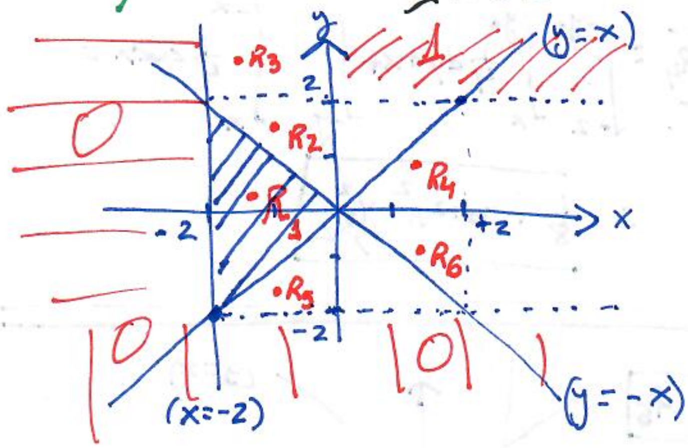
CONTROL 08/11/2021 (Soluciones propuestas)

EJERCICIO 1.-

Sea $(X,Y) \rightsquigarrow U(R)$ con $R=\{(x,y)\in\mathbb{R}^2 / -2\le x\le y\le -x\}$

a) Densidad conjunta.-



Como la distribución es uniforme en este recinto, $f(x,y)=\frac{1}{\text{área}(R)}$

$$\text{área}(R)=\int_{-2}^{0}\int_{x}^{-x} 1\,dy\,dx=\int_{-2}^{0}-2x\,dx=$$

$$=\left(-x^2\right]_{-2}^{0}=4$$

Por tanto:

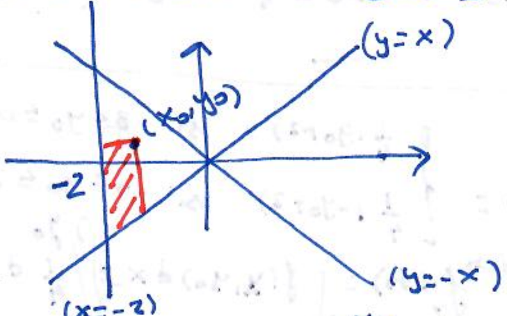
$$\boxed{f(x,y)=\begin{cases}1/4 & \text{si } -2\le x\le y\le -x\\ 0 & \text{———}\end{cases}}$$

b) Función de distribución conjunta.-

$$F(x_0,y_0)=P[X\le x_0,\ Y\le y_0]=\begin{cases}0 \longrightarrow x_0\le -2 \text{ ó } x_0\ge -2;\ y_0\le -2\\ R_1 \longrightarrow -2\le x_0\le y_0\le -x_0\\ R_2 \longrightarrow -2\le x_0\le 0;\ y_0\ge -x_0\\ R_3 \longrightarrow -2\le x_0\le 0;\ y_0\ge 2\\ R_4 \longrightarrow x_0\ge 0;\ -x_0\le y_0\le x_0\\ R_5 \longrightarrow -2\le y_0\le 0;\ y_0\le x_0\\ R_6 \longrightarrow -2\le y_0\le 0;\ -x_0\le y_0\\ 1 \longrightarrow x_0\ge 0;\ y_0\ge 2\end{cases}$$

Calculamos las integrales

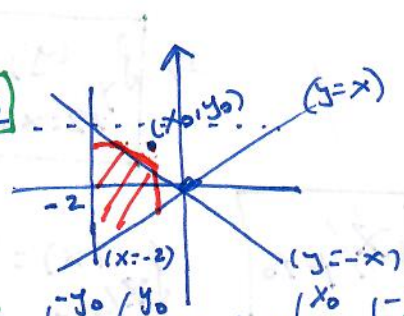
$\boxed{R_1}$



$$R_1=\int_{-2}^{x_0}\int_{x}^{y_0}\frac14\,dy\,dx=\int_{-2}^{x_0}\frac14(y_0-x)\,dx=$$

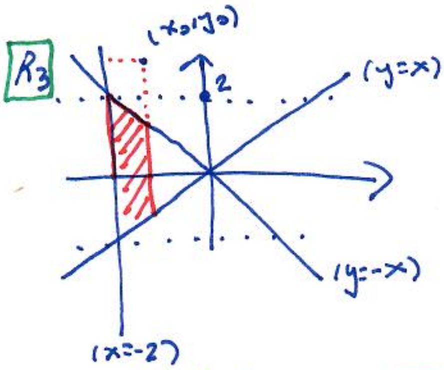
$$=\left(-\frac14\frac{(y_0-x)^2}{2}\right]_{-2}^{x_0}=\boxed{-\frac18(y_0-x_0)^2+\frac18(y_0+2)^2}$$

$\boxed{R_2}$

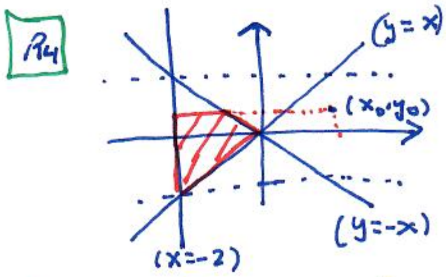


$$R_2=\int_{-2}^{-y_0}\int_{x}^{y_0}\frac14\,dy\,dx+\int_{-y_0}^{x_0}\int_{x}^{-x}\frac14\,dy\,dx=$$

$$=\boxed{-\frac18(4y_0^2)+\frac18(y_0+2)^2-\frac{x_0^2}{4}+\frac{y_0^2}{4}}$$

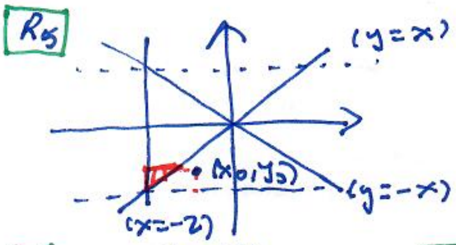


$$\boxed{R_3} = \int_{-2}^{x_0}\int_{x}^{-x} \tfrac{1}{4}\,dy\,dx = \boxed{-\frac{x_0^2}{4}+1}$$

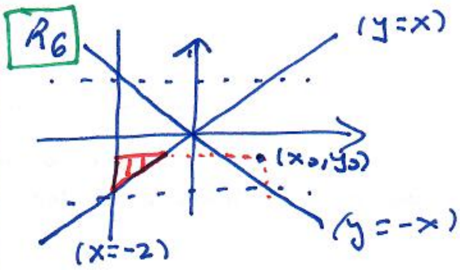


$$\boxed{R_4}: \int_{-2}^{-y_0}\int_{x}^{y_0} \tfrac{1}{4}\,dy\,dx + \int_{-y_0}^{0}\int_{x}^{-x} \tfrac{1}{4}\,dy\,dx =$$

$$= \boxed{\tfrac{1}{8}(y_0+2)^2 - y_0^2/2}$$



$$\boxed{R_5} = \int_{-2}^{y_0}\int_{x}^{y_0} \tfrac{1}{4}\,dy\,dx = \boxed{-\frac{y_0^2}{2}+\frac{1}{8}(y_0+2)^2}$$



$$\boxed{R_5 = R_6}$$

c) Distribuciones condicionadas.

$\boxed{Y/_{X=x_0}}$

$$f_{Y/_{X=x_0}}(y) = \frac{f(x_0,y)}{f_X(x_0)} = \frac{1/4}{-x_0/2} = -1/2x_0 \Rightarrow$$

$$\left(f_X(x_0) = \int_{\mathbb{R}} f(x_0,y)\,dy = \int_{x_0}^{-x_0} \tfrac{1}{4}\,dy = \left(\tfrac{1}{4}y\right]_{x_0}^{-x_0} = \frac{-x_0}{2} \quad (-2 \le x_0 \le 0) \right)$$

$$\Rightarrow \boxed{f_{Y/_{X=x_0}}(y) = \begin{cases} -\frac{1}{2x_0} & \text{si} \quad x_0 \le y \le -x_0 \quad (-2 \le x_0 \le 0) \\ 0 & \text{———} \end{cases}}$$

(es fácil comprobar que es función de densidad pq:
$$\int_{x_0}^{-x_0} \frac{-1}{2x_0}\,dy = -\frac{1}{2x_0}(-x_0-x_0) = 1 \Big)$$

$\boxed{X/_{Y=y_0}}$

$$f_{X/_{Y=y_0}}(x) = \frac{f(x,y_0)}{f_Y(y_0)} = (*) \Rightarrow$$

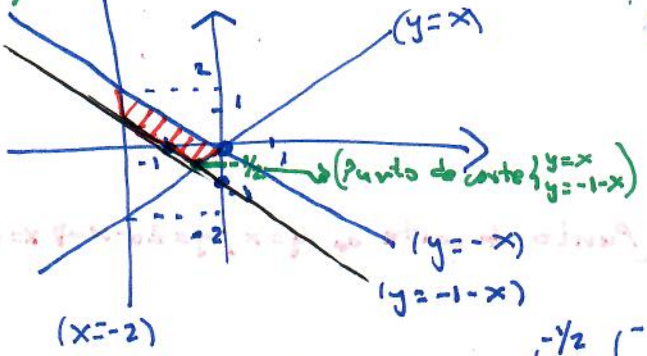
$$\left(f_Y(y_0) = \int_{\mathbb{R}} f(x,y_0)\,dx = \begin{cases} \int_{-2}^{y_0} \frac{1}{4}\,dy = \frac{1}{4}(y_0+2) & \text{si } -2 \le y_0 \le 0 \\ \int_{-2}^{-y_0} \frac{1}{4}\,dy = \frac{1}{4}(-y_0+2) & \text{si } 0 \le y_0 \le 2 \end{cases} \right)$$

$$\Rightarrow (*) = \begin{cases} \dfrac{1/4}{1/4(y_0+2)} & -2 \le x \le y_0 \quad (-2 \le y_0 \le 0) \\[2ex] \dfrac{1/4}{1/4(-y_0+2)} & -2 \le x \le -y_0 \quad (0 \le y_0 \le 2) \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{f_{X/_{Y=y_0}}(x) = \begin{cases} 1/_{y_0+2} & -2 \le x \le y_0 \quad (-2 \le y_0 \le 0) \\ 1/_{2-y_0} & -2 \le x \le -y_0 \quad (0 \le y_0 \le 2) \end{cases}}$$

(Nuevamente, es fácil comprobar que integra 1 en cada parte dependiendo de y_0 p.q: $\int_{-2}^{y_0} 1/_{y_0+2}\,dx = 1$ $(-2 \le y_0 \le 0)$ y $\int_{-2}^{-y_0} \frac{1}{2-y_0}\,dx = 1$ $(0 \le y_0 \le 2)$)

d) $P[X+Y+1 \geq 0]$



Dibujo la recta $x+y=-1$ $(y=-1-x)$

x	y
0	-1
-1	0

Por tanto $P[X+Y+1\geq 0] = \int_{-2}^{-1/2}\int_{-1-x}^{-x} \frac{1}{4}\,dy\,dx + \int_{-1/2}^{0}\int_{x}^{-x} \frac{1}{4}\,dy\,dx =$

$= \int_{-2}^{-1/2} \frac{1}{4}(\cancel{-x}+1\cancel{+x})\,dx + \int_{-1/2}^{0} -\frac{x}{2}\,dx =$

$= \left(\frac{1}{4}x\right]_{-2}^{-1/2} - \left(\frac{x^2}{4}\right]_{-1/2}^{0} = -\frac{1}{8}+\frac{1}{2}+\frac{1}{16} = 7/16$

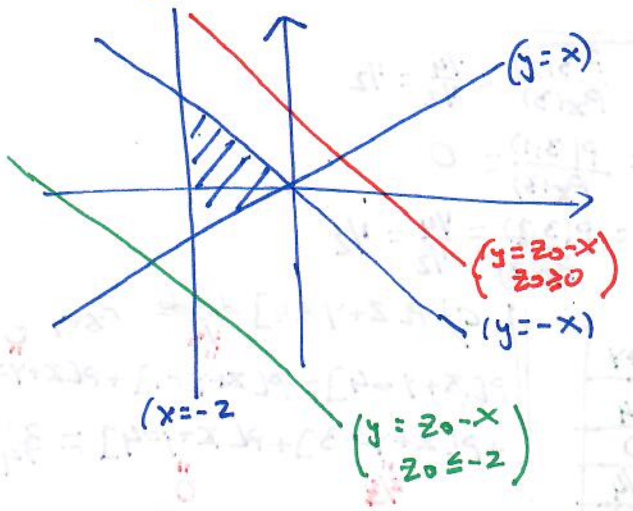
$$\boxed{P[X+Y+1\geq 0] = 7/16}$$

c) Distribución marginal de $Z = X+Y$

Se puede plantear el cambio de variable bidimensional $(Z,T) = (X+Y, Y)$, por ejemplo, obtener su función de densidad conjunta y a partir de ella obtener la marginal de Z pedida.

Sin embargo, en este caso optamos por proporcionar la función de distribución de la variable aleatoria $Z = X+Y$ a partir de la definición de función de distribución.

$F_Z(z_0) = P[Z \leq z_0] = P[X+Y \leq z_0]$

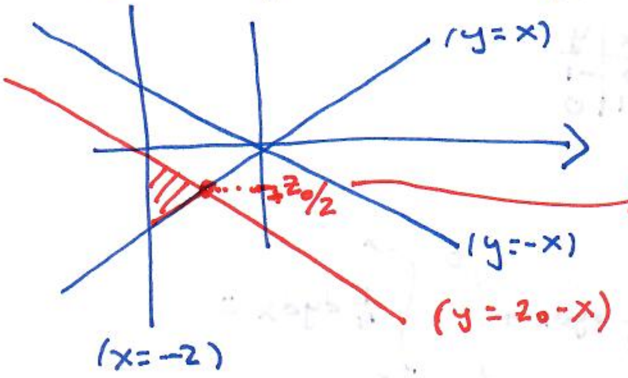


Procedemos de la misma forma que el apartado anterior.

Dibujamos las rectas $X+Y=z_0$ $(y=z_0-x)$

- Si $\boxed{z_0 \geq 0}$ está por encima de $y=-x$ y en este caso $\boxed{P[X+Y\leq z_0]=1}$, p.q sería integrar $f(x,y)$ en todo el recinto.
- $\boxed{\text{Si } z_0 \leq -2}$, están fuera del recinto (por debajo) de modo que integramos 0 y en este caso $\boxed{P[X+Y\leq z_0]=0}$

• Si $-2 < z_0 < 0$, es el único caso en que están dentro del dominio, siendo paralelas a $y = -x$.



(Punto de corte de $y = x$; $y = z_0 - x \Rightarrow x = z_0 - x \Rightarrow x = z_0/2$)

De modo que $P[Z \leq z_0] = P[X + Y \leq z_0] = \int_{-2}^{z_0/2} \int_{x}^{z_0 - x} \frac{1}{4}\, dy\, dx =$

$$= \int_{-2}^{z_0/2} \frac{1}{4}(z_0 - x) - \frac{1}{4}x\, dx = \int_{-2}^{z_0/2} \frac{1}{4}z_0 - \frac{1}{2}x\, dx = \left(\frac{1}{4}z_0 x - \frac{1}{4}x^2\right]_{-2}^{z_0/2} =$$

$$= \frac{z_0^2}{8} - \frac{z_0^2}{16} + \frac{z_0}{2} + 1 \Rightarrow P[Z \leq z_0] = 1 + \frac{z_0}{2} + \frac{z_0^2}{16}$$

En conclusión:

$$\boxed{F_Z(z_0) = P[Z \leq z_0] = \begin{cases} 0 & z_0 \leq -2 \\ 1 + \frac{z_0}{2} + \frac{z_0^2}{16} & -2 \leq z_0 \leq 0 \\ 1 & z_0 \geq 0 \end{cases}}$$

EJERCICIO 2.- Sea (X, Y) con f.m.p:

X \ Y	0	1	2
1	1/4	0	0
2	0	1/4	0
3	1/4	0	1/4

a) Distribuciones condicionadas a todos los valores de X

$\boxed{Y/X=1}$

$$P_{Y/X=1}(0) = \frac{P(1,0)}{P_X(1)} = \frac{1/4}{1/4} = 1$$

$$P_{Y/X=1}(1) = \frac{P(1,1)}{P_X(1)} = 0$$

$$P_{Y/X=1}(2) = \frac{P(1,2)}{P_X(1)} = 0$$

$\boxed{Y/X=2}$

$$P_{Y/X=2}(0) = \frac{P(2,0)}{P_X(2)} = 0$$

$$P_{Y/X=2}(1) = \frac{P(2,1)}{P_X(2)} = \frac{1/4}{1/4} = 1$$

$$P_{Y/X=2}(2) = \frac{P(2,2)}{P_X(2)} = 0$$

$\boxed{Y/X=3}$

$$P_{Y/X=3}(0) = \frac{P(3,0)}{P_X(3)} = \frac{1/4}{1/2} = 1/2$$

$$P_{Y/X=3}(1) = \frac{P(3,1)}{P_X(3)} = 0$$

$$P_{Y/X=3}(2) = \frac{P(3,2)}{P_X(3)} = \frac{1/4}{1/2} = 1/2$$

b) f.m.p de $(X+Y, X-Y)$

X+Y \ X-Y	-1	0	1	2	3	P_{X+Y}
1	0	0	1/4	0	0	1/4
2	0	0	0	0	0	0
3	0	0	1/4	0	1/4	2/4
4	0	0	0	0	0	0
5	0	0	1/4	0	0	1/4
P_{X-Y}	0	0	3/4	0	1/4	$\Sigma=1$

c) $P[X+Y \leq 4]$ desde P_{X+Y}

$$P[X+Y \leq 4] = \underbrace{P[X+Y=1]}_{1/4} + \underbrace{P[X+Y=2]}_{0} + \underbrace{P[X+Y=3]}_{2/4} + \underbrace{P[X+Y=4]}_{0} = 3/4$$

$$\boxed{P[X+Y \leq 4] = 3/4}$$

EJERCICIO 3.-

a) Deducir la FGM de una distribución Gamma.

Sea X v.a. $\rightsquigarrow \Gamma(u,\lambda)$; $u,\lambda \in \mathbb{R}_+$

Su función de densidad es $f_X(x) = \frac{\lambda^u}{\Gamma(u)} x^{u-1} e^{-\lambda x}$, $x>0$

Vamos a demostrar que $M_X(t) = \left(1-\frac{t}{\lambda}\right)^{u}$, $t<\lambda$

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f_X(x)\,dx = \int_0^{+\infty} e^{tx} \frac{\lambda^u}{\Gamma(u)} x^{u-1} e^{-\lambda x}\,dx =$$

$$= \frac{\lambda^u}{\Gamma(u)} \int_0^{+\infty} e^{x(t-\lambda)} x^{u-1}\,dx = \left[\begin{array}{l} y=(\lambda-t)x \\ dy=(\lambda-t)dx \\ \text{cambio de variable} \end{array}\right] = \int_0^{+\infty} \frac{e^{-y}}{\lambda-t}\left(\frac{y}{\lambda-t}\right)^{u-1} dy =$$

$$= \frac{\lambda^u}{\cancel{\Gamma(u)}} \cdot \frac{1}{(\lambda-t)^{u-1}(\lambda-t)} \underbrace{\int_0^{+\infty} e^{-y} y^{u-1}\,dy}_{\cancel{\Gamma(u)}} = \frac{\lambda^u}{(\lambda-t)^u} = \left(\frac{\lambda}{\lambda-t}\right)^u = \left(\frac{\lambda-t}{\lambda}\right)^{-u} =$$

$$= \left(1-\frac{t}{\lambda}\right)^{-u}$$

Como $\lambda - t = 0$ sii $t=\lambda \Rightarrow M_X(t)$ se define si $t<\lambda$, $\exists$ debe ser un intervalo que contenga al 0 y $\lambda \in \mathbb{R}^+$

En conclusión:

$$\boxed{M_X(t) = \left(1-\frac{t}{\lambda}\right)^{-u} ;\ t<\lambda}$$

y evidentemente, no se define para todos los números reales.

b) Obtener el momento de orden 2 centrado en la media desde FGM

$\mu_2 = E[(X-E[X])^2] = m_2 - m_1^2 = E[X^2] - (E[X])^2 \rightarrow$ "Es la varianza"

$$E[X] = m_1 = \frac{dM_X(t)}{dt}\Big|_{t=0} = -u\left(1-\frac{t}{\lambda}\right)^{-u-1} \cdot (-1/\lambda)\Big|_{t=0} = u/\lambda$$

$$E[X^2] = m_2 = \frac{d^2M_X(t)}{dt^2}\Big|_{t=0} = \frac{d}{dt}\left(\frac{u}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-u-1}\right)_{t=0} = \frac{(u+1)u}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-u-2}\Big|_{t=0} =$$

$$= \frac{(u+1)u}{\lambda^2}$$

Por tanto $\mu_2 = \frac{(u+1)u}{\lambda^2} - \left(\frac{u}{\lambda}\right)^2 = \frac{u^2+u}{\lambda^2} - \frac{u^2}{\lambda^2} = \frac{u}{\lambda^2} \Rightarrow$

$$\Rightarrow \boxed{\mu_2 = Var[X] = u/\lambda^2}$$