③ (Rel. 2). $\mathbb{Z}[\sqrt{3}]$, $(3+\sqrt{3}, 2)$, $[3+\sqrt{3}, 2]$

$$N(a+b\sqrt{3}) = (a+b\sqrt{3})(a-b\sqrt{3}) = a^2 - 3b^2$$

$$N(3+\sqrt{3}) = 3^2 - 3 = 6, \quad N(2) = 4$$

$3+\sqrt{3}$	1	0
2	0	1
$1+\sqrt{3}$	1	-1
0	$-$	$-$

$$\frac{3+\sqrt{3}}{2} = \frac{3}{2} + \frac{1}{2}\sqrt{3} \; ; \; \overset{3+\sqrt{3}}{\overset{\|}{2 \cdot 1 + (1+\sqrt{3})}}$$

$$\frac{2}{1+\sqrt{3}} = \frac{2-2\sqrt{3}}{1-3} = \frac{2(1-\sqrt{3})}{-2} = -1+\sqrt{3}$$

$$(3+\sqrt{3}, 2) = 1+\sqrt{3} \qquad [3+\sqrt{3}, 2] = \frac{(3+\sqrt{3}) \cdot 2}{1+\sqrt{3}} =$$

$$\frac{(3+\sqrt{3})\, 2 \cdot (1-\sqrt{3})}{-2} = (3+\sqrt{3})(-1-\sqrt{3})$$

⑧ (Rel 2) $e+d = 12$, $e\,pe + d\,pd = 1200$ $\quad pe = pd + 30$

$e \to$ cajas de leche entera $\qquad pe \to$ precio entera

$d \to$ cajas de leche desnatada. $\quad pd \to$ precio desnatada

$$e\,pe + d\,pd = 1200 \Rightarrow (12-d)(pd+30) + d\,pd = 1200$$

$$\Rightarrow 12pd + 12\cdot 30 - \cancel{d\,pd} - 30d + \cancel{d\,pd} = 1200$$

$$12pd - 30d = 1200 - 360 = 840$$

calc. $(-30, 12)$.

-30	1	0
12	0	1
6	1	3
0	$-$	$-$

$\cancel{30 = 12\cdot 2 + 6}$ $\quad -30 = 12\cdot(-3) + 6$

$\cancel{12 = 6\cdot 2 + 0}$ $\quad 12 = 6\cdot 2 + 0$

$6 = (-30)(1) + 12(3)$

$840 = 6\cdot 140 \Rightarrow$ la ecuación tiene solución.

$$6 = (-30)\cdot(1) + 12(3) \Rightarrow 840 = (-30)(140) + 12(420)$$

Es decir, una solución particular es:

$$\begin{cases} d_0 = 140 \\ pd_0 = 420 \end{cases} \Rightarrow \begin{aligned} d &= 140 + \tfrac{12}{6}K = 140 + 2K \\ pd_0 &= 420 + \tfrac{30}{6}K = 420 + 5K \end{aligned}$$

$d = 140 + 2K > 0 \Leftrightarrow 140 > -2K \Leftrightarrow K > -70 \Rightarrow K = -69$

$$\begin{aligned} d &= 140 + 2(-69) = 2 \\ pd_0 &= 420 + 5(-69) = 75 \end{aligned} \Rightarrow \begin{aligned} e &= 10 \\ pe &= 105 \end{aligned}$$

La solución es:

$$\left[\begin{array}{ll} d = 2 & e = 10 \\ pd = 75\,€ & pe = 105\,€ \end{array}\right]$$

Asumiendo que compra al menos 1 caja de desnatada

Relación 8 $\qquad \sum_{i=0} a_i x^i$

Sea $A = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$. Div. el. formas canónicas

$$R = xI - A = \begin{pmatrix} x+1 & -3 & 0 \\ 0 & x-1 & 0 \\ 0 & -2 & x+1 \end{pmatrix} \overset{E_{12}}{\sim} \begin{pmatrix} -3 & x+1 & 0 \\ x-1 & 0 & 0 \\ -2 & 0 & x+1 \end{pmatrix}$$

$$\overset{C_2(3)}{\sim} \begin{pmatrix} -3 & 3(x+1) & 0 \\ x-1 & 0 & 0 \\ -2 & 0 & x+1 \end{pmatrix} \overset{E_{21}(x+1)}{\sim} \begin{pmatrix} -3 & 0 & 0 \\ x-1 & x^2-1 & 0 \\ -2 & -2(x+1) & x+1 \end{pmatrix}$$

$$\overset{F_3(-1/3)}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ x-1 & x^2-1 & 0 \\ -2 & -2(x+1) & x+1 \end{pmatrix} \underset{F_{31}(+2)}{\overset{F_{21}(-(x-1))}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-1 & 0 \\ 0 & -2(x+1) & x+1 \end{pmatrix} \overset{C_{23}(2)}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-1 & 0 \\ 0 & 0 & x+1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & x^2-1 \end{pmatrix} \qquad \begin{aligned} d_1 &= x+1 \\ d_2 &= x^2-1 = (x+1)(x-1) \end{aligned}$$

- Los factores invariantes son $x+1$, x^2-1.
- La forma canónica racional es: (Frobenius)

$$M_{x+1} \oplus M_{x^2-1} = \left(\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

Los divisores son $(x+1, x-1, x+1)$

- La forma canónica racional primaria es: Weierstrass. El orden da igual.

$$M_{x+1} \oplus M_{x-1} \oplus M_{x+1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $p_A(x) = (x+1)(x^2-1) = x^3 + x^2 - x - 1$
- $m_A(x) = d_r = x^2 - 1$

$$A = \begin{pmatrix} 3 & -4 & 6 & -14 \\ 1 & -1 & 1 & -5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

sale

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^2-2x+1 & 0 \\ 0 & 0 & 0 & x^2-2x+1 \end{pmatrix}$$

- Los fact. div. son x^2-2x+1, x^2-2x+1
- La f.c.r. (Frobenius):

$$M_{x^2-2x+1} \oplus M_{x^2-2x+1} = \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 2 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & 1 & 2 \end{array}\right)$$

- Los ~~factores~~ divisores el. son $(x-1)^2, (x-1)^2$ (lo mismitico)

La f.c.r.p. (Weierstrass) es la misma.

La matriz sí tiene f.c de Jordan: $\begin{pmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{pmatrix}$

$$J_{(x-1)^2} \oplus J_{(x-1)^2} = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 1 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 1 & 1 \end{array}\right)$$

④ $T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x,y,z) = (7x-2y+z, -2x+10y+2z, x-2y+7z)$

Fijamos $B_U = (e_i)$.

$$A = M_{B_U}(T) = \begin{pmatrix} 7 & -2 & 1 \\ -2 & 10 & 2 \\ 1 & -2 & 7 \end{pmatrix} \rightsquigarrow R = xI - A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3-24x^2+188x-480 \end{pmatrix}$$

- $f.car.(T) = (x^3 - 24x^2 + 188x - 480)$
- $f.C.R(A) = \begin{pmatrix} 0 & 0 & 480 \\ 1 & 0 & -188 \\ 0 & 1 & 24 \end{pmatrix}$
- $p_A(x) = m_A(x) = x^3 - 24x^2 + 188x - 480$
- div. el. son $(x-10), (x-6), (x-8)$
- $FCRP = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ = Forma Jordan

② $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\mathcal{M}_{x^2-1} \oplus \mathcal{M}_{x-1} \qquad \mathcal{M}_{x-1} \oplus \mathcal{M}_{x-1} \oplus \mathcal{M}_{x+1}$$

$$\simeq \qquad \not\simeq$$

$$\mathcal{M}_{x+1} \oplus \mathcal{M}_{x-1} \oplus \mathcal{M}_{x-1}$$

son semejantes (igual f.c.r.p.)

③. Los ~~div. elem~~ f. inv. son $f\,inv(A) = (x+1, x^2+2x+1, x^4+2x^3-x^2-4x-2)$

A racional.

$$FCR(A) = \begin{pmatrix} -1 & & & & & & \\ & 0 & -1 & & & & \\ & 1 & -2 & & & & \\ & & & 0 & 0 & 0 & 2 \\ & & & 1 & 0 & 0 & 4 \\ & & & 0 & 1 & 0 & 1 \\ & & & 0 & 0 & 1 & -2 \end{pmatrix}$$

$x^2+2x+1 = (x+1)^2$

$$\begin{array}{l|l} x^4+2x^3-x^2-4x-2 & x^2+2x+1 \\ 0 \quad 0 \quad -2x^2-4x-2 & x^2-2 \\ \quad 0 & \end{array}$$

$x^4+2x^3-x^2-4x-2 = (x^2+2x+1)(x^2-2)$

Los div. el son $(x+1), (x+1)^2, (x+1)^2, (x^2-2)$.

$$FCRP(A) = \begin{pmatrix} -1 & & & & & & \\ & 0 & -1 & & & & \\ & 1 & -2 & & & & \\ & & & 0 & -1 & & \\ & & & 1 & -2 & & \\ & & & & & 0 & 2 \\ & & & & & 1 & 0 \end{pmatrix}$$

como A es racional no podemos hacer $(x-\sqrt{2})(x+\sqrt{2})$ $(\sqrt{2} \notin \mathbb{Q})$

Hallar las matrices $A \in \mathcal{M}_3(\mathbb{R})$ salvo semejanza tales que $P_A(x) = (x-2)^3$

Los pos. div. el son

$(x-2, x-2, x-2) \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$(x-2, (x-2)^2) \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

$((x-2)^3) \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

⑤ $P_A(x) = (x^4-1)(x^2-1)$. Mat. coef. reales. salvo semejanza

$= (x^2+1)(x+1)^2(x-1)^2$

Hallamos las formas canónicas y ya está.

$(x-1, x-1, x+1, x+1, x^2+1)$

$((x-1)^2, x+1, x+1, x^2+1)$

$(x-1, x-1, (x+1)^2, x^2+1)$

$((x-1)^2, (x+1)^2, x^2+1)$

a^2, a^2

a, a, a, a

a^2, a, a

a^3, a

a^4

Hallar $A \in GL_3(\mathbb{R})$: $A^3 = A^{-1}$

Es equiv. a $A^4 = I$ A satisface x^4-1 $(P(A)=0)$

Es decir, $m_A(x) \mid x^4-1$. $gr(m_A(x)) \leq 3$

$x^4-1 = (x^2+1)(x-1)(x+1)$

Sus divisores de grado menor o igual que 3 son:

1°) $x-1$ 3°) x^2+1 5°) $(x^2+1)(x+1)$

2°) $x+1$ 4°) x^2-1 6°) $(x^2+1)(x-1)$

1) $m_A(x) = x-1$

como $m_A(x) = d_3$, $d_1 = d_2 = d_3 = x-1$ (son $\neq 1$ y todos div. d_3)

En tal caso $A = I$

2) $m_A(x) = x+1$

The same. $A = -I$

3) $m_A(x) = x^2+1$ No hay porque la suma de los grados tiene que ser 3.

4) $m_A(x) = x^2-1$

$(x-1, x^2-1)$ $\longrightarrow$ $\left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array}\right)$

$(x+1, x^2-1)$ $\longrightarrow$ $\left(\begin{array}{c|cc} -1 & 0 & \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array}\right)$

5) $m_A(x) = (x^2+1)(x+1)$ como es de grado 3, $d_1 = m_A$ y es el único

$A = \begin{vmatrix} 0 & 0 \\ & \end{vmatrix}$

Igual para 6

⑥ Hallar $A \in \mathcal{M}_3(\mathbb{R})$: $A^6 = I$

$f(x) = x^6 - 1$. $f(A) = 0 \Rightarrow m_A(x) \mid f(x)$, $gr(m_A(x)) \leq 3$

$x^6 - 1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$

Sus divisores de grado menor o igual que 3.

$I \leftarrow$ 0) $x-1$ $\quad$ ~~2)~~ x^2-x+1 $\quad$ 1 0) $(x-1)(x^2-x+1)$

$-I \leftarrow$ 0) $x+1$ $\quad$ ~~2)~~ x^2+x+1 $\quad$ 1 0) $(x-1)(x^2+x+1)$

2 0) x^2-1 $\quad$ 1 0) $(x+1)(x^2-x+1)$

1 0) $(x+1)(x^2+x+1)$

⑧. $A \in \mathcal{M}_n(\mathbb{Q})$ $A^3 + A^2 - I = 0$

a) Razonar que $3 \mid n \Leftrightarrow \exists A : A^3 + A^2 - I = 0$ $(A \in \mathcal{M}_n(\mathbb{Q}))$

$\Leftarrow$) $f(x) = x^3 + x^2 - 1$. $f(A) = 0$. $m_A(x) \mid x^3 + x^2 - 1$

f es irreducible (no tiene raíces) $\Rightarrow m_A(x) = f(x)$

$\forall j$ $d_j = m_A(x) \Rightarrow n = \sum gr(d_j) = 3 \cdot m$.

$\Rightarrow$) $n =$

Ejemplo: Factorizar $3+i$ en $\mathbb{Z}[i]$

$N(3+i) = 10 \Rightarrow 3+i$ no es primo.

$10 = 2 \cdot 5 \Rightarrow \exists \, \alpha_1, \alpha_2 : N(\alpha_1) = N(2), N(\alpha_2) = N(5)$

Para que $N(\alpha_1) = 2$, $\alpha_1 = a + bi$, $a^2 + b^2 = 2 \Rightarrow |a| = |b| = 1$

Tomamos $\alpha_1 = 1+i$ (todos son asociados)

$$\frac{3+i}{1+i} = \frac{(3+i)(1-i)}{2} = \frac{3-3i+i+1}{2} = 2-i$$

bxm

$(ax^e +$

$a^{-1}bxm-i$

Es decir: $3+i = (1+i)(2-i)$

Ej. Factorizar $11+7i$ en $\mathbb{Z}[i]$

$N(11+7i) = 121 + 49 = 170 = 2 \cdot 5 \cdot 17$

170 no es primo o $\pm$ cuadrado de un primo.

Si $\exists$ div. de α / $N(\alpha) = 2$: $1+i = \alpha$

Probamos:

$$\frac{11+7i}{1+i} = \frac{(11+7i)(1-i)}{2} = \frac{11+7i-11i+7}{2} = 9-2i$$

$11+7i = (1+i)(9-2i)$

$N(p)$ $\pm p^2$

Factorizamos $9-2i$: $N(9-2i) = 5 \cdot 17$, no es primo.

Si $\exists$ div de $9-2i$ / $N(\alpha) = 5$. $N(\pm p^2$

$a^2+b^2 = p$

$a^2+b^2 = \pm p^2$

$a^2 + b^2 = 5 \Rightarrow a^2 = 1, b^2 = 4$ ó $a^2 = 4, b^2 = 1$

Es decir, las soluciones son:

- $1+2i$
- $1-2i$
- $-1+2i$
- $-1-2i$
- $2+i$
- $2-i$
- $-2+i$
- $-2-i$

$$\frac{9+2i}{1+2i} = \frac{(9-2i)(1-2i)}{5} = \frac{9-18i-2i-4}{5} = 1-4i$$

$N(1-4i) = 17$

$11+7i = (1+i)(1-4i)(1+2i)$

$(2i, 11+7i) = (2, 11+7i)$

4 $\quad 2^2$

$b^2 = 3$ $\quad p=2$

$\frac{2}{1+i} = 1+i \Rightarrow (2, 11+7i) = 1+i$ $\quad b^2 = p^2 - 1$

$b^2 = (p+1)(p-1)$

$a^2 + b^2 = p$

$[2i, 11+7i]$ $\quad 1 + n^2 i$ $\quad b^2 = p - 1$

$1 + 13i$

Factorizar en producto de irreducibles 180 en $\mathbb{Z}[\sqrt{-2}]$

180	2
90	2
45	3
15	3
5	5

$180 = 2^2 \cdot 3^2 \cdot 5$

$a^2 + 2b^2 = 2 \Leftrightarrow \begin{cases} b = \pm 1 \\ a = 0 \end{cases} \Leftrightarrow |a+bi| = i\sqrt{2}$ ó $-i\sqrt{2}$

$\frac{2}{i\sqrt{2}} = \frac{2(-i\sqrt{2})}{2} = -i\sqrt{2} \Rightarrow 2 = (i\sqrt{2})(-i\sqrt{2})$

$N(3) = 9. \quad a^2 + 2b^2 = 3 \Rightarrow \begin{cases} b = \pm 1 \\ a = \pm 1 \end{cases}$: $\boxed{1+i\sqrt{2}}$, $\boxed{1-i\sqrt{2}}$, $-1+i\sqrt{2}$, $-1-i\sqrt{2}$

$\frac{3}{1+i\sqrt{2}} = \frac{3(1+i\sqrt{2})}{3} = 1 - i\sqrt{2}$

$3 = (1+i\sqrt{2})(1-i\sqrt{2})$

$N(5) = 25 \quad a^2 + 2b^2 = 5 \Rightarrow \{$ No hay solución.

Es decir, 5 es irreducible (primo).

$180 = (i\sqrt{2})^2(-i\sqrt{2})^2(1+i\sqrt{2})^2(1-i\sqrt{2})^2 \cdot 5$