1 Donnex des exemples et contre-exemples d'espaces complets.

Examples of complete spaces include:

- The set of real numbers with the Euclidean metric, also known as the real line (R, |·|)
- The set of all convergent sequences of real numbers with the metric of pointwise convergence, also known as the space of real-valued sequences (l^p, d_p) where p is any 1<=p<=infinity.

Counter-examples of complete spaces include:

- The set of rational numbers with the Euclidean metric, also known as the rational line (Q, |·|)
- The set of all Cauchy sequences of real numbers that do not converge, also known as the space of Cauchy sequences (C, d_C)
- The space of bounded sequences (l^∞, d_∞) is not complete because it does not contain all its Cauchy sequences. A Cauchy sequence in this space is a sequence of real numbers such that for every ε > 0, there exists a natural number N such that for all m,n > N, the difference between the mth and nth term of the sequence is less than ε. However, not all Cauchy sequences in this space converge to a point in the space. For example, the sequence (a_n) where a_n = 1 for all n is a Cauchy sequence in this space but does not converge to any point in the space because the limit of the sequence is not a point in the space.

2 Donnex des exemples de sous-espaces vectoriels denses

Examples of dense vector subspaces include:

- The set of all polynomials with real coefficients, also known as the space of polynomials (P(R), +, .), is a dense subspace of the space of continuous functions (C(R), +, .), where C(R) is the set of all continuous functions from the real numbers to the real numbers. The reason is that for every continuous function f(x) on the real numbers, we can approximate it arbitrarily well by a polynomial function. This is because the Weierstrass approximation theorem states that for any continuous function f(x) on a closed interval [a,b], and for any ε > 0, there exists a polynomial function p(x) such that the difference |f(x) - p(x)| < ε for all x in [a,b].

Therefore, for any point in the space of continuous functions (C(R), +, .), we can always find a sequence of polynomials that converges to it. This means that the space of polynomials (P(R), +, .) is dense in the space of continuous functions (C(R), +, .), so it is a dense subspace.

- The set of all rational numbers, also known as the space of rationals (Q, +, .), is a dense subspace of the space of real numbers (R, +, .).
- The set of all trigonometric polynomials, also known as the space of trigonometric polynomials (TP(R), +, .), is a dense subspace of the space of square-integrable functions (L^2(R), +, .), where L^2(R) is the set of all square-integrable functions from the real numbers to the real numbers.

A subspace is called dense if for every point in the ambient space, there exists a sequence of points in the subspace that converges to it.

3. Could you explain intuitively the concept of a subset separable?

A subset of a topological space is called separable if it contains a countable dense subset. Intuitively, this means that the subset can be "approximated" by a countable number of points that are "arbitrarily close" to every point in the subset.

For example, the set of real numbers R is separable because it contains the countable set of rational numbers Q, which is dense in R. This means that for every real number x, we can find a sequence of rational numbers that converges to x.

Similarly, the space of continuous functions on a compact interval is separable because it contains the set of polynomials with rational coefficients, which is dense in the space of continuous functions.

In other words, the idea behind separability is that we can approximate the subset by a countable set of points that are "densely" distributed throughout the subset, allowing us to approximate any point in the subset.

4. Définissez les notion d'ensemble rare et maigre et illustrez-la à l'aide d'un dessin

For example, the set of all irrational numbers in the real line is a nowhere dense set, because its closure is the set of all real numbers, which has non-empty interior. Therefore, the set of all irrational numbers is not rare.

On the other hand, the set of all rational numbers in the real line is not nowhere dense, because its closure is the set of all real numbers and it has non-empty interior. Therefore, the set of all rational numbers is not rare.

A set that is rare is illustrated in a Venn diagram as a set that can be expressed as a countable union of sets with a small area (or volume) in relation to the ambient space, while the closure of each of these sets is a set with an empty area (or volume).

5. Illustrez par des exemples les notions de sous-espace vectorial rare et maigre

A vector subspace is called rare or meager if it can be expressed as the countable union of nowhere dense vector subspaces.

For example, consider the vector subspace of all polynomials with rational coefficients in the space of all polynomials with real coefficients. This vector subspace is not rare or meager, because it is dense in the ambient space of all polynomials with real coefficients.

On the other hand, consider the vector subspace of all polynomials with coefficients that are roots of unity in the space of all polynomials with complex coefficients. This vector subspace is rare or meager, because it can be expressed as a countable union of vector subspaces of polynomials of the form (a_0 + a_1 z + a_2 z^2 + ... + a_n z^n) where z is a root of unity and the coefficients a_i are complex numbers. Each of these vector subspaces is nowhere dense in the ambient space of all polynomials with complex coefficients because its closure does not contain any non-zero polynomials.

It's worth to note that some authors use the term "meager" instead of "rare" to name this kind of sets.

Another example of a rare or meager vector subspace is the space of all convergent sequences of real numbers in the space of all Cauchy sequences of real numbers. This vector subspace can be expressed as a countable union of vector subspaces of the form $\{(x_n) \mid x_n = a$ for all n, where a is a real number$\}$. Each of these vector subspaces is nowhere dense in the ambient space of all Cauchy sequences of real numbers because its closure does not contain any non-constant sequences.

This is a good example to illustrate that the fact of a vector subspace being rare or meager doesn't imply that it's a small subspace. Even though the space of convergent sequences is rare, it's a quite important subset, it's the subset of Cauchy sequences that converge to a limit.

6. Interpretation of Baire Category theorem:

The Baire Category Theorem is a fundamental result in topology and analysis that states that a complete metric space cannot be expressed as the countable union of nowhere dense sets. In other words, if a complete metric space X is the countable union of nowhere dense sets, then at least one of these sets must have non-empty interior.

The Baire Category Theorem can be used to prove that certain sets are not rare or meager. For example, if we can prove that a set A is dense in a complete metric space X, then by the Baire Category Theorem we can conclude that A is not rare or meager.

More intuitively:

The Baire Category Theorem can be understood intuitively as a statement about "generic" points in a complete metric space. A complete metric space is one in which every Cauchy sequence converges to a point in the space.

The theorem states that if a complete metric space is a countable union of nowhere dense sets, then there must be a "generic" point in the space that is not in any of these nowhere dense sets. In other words, the set of "non-generic" points, which are the points that are in a nowhere dense set, is very small in comparison to the whole space.

The intuition behind this is that a nowhere dense set is a set that has no "interior", it cannot contain any open sets. So, if we were to divide a complete metric space into countable many subsets, each of which is nowhere dense, it would mean that the whole space would be a "fractal-like" set, with no "big" open sets. However, since the space is complete, it must contain a "big" open set (or at least a point that is arbitrarily close to an open set) and that is what the theorem states.

In other words, the Baire Category Theorem tells us that complete metric spaces are not "pathological" in the sense that they can't be divided into small and insignificant parts. Instead, they have to have "big" and important parts, even if they are just a point.

Rare sets: or nowhere dense sets are those whose closure has empty interior

7. Banach spaces:

A Banach space is a complete normed vector space, which means that it is a vector space that is equipped with a norm (a function that assigns a non-negative real number to each vector in the space) and is complete in the sense that every Cauchy sequence in the space converges to a point in the space.

Some examples of Banach spaces are:

- The space of all continuous functions on a closed interval of the real numbers equipped with the supremum norm.
- The space of all square-summable sequences of real numbers equipped with the Euclidean norm.
- The space of all complex-valued functions on the unit disk in the complex plane equipped with the supremum norm.

A Hilbert space is a special type of Banach space that is also an inner product space.