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Devoir ④

$$V: \ell^2(\mathbb{N}, \mathbb{R}) \to \ell^2(\mathbb{N}, \mathbb{R}) \qquad V(\delta_n) = \delta_{n+1} \quad \forall n \in \mathbb{N}.$$

We know that $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$, then

$$(a_n)_{n \in \mathbb{N}} = \sum_{i=1}^{\infty} a_i \cdot \delta_i.$$ So applying our operator to a suite of $\ell^2(\mathbb{N}, \mathbb{R})$ means:

$$V\left((a_n)_{n \in \mathbb{N}}\right) = \sum_{i=1}^{\infty} a_i \cdot V(\delta_i) = \sum_{i=1}^{\infty} a_i \cdot \delta_{i+1} = (0, a_0, a_1 \underline{\quad})$$

a) An application defined between two metric spaces is an isometry if it preserves distance. In our case, as V is defined as $V: \ell^2(\mathbb{N}, \mathbb{R}) \to \ell^2(\mathbb{N}, \mathbb{R})$, being $\ell^2(\mathbb{N}, \mathbb{R})$ a metric space, V would be an isometry if:

$$\| V((a_n)_{n \in \mathbb{N}}) - V((b_n)_{n \in \mathbb{N}}) \|_{\ell^2(\mathbb{N}, \mathbb{R})} = \| (a_n)_{n \in \mathbb{N}} - (b_n)_{n \in \mathbb{N}} \|_{\ell^2(\mathbb{N}, \mathbb{R})}$$

Therefore we have to prove:

$$\| V((a_n)_{n \in \mathbb{N}}) - V((b_n)_{n \in \mathbb{N}}) \|_2 = \| (a_n)_{n \in \mathbb{N}} - (b_n)_{n \in \mathbb{N}} \|_2$$

$$\| V((a_n)_{n \in \mathbb{N}}) - V((b_n)_{n \in \mathbb{N}}) \|_2 = \| (0, a_0, a_1 \underline{\quad}) - (0, b_0, b_1 \underline{\quad}) \|_2 =$$

$$= \| (0-0, a_0-b_0, a_1-b_1, \underline{\quad}) \|_2 = \left(\sum_{i=0}^{\infty} (a_i - b_i)^2 + (0-0) \right)^{1/2} =$$

$$= \left(\sum_{i=0}^{\infty} (a_i - b_i)^2 \right)^{1/2} = \| (a_n)_{n \in \mathbb{N}} - (b_n)_{n \in \mathbb{N}} \|_2$$

Then we conclude by saying that V is an isometry, and Trivially $\| V((a_n)_{n \in \mathbb{N}}) \|_2 = \| (a_n)_{n \in \mathbb{N}} \|_2$ what means $\| V \|_2 = 1$

b) The kernel of our linear operator is defined as:

$$\ker(V) = \{ (a_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N}, \mathbb{R}) : V((a_n)_{n \in \mathbb{N}}) = (0)_{n \in \mathbb{N}} \}.$$

Where $(0)_{n \in \mathbb{N}}$ denotes the zero suite.

And Trivially we get that $\ker(V) = \{ (0)_{n \in \mathbb{N}} \}$ as

$$V((a_n)_{n \in \mathbb{N}}) = (0, a_0, a_1 \underline{\quad\quad}) = (0)_{n \in \mathbb{N}}$$

The image of V is defined as all the Suites from $\ell^2(\mathbb{N}, \mathbb{R})$ that have a 0 in Their first component, because for an arbitrary $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$ $V((a_n)_{n \in \mathbb{N}}) = (0, a_0, a_1 \underline{\quad\quad})$ and $(0, a_0, a_1, \underline{\quad\quad}) \in \ell^2(\mathbb{N}, \mathbb{R})$. Therefore:

$$\text{Img}(V) = \{ (a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R}) : a_0 = 0 \}$$

c) ~~The~~ Applying the definition of adjoint we know That:

$$\forall (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$$

$$(V((a_n)_{n \in \mathbb{N}}) \mid (b_n)_{n \in \mathbb{N}}) = ((a_n)_{n \in \mathbb{N}} \mid V^*(b_n)_{n \in \mathbb{N}}).$$

V^* denotes adjoint of V. Lets get an expression for it:

$$(V((a_n)_{n \in \mathbb{N}}) \mid (b_n)_{n \in \mathbb{N}}) = (\overbrace{(0, a_0, a_1 \underline{\quad})}^{(a'_n)_{n \in \mathbb{N}}} \mid (b_0, b_1, b_2 \underline{\quad})) =$$

$$= \sum_{i=0}^{\infty} \text{\st{a_i b_i}} \, a'_i \cdot b_i = \sum_{i=0}^{\infty} a_i \cdot b_{i+1} + 0 \cdot b_0 =$$

$$= \sum_{i=0}^{\infty} a_i \cdot b_{i+1} = ((a_n)_{n \in \mathbb{N}} \mid (b_1, b_2, \underline{\quad})).$$

Therefore $V^* : \ell^2(\mathbb{N}, \mathbb{R}) \to \ell^2(\mathbb{N}, \mathbb{R})$ it is defined as

$$V^*(x_n) = x_{n-1} \quad \forall n \in \mathbb{N}.$$

So, applyed to a $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$:

$$V^*((a_n)_{n\in\mathbb{N}}) = (a_1, a_2, \underline{\quad}).$$ For as $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R})$

$$V^*(V((a_n)_{n\in\mathbb{N}})) = V^*((0, a_0, a_1, \underline{\quad})) =$$

$$= (a_0, a_1, a_2, \underline{\quad}) = (a_n)_{n\in\mathbb{N}} \quad \text{What implies that}$$

$$V^*V = Id_{\ell^2(\mathbb{N},\mathbb{R})}$$

$$V(V^*((a_n)_{n\in\mathbb{N}})) = V((a_1, a_2, a_3, \underline{\quad})) = (0, a_1, a_2, \underline{\quad})$$

what leads us to: $V^*V = Id_{\ell^2(\mathbb{N},\mathbb{R})} \neq VV^*$.

d) Taking in count the definition of eigenvalue we know that being V a linear transformation, ~~between a vector space and the same space~~ an eigenvalue it is a $\lambda \in \mathbb{R}$ such that

$$V((a_n)_{n\in\mathbb{N}}) = \lambda (a_n)_{n\in\mathbb{N}} \quad \text{with } (a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R}). \text{ and}$$
$$(a_n)_{n\in\mathbb{N}} \neq (0)_{n\in\mathbb{N}}$$

In our case the eigenvalues of V would be:

$$V((a_n)_{n\in\mathbb{N}}) = \lambda (a_n)_{n\in\mathbb{N}}$$
$$(0, a_0, a_1, \underline{\quad}) = \lambda (a_0, a_1, a_2 \underline{\quad})$$

$$\begin{cases} 0 = \lambda a_0 \\ a_0 = \lambda a_1 \\ a_1 = \lambda a_2 \\ \vdots \end{cases}$$

If $\lambda = 0$ Then $a_0 = a_1 = a_2 \underline{\quad} = 0$ what leads to absurd. If $a_0 = 0$ Then as $\lambda \neq 0$, $a_1 = 0$ and Then $a_2 = 0$ what leads us again To an absurd.

Therefore our operator V has not any eigenvalue.

e) We know that for every $\lambda \in \sigma(V^*)$ (spectrum of V^*)
$\|\lambda\| \leq \|V^*\|_{\ell^2(\mathbb{N}, \mathbb{R})} = 1$. Therefore, $\sigma(V^*) \subseteq [-1, 1]$.
Also we have to take in count the fact that
the spectrum of an operator is a closed subset,
as the solutions of the resolvent of an operator
~~af~~ are an open subset.
Lets start by searching the eigenvalues of V^*
because if $\lambda \in \sigma(V^*)$ an it is an eigenvalue then
λ is an approximate eigenvalue:
$$V^*((a_n)_{n \in \mathbb{N}}) = \lambda (a_n)_{n \in \mathbb{N}} \quad (a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R}) \text{ and}$$
$$(a_n)_{n \in \mathbb{N}} \neq (0)_{n \in \mathbb{N}}.$$

$$(a_1, a_2, a_3 \text{---}) = \lambda(a_0, a_1, a_2\text{---})$$

$$\begin{cases} a_1 = \lambda a_0 \\ a_2 = \lambda a_1 \\ a_3 = \lambda a_2 \\ \vdots \end{cases}$$
So we get that $a_{n+1} = \lambda a_n$,
what leads us to $(a_n)_{n \in \mathbb{N}} = (\lambda^n a_0)_{n \in \mathbb{N}}$.

Lets study the possible values of λ.
If $|\lambda| = 1$, the suite $(a_0 \lambda^n)_{n \in \mathbb{N}} \notin \ell^2(\mathbb{N})$ as.
$$\sum_{i=0}^{\infty} |a_0 \lambda^i|^2 = \sum_{i=0}^{\infty} |a_0| \text{ diverge.}$$

If $|\lambda| \lessgtr 1$ then $\sum_{i=0}^{\infty} |a_0 \lambda^i|^2 \leq \sum_{i=0}^{\infty} |a_0|^2 |\lambda^i|^2 =$

$$= |a_0|^2 \sum_{i=0}^{\infty} |\lambda^i|^2 < \infty \quad \text{as } |\lambda| < 1 \text{ and then } (a_0\lambda^n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$$

Therefore the values ~~&~~ in the interval $]-1, 1[$ are
all the eigenvalues of V^* and as consequence,
for every $\lambda \in]-1, 1[$ λ is an approximate eigenvalue.

e) So we have that $]-1,1[\subseteq \sigma(V^*) \subseteq [-1,1]$,
and as $\sigma(V^*)$ is a closed subset then
$\sigma(V^*) = [-1,1]$.

To get $\sigma(V)$ we will prove that
being V^* the adjoint of V, $\sigma(V^*) = \sigma(V)$.

We will begin for the definition of
spectrum.

$\lambda \in \sigma(V^*)$ if and only if $V^* - \lambda Id$ has
no inverse.

Then $\lambda \notin \sigma(V^*)$, the resolvent $(V^* - \lambda Id)$ can be
inverted. Lets denote T^* the inverse of $(V^* - \lambda Id)$.
Then $(T^*)^* = T$ is the inverse of $(V^* - \lambda Id)^* =$
$= (V - \lambda Id)$. Therefore $V - \lambda Id$ can be inverted
Then $\lambda \notin \sigma(V)$. This means that $\sigma(V) = \sigma(V^*)$
So, Therefore $\sigma(V) = \{ \lambda \in \mathbb{R} : |\lambda| \le 1 \}$.

f) We know by the exercise before that
$\sigma(V) = [-1,1]$
$\overset{\circ}{\sigma}(V) =]-1,1[\qquad \overline{\sigma}(V) = \{-1,1\}$

For every $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$ we get that
$\| V((a_n)_{n \in \mathbb{N}}) \|_2 = \| (a_n)_{n \in \mathbb{N}} \|_2$ (as we saw in a))

$\| V((a_n)_{n \in \mathbb{N}}) - \lambda (a_n)_{n \in \mathbb{N}} \|_2 \ge \| V((a_n)_{n \in \mathbb{N}}) \|_2 - |\lambda| \|(a_n)_{n \in \mathbb{N}} \|_2 =$

$= (1 - |\lambda|)(a_n)_{n \in \mathbb{N}} \quad (*)$

Lets suppose now that λ is an approximate
eigenvalue.

Then by definition of approximated eigenvalue.
exist $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R})$ such that

$\| (a_n)_{n \in \mathbb{N}} \|_2 = 1$ and

$$(V((a_n)_{n \in \mathbb{N}}) - \lambda (a_n)_{n \in \mathbb{N}}) \xrightarrow[n \to \infty]{} 0$$

From $(*)$ we get that $(1 - |\lambda|) \| (a_n)_{n \in \mathbb{N}} \|_2 \xrightarrow[n \to \infty]{} 0$
Since $\| (a_n)_{n \in \mathbb{N}} \|_2 = 1$ we obtain $|\lambda| = 1$.
So the unique possible values of λ are $\{-1, 1\} = \bar\sigma(V)$.
(being λ an approximated eigenvalue).

Therefore we can discard that any
eigenvalue or approximated eigenvalue are
in the $\bar\sigma(V) =]-1, 1[$.

To conclude, lets confirm that $\lambda = \{-1, 1\}$ are
eigenvalue. We know that if λ is not
an approximate eigenvalue if and only if
$(V - \lambda Id)$ has an inverse. But $\{-1, 1\} \in \sigma(V) = [-1, 1]$
and that means that for $|\lambda| = 1$
$(V - \lambda Id)$ has no inverse. Therefore we
conclude saying that $\lambda \in \{-1, 1\} = \bar\sigma(V)$ are
approximated eigenvalues

$\square$

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