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Probabilidad

2º Grado en Matemáticas

Facultad de Ciencias
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1) Función generatriz de momentos y calcular matriz de varianzas/covarianzas.

X\Y	-4	0
2	0	1/16
4	1/4	1/16
6	1/8	1/2

$$M_X(t_1,t_2) = \sum_{i=1}^{3}\sum_{j=1}^{2} e^{t_1 x_i + t_2 y_j} \cdot P[X=x_i, Y=y_j] =$$

$$= \frac{e^{2t_1}}{16} + \frac{e^{4t_1-4t_2}}{4} + \frac{e^{4t_1}}{16} + \frac{e^{6t_1-4t_2}}{8} + \frac{e^{6t_1}}{2}$$

$$M_X(0,t_2) = \frac{1}{16} + \frac{e^{-4t_2}}{4} + \frac{1}{16} + \frac{e^{-4t_2}}{8} + \frac{1}{2}$$

$$M_X(t_1,0) = \frac{e^{2t_1}}{16} + \frac{e^{4t_1}}{4} + \frac{e^{4t_1}}{16} + \frac{e^{6t_1}}{8} + \frac{e^{6t_1}}{2} = \frac{e^{2t_1}}{16} + \frac{5e^{4t_1}}{16} + \frac{5e^{6t_1}}{8}$$

$$E[X] = [E[X], E[Y]]$$

$$m_{10} = \frac{\partial M_X(t_1,t_2)}{\partial t_1}\Big|_{t_1=t_2=0} = \frac{2e^{2t_1}}{16} + \frac{\cancel{4}e^{4t_1-4t_2}}{\cancel{4}} + \frac{4e^{4t_1}}{16} + \frac{6e^{6t_1-4t_2}}{8} + \frac{6e^{6t_1}}{2} = \frac{41}{8}$$

$$m_{01} = \frac{\partial M_X(t_1,t_2)}{\partial t_2}\Big|_{t_1=t_2=0} = \frac{e^{4t_1-4t_2}}{4}(-4) + \frac{e^{6t_1-4t_2}}{8}(-4) = \frac{-3}{2}$$

$$E[X] = \left[\frac{41}{8}, \frac{-3}{2}\right]$$

$$E[X\cdot Y] = \frac{\partial}{\partial t_1}\left(\frac{-4e^{4t_1-4t_2}}{4} + \frac{-4e^{6t_1-4t_2}}{8}\right)\Big|_{t_1=t_2=0} = \frac{\partial}{\partial t_1}\left(-e^{4t_1-4t_2} - \frac{e^{6t_1-4t_2}}{2}\right)\Big|_{t_1=t_2=0} =$$

$$= \left(-4e^{t_1-4t_2} - \frac{6e^{6t_1-4t_2}}{2}\right)\Big|_{t_1=t_2=0} = -4 - \frac{6}{2} = -7$$

$$Var[X] = E[X^2] - E[X]^2 \Rightarrow$$

- $E[X^2] = \frac{4e^{t_1}}{16} + 4e^{4t_1-4t_2} + \frac{16e^{t_1}}{16} + \frac{36e^{6t_1-4t_2}}{8} + \frac{36e^{6t_1}}{2}\Big|_{t_1=t_2=0} = \frac{111}{4}$
- $E[Y^2] = \frac{16}{4}e^{4t_1-4t_2} + \frac{16}{8}e^{6t_1-4t_2} = 6$

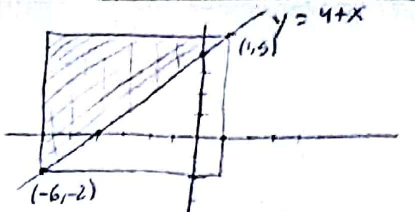
$$Var[X] = E[X^2] - E[X]^2 = \frac{111}{4} - \left(\frac{41}{8}\right)^2 = \frac{95}{64}$$

$$Var[Y] = E[Y^2] - E[Y]^2 = 6 - \left(-\frac{3}{2}\right)^2 = \frac{15}{4}$$

$$Cov[XY] = E[XY] - E[X]E[Y] = -7 - \left(\frac{41}{8}\right)\left(\frac{-3}{2}\right) = \frac{11}{16}$$

$$\boxed{Cov_X = \begin{pmatrix} \frac{95}{64} & \frac{11}{16} \\ \frac{11}{16} & \frac{15}{4} \end{pmatrix}} \quad //$$

② $f(x,y) = \frac{2}{49}$, $-6 < x < 1$, $4+x < y < 5$



a) Curva de regresión de Y sobre X:

Curva regresión Y/X

$$E[Y/X=x_0] = \int_{-\infty}^{\infty} y \cdot f(y/X=x_0)\,dy = \int_{4+x}^{5} y \cdot \frac{1}{1-x_0}\,dy = \frac{1}{1-x_0}\int_{4+x}^{5} y\,dy = \frac{1}{1-x_0}\left.\frac{y^2}{2}\right]_{4+x}^{5} = \frac{25-(4+x_0)^2}{2(1-x_0)}, \quad -6<x_0<1$$

$$f(y/X=x_0) = \frac{f(x_0,y)}{f_1(x_0)} = \frac{\cancel{2/49}}{\cancel{2/49}(1-x_0)} = \frac{1}{1-x_0}, \quad 4+x_0<y<5, \; -6<x_0<1$$

$$\frac{-x^2-8x+9}{2(1-x)} = \frac{x+9}{2}.$$

Como es una recta, coincide con la recta de regresión.

$$f_1(x) = \int_{4+x}^{5} \frac{2}{49}\,dy = \frac{2}{49}\,y\Big]_{4+x}^{5} = \frac{2}{49}(5-4-x) = \frac{2}{49}(1-x), \quad -6<x<1$$

b) Calculamos recta de regresión de Y sobre X.

$$y = ax+b, \quad a = \frac{Cov(X,Y)}{Var[X]}, \quad b = E[Y] - a\cdot E[X]$$

Nota: (la calculamos igualmente para practicar)
$Cov(X,Y) = E[XY] - E[X]\,E[Y]$
$Var[X] = E[X^2] - E[X]^2$

- $E[XY] = \int_{-6}^{1}\int_{4+x}^{5} \frac{2}{49}xy\,dy\,dx = \frac{2}{49}\int_{-6}^{1} x\,dx \int_{4+x}^{5} y\,dy = \frac{\cancel{2}}{49}\int_{-6}^{1} x\frac{(-x^2-8x+9)}{\cancel{2}}\,dx = \frac{1}{49}\left(-\frac{4949}{12}\right) = -\frac{101}{12}$

- $E[X] = \frac{2}{49}\int_{-6}^{1} x\,dx \int_{4+x}^{5} dy = \frac{2}{49}\int_{-6}^{1} x\cdot[y]_{4+x}^{5}\,dx = \frac{2}{49}\int_{-6}^{1} x - x^2\,dx = \frac{2}{49}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-6}^{1} = -\frac{11}{3}$

- $E[X^2] = \int_{-6}^{1}\int_{4+x}^{5}\frac{2}{49}x^2 y\,dy\,dx = \frac{2}{49}\int_{-6}^{1} x^2[y]_{4+x}^{5}\,dx = \frac{2}{49}\int_{-6}^{1} x^2 - x^3\,dx = \frac{2}{49}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{-6}^{1} = \frac{2}{49}\cdot\frac{4753}{12} = \frac{97}{6}$

- $E[Y] = \int_{-6}^{1}\int_{4+x}^{5}\frac{2}{49}y\,dy\,dx = \frac{2}{49}\int_{-6}^{1} dx \int_{4+x}^{5} y\,dy = \frac{1}{49}\int_{-6}^{1} -x^2-8x+9\,dx = \frac{1}{49}\left[\frac{-x^3}{3} - \frac{8x^2}{2} + 9x\right]_{-6}^{1} = \frac{1}{49}\cdot\left(\frac{392}{3}\right) = \frac{8}{3}$

- $E[Y^2] = \int_{-6}^{1}\int_{4+x}^{5}\frac{2}{49}y^2\,dy\,dx = \frac{2}{49}\int_{-6}^{1} dx \int_{4+x}^{5} y^2\,dy = \frac{2}{49}\int_{-6}^{1}\left.\frac{y^3}{3}\right]_{4+x}^{5} = \frac{2}{49}\cdot\frac{1}{3}\int_{-6}^{1}\left(5^3 - (4+x)^3\right)dx = \frac{59}{6}$

luego $Cov(X,Y) = -\frac{101}{12} - \left(-\frac{11}{3}\right)\cdot\left(\frac{8}{3}\right) = \frac{49}{36}$

$Var[X] = \frac{97}{6} - \left(-\frac{11}{3}\right)^2 = \frac{49}{18}$

$a = \frac{49/36}{49/18} = \frac{1}{2}$

$b = \frac{8}{3} - \frac{1}{2}\cdot\left(-\frac{11}{3}\right) = \frac{9}{2}$

$\Rightarrow \boxed{y = \frac{1}{2}x + \frac{9}{2}}$ Recta de regresión Y/X

c) Coeficiente de determinación lineal:

$$\rho = \frac{Cov(X,Y)^2}{Var[X]\,Var[Y]}, \quad Var[Y] = E[Y^2] - E[Y]^2 = \frac{59}{6} - \left(\frac{8}{3}\right)^2 = \frac{49}{18}$$

La $Var[X]$ ya la teníamos, luego:

$$\boxed{\rho = \frac{(49/36)^2}{\frac{49}{18}\cdot\frac{49}{18}} = \frac{1}{4}}$$

d) Razón de correlación

$$\eta^2_{Y/X} = \frac{Var[E[Y/X]]}{Var[Y]} = 1 - \frac{E[Var[Y/X]]}{Var[Y]}$$

1° Calculamos $Var[Y/X] = \underbrace{E[Y^2/X]}_{\text{la calculamos}} - \underbrace{E[Y/X]}_{\text{ya la tenemos}}^2$

$$E[Y^2/X] = \int_{4+x}^{5} y^2\cdot\frac{1}{1-x}\,dy = \frac{1}{1-x}\left[\frac{y^3}{3}\right]_{4+x}^{5} = \frac{5^3-(4+x)^3}{3(1-x)}$$

$$Var[Y/X] = \frac{125-(4+x)^3}{3(1-x)} - \left(\frac{25-(4+x)^2}{2(1-x)}\right)^2 = \dots = \frac{1}{12}(x-1)^2$$

$$E[Var[Y/X]] = \int_{-6}^{1}\frac{1}{12}(x-1)^2\cdot\frac{2}{49}(1-x)\,dx = \frac{2}{588}\int_{-6}^{1}(x-1)^2(1-x)\,dx = \frac{2}{588}\int_{-6}^{1} -x^3+3x^2-3x+1\,dx =$$

$$= \frac{2}{588}\left[-\frac{x^4}{4} + \frac{3x^3}{3} - \frac{3x^2}{2} + x\right]_{-6}^{1} = \frac{49}{24} \,/\!/$$

$$\boxed{\eta^2_{Y/X} = 1 - \frac{49/24}{49/18} = 1 - \frac{3}{4} = \frac{1}{4}}$$ dependencia media – baja