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(1) One measurable function $u : \mathbb{R}^d \to \mathbb{R}$ is a B.M.O. if

$$\|u\|_{BMO(\mathbb{R}^d,\mathbb{R})} = \sup_{\substack{a\in\mathbb{R}^d \\ r\in(0,1)}} \frac{1}{\mu(\mathbb{B}^d[a,r])} \int_{\mathbb{B}^d[a,r]} \frac{1}{\mu(\mathbb{B}^d[a,r])} \int_{\mathbb{B}^d[a,r]} |u(x)-u(y)|\, d\mu(x)\, d\mu(y) < \infty$$

where μ is Lebesgue's measure on $\mathbb{R}^d$.

<u>SOLUTION</u>

i) f measurable and delimit $\Rightarrow |f(x)| \leq M \ : M\in\mathbb{R} \ \forall x \in \mathbb{B}^d[a,r]$.

$$\int_{\mathbb{B}^d[a,r]} |f(x)| \leq M \cdot \mu(\mathbb{B}^d[a,r]) \Rightarrow$$

$$\Rightarrow \|f\|_{BMO(\mathbb{R}^d,\mathbb{R})} \leq \left(\frac{1}{\mu(\mathbb{B}^d[a,r])}\right)^2 \cdot K \cdot \left(\mu(\mathbb{B}^d[a,r])\right)^2 < \infty \quad (K\in\mathbb{R}) \Rightarrow$$

$\Rightarrow f \in BMO(\mathbb{R}^d, \mathbb{R})$. $\boxed{\text{True}}$

ii) f constant $\Rightarrow f$ measurable and delimit $\Rightarrow$ True.

(Also f constant $\Rightarrow \|f\|_{BMO(\mathbb{R}^d,\mathbb{R})} = 0 \Rightarrow \boxed{\text{True}}$.)

iii) $u, v \in BMO(\mathbb{R}^d, \mathbb{R})$ $\lambda, \beta \in \mathbb{R}$: Then:

$$\|\lambda u + \beta v\|_{BMO(\mathbb{R}^d,\mathbb{R})} \leq |\lambda| \cdot \|u\|_{BMO(\mathbb{R}^d,\mathbb{R})} + |\beta|\, \|v\|_{BMO(\mathbb{R}^d,\mathbb{R})}$$

We know that $\|u\|_{BMO(\mathbb{R}^d,\mathbb{R})} < \infty$ and $\|v\|_{BMO(\mathbb{R}^d,\mathbb{R})} < \infty$

so $\|\lambda u + \beta v\|_{BMO(\mathbb{R}^d,\mathbb{R})} < \infty$ Then $(\lambda u + v\cdot\beta) \in BMO(\mathbb{R}^d, \mathbb{R})$.

Therefore, <u>They build a vectorial space</u>, and because of ii) ($\|f\|_{BMO(\mathbb{R}^d,\mathbb{R})} = 0$ if f constant)

~~the~~ $\|\cdot\|_{BMO(\mathbb{R}^d,\mathbb{R})}$ is a <u>semi-norme</u>.