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Devoir 4

$V: \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R}) \qquad V(\delta_n) = \delta_{n+1} \quad \forall n \in \mathbb{N}.$

We know that $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$, then

$(a_n)_{n\in\mathbb{N}} = \sum_{i=1}^{\infty} a_i \cdot \delta_i$. So applying our operator to a suite of $\ell^2(\mathbb{N},\mathbb{R})$ means:

$$V((a_n)_{n\in\mathbb{N}}) = \sum_{i=1}^{\infty} a_i \, V(\delta_i) = \sum_{i=1}^{\infty} a_i \, \delta_{i+1} = (0, a_0, a_1 \text{ ---})$$

a) An application defined between two metric spaces is an isometry if it preserves distance. In our case, as V is defined as $V: \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R})$, being $\ell^2(\mathbb{N},\mathbb{R})$ a metric space, V would be an isometry if:

$$\| V((a_n)_{n\in\mathbb{N}}) - V((b_n)_{n\in\mathbb{N}}) \|_{\ell^2(\mathbb{N},\mathbb{R})} = \| (a_n)_{n\in\mathbb{N}} - (b_n)_{n\in\mathbb{N}} \|_{\ell^2(\mathbb{N},\mathbb{R})}$$

Therefore we have to prove:

$$\| V((a_n)_{n\in\mathbb{N}}) - V((b_n)_{n\in\mathbb{N}}) \|_2 = \| (a_n)_{n\in\mathbb{N}} - (b_n)_{n\in\mathbb{N}} \|_2$$

$$\| V((a_n)_{n\in\mathbb{N}}) - V((b_n)_{n\in\mathbb{N}}) \|_2 = \| (0, a_0, a_1 \text{ ---}) - (0, b_0, b_1 \text{ ---}) \|_2 =$$

$$= \| (0-0, a_0-b_0, a_1-b_1, \text{ ---}) \|_2 = \left(\sum_{i=0}^{\infty} (a_i-b_i)^2 + (0-0) \right)^{1/2} =$$

$$= \left(\sum_{i=0}^{\infty} (a_i-b_i)^2 \right)^{1/2} = \| (a_n)_{n\in\mathbb{N}} - (b_n)_{n\in\mathbb{N}} \|_2$$

Then we conclude by saying that V is an isometry, and trivially $\| V((a_n)_{n\in\mathbb{N}}) \|_2 = \| (a_n)_{n\in\mathbb{N}} \|_2$ what means $\| V \|_2 = 1$

b) The kernel of our linear operator is defined as:

$$Ker(V) = \{(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R}) : V((a_n)_{n\in\mathbb{N}}) = (0)_{n\in\mathbb{N}}\}.$$

Where $(0)_{n\in\mathbb{N}}$ denotes the zero suite.

And trivially we get that $Ker(V) = \{(0)_{n\in\mathbb{N}}\}$ as $V((a_n)_{n\in\mathbb{N}}) = (0, a_0, a_1, \text{——}) = (0)_{n\in\mathbb{N}}$.

The image of V is defined as all the suites from $\ell^2(\mathbb{N},\mathbb{R})$ that have a 0 in their first component, because for an arbitrary $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$ $V((a_n)_{n\in\mathbb{N}}) = (0, a_0, a_1, \text{——})$ and $(0, a_0, a_1, \text{——}) \in \ell^2(\mathbb{N},\mathbb{R})$. Therefore:

$$Img(V) = \{(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R}) : a_0 = 0\}$$

c) ~~The~~ Applying the definition of adjoint we know that:

$\forall (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$

$$(V((a_n)_{n\in\mathbb{N}}) \mid (b_n)_{n\in\mathbb{N}}) = ((a_n)_{n\in\mathbb{N}} \mid V^*(b_n)_{n\in\mathbb{N}}).$$

V^* denotes adjoint of V. Lets get an expression for it:

$$(V((a_n)_{n\in\mathbb{N}}) \mid (b_n)_{n\in\mathbb{N}}) = (\underbrace{(0, a_0, a_1 \text{——})}_{(a'_n)_{n\in\mathbb{N}}} \mid (b_0, b_1, b_2 \text{——})) =$$

$$= \sum_{i=0}^{\infty} a'_i \cdot b_i = \sum_{i=0}^{\infty} a_i \cdot b_{i+1} + 0 \cdot b_0 =$$

$$= \sum_{i=0}^{\infty} a_i \cdot b_{i+1} = ((a_n)_{n\in\mathbb{N}} \mid (b_1, b_2, \text{——})).$$

Therefore $V^* : \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R})$ it is defined as

$V^*(\delta_n) = \delta_{n-1}$ $\forall n \in \mathbb{N}$.

So, applyed to a $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$:

$V^*((a_n)_{n\in\mathbb{N}}) = (a_1, a_2, \_\_\_)$. For a $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$

$V^*(V((a_n)_{n\in\mathbb{N}})) = V^*((0, a_0, a_1, \_\_\_)) =$
$= (a_0, a_1, a_2, \_\_\_) = (a_n)_{n\in\mathbb{N}}$ what implies that

$V^* V = Id_{\ell^2(\mathbb{N},\mathbb{R})}$

$V(V^*((a_n)_{n\in\mathbb{N}})) = V((a_1, a_2, a_3, \_\_\_)) = (0, a_1, a_2, \_\_\_)$

what leads us to: $V^* V = Id_{\ell^2(\mathbb{N},\mathbb{R})} \neq V V^*$.

d) Taking in count the definition of eigenvalue we know that being V a linear transformation,
~~[illegible]~~
an eigenvalue it is a $\lambda \in \mathbb{R}$ such that

$V((a_n)_{n\in\mathbb{N}}) = \lambda (a_n)_{n\in\mathbb{N}}$ with $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$. and $(a_n)_{n\in\mathbb{N}} \neq (0)_{n\in\mathbb{N}}$

In our case the eigenvalues of V would be:

$V((a_n)_{n\in\mathbb{N}}) = \lambda (a_n)_{n\in\mathbb{N}}$
$(0, a_0, a_1, \_\_\_) = \lambda (a_0, a_1, a_2 \_\_\_)$

$$\begin{cases} 0 = \lambda a_0 \\ a_0 = \lambda a_1 \\ a_1 = \lambda a_2 \\ \vdots \end{cases}$$

If $\lambda = 0$ Then $a_0 = a_1 = a_2 = \_\_\_ = 0$ what leads to absurd. If $a_0 = 0$ Then as $\lambda \neq 0$, $a_1 = 0$ and then $a_2 = 0$ what leads us again to an absurd.

Therefore our operator V has not any eigenvalue.

e) We know that for every $\lambda \in \sigma(V^*)$ (spectrum of V^*) $\|\lambda\| \leq \|V^*\|_{\ell^2(\mathbb{N},\mathbb{R})} = 1$. Therefore, $\sigma(V^*) \subseteq [-1,1]$.

Also we have to take in count the fact that the spectrum of an operator is a closed subset, as the solutions of the resolvent of an operator are an open subset.

Lets start by searching the eigenvalues of V^* because if $\lambda \in \sigma(V^*)$ an it is an eigenvalue then λ is an approximate eigenvalue:

$$V^*((a_n)_{n\in\mathbb{N}}) = \lambda (a_n)_{n\in\mathbb{N}} \qquad (a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R}) \text{ and } (a_n)_{n\in\mathbb{N}} \neq (0)_{n\in\mathbb{N}}.$$

$$(a_1, a_2, a_3 \ldots) = \lambda (a_0, a_1, a_2 \ldots)$$

$$\begin{cases} a_1 = \lambda a_0 \\ a_2 = \lambda a_1 \\ a_3 = \lambda a_2 \\ \vdots \end{cases}$$

So we get that $a_{n+1} = \lambda a_n$, what leads us to $(a_n)_{n\in\mathbb{N}} = (\lambda^n a_0)_{n\in\mathbb{N}}$.

Lets study the possible values of λ.

If $|\lambda| = 1$, the suite $(a_0 \lambda^n)_{n\in\mathbb{N}} \notin \ell^2(\mathbb{N})$ as
$$\sum_{i=0}^{\infty} |a_0 \lambda^i|^2 = \sum_{i=0}^{\infty} |a_0| \text{ diverge.}$$

If $|\lambda| < 1$ then $\sum_{i=0}^{\infty} |a_0 \lambda^i|^2 = \sum_{i=0}^{\infty} |a_0|^2 |\lambda^i|^2 =$

$= |a_0|^2 \sum_{i=0}^{\infty} |\lambda^i|^2 < \infty$ as $|\lambda| < 1$ and then $(a_0\lambda^n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$

Therefore the values in the interval $]-1,1[$ are all the eigenvalues of V^* and as consequence, for every $\lambda \in]-1,1[$ λ is an approximate eigenvalue.

e) So we have that $]-1,1[\subseteq \sigma(V^*) \subseteq [-1,1]$, and as $\sigma(V^*)$ is a closed subset then $\sigma(V^*) = [-1,1]$.

To get $\sigma(V)$ we will prove that being V^* the adjoint of V, $\sigma(V^*) = \sigma(V)$.

We will begin for the definition of spectrum.

$\lambda \in \sigma(V^*)$ if and only if $V^* - \lambda Id$ has no inverse.

Then $\lambda \notin \sigma(V^*)$, the resolvent $(V^* - \lambda Id)$ can be inverted. Lets denote T^* the inverse of $(V^* - \lambda Id)$.

Then $(T^*)^* = T$ is the inverse of $(V^* - \lambda Id)^* =$
$= (V - \lambda Id)$. Therefore $V - \lambda Id$ can be inverted

Then $\lambda \notin \sigma(V)$. This means that $\sigma(V) = \sigma(V^*)$

So, Therefore $\sigma(V) = \{\lambda \in \mathbb{R} : |\lambda| \leq 1\}$.

f) We know by the exercise before that

$\sigma(V) = [-1,1]$

$\mathring{\sigma}(V) =]-1,1[\qquad \bar{\sigma}(V) = \{-1,1\}$

For every $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$ we get that

$\| V((a_n)_{n\in\mathbb{N}}) \|_2 = \| (a_n)_{n\in\mathbb{N}} \|_2$ (as we saw in a))

$$\| V((a_n)_{n\in\mathbb{N}}) - \lambda (a_n)_{n\in\mathbb{N}} \|_2 \geq \| V((a_n)_{n\in\mathbb{N}} \|_2 - |\lambda| \, \|(a_n)_{n\in\mathbb{N}} \|_2 =$$

$$= (1 - |\lambda|)(a_n)_{n\in\mathbb{N}} \quad (*)$$

Lets suppose now that λ is an approximate eigenvalue.

Then by definition of approximated eigenvalue.
exist $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$ such that
$\|(a_n)_{n\in\mathbb{N}}\|_2 = 1$ and

$$(V((a_n)_{n\in\mathbb{N}}) - \lambda (a_n)_{n\in\mathbb{N}}) \xrightarrow[n\to\infty]{} 0$$

From $(*)$ we get that $(1-|\lambda|)\,\|(a_n)_{n\in\mathbb{N}}\|_2 \xrightarrow[n\to\infty]{} 0$
Since $\|(a_n)_{n\in\mathbb{N}}\|_2 = 1$ we obtain $|\lambda| = 1$.
So the unique possible values of λ are $\{-1,1\} = \bar{\sigma}(V)$.
(being λ an approximated eigenvalue).

Therefore we can discard that any eigenvalue or approximated eigenvalue are in the $\mathring{\sigma}(V) =]-1,1[$.

To conclude, lets confirm that $\lambda = \{-1,1\}$ are eigenvalue. We know that if λ is not an approximate eigenvalue if and only if $(V-\lambda Id)$ has an inverse. But $\{-1,1\} \in \sigma(V) = [-1,1]$ and that means that for $|\lambda| = 1$ $(V-\lambda Id)$ has no inverse. Therefore we conclude saying that $\lambda \in \{-1,1\} = \bar{\sigma}(V)$ are approximated eigenvalues

$\square$