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We have to prove that the function

$$l : L^3((0,1), dx) \longrightarrow \mathbb{R}$$

$$l(u) = \int_0^1 \frac{u(x)}{|x|^\alpha}\, dx$$

is an element of the dual space of $L^3((0,1), dx)$.

This dual space is:

$$L^3((0,1), dx)^* = \mathcal{L}(L^3((0,1), dx), \mathbb{R}).$$

We will base our proof in the Syllabus' Theorem 14.20 and in the Hölders' inequality.

Applying 14.20 theorem in our case we have that.

$\forall\ l \in L^3((0,1), dx)^*$ exists an unique $g \in L^{3/2}((0,1), dx)$ such that for every $u \in L^3((0,1), dx)$:

$$\langle l, u\rangle = l(u) = \int_0^1 g \cdot u \, dx.$$

Then the values of α that achieve that $l \in L^3((0,1), dx)^*$ are those which holds that

$$l(u) \leq \infty.$$

We trivially can assume that the function g in 14.20 theorem, in our case is $g(x) = \frac{1}{|x|^\alpha} = \frac{1}{x^\alpha}$ (as we work with $x \in (0,1)$)

As follows from the theorem, the fuction $g \in L^{3/2}((0,1), dx)$ and That means that ~~...~~ $\|g\|_{L^{3/2}((0,1),dx)} < \infty$ and g is measurable. (for some $\alpha \in \mathbb{R}$)

$\|g\|_{L^{3/2}((0,1),dx)} = \left(\int_0^1 \left(\frac{1}{|x|^\alpha}\right)^{3/2} dx \right)^{2/3}$. We can

just focus in ~~the~~ The integral, to compute those αs

$$\int_0^1 \left(\frac{1}{|x|^\alpha}\right)^{3/2} dx = \int_0^1 \frac{1}{|x|^{\frac{3}{2}\alpha}} dx$$ and we

know that $\frac{1}{x^a}$ is integrable if and only if $a < 1$ in $(0,1)$. Therefore: from $\int_0^1 \frac{1}{|x|^{\frac{3}{2}\alpha}}$ we get that

for $\alpha < \frac{2}{3}$ $\frac{1}{|x|^{\frac{3}{2}\alpha}}$ is integrable and Then

$\int_0^1 \frac{1}{|x|^{\frac{3}{2}\alpha}} dx < \infty$. So The unique possible values for which $g \in L^{3/2}((0,1), dx)$ are $\alpha \in \left]-\infty, \frac{2}{3}\right[$.

And Then we conclude with

$$\ell(u) = \int_0^1 \frac{u(x)}{|x|^\alpha} dx \le [\text{by Hölder's inequality}] \le$$

$$\le \left(\int_0^1 (u(x))^3 dx\right)^{1/3} \cdot \left(\int_0^1 \left(\frac{1}{|x|^\alpha}\right)^{3/2} dx\right)^{2/3} = \|u\|_{L^3((0,1),dx)} \cdot \|g\|_{L^{3/2}((0,1),dx)} < \infty$$

Because ~~...~~ $\|u\|_{L^3((0,1),dx)} < \infty$ Trivially and $\|g\|_{L^{3/2}((0,1),dx)} < \infty$ for every $\alpha \in \left]-\infty, \frac{2}{3}\right[$ so $\ell(u) = \int_0^1 \frac{u(x)}{|x|^\alpha} dx$ is an element of $L^3((0,1), dx)^*$ for those α $\square$