(1)

(1) $\forall m \in \mathbb{N}$ ó $m = 0$

ó $m \neq 0 \Rightarrow \exists_1 \, n \in \mathbb{N} : m = \sigma(n)$

$\square$

(2) Sea $m = 0$ $\quad 0 + 0 = 0$

Si $m + 0 = 0 + m = m$ entonces:

$\sigma(m) + 0 = \sigma(m) = \sigma(m+0) = \sigma(0+m) = 0 + \sigma(m)$ $\square$

(3) Sea $m = 0$: $\quad 1 + 0 = 1 \Rightarrow 1 + 0 = \sigma(0)$

$0 + 1 = 0 + \sigma(0) = \sigma(0+0) = \sigma(0) = 1 + 0 = 1$

Si $m + 1 = 1 + m = \sigma(m)$ entonces:

$1 + \sigma(m) = \sigma(1+m) = \sigma(m+1) = \sigma(\sigma(m))$ $\square$

(4) Sabemos: $(m+n) + 1 = \sigma(m+n) = m + \sigma(n) = m + (n+1)$

Si $(m+n) + p = m + (n+p)$ entonces:

$(m+n) + \sigma(p) = \sigma((m+n)+p) = \sigma(m + (n+p)) = m + \sigma(n+p) =$

$= m + (n + \sigma(p))$ $\square$

(5) Sabemos que para $m = 1$ se cumple:

$n + 1 = 1 + n$.

Si $m + n = n + m$

$\sigma(m) + n = (m+1) + n = m + (1+n) = m + (n+1) =$

$= (m+n) + 1 = (n+m) + 1 = \sigma(n+m) = n + \sigma(m)$

$\square$

(6) Si $p=0 \Rightarrow m+0 = n+0$

$$\underset{m}{\overset{\shortparallel}{m+0}} = \underset{n}{\overset{\shortparallel}{n+0}} \Rightarrow m=n$$

Si $m+p = n+p \Rightarrow m=n$, entonces:

$\Rightarrow m+\sigma(p) = \sigma(m+p) = \sigma(n+p) = n+\sigma(p) \Rightarrow m=n \;\; \forall m,n \in \mathbb{N}$ $\square$

(7) Si $n=0$ $\quad m+0=0 \Rightarrow m=0$

Si $m+n=0 \Rightarrow m=n=0$, entonces:

$m+\sigma(n) = \sigma(m+n) = \sigma(0) \Rightarrow m+n=0 \Leftrightarrow m=n=0$ $\square$

(8) ~~Si $m+0=0 \Rightarrow 0 \cdot m = 0$~~

$m \cdot 0 = 0$. Comprobemos $0 \cdot m = 0$.

Para $m=0$ se cumple: $0 \cdot 0 = 0$.

Si $0 \cdot m = 0 \Rightarrow 0 \cdot \sigma(m) = 0 \cdot m + 0 = 0+0=0$ $\square$

(9) Si $m=0$ se cumple (apartado anterior)

Si $1 \cdot m = m \Rightarrow 1 \cdot \sigma(m) = 1 \cdot m + 1 = m+1 = \sigma(m) = \sigma(m) \cdot 1$. $\square$

(10) Si $p=0$ $\quad (m+n) \cdot 0 = 0 = m \cdot 0 + n \cdot 0 = 0$

Si $(a+b) \cdot c = a \cdot c + b \cdot c \Rightarrow$

$\Rightarrow (a+b) \cdot \sigma(c) = (a+b) \cdot c + (a+b) = (a \cdot c + b \cdot c) + (a+b) =$

$= (a \cdot c + a) + (b c + b) = a \cdot \sigma(c) + b \cdot \sigma(c)$ $\square$

(11) Si $n=0$ se cumple $(m\cdot 0 = 0\cdot m = 0)$

Si $m\cdot n = n\cdot m \Rightarrow$

$\Rightarrow \sigma(n)\cdot m = (n+1)\cdot m = n\cdot m + 1m = m\cdot n + m = m\cdot\sigma(n)$

$\square$

(12) Si $p=0$ se cumple $(m\cdot n)\cdot 0 = m\cdot(n\cdot 0) = 0$

Si $m\cdot(n\cdot p) = (mn)\cdot p \Rightarrow$

$\Rightarrow m\cdot(n\cdot\sigma(p)) = m\cdot(n\cdot p + n) = m(n\cdot p) + m\cdot n =$

$= (m\cdot n)\cdot p + m\cdot n = (m\cdot n)\cdot\sigma(p)$

$\square$

(13) Si $m=0$ $0\cdot n = 0$

Si $n=0$ $m\cdot 0 = 0$

Si $m\cdot n = 0 \Leftrightarrow m=0 \wedge n=0 \Rightarrow$

$\Rightarrow m\cdot\sigma(n) = \underbrace{m\cdot n}_{0} + m = m$

$\square$

(14) $0\in\mathbb{N} \Rightarrow m=0$ $m^0 = 1$

$\square$

(15) $n=1$ $0^1 = 0$

Si $0^n = 0 \Rightarrow$

$\Rightarrow 0^{\sigma(n)} = 0^n\cdot 0 = 0\cdot 0 = 0$

$\square$

(16) $n=1$ $1^1 = 1$

Si $1^n = 1 \Rightarrow$

$\Rightarrow 1^{\sigma(n)} = 1^n\cdot 1 = 1\cdot 1 = 1$

$\square$

(17) Si $m = 0$: $0^{n+p} = 0 = 0^n \cdot 0^p = 0 \cdot 0 = 0$

Si $m^{n+p} = m^n \cdot m^p \Rightarrow$

$\Rightarrow m^{\sigma(n+p)} = m^{n+p} \cdot m = m^n \cdot m^p \cdot m$ $\square$

(18) Si $m = 0$ $0^{n \cdot p} = 0 = (0^n)^p = 0^p = 0$

Si $m^{n \cdot p} = (m^n)^p \Rightarrow$

$\Rightarrow m^{\sigma(n \cdot p)} = m^{n \cdot p} \cdot m = (m^n)^p \cdot m$ $\square$

②

1) $m \leq m$

$m \leq m$ ya que existe $0 \in \mathbb{N}$ que cumple $m+0=m$

2) Si $m \leq n$ y $n \leq m$ entonces $m=n$

Si $m \leq n \Rightarrow \exists\, x \in \mathbb{N}: m+x=n$

Si $n \leq m \Rightarrow \exists\, x' \in \mathbb{N}: n+x'=m$ $\Big\} \Rightarrow$

$\Rightarrow m+x+x'=m \Rightarrow x+x'=0 \Rightarrow x=x'=0 \Rightarrow m=n$

3) Si $m \leq n$ y $n \leq p$ entonces $m \leq p$

$m \leq n \Rightarrow \exists\, x \in \mathbb{N}: m+x=n$

$n \leq p \Rightarrow \exists\, x' \in \mathbb{N}: n+x'=p$ $\Big\} \Rightarrow m+(x+x')=p \Rightarrow$

$\Rightarrow \exists\, x+x' \in \mathbb{N}: m+(x+x')=p \Rightarrow m \leq p$

4) $m \leq n$ ó $n \leq m$

- Si $\exists\, x \in \mathbb{N}: m+x=n \Rightarrow m \leq n$
- Si $\nexists\, x \in \mathbb{N}: m+x=n \Rightarrow \forall x \in \mathbb{N}\ m+x \neq n \Rightarrow$

$\Rightarrow \exists\, x' \in \mathbb{N}: n+x'=m \Rightarrow n \leq m.$

5) Si $m \leq n \Rightarrow \exists_1\, p \in \mathbb{N}: m+p=n$

$m \leq n \Rightarrow \exists\, x \in \mathbb{N}: m+x=n.$

Supongamos que $\exists\, x'$: $m+x'=n$

$m+x=n$
$m+x'=n$ $\Rightarrow m+x=m+x' \Leftrightarrow x=x'$

6) Si $m \leq n \Rightarrow m+p \leq n+p$

$m \leq n \Rightarrow \exists x \in \mathbb{N} : m+x=n \Rightarrow m+p+x = n+p \Rightarrow$

$\Rightarrow m+p \leq n+p$

7) Si $m \leq n \Rightarrow m \cdot p \leq n \cdot p$

$m \leq n \Rightarrow \exists x \in \mathbb{N}: m+x=n \Rightarrow$

$\Rightarrow m \cdot p + x \cdot p = n \cdot p \Rightarrow m \cdot p \leq n \cdot p.$

8) Si $m \cdot p \leq n \cdot p$ y $p \neq 0 \Rightarrow m \leq n$

Si $m \cdot p \leq n \cdot p \Rightarrow \exists x \in \mathbb{N}: m \cdot p + x = n \cdot p.$

Tomando $p=1$ $m+x=n \Rightarrow m \leq p$

Tomando $p=p+1$ $m \cdot p + m + x = n \cdot p + \cancel{m} \Rightarrow$

$\Rightarrow m \cdot p + x + \cancel{m} = n \cdot p + \cancel{m} \Rightarrow m \cdot p + x = n \cdot p$

9) Si $m \cdot p = n \cdot p$ y $p \neq 0 \Rightarrow m = n.$

Si $m \cdot p = n \cdot p \Rightarrow \exists x = 0 \in \mathbb{N}: m \cdot p + x = n \cdot p \Rightarrow$

$\Rightarrow$ Tomando $p=1$ $m+0=n \Rightarrow m=n$

Tomando $p=p+1$ $mp+m+0 = np+m \Rightarrow$

$\Rightarrow mp = n \cdot p$

③

1) $\forall\, n \geq 1 \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$k = 1 \qquad 1 = \frac{1\,(1+1)}{2} = 1$

Supouemos para n:

$k = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$k = n+1 \qquad \sum_{k=1}^{n+1} k = \frac{(n+1)(n+1+1)}{2} = \frac{n\cdot(n+1)+(n+1)+(n+1)}{2} =$

$= \frac{n(n+1)}{2} + (n+1)$

$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1)$

2) $\forall\, n \geq 1 \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$k=1 \qquad 1^2 = \frac{1\,(2)\,(2+1)}{6} = 1$

Supouemos para n:

$k=n \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)\,(2n+1)}{6}$

$k = n+1 \qquad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+1+1)\,(2n+2+1)}{6} =$

$= \frac{(n(n+1) + (n+1) + (n+1))\,((2n+2)+1)}{6} = \frac{n(n+1)(2n+2)}{6} + \frac{(2n+1)\,(n+1) + 2n+1}{}$

$\frac{(n+1) + 2n(n+1) + 4(n+1)}{6} = \frac{n(n+1)(2n+1)}{6} + \frac{(6n+6)(n+1)}{6} =$

$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n\cdot(n+1)(2n+1)}{6} + (n+1)^2$

3) $\forall n \geq 1 \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$k=1 \qquad 1^3 = \left(\frac{1(2)}{2}\right)^2 = 1$

Suponemos para n:

$k=n \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$k=n+1 \qquad \sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+1+1)}{2}\right)^2 = \left(\frac{n\cdot(n+1)}{2} + (n+1)\right)^2 =$

$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^2 + n\cdot(n+1)^2 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$

$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$

4) $\forall n \geq 1 \quad \sum_{k=1}^{n} k^5 + \sum_{k=1}^{n} k^7 = 2\left(\frac{n(n+1)}{2}\right)^4$

$k=1 \qquad 1 + 1 = 2\left(\frac{1(2)}{2}\right)^4 = 2$

Suponemos para n.

$k=n+1 \qquad \sum_{k=1}^{n+1} k^5 + \sum_{k=1}^{n+1} k^7 = 2\left(\frac{(n+1)(n+1+1)}{2}\right)^4 = 2\left(\frac{n\cdot(n+1)}{2} + (n+1)\right)^4 =$

$= 2\left(\left(\frac{(n+1)n}{2}\right)^4 + 4\left(\frac{(n+1)n}{2}\right)^3(n+1) + 6\left(\frac{(n+1)\cdot n}{2}\right)^2(n+1)^2 + 4\left(\frac{(n+1)\cdot n}{2}\right)^2\cdot(n+1)^3 + \right.$

$\left. + (n+1)^4\right) = 2\left(\frac{(n+1)\cdot n}{2}\right)^4 + (n+1)^4\cdot n^3 + 3(n+1)^4 + n^2 + 4(n+1)^4\cdot n + 2(n+1)^4 =$

$= 2\left(\frac{(n+1)n}{2}\right)^4 + (n+1)^4\left(n^3 + 3n^2 + \cancel{4n+2}\right) 3n + 1 + n + 1\right) =$

$= 2\left(\frac{(n+1)n}{2}\right)^4 + (n+1)^5 + (n+1)^4\left(n^3 + 3n^2 + 3n + 1\right) =$

$= 2\left(\frac{(n+1)n}{2}\right)^4 + (n+1)^5 + (n+1)^7$

$\sum_{k=1}^{n+1} k^5 + \sum_{k=1}^{n+1} k^7 = \sum_{k=1}^{n} k^5 + \cancel{\sum_{k=1}^{n} k^7} + (n+1)^5 + \sum_{k=1}^{n} k^7 + (n+1)^7 =$

5) $\forall n \geq 0 \quad \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1} \quad a \neq 1$

$k=0 \quad 1 = \frac{a-1}{a-1} = 1$

Suponemos para n:

$k=n \quad \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$

$k=n+1$:

$$\sum_{k=0}^{n+1} a^k = \frac{a^{n+1+1}-1}{a-1} = \frac{a^{n+1} \cdot a - 1}{a-1} =$$

$$= \frac{a \cdot a^{n+1}}{a-1} - \frac{1}{a-1} + \frac{a^{n+1}}{a-1} - \frac{a^{n+1}}{a-1} = \frac{a \cdot a^{n+1} - a^{n+1}}{a-1} +$$

$$+ \frac{a^{n+1}-1}{a-1} = \frac{a^{n+1}(a-1)}{(a-1)} + \frac{a^{n+1}-1}{a-1} = \frac{a^{n+1}-1}{a-1} + a^{n+1}$$

$$\sum_{k=0}^{n+1} a^k = \sum_{k=0}^{n} a^k + a^{n+1} = \frac{a^{n+1}-1}{a-1} + a^{n+1}$$

6) $\forall n \geq 1 \quad \sum_{k=0}^{n} (k \cdot k!) = (n+1)! - 1$

$k=1 \quad 1 = (1+1)! - 1 = 1$

Suponemos para n:

$k = n+1$

$$\sum_{k=1}^{n+1} (k \cdot k)! = (n+1+1)! - 1 = (n+2)! - 1 =$$

$$= (n+2)(n+1)! - 1 = (n+2)(n+1)! - (n+1)! + (n+1)! - 1 =$$

$$= (n+1)!((n+2)-1) + (n+1)! - 1 = (n+1)!(n+1) + (n+1)! - 1$$

$$\sum_{k=1}^{n+1} (k \cdot k!) = \sum_{k=1}^{n} (k \cdot k!) + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1)(n+1)!$$

7) $\forall n \geq 2 \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$

$k=2 \quad \sum_{k=1}^{2} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+2}{2} > \sqrt{2}$

Suponemos para n:

$k = n \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$

$k = n+1 \quad \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} > \sqrt{n+1} \quad ?$

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \sqrt{n} + \frac{\sqrt{n+1}}{n+1} =$$

$$= \frac{(n+1)\sqrt{n} + \sqrt{n+1}}{(n+1)} = \frac{(n+1)\sqrt{n} + \sqrt{n+1}}{(n+1)} + \sqrt{n+1} - \sqrt{n+1} =$$

$$= \underbrace{\frac{(n+1)\sqrt{n} + \sqrt{n+1} - (n+1)\sqrt{n+1}}{(n+1)}}_{>0} + \sqrt{n+1} > \sqrt{n+1} \quad ✓$$

8) $\forall n \geq 4 : \quad 2^n \geq n^2$

$k=4 \quad 2^4 = 16 \geq 16$

Suponemos para n: $\quad 2^n \geq n^2$

$k = n+1 \quad 2^{n+1} \geq (n+1)^2$

$$2^{n+1} = 2^n \cdot 2 \geq 2 \cdot n^2 \geq (n+1)^2 \quad ?$$

$2n^2 \geq (n+1)^2 = n^2 + 2n + 1 \iff n^2 \geq 2n+1 \iff n^2 - 2n - 1 \geq 0 \iff$

$\iff (n + \sqrt{2} - 1)(n - \sqrt{2} - 1) \geq 0$ ⇐ (se cumple para $\forall n \geq 4$)

✓

a) $\forall n \geq 4 \quad n! > 2^n$

$k=4 \quad 4! = 24 > 16$

Suponemos para n: $n! > 2^n$

$k = n+1 \quad (n+1)! > 2^{n+1}$?

$(n+1)! = (n+1) \cdot n! > (n+1) 2^n \overset{?}{>} 2^{n+1}$

$(n+1) 2^n > 2^{n+1} = 2^n \cdot 2 \iff n+1 > 2$

$\forall n \geq 4 \quad n+1 > 2 \Rightarrow (n+1) \cdot 2^n > 2^{n+1}$

(4)

a) $3^{2n}-2^n$ divisible por 7.

$n=0 \Rightarrow 0$ divisible x7

$n=1 \Rightarrow 3^2-2=7 \Rightarrow$ divi. x7

$n=n \Rightarrow$ supovemos

$n=n+1 \qquad 3^{2(n+1)}-2^{n+1}=(3^2)^{n+1}-2^{n+1}=$

$=2^{n+1}-2^{n+1}=0$

b) $3^{2n+1}+2^{n+2}$ divisible por 7.

$3^{2n+1}+2^{n+2}$ es div. x7 si $3^{2n+1}+2^{n+2}$ es 0 en $\mathbb{Z}_7$

$3^{2n+1}+2^{n+2}=3^{2n}\cdot 3+2^n+4=2^n\cdot 3+2^n\cdot 4=$

$=2^n\cdot 7=0$

c) $3^{2n+2}+2^{6n+1}$ divisible por 11.

$3^{2n+2}+2^{6n+1}$ es divisible x7 si es 0 en $\mathbb{Z}_{11}$

$3^{2n+2}+2^{6n+1}=3^{2n}\cdot 3^2+(2^6)^n\cdot 2=9^n\cdot 9+9^n\cdot 2=$

$=9^n\cdot 11=0$

d) $3\cdot 5^{2n+1}+2^{3n+1}$ divisible por 17

$3\cdot 5^{2n+1}+2^{3n+1}$ divisible por 17 si es 0 en $\mathbb{Z}_{17}$

$3\cdot 5^{2n+1}+2^{3n+1}=3\cdot 5\cdot(5^2)^n+2(2^3)^n=$

$=15\cdot(25)^n+2\cdot(8)^n=15\cdot 8^n+2\cdot 8^n=8^n\cdot 17=0$

e) $n(n^2+2)$ es múltiplo de 3.

$n=0 \Rightarrow 0$ múltiplo de 3.

$n=n \Rightarrow n(n^2+2)$ lo suponemos

$n=n+1 \Rightarrow (n+1)((n+1)^2+2) = (n+1)(n^2+1+2n+2) =$

$= (n+1)((n^2+2)+(2n+1)) = n\cdot(n^2+2) + n(2n+1)+(n^2+2)+(2n+1) =$

$= n(n^2+2) + 2n^2 + n + n^2 + 2 + 2n + 1 =$

$= \underbrace{n(n^2+2)}_{\text{múltiplo}} + \underbrace{3(n^2+n+1)}_{\text{múltiplo}}$

g) $7^{2n} + 16n - 1$ es múltiplo de 64.

$7^{2n} + 16n - 1 = 64\cdot K \Rightarrow 7^{2n} = 64K - 16n + 1$

$n=0 \quad 1 = 1$

$n=n$ lo suponemos $\; 7^{2n} = 64K - 16n + 1$

$n=n+1 \quad 7^{2n+2} + 16n + 16 - 1 = 49\cdot 7^{2n} + 16n + 15 =$

$= 49(64K - 16n + 1) + 16n + 15 =$

$= 49\cdot 64K - 768n + 64 = 64(49K - 12n + 1)$

h) $(n+1)(n+2)\dots(n+n)$ es múltiplo de 2^n

$n=0 \Rightarrow (0+1)(0+2)\dots(0+0)=0 = 2^0 \cdot 0$

$n=n$ Suponemos que se cumple:

$$(n+1)(n+2)\dots(n+n) = k\cdot 2^n$$

$n=n+1$: $(n+1)(n+2)(n+3)(2n+2) = 2^n k(2n+2) \Rightarrow$

$$\Rightarrow (n+2)(n+3)\dots(2n+2) = \frac{2^n \cdot k(2n+2)}{(n+1)} \Rightarrow$$

$$\Rightarrow \frac{2^{n+1}\cdot n\cdot k + 2^{n+1} k}{n+1} = \frac{k\cdot 2^{n+1}(n+1)}{(n+1)}$$

i) $4^{2n} - 2^n$ divisible por 7.

Equivale a 0 en $\mathbb{Z}_7$

$$4^{2n} - 2^n = (4^2)^n - 2^n = 16^n - 2^n = 2^n - 2^n = 0$$

j) $2^{3n} - 14^n$ es divisible por 6

Equivale a 0 en $\mathbb{Z}_6$

$$2^{3n} - 14^n = (2^3)^n - 14^n = 8^n - 14^n = 2^n - 2^n = 0$$

5

1) $1 + 3 + 5 + \dots + (2n-1) = n^2$

Se cumple para el primer elemento (1):

$1 = 1^2$

Suponemos que se cumple para n:

$1 + 3 + 5 + \dots + (2n-1) = n^2$

Comprobamos que se cumple para n+1:

$$\underbrace{1 + 3 + 5 + \dots + (2n-1)}_{n^2} + (2n+1) = (n^2+1)^2$$

$$n^2 + 2n + 1 = n^2 + 1 + 2n$$

Luego se verifica la propiedad del enunciado

2) $2^{2n} \equiv 1 \mod 3$:

Se cumple para el primer término (0):

$2^0 \equiv 1 \mod 3$.

Suponemos que se cumple para n:

$2^{2n} \equiv 1 \mod 3$:

Comprobamos que se cumple para n+1:

$2^{2n+2} = 2^{2n} \cdot 2^2 \equiv 1 \cdot 1 \mod 3$

Luego se verifica la propiedad del enunciado.

3) $2^{2n-1} \equiv 2 \mod 3$:

Se cumple para el primer término (1):

$$2^1 \equiv 2 \mod 3$$

Suponemos que se cumple para n:

$$2^{2n-1} \equiv 2 \mod 3$$

Comprobamos que se cumple para n:

$$2^{2n+1} = 2^{2n-1} \cdot 2^2 \equiv 2 \cdot 1 \mod 3$$

Luego se cumple la propiedad del enunciado.