Devoir 02 Juan Valentin Gerrero Caro

[Exercise 1] Lets start by giving definitions

- $C_b(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : f$ continue and bounded $\}$
- $C_0(\mathbb{R}) = \{ f \in C_b(\mathbb{R}) : \lim\limits_{x \to \pm\infty} \sup |f(x)| = 0 \}$

- $u \in C_0(\mathbb{R}) \Rightarrow \lim\limits_{x \to \pm\infty} \sup |u(x)| = 0$
- $\| \cdot \|_\infty = \sup\limits_{x \in \mathbb{R}} | \cdot |$. Ex.: $\| f \|_\infty = \sup\limits_{x \in \mathbb{R}} |f(x)|$
- Linear function $A : (C_0(\mathbb{R}), \| \cdot \|_\infty) \to \mathbb{R}$

$$u \longrightarrow \int_{\mathbb{R}} \frac{u(x)}{1+x^2} \, dx$$

- $\|A\|_{\mathcal{L}(C_0(\mathbb{R}), \mathbb{R})} = \sup\limits_{\|u\|_\infty \leq 1} |A(u)|$

So we have defined all the elements in the exercise's heading. Now we can start proving the continuity of the <u>linear functional</u> A.

We know that if $A \in \mathcal{L}(C_0(\mathbb{R}), \mathbb{R})$ Then A is bounded and linear, and by Proposition 4.9, A is uniformly continuous what means A is continuous. Then, we have to prove that $A \in \mathcal{L}(C_0(\mathbb{R}), \mathbb{R})$ what means that A is linear and bounded. We already know that A is linear so we just have to prove that it is bounded.

$$|A(u)| = \left| \int_{\mathbb{R}} \frac{u(x)}{1+x^2} dx \right| \leq \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2} dx \leq \int_{\mathbb{R}} \frac{\|u(x)\|_\infty}{1+x^2} dx$$

Then we can extract $\|u\|_\infty$ from the integral:

$$\int_{\mathbb{R}} \frac{\|u(x)\|_\infty}{1+x^2} dx = \|u(x)\|_\infty \int_{\mathbb{R}} \frac{1}{1+x^2} dx =$$

$$= \|u(x)\|_\infty \cdot \left[\arctg(x) \right]_{-\infty}^{+\infty} = \|u\|_\infty \cdot \pi \Rightarrow$$

$$\Rightarrow |A(u)| \leq \pi \|u\|_\infty \quad \forall u \in C_0(\mathbb{R}) \Rightarrow A \text{ is } \underline{\text{bounded}}.$$

Also we know by $|A(u)| \leq \pi \|u\|_\infty$ that $\|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} \leq \pi$, because $|A(u)| = \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} \cdot \|u\|_\infty$

So, as A is bounded and a linear functional then A is <u>contiuve</u>

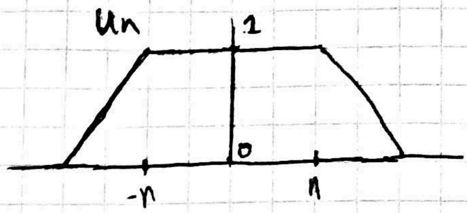
Having proved that A is contiuve and $\|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} \leq \pi$ lets try to give $\|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})}$ by proving that $\pi \leq \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})}$

Then we will conclude by saying that $\|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} = \pi$.

As we said at the beginning of the exercise.

$$\|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} = \sup_{\|u\|_\infty \leq 1} |A(u)| \Rightarrow u \in B(0,1)$$

We know that u with $\|u\|_\infty \leq 1$ that allows us to reach $\sup |A(u)|$ is $u^* = 1$ (cte) but $u^* \notin C_0(\mathbb{R})$ cause $\limsup_{x \to \pm\infty} u^*(x) = 1 \neq 0$. Then we will create a suite $\{u_n\} \to u^*$ and $u_n \in C_0(\mathbb{R})$ $\forall n \in \mathbb{N}$.



$$u_n(x) = \begin{cases} 1 & \text{if } x \in [-n, n] \\ 1+n-x & \text{if } x \in]n, n+1[\\ 1-n-x & \text{if } x \in]-n-1, -n[\\ 0 & \text{if } x \in]-\infty, -n-1] \cup [n+1, +\infty[\end{cases}$$

$\{u_n\} \to u^*$ and $\{u_n\}$ is growing:

$u_{n+1}(x) - u_n(x) \geq 0$:

$\forall x \in [-n, n]$ $\quad u_{n+1}(x) - u_n(x) = 1 - 1 = 0 \geq 0$

$\forall x \in [-n-1, -n[$ $\quad u_{n+1}(x) - u_n(x) = 1 - 1 + n + x \geq 0$

$\forall x \in]n, n+1]$ $\quad u_{n+1}(x) - u_n(x) = 1 - 1 - n + x = -n + x \geq 0$

$\forall x \in]-\infty, -n-1] \cup [n+1, +\infty[$ $\quad u_{n+1}(x) - u_n(x) \geq 0$

Also we know that $\displaystyle\int_{\mathbb{R}} \frac{u^*}{1+x^2}\, dx = \pi < \infty$

Then, by the monotone convergence's theorem, we know that our $\{u_n\} \to u^*$ achieve that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{u_n}{1+x^2}\, dx = \int_{\mathbb{R}} \frac{u^*}{1+x^2}\, dx = \pi$$

Then we know that:

$$\pi = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{u_n}{1+x^2}\,dx \leq \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} \frac{u_n}{1+x^2}\,dx \leq$$

$$\leq \sup_{\|u\|_\infty \leq 1} \int \frac{u}{1+x^2}\,dx = \sup_{\|u\|_\infty \leq 1} A(u) \leq$$

$$\leq \sup_{\|u\|_\infty \leq 1} |A(u)| = \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})}.$$

So finally we have arrived at $\pi \leq \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})}$

So:

$$\begin{cases} \pi \geq \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} \\ \pi \leq \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})} \end{cases} \Rightarrow \boxed{\pi = \|A\|_{\mathcal{L}(C_0(\mathbb{R}),\mathbb{R})}}$$