 ID, IEN, and LM arrays for the mesh of Figure 2.10.1.

Example 2

As a final example, we consider a typical four-node, elasticity element in some lar mesh; see Fig. 2.10.3. We assume the pertinent entries of the ID array are given follows:

$$\left.\begin{aligned} \mathrm{ID}(1,\ 32) &= 0 \\ \mathrm{ID}(2,\ 32) &= 0 \\ \mathrm{ID}(1,\ 59) &= 115 \\ \mathrm{ID}(2,\ 59) &= 116 \\ \mathrm{ID}(1,\ 164) &= 0 \\ \mathrm{ID}(2,\ 164) &= 325 \\ \mathrm{ID}(1,\ 168) &= 332 \\ \mathrm{ID}(2,\ 168) &= 333 \end{aligned}\right\} \qquad (2.10.$$

The entries of IEN follow from Fig. 2.10.3:

$$\left.\begin{aligned} \mathrm{IEN}(1,\ e) &= 164 \\ \mathrm{IEN}(2,\ e) &= 32 \\ \mathrm{IEN}(3,\ e) &= 168 \\ \mathrm{IEN}(4,\ e) &= 59 \end{aligned}\right\} \qquad (2.10.$$

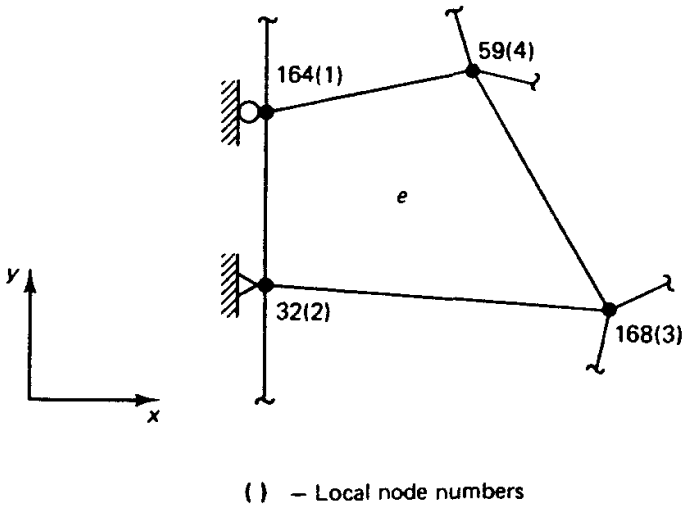


Figure 2.10.3 Typical four-node elasticity element; global and local node numbers.

Combining (2.10.4) and (2.10.5), by way of (2.10.1), yields entries of the LM array:

$$\left.\begin{aligned}
\mathrm{LM}(1,\ 1,\ e) &= \ 0\\
\mathrm{LM}(2,\ 1,\ e) &= 325\\
\mathrm{LM}(1,\ 2,\ e) &= \ 0\\
\mathrm{LM}(2,\ 2,\ e) &= \ 0\\
\mathrm{LM}(1,\ 3,\ e) &= 332\\
\mathrm{LM}(2,\ 3,\ e) &= 333\\
\mathrm{LM}(1,\ 4,\ e) &= 115\\
\mathrm{LM}(2,\ 4,\ e) &= 116
\end{aligned}\right\} \qquad (2.10.6)$$

The contribution to the global arrays may be deduced from LM:

Stiffness (due to symmetry, only the upper triangular portion need be assembled.)

$$\left.\begin{aligned}
K_{115,\,115} &\leftarrow K_{115,\,115} + k^e_{77}\\
K_{115,\,116} &\leftarrow K_{115,\,116} + k^e_{78}\\
K_{115,\,325} &\leftarrow K_{115,\,325} + k^e_{72}\\
K_{115,\,332} &\leftarrow K_{115,\,332} + k^e_{75}\\
K_{115,\,333} &\leftarrow K_{115,\,333} + k^e_{76}\\
K_{116,\,116} &\leftarrow K_{116,\,116} + k^e_{88}\\
K_{116,\,325} &\leftarrow K_{116,\,325} + k^e_{82}\\
K_{116,\,332} &\leftarrow K_{116,\,332} + k^e_{85}\\
K_{116,\,333} &\leftarrow K_{116,\,333} + k^e_{86}\\
K_{325,\,325} &\leftarrow K_{325,\,325} + k^e_{22}\\
K_{325,\,332} &\leftarrow K_{325,\,332} + k^e_{25}\\
K_{325,\,333} &\leftarrow K_{325,\,333} + k^e_{26}\\
K_{332,\,332} &\leftarrow K_{332,\,332} + k^e_{55}\\
K_{332,\,333} &\leftarrow K_{332,\,333} + k^e_{56}\\
K_{333,\,333} &\leftarrow K_{333,\,333} + k^e_{66}
\end{aligned}\right\} \qquad (2.10.7)$$

Force

$$\left.\begin{aligned}
F_{115} &\leftarrow F_{115} + f^e_7\\
F_{116} &\leftarrow F_{116} + f^e_8\\
F_{325} &\leftarrow F_{325} + f^e_2\\
F_{332} &\leftarrow F_{332} + f^e_5\\
F_{333} &\leftarrow F_{333} + f^e_6
\end{aligned}\right\} \qquad (2.10.8)$$

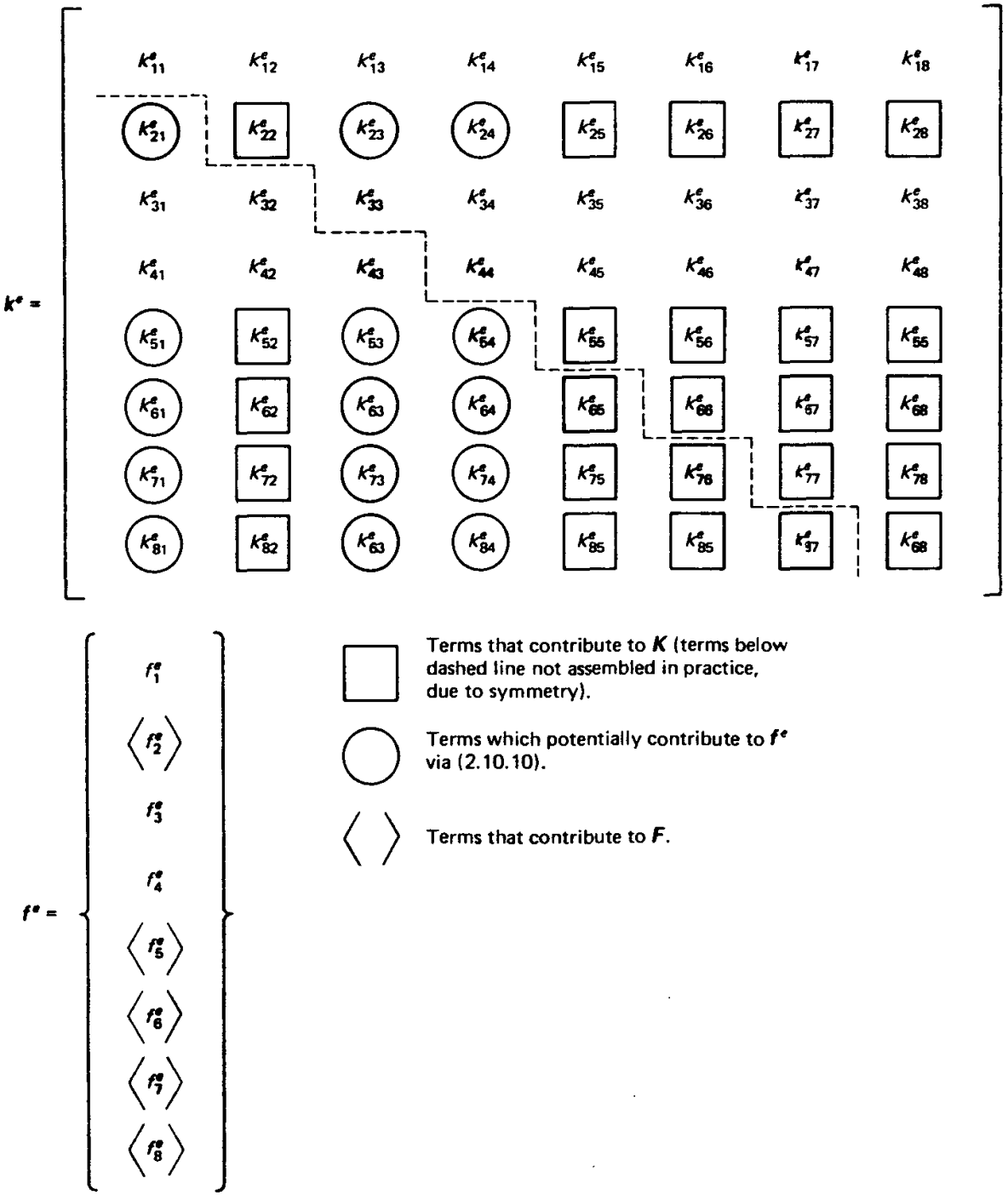
where

$$f_p^e = \cdots - \sum_{q=1}^{n_{ec}} k_{pq}^e q_q^e \tag{2.10.9}$$

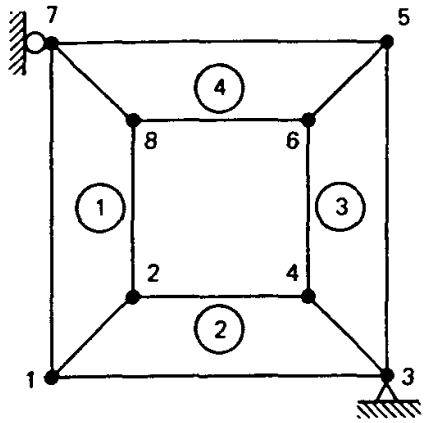
(We have omitted the first two terms in the right-hand side of (2.9.5) in writing (2.10.9).) In the present example, only q_1^e, q_3^e and q_4^e may be nonzero. Therefore (2.10.9) may be simplified to

$$f_p^e = \cdots - k_{p1}^e q_1^e - k_{p3}^e q_3^e - k_{p4}^e q_4^e \tag{2.10.10}$$

The multiplications indicated in (2.10.10) are only performed in practice if the q_p^e's are nonzero. A schematic representation of the contributions of $\boldsymbol{k}^e$ and $\boldsymbol{f}^e$ to $\boldsymbol{K}$ and $\boldsymbol{F}$ is shown in Figure 2.10.4.



Exercise 1. Consider a two-dimensional elastostatic boundary-value problem. Set up the ID, IEN, and LM arrays for the following mesh of four-node quadrilaterals:



2.11 SUMMARY OF IMPORTANT EQUATIONS FOR PROBLEMS CONSIDERED IN CHAPTERS 1 AND 2

1. *Classical Linear Elastostatics*

(S)

$$\sigma_{ij,j} + f_i = 0 \quad \text{on } \Omega$$

$$u_i = g_i \quad \text{on } \Gamma_{g_i}$$

$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_{h_i}$$

where $\sigma_{ij} = c_{ijkl}\epsilon_{kl} = c_{ijkl}u_{(k,l)}$

(W)[19] Find $u \in \mathcal{S}$, such that $\forall\, w \in \mathcal{V}$

$$a(w, u) = (w, \ell) + (w, h)_\Gamma$$

where

$$a(w, u) = \int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega$$

$$(w, \ell) = \int_\Omega w_i \ell_i\, d\Omega$$

$$(w, h)_\Gamma = \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Omega \right)$$

(G) Find $v^h \in \mathcal{V}^h$, such that $\forall\, w^h \in \mathcal{V}^h$

$$a(w^h, v^h) = (w^h, \ell) + (w^h, h)_\Gamma - a(w^h, g^h)$$

[19] The notation $\forall$ means "for all."

(*M*) $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$, where $\boldsymbol{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}}(\boldsymbol{k}^e)$, $\boldsymbol{F} = \boldsymbol{F}_{\text{nodal}} + \mathop{\mathbf{A}}_{e=1}^{n_{el}}(\boldsymbol{f}^e)$[20]

$$k^e_{pq} = \boldsymbol{e}_i^T \boldsymbol{k}^e_{ab} \boldsymbol{e}_j, \qquad \boldsymbol{k}^e_{ab} = \int_{\Omega^e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega$$

$$f^e_p = \int_{\Omega^e} N_a \ell_i \, d\Omega + \int_{\Gamma^e_{h_i}} N_a h_i \, d\Gamma - \sum_{q=1}^{n_{el}} k^e_{pq} q^e_q \qquad \text{(no sum on } i\text{)}$$

$$p = n_{ed}(a - 1) + i$$

$$q = n_{ed}(b - 1) + j$$

Stress at a point: $\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) \boldsymbol{d}^e_a$

2. *Classical Linear Heat Conduction*

(*S*)
$$q_{i,i} = \ell \quad \text{in } \Omega$$
$$u = g \quad \text{on } \Gamma_g$$
$$-q_i n_i = h \quad \text{on } \Gamma_h$$

where $q_i = \kappa_{ij} u_{,j}$

(*W*) Find $u \in \mathcal{S}$, such that $\forall\, w \in \mathcal{V}$

$$a(w, u) = (w, \ell) + (w, h)_\Gamma$$

where

$$a(w, u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} \, d\Omega$$

$$(w, \ell) = \int_\Omega w \ell \, d\Omega$$

$$(w, h)_\Gamma = \int_\Gamma w h \, d\Gamma$$

(*G*) Find $v^h \in \mathcal{S}^h$, such that $\forall\, w^h \in \mathcal{V}^h$

$$a(w^h, v^h) = (w^h, \ell) + (w^h, h)_\Gamma - a(w^h, g^h)$$

[20]In defining $\boldsymbol{F}$ we have added to the element contributions the term $\boldsymbol{F}_{\text{nodal}}$, which is a vector of nodal applied forces. The reason for this is that it is often easier in practice to directly input concentrated forces at nodes rather than go through the element-by-element form and assemble procedure. The expression for $\boldsymbol{F}$ then emphasizes that both modes of constructing $\boldsymbol{F}$ are to be accommodated in the computer implementation of problems of this type.

(M) $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$, where $\boldsymbol{K} = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}}(\boldsymbol{k}^e)$, $\boldsymbol{F} = \boldsymbol{F}_{\text{nodal}} + \underset{e=1}{\overset{n_{el}}{\mathbf{A}}}(\boldsymbol{f}^e)^{20}$

$$k^e_{ab} = \int_{\Omega^e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega$$

$$f^e_a = \int_{\Omega^e} N_a \ell \, d\Omega + \int_{\Gamma^e_h} N_a h \, d\Gamma - \sum_{b=1}^{n_{el}} k^e_{ab} q^e_b$$

Heat flux vector at a point: $\boldsymbol{q}(x) = -\boldsymbol{D}(x) \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(x) d^e_a$

3. *One-Dimensional Model Problem*

(S)
$$u_{,xx} + \ell = 0 \qquad \text{on } \Omega =]0, 1[$$
$$u(1) = q \qquad (\Gamma_q = \{1\})$$
$$-u_{,x}(0) = h \qquad (\Gamma_h = \{0\})$$

(W) Find $u \in \mathcal{S}$, such that $\forall\, w \in \mathcal{V}$

$$a(w, u) = (w, \ell) + wh(0)$$

where

$$a(w, u) = \int_0^1 w_{,x} u_{,x} \, dx$$

$$(w, \ell) = \int_0^1 w \ell \, dx$$

(G) Find $v^h \in \mathcal{V}^h$, such that $\forall\, w^h \in \mathcal{V}^h$

$$a(w^h, v^h) = (w^h, \ell) + w^h(0)h - a(w^h, q^h)$$

(M) $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$, where $\boldsymbol{K} = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}}(\boldsymbol{k}^e)$, $\boldsymbol{F} = \boldsymbol{F}_{\text{nodal}} + \underset{e=1}{\overset{n_{el}}{\mathbf{A}}}(\boldsymbol{f}^e)^{20}$

$$k^e_{ab} = \int_{\Omega^e} N_{a,x} N_{b,x} \, dx$$

$$f^e_a = \int_{\Omega^e} N_a \ell \, dx + \begin{cases} h\delta_{a1} & e = 1 \\ 0 & e = 2, \ldots, n_{el} - 1 \\ -q k^e_{2a} & e = n_{el} \end{cases}$$

2.12 AXISYMMETRIC FORMULATIONS AND ADDITIONAL EXERCISES

Axisymmetric formulations are expressed in terms of cylindrical coordinates:

$$x_1 = r = \text{the radial coordinate}$$

$$x_2 = z = \text{the axial coordinate}$$

$$x_3 = \theta = \text{the circumferential coordinate}$$

The basic hypothesis of axisymmetry is that all functions under consideration are independent of θ. That is, they are functions of r and z only. Thus three-dimensional problem classes are reduced to two-dimensional ones.

Heat Conduction

The axisymmetric formulation for heat conduction is almost identical to the two-dimensional Cartesian case considered previously. The only difference is that a factor of $2\pi r$ needs to be included in each integrand of the variational equation to account for the correct volumetric weighting (e.g., $d\Omega = 2\pi r\, dr\, dz$ replaces $d\Omega = dx_1\, dx_2$). Since the constant 2π is common to all terms, it may be cancelled throughout if desired.

Elasticity

The displacement components in cylindrical coordinates are:

$$u_1 = u_r = \text{the radial displacement}$$

$$u_2 = u_z = \text{the axial displacement}$$

$$u_3 = u_\theta = \text{the circumferential displacement}$$

In addition to the basic hypothesis of axisymmetry, we further assume that $u_\theta = 0$, and thus

$$\epsilon_{r\theta} = \epsilon_{z\theta} = 0 \tag{2.12.1}$$

Note that $\epsilon_{\theta\theta} = u_r/r$ and therefore it is generally not zero. The constitutive moduli are assumed to be such that the preceding kinematical hypotheses result in

$$\sigma_{r\theta} = \sigma_{z\theta} = 0 \tag{2.12.2}$$

The preceding assumptions lead to what is called the ***torsionless axisymmetric case.*** This formulation is similar to but somewhat more complicated than the two-dimensional cases of plane strain and plane stress. Here there are four nonzero components of stress and strain:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{33} \end{Bmatrix} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{rz} \\ \sigma_{\theta\theta} \end{Bmatrix} \tag{2.12.3}$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ \epsilon_{33} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ 2\epsilon_{rz} \\ \epsilon_{\theta\theta} \end{Bmatrix} \tag{2.12.4}$$

The ordering emanates from the following generalization of Table 2.7.1:

I / J	i / k	j / l
1	1	1
3	1	2
3	2	1
2	2	2
4	3	3

The $\boldsymbol{D}$ array takes on the following form:

$$\boldsymbol{D} = [D_{IJ}] = \underbrace{\begin{bmatrix} \boldsymbol{D}_{33} & \boldsymbol{D}_3 \\ \boldsymbol{D}_3^T & D_{44} \end{bmatrix}}_{4 \times 4} \tag{2.12.5}$$

$$\boldsymbol{D}_{33} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{symmetric} & & D_{33} \end{bmatrix} \tag{2.12.6}$$

$$\boldsymbol{D}_3 = \begin{Bmatrix} D_{14} \\ D_{24} \\ D_{34} \end{Bmatrix} \tag{2.12.7}$$

where $D_{IJ} = c_{ijkl}$, in which the indices are related by the table. The $\boldsymbol{B}_a$-matrix takes on the form

$$\boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \\ \hline \dfrac{N_a}{r} & 0 \end{bmatrix} \tag{2.12.8}$$

Again, a factor of $2\pi r$ needs to be included in all integrands.

Remark

The ***plane strain*** case may be obtained from the axisymmetric formulation by

i. Ignoring the $2\pi r$ factors; and
ii. Ignoring the fourth row of $\boldsymbol{B}_a$ and the fourth row and column of $\boldsymbol{D}$.

Furthermore, the ***plane stress*** case may be similarly obtained if, in addition, $\boldsymbol{D}_{33}$ is replaced by

$$\boxed{\boldsymbol{D}_{33} - \boldsymbol{D}_3 \boldsymbol{D}_{44}^{-1} \boldsymbol{D}_3^T} \tag{2.12.9}$$

which directly follows from the plane stress condition, $\sigma_{44} = 0$. (References [6] and [7] may be consulted for further elaboration on the physical ideas.) Sometimes (2.12.9) is referred to as the ***statically condensed*** elastic coefficient matrix.

Consequently, in programming the axisymmetric case, for a small amount of additional effort both plane strain and plane stress may also be included.

Exercise 1. Under the assumption of isotropy, show that $\boldsymbol{D}_{33}$ is the same as the $\boldsymbol{D}$-matrix in (2.7.34). Furthermore, show that $D_{44} = \lambda + 2\mu$ and

$$\boldsymbol{D}_3 = \begin{Bmatrix} \lambda \\ \lambda \\ 0 \end{Bmatrix} \tag{2.12.10}$$

Exercise 2. Verify that for the isotropic case, (2.12.9) achieves a similar end to the procedure described in Remark 9 of Sec. 2.7.

Exercise 3. Consider the one-dimensional model problem discussed previously. Obtain exact expressions for $f = \{f_a^e\}$, $a = 1, 2$, for the following cases (ignore q and h contributions):

i. ℓ = constant.
ii. $\ell = \delta(x - \bar{x})$, the delta function, where $x_1^e \le \bar{x} \le x_2^e$. Specialize for the cases $\bar{x} = x_b^e$ and $\bar{x} = (x_1^e + x_2^e)/2$.

Solution

$$f_a^e = \int_{x_1^e}^{x_2^e} N_a(x)\ell(x)\, dx$$

i. $$f_a^e = \ell \int_{x_1^e}^{x_2^e} N_a(x)\, dx = \frac{\ell h^e}{2} \underbrace{\int_{-1}^{+1} N_a(\xi)\, d\xi}_{1}$$

$$f^e = \frac{\ell h^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

ii. $f_a^e = \int_{x_1^e}^{x_2^e} N_a(x)\delta(x - \bar{x})\, dx = N_a(\bar{x})$

For $x = x_b^e$,

$$f_a^e = N_a(\bar{x}) = N_a(x_b^e) = \delta_{ab} \qquad \text{(Kronecker delta)}$$

$$f^e = \begin{Bmatrix} \delta_{1b} \\ \delta_{2b} \end{Bmatrix}$$

For $\bar{x} = (x_1^e + x_2^e)/2$,

$$f_a^e = N_a(\bar{x}) = N_a\left(\frac{x_1^e + x_2^e}{2}\right) = \frac{1}{2}$$

Therefore,

$$f^e = \frac{1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Exercise 4. Consider the boundary-value problem for classical linear elastostatics discussed previously. In the linearized theory of ***small displacements superposed upon large***, the stiffness term in the variational equation,

$$\int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega$$

is replaced by

$$\int_\Omega w_{i,j} d_{ijkl} u_{k,l}\, d\Omega$$

where

$$d_{ijkl} = c_{ijkl} + \delta_{ik}\sigma_{jl}^0$$
$$\sigma_{jl}^0 = \sigma_{lj}^0$$

and the σ_{jl}^0's (i.e., ***initial stresses***) are given functions of $\boldsymbol{x} \in \Omega$. It follows from the symmetries of c_{ijkl} and σ_{jl}^0 that

$$d_{ijkl} = d_{klij}$$

Assume $n_{sd} = 2$. An index-free formulation of the stiffness term is given by

$$\int_\Omega \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \\ w_{1,2} - w_{2,1} \end{Bmatrix}^T \underbrace{\boldsymbol{D}}_{4 \times 4} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \\ u_{1,2} - u_{2,1} \end{Bmatrix} d\Omega$$

which leads to the following definition of the element stiffness matrix:

$$k_{pq}^e = e_i^T \int_{\Omega^e} \underbrace{\boldsymbol{B}_a^T}_{2 \times 4} \underbrace{\boldsymbol{D}}_{4 \times 4} \underbrace{\boldsymbol{B}_b}_{4 \times 2}\, d\Omega\, e_j$$

Set up $\boldsymbol{B}_a$ in terms of the shape function N_a. Define the components of $\boldsymbol{D}$ in terms of the d_{ijkl}'s. (The σ_{jl}^0-contribution to the stiffness is sometimes called the ***initial-stress stiffness matrix***. It is important to account for it in the solution of many nonlinear problems.)

Exercise 5. Let Ω be a region in $\mathbb{R}^2$ and let its boundary $\Gamma = \overline{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}$ where $\Gamma_1, \ldots, \Gamma_4$ are nonoverlapping subregions of Γ. Let $\boldsymbol{n}$ be the unit outward normal vector to Γ such that $\boldsymbol{s}$ and $\boldsymbol{n}$ form a right-hand rule basis; see Fig. 2.12.1. Consider the following boundary-value problem in classical linear elastostatics: Given $f_i : \Omega \to \mathbb{R}$; $g_i : \Gamma_1 \to \mathbb{R}$; $h_i : \Gamma_2 \to \mathbb{R}$; g_n and $h_s : \Gamma_3 \to \mathbb{R}$; and g_s and $h_n : \Gamma_4 \to \mathbb{R}$; find $u_i : \overline{\Omega} \to \mathbb{R}$ such that

$$
\begin{aligned}
\sigma_{ij,j} + f_i &= 0 && \text{in } \Omega \\
u_i &= g_i && \text{on } \Gamma_1 \\
\sigma_{ij} n_j &= h_i && \text{on } \Gamma_2 \\
\left.\begin{aligned} u_i n_i &= g_n \\ \sigma_{ij} n_j s_i &= h_s \end{aligned}\right\} && \text{on } \Gamma_3 && \begin{pmatrix}\text{normal displacement} \\ \text{tangential traction}\end{pmatrix} \\
\left.\begin{aligned} u_i s_i &= g_s \\ \sigma_{ij} n_j n_i &= h_n \end{aligned}\right\} && \text{on } \Gamma_4 && \begin{pmatrix}\text{tangential displacement} \\ \text{normal traction}\end{pmatrix}
\end{aligned}
$$

where $\sigma_{ij} = c_{ijkl} u_{(k,l)}$. Establish a weak formulation for this problem in which all "g-type" boundary conditions are essential and all "h-type" boundary conditions are natural. State all requirements on the spaces $\mathcal{S}$ and $\mathcal{V}$. *Hint:* $\boldsymbol{w} = w_n \boldsymbol{n} + w_s \boldsymbol{s}$; i.e., $w_i = w_n n_i + w_s s_i$.

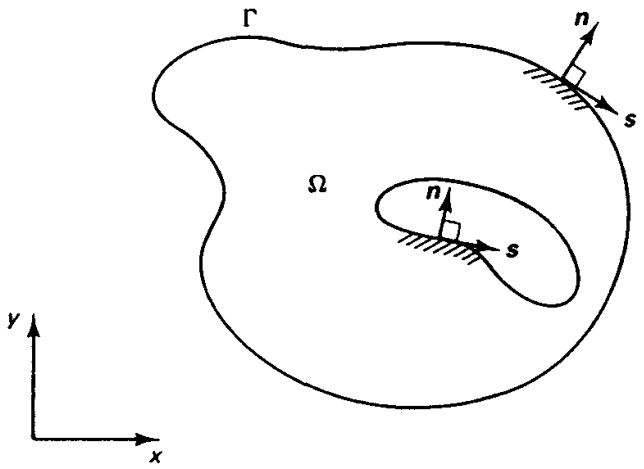


Figure 2.12.1

Exercise 6. In practice, it is often useful to generalize the constitutive equation of classical elasticity to the form

$$\sigma_{ij} = c_{ijkl}(\epsilon_{kl} - \epsilon^0_{kl}) + \sigma^0_{ij} \tag{2.12.11}$$

where ϵ^0_{ij} and σ^0_{ij} are the ***initial strain*** and ***initial stress***, both given functions of $\boldsymbol{x}$. The initial strain term may be used to represent thermal expansion effects by way of

$$\epsilon^0_{kl} = -\theta c_{kl} \tag{2.12.12}$$

where θ is the ***temperature*** and the c_{kl}'s are the ***thermal expansion coefficients*** (both given functions). Clearly, (2.12.11) will in no way change the stiffness matrix. However there will be additional contributions to f^e_p. Generalize the definition of f^e_p to account for these additional terms.

Solution We begin with the weak form

$$\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \right)$$

Substituting (2.12.11) and (2.12.12) into the weak form leads to

$$\int_\Omega w_{(i,j)} c_{ijkl} \epsilon_{kl}\, d\Omega = \int_\Omega w_i \ell_i\, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \right)$$

As before

New contributions to right-hand side:

$$+ \int_\Omega w_{(i,j)} c_{ijkl} \epsilon^0_{kl}\, d\Omega$$

$$- \int_\Omega w_{(i,j)} \sigma^0_{ij}\, d\Omega$$

from which the additional terms in f^e_p may be deduced:

$$f^e_p = \cdots + e^T_i \int_{\Omega^e} B^T_a D \theta c\, d\Omega - e^T_i \int_{\Omega^e} B^T_a \sigma^0\, d\Omega$$

Anywhere inside integral

where

$$c = \begin{Bmatrix} c_{11} \\ c_{22} \\ 2c_{12} \end{Bmatrix}; \qquad \sigma^0 = \begin{Bmatrix} \sigma^0_{11} \\ \sigma^0_{22} \\ \sigma^0_{12} \end{Bmatrix}; \ldots$$

Without loss of generality, we may assume symmetry, i.e., $c_{12} + c_{21} = 2c_{12}$

Exercise 7. Consider the following boundary-value problem:

$$u_{,xx} - p(p-1)x^{p-2} = 0, \qquad 0 < x < 1$$

$$-u_{,x}(0) = 0$$

$$u(1) = 1$$

where p is a given constant.

i. Obtain the exact solution to this problem for $p = 5$. Sketch.
ii. State the weak formulation of the problem.
iii. State the Galerkin formulation.
iv. State the matrix formulation.
v. Solve the matrix problem assuming $p = 5$ and using the piecewise linear finite element space for the following cases:
   a. one element
   b. two equal-length elements
vi. Compare the exact value of $u_{,x}(1)$ with the approximate values computed in part v. Explain why it is impossible for these results to compare favorably.

Exercise 8. In heat conduction, it is often of interest to accurately calculate the boundary heat flux over a portion of the boundary where temperature is specified. Suppose we use the usual Galerkin finite element formulation to calculate the temperature. However, instead of calculating the heat flux in the usual way (i.e., by differentiating the temperature), we introduce a *post-processing* which derives from the following weak formulation:

Find $u \in \mathcal{S}$ and $h \in L_2(\Gamma_q)$ such that for all $w \in \mathcal{V}$,

$$-\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w\ell \, d\Omega + \int_{\Gamma_h} wh \, d\Gamma + \int_{\Gamma_q} wh \, d\Gamma$$

where h is the *unknown* heat flux on Γ_q

(*Note:* In this formulation, it is *not* assumed that $w = 0$ on Γ_q!)

i. Show, in addition to the usual differential equations and boundary conditions, that

$$h = -q_i n_i \quad \text{on} \quad \Gamma_q$$

arises naturally from the new weak formulation.

ii. State the Galerkin and matrix formulations corresponding to the new weak formulation assuming h is approximated in the usual way, namely

$$h^h(x) = \sum_{A \in \eta_q} N_A(x) h_A$$

(*Hint:* The equations governing the temperature are unchanged.)

iii. Specialize this formulation to the one-dimensional problem described in Exercise 7 and calculate the boundary flux at $x = 1$ by the new procedure (cf. parts v and vi of Exercise 7).

(*Hint:* The new method should produce exact results for these cases.)

iv. Develop a counterpart of the new formulation for elasticity. That is, introduce the ith component of traction as an independent unknown on Γ_{q_i} and carefully state the weak formulation.

v. *Prove* that the new method is exact for the one-dimensional model problem of Chapter 1.

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3

Isoparametric Elements and Elementary Programming Concepts

3.1 PRELIMINARY CONCEPTS

We wish to define the shape functions in such a way that, as the finite element mesh is refined, the approximate Galerkin solution converges to the exact solution. The following question arises: What conditions must the shape functions satisfy so that this property is guaranteed? We shall be content, for the time being, to state sufficient conditions for convergence. These conditions are possessed by the most prevalent and important finite element shape functions. However, we note that convergent elements can be constructed from shape functions which do not satisfy all these requirements. Nevertheless these conditions may be considered basic in that they provide the simplest criteria to ensure convergence for a wide class of problems.

The basic convergence requirements are that the shape functions be

C1. smooth (i.e., at least C^1) on each element interior, Ω^e;
C2. continuous across each element boundary Γ^e: and
C3. complete.

Remarks

1. Conditions C1 and C2 guarantee that first derivatives of the shape functions have, at worst, finite jumps across the element interfaces; see Fig. 3.1.1. This ensures that all integrals necessary for the computation of element arrays (see Sec. 2.11) are well defined, since at most first derivatives appear in the integrands. If we permit finite discontinuities in the shape functions on element boundaries, the derivatives possess delta functions (cf. Sec. 1.10) and we are unable to make sense out of the squares of these quantities that would appear in the stiffness integrands.