

# *Introduction to Cryptography*

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MAT2450 – Lecture 3



## Reminder: security against eavesdropper

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Security experiment:

Given  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$ , and adversary  $\mathcal{A}$ , define the following experiment  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$ :

1.  $\mathcal{A}$  outputs  $m_0, m_1 \in \mathcal{M}$
2. Choose  $k \leftarrow \mathcal{K}$  and  $b \leftarrow \{0, 1\}$ , and send  $\text{Enc}_k(m_b)$  to  $\mathcal{A}$
3.  $\mathcal{A}$  outputs  $b'$
4. Define  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}} := 1$  iff  $b = b'$



## Reminder: Building Encryption Schemes

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Suppose  $G$  is a pseudorandom generator with expansion factor  $l$

Let  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  be:

- ▶  $\text{Gen}(1^n)$  outputs uniformly random  $k$  from  $\{0, 1\}^n$
- ▶  $\text{Enc}$ , on input  $m \in \{0, 1\}^{l(n)}$  and  $k \in \{0, 1\}^n$ , provides  $c := m \oplus G(k)$
- ▶  $\text{Dec}$ , on input  $c \in \{0, 1\}^{l(n)}$  and  $k \in \{0, 1\}^n$ , provides  $m := c \oplus G(k)$



# Multiple encryption

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So far, we have seen how to encrypt one *single* message  
How can we extend this to several messages?

- ▶ Repeat the same process?
  - ▶  $c_1 = G(k) \oplus m_1$
  - ▶  $c_2 = G(k) \oplus m_2$

Quizz: would that work?

1. Yes
2. No

**Answer: no**

- ▶ For example, anyone can compute  $c_1 \oplus c_2 = m_1 \oplus m_2$



# Multiple encryption

---

So far, we have seen how to encrypt one *single* message  
How can we extend this to several messages?

- ▶ Repeat the same process?
  - ▶  $c_1 = G(k) \oplus m_1$
  - ▶  $c_2 = G(k) \oplus m_2$
  - ▶ But then,  $c_1 \oplus c_2 = m_1 \oplus m_2$  !
  - ▶ A very bad idea
- ▶ Use a different key for each message?
  - ▶ But how do we transmit it?
- ▶ Use a different part of the pseudorandom stream?
  - ▶ But how?

First, we need to define the security we want to achieve



## Secure multiple encryption

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Define the multiple-message eavesdropping experiment  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{mult}}(n)$

1.  $\mathcal{A}$  outputs  $M_0 = (m_0^1, \dots, m_0^t), M_1 = (m_1^1, \dots, m_1^t)$
2. Choose  $k \leftarrow \text{Gen}(1^n)$  and  $b \leftarrow \{0, 1\}$ , and send  $(\text{Enc}_k(m_b^1), \dots, \text{Enc}_k(m_b^t))$  to  $\mathcal{A}$
3.  $\mathcal{A}$  outputs  $b'$
4. Define  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{mult}}(n) := 1$  iff  $b = b'$



## Secure multiple encryption

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$\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  has *indistinguishable multiple encryption* in the presence of eavesdroppers if

$\forall$  PPT  $\mathcal{A}$ ,  $\exists \epsilon :$

$$\Pr[\text{PrivK}_{\mathcal{A}, \Pi}^{\text{mult}}(n) = 1] \leq \frac{1}{2} + \epsilon(n)$$



## Secure multiple encryption

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Does secure (single) encryption imply secure multiple encryption?

- ▶ Of course not! (see previous attempt)

But can we prove it?

- ▶ That is, taking the naive “repetition” idea of slide 5, can we build an adversary that wins  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{mult}}$  with non-negligible probability?
- ▶ Easy example:  $\mathcal{A}$  outputs

$$M_0 = (0 \dots 0, 0 \dots 0), M_1 = (0 \dots 0, 1 \dots 1)$$





# Probabilistic encryption

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Observation: we cannot achieve acceptable security with a *deterministic* scheme<sup>1</sup>

We need *probabilistic encryption*

- ▶ The same message, encrypted with the same key, yields different results
- ▶ Of course, decryption remains deterministic

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<sup>1</sup>At least, not without maintaining a state between encryptions.



## Remark

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Encrypting the same message twice is not the only problem  
As we have seen, if we encrypt two *different* messages as

- ▶  $c_1 = G(k) \oplus m_1$
- ▶  $c_2 = G(k) \oplus m_2$

then the adversary learns  $c_1 \oplus c_2 = m_1 \oplus m_2$

This is a lot of information!

- ▶ If one of the messages is (partially) known
- ▶ If both messages are in English
- ▶ ...

A frequent and devastating mistake



# Security against Chosen-Plaintext Attacks (CPA)

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So far, we have only considered *passive* adversaries

- ▶ eavesdrops on ciphertext
- ▶ must recover (some info on) plaintext

But a real-world adversary could have access to additional information

- ▶ previous encryptions (with same key) of messages he knows
- ▶ previous encryptions (with same key) of messages he has chosen
- ▶ ...

Can we capture these notions?



## The new adversary

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Let us define a more powerful adversary

- ▶ Granted access to an *encryption oracle*  $\text{Enc}_k(\cdot)$  that will encrypt messages of his choice
- ▶ Allowed to call oracle adaptively, before and after submitting two challenge messages  $m_0, m_1$  of his choice
- ▶ As before, must tell whether he receives  $\text{Enc}_k(m_0)$  or  $\text{Enc}_k(m_1)$



## More formally...

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Given  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$ , and adversary  $\mathcal{A}$ , define the following experiment  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n)$ :

1. Choose  $k \leftarrow \text{Gen}(1^n)$
2.  $\mathcal{A}$  is given oracle access to  $\text{Enc}_k(\cdot)$
3.  $\mathcal{A}$  outputs  $m_0, m_1 \in \mathcal{M}$
4. Choose  $b \leftarrow \{0, 1\}$  and send  $\text{Enc}_k(m_b)$  to  $\mathcal{A}$
5.  $\mathcal{A}$  is again given oracle access to  $\text{Enc}_k(\cdot)$
6.  $\mathcal{A}$  outputs  $b'$
7. Define  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n) := 1$  iff  $b = b'$



## Security against CPA

---

$\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  has *indistinguishable encryption under a chosen-plaintext attack* if  $\forall$  PPT  $\mathcal{A}$ ,  $\exists \epsilon$  :

$$\Pr[\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n) = 1] \leq \frac{1}{2} + \epsilon(n)$$

Remark: what is the relationship between  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}(n)$  and  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n)$ ?

- ▶ CPA-security  $\Rightarrow$  security against an eavesdropper



## Quiz: why not this?

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Given  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$ , and adversary  $\mathcal{A}$ , define the following experiment  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n)$ :

1. Choose  $k \leftarrow \text{Gen}(1^n)$
2.  $\mathcal{A}$  is given oracle access to  $\text{Enc}_k(\cdot)$
3.  $\mathcal{A}$  outputs  $m_0, m_1 \in \mathcal{M}$
4. Choose  $b \leftarrow \{0, 1\}$  and send  $\text{Enc}_k(m_b)$  to  $\mathcal{A}$
5.  $\mathcal{A}$  is again given oracle access to  $\text{Enc}_k(\cdot)$ , but cannot ask for  $\text{Enc}_k(m_0)$  or  $\text{Enc}_k(m_1)$
6.  $\mathcal{A}$  outputs  $b'$
7. Define  $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{cpa}}(n) := 1$  iff  $b = b'$



## Quizz

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Why don't we prevent the adversary from choosing  $m_0$  or  $m_1$  after the challenge phase ?

**Answer:**

- ▶ Because we really want this case to be taken into account, and ensure that the adversary will lose, even then.
- ▶ We want an encryption scheme that prevents from recognizing whether two ciphertexts correspond to the same plaintext (i.e. probabilistic encryption).





## *Extending the definition*

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As before, we can extend this notion to multiple encryption

Definition extension is straightforward (try it!)

Good news:

- ▶ CPA security for single encryption  $\Rightarrow$  CPA security for multiple encryption



## *Relation between definitions*

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We have:

CPA  $\Rightarrow$  Multiple message eavesdropper  $\Rightarrow$  Eavesdropper

CPA  $\not\Rightarrow$  Multiple message eavesdropper  $\not\Rightarrow$  Eavesdropper



## How to perform CPA-secure encryption?

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Idea for a probabilistic encryption scheme:

Change Enc as follows:

- ▶ Pick  $r \leftarrow \{0, 1\}^n$  and encrypt  $m$  as  $\langle r, G(k \| r) \oplus m \rangle$

Does it work?

- ▶  $G$  only guaranteed to output pseudorandom values with *secret* seed.
- ▶ Let  $G$  be a PRG. Define the PRG  $G'(s_1 \| s_2) = s_1 \| G(s_2)$ :
  - ▶ It is still length increasing
  - ▶ If the output of  $G$  is pseudorandom, then so is the one of  $G'$ .

But using  $G'$  would leak  $k$  immediately!



## *How to perform CPA-secure encryption?*

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We need something stronger than just a PRG:

We need a  $F$  such that:

- ▶ When  $r$  is public (but not  $k$ ),  $F(k, r)$  is pseudorandom
- ▶ For two randomly-chosen public  $r_1, r_2$ ,  $F(k, r_1)$  and  $F(k, r_2)$  are pseudorandom



# Pseudorandom functions

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First, what is a random function?

- ▶ A function can be described as a big (input,output) table
- ▶ A random function is one such table, with all outputs chosen at random

A pseudorandom function is one that cannot be efficiently distinguished from a truly random one

- ▶ Does not make much sense to say that a *fixed* function is pseudorandom
- ▶ Instead, we will consider *distributions* on functions, using *keyed functions*



# Pseudo-random functions

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A keyed function  $F$  is a function

$$F : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^* : (k, x) \rightarrow F(k, x)$$

$F$  is said to be efficient if there is a deterministic polynomial-time algorithm to compute  $F(k, x)$  given  $k, x$

A keyed function introduces a distribution on functions

- ▶ Choose a random  $k$
- ▶ Define  $F_k(x) := F(k, x)$

We say that  $F$  is pseudorandom if, for random  $k$ ,  $F_k$  cannot be distinguished from a random function

Note: we will focus on *length-preserving* functions



## *How dense is the distribution of $F_k$ ?*

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A random function  $\{0, 1\}^n \rightarrow \{0, 1\}^n$  can be described as a sorted table of outputs

- ▶  $2^n$  entries in the table
- ▶ Each entry is  $n$  bit long
- ▶ Full table has size  $n \cdot 2^n$  bits

Conversely, each such table describes a valid function

$\implies$  There are thus  $2^{n \cdot 2^n}$  possible functions

On the other hand, for a given  $F$ , there are only  $2^n$  possible functions  $F_k(\cdot \dots)$

$\implies F$  generates only a very small part of the full space



# Indistinguishability

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How can we formalize the notion “indistinguishable from a random function”?

- ▶ Attempt 1: a distinguisher  $D$  receiving a challenge function  $g$  cannot, in polynomial time, tell whether  $g$  is a true random function  $f$  or a PRF  $F_k$  for some  $k$
- ▶ Problem: describing  $g$  requires  $n \cdot 2^n$  bits: this is not polynomial
- ▶ So  $D$  cannot “read”  $g$  in polynomial time

Instead, we will give  $D$  *oracle access* to  $g$

- ▶  $D$  can query  $g$  with values  $x$  of his choice and get corresponding results  $g(x)$





# Indistinguishability

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$F : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$  is a pseudorandom function if  $\forall$  PPT  $D$ ,  $\exists$  negl.  $\epsilon$ :

$$\left| \Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1] \right| \leq \epsilon(n)$$

## Remarks

- ▶  $k$  is chosen uniformly at random in  $\{0, 1\}^n$
- ▶  $D$  is not given the key  $k$
- ▶ As  $D$  is PPT, he can only do a polynomially bounded number of oracle queries



# *Do pseudorandom functions exist?*

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In fact, we do not know

- ▶ It has been shown that PRF functions exist iff pseudorandom generators exist
- ▶ In practice, we have good candidates (discussed later)



## Building a CPA-secure scheme from a PRF

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Define  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  as:

- ▶ Gen: choose random  $k \leftarrow \{0, 1\}^n$
- ▶ Enc: on input  $m, k \in \{0, 1\}^n$ ,
  - ▶ choose random  $r \leftarrow \{0, 1\}^n$
  - ▶  $c := \langle r, F_k(r) \oplus m \rangle$
- ▶ Dec: on input  $k \in \{0, 1\}^n$  and  $c = \langle r, s \rangle$ , output  $m := s \oplus F_k(r)$



## Security of this construction

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**Theorem:** if  $F$  is a pseudorandom function, this construction has indistinguishable encryption under a chosen-plaintext attack

**Proof:** in two steps:

- ▶ Prove that the scheme is secure if  $F_k$  is replaced by a truly random function
- ▶ Prove that if the scheme (with  $F_k$ ) were insecure, we could distinguish  $F_k$  from a truly random function



## Step 1: idealized scheme

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Initial scheme:

Define  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  as:

- ▶ Gen: choose random  $k \leftarrow \{0, 1\}^n$
- ▶ Enc: on input  $m, k \in \{0, 1\}^n$ ,
  - ▶ choose random  $r \leftarrow \{0, 1\}^n$
  - ▶  $c := \langle r, F_k(r) \oplus m \rangle$
- ▶ Dec: on input  $k \in \{0, 1\}^n$  and  $c = \langle r, s \rangle$ , output  $m := s \oplus F_k(r)$



## Step 1: idealized scheme

---

Idealized version

Define  $\widetilde{\Pi} := \langle \widetilde{\text{Gen}}, \widetilde{\text{Enc}}, \widetilde{\text{Dec}} \rangle$  as:

- ▶  $\widetilde{\text{Gen}}$ : choose random  $f$
- ▶  $\widetilde{\text{Enc}}$ : on input  $m, k \in \{0, 1\}^n$ ,
  - ▶ choose random  $r \leftarrow \{0, 1\}^n$
  - ▶  $c := \langle r, f(r) \oplus m \rangle$
- ▶  $\widetilde{\text{Dec}}$ : on input  $k \in \{0, 1\}^n$  and  $c = \langle r, s \rangle$ , output  $m := s \oplus f(r)$

This is *almost* a one-time pad



## Step 1: idealized scheme

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Consider a CPA-adversary  $\mathcal{A}$

- ▶ Uses oracle  $\widetilde{\text{Enc}}_k(\cdot)$  to encrypt messages of his choice
- ▶ Outputs  $m_0, m_1$
- ▶ Receives  $\widetilde{\text{Enc}}_k(m_b) = \langle r, f(r) \oplus m_b \rangle$  with random  $b$
- ▶ Uses oracle  $\widetilde{\text{Enc}}_k(\cdot)$  to encrypt messages of his choice
- ▶ Must tell whether  $b = 0$  or  $1$

(Note:  $\mathcal{A}$  makes at most  $q(n)$  oracle queries in total)

What are his chances of success?



## Chances of success

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Two cases:

- ▶ Case 1:  $r$  is never used by oracle except for  $m_b$ 
  - ▶ Then  $\mathcal{A}$  learns nothing useful from oracle queries
    - ▶ ( $f$  random  $\Rightarrow f(r) \perp f(r')$  for  $r \neq r'$ )

Quizz: What are  $\mathcal{A}$ 's chances of success then?

1.  $\frac{1}{2}$
2.  $\frac{1}{2} + \epsilon(n)$
3.  $\epsilon(n)$

**Answer: 1**

- ▶ This corresponds to a OTP





## Chances of success

---

Two cases:

- ▶ Case 1:  $r$  is never used by oracle except for  $m_b$ 
  - ▶ Then  $\mathcal{A}$  learns nothing useful from oracle queries
    - ▶ ( $f$  random  $\Rightarrow f(r) \perp f(r')$  for  $r \neq r'$ )
  - ▶ Same as one-time pad
  - ▶  $\mathcal{A}$  succeeds with probability  $\frac{1}{2}$
- ▶ Case 2:  $r$  has been used at least once by oracle
  - ▶  $\mathcal{A}$  has  $\langle r, f(r) \oplus m_b \rangle$  and  $\langle r, f(r) \oplus m' \rangle$ , for some  $m'$  he knows
  - ▶  $\mathcal{A}$  can easily deduce  $f(r)$  and hence the value of  $b$
  - ▶  $\mathcal{A}$  always wins in this case
  - ▶ But the probability for this case is only  $\frac{q(n)}{2^n}$



## Chances of success

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So,

$$\begin{aligned}\Pr[\text{PrivK}_{\mathcal{A}, \tilde{\Pi}}^{\text{cpa}}(n) = 1] &= \Pr[\text{PrivK}_{\mathcal{A}, \tilde{\Pi}}^{\text{cpa}}(n) = 1 \wedge \text{Repeat}] \\ &\quad + \Pr[\text{PrivK}_{\mathcal{A}, \tilde{\Pi}}^{\text{cpa}}(n) = 1 \wedge \overline{\text{Repeat}}] \\ &\leq \Pr[\text{Repeat}] \\ &\quad + \Pr[\text{PrivK}_{\mathcal{A}, \tilde{\Pi}}^{\text{cpa}}(n) = 1 | \overline{\text{Repeat}}] \\ &\leq \frac{q(n)}{2^n} + \frac{1}{2}\end{aligned}$$

We have thus showed that, if a truly random  $f$  is used, the scheme is secure



## *Step 2: real scheme is as good as idealized one*

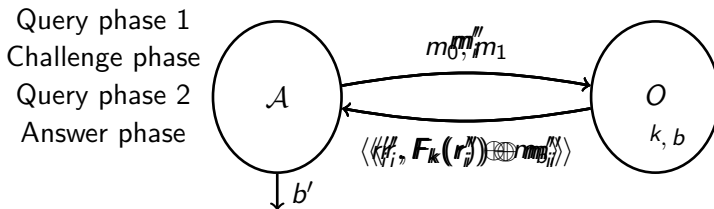
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Suppose that there exists  $\mathcal{A}$  that can break the (real) scheme with non-negligible probability  $\frac{1}{2} + \eta(n)$

We will show that  $\mathcal{A}$  can distinguish  $F_k$  from a truly random function with non-negligible probability



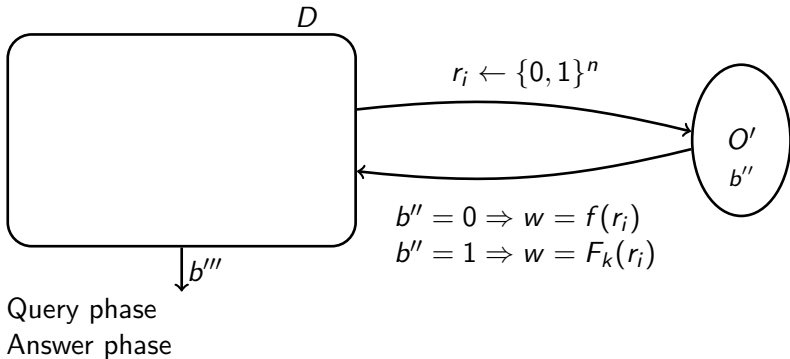
# What does the adversary look like?



By assumption,  $\Pr[b = b'] = \frac{1}{2} + \eta(n)$ , with  $\eta$  non-negligible.

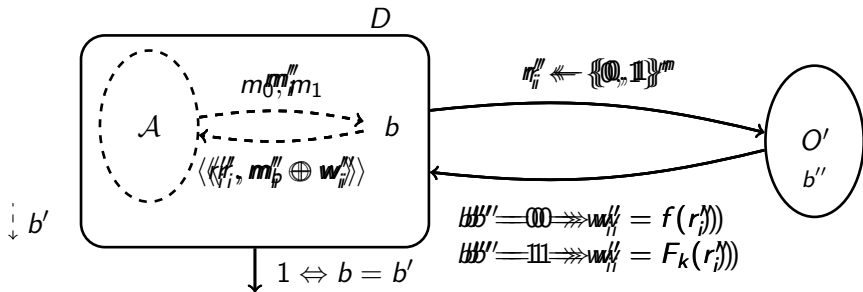


## What distinguisher must we build?



# Reduction

Query phase 2  
(identical to query phase 1)



## Step 2: real scheme is as good as idealized one

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On the one hand,

- ▶ If  $O'$  is the function  $F_k$ , then  $\mathcal{A}$  is de facto interfaced with  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$
- ▶ By assumption,  $\mathcal{A}$  will then win with non-negligible probability

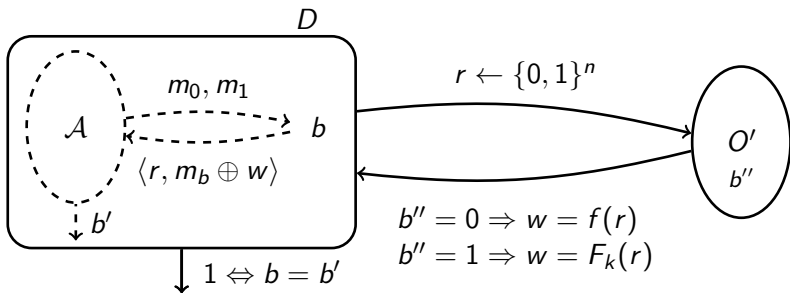
On the other hand,

- ▶ If  $O'$  is a random  $f$ , then  $\mathcal{A}$  is de facto interfaced with  $\widetilde{\Pi} := \langle \widetilde{\text{Gen}}, \widetilde{\text{Enc}}, \widetilde{\text{Dec}} \rangle$
- ▶ And we have showed that, in this case,  $\mathcal{A}$  *cannot* win with non-negligible probability

$\Rightarrow$  Let us run  $\mathcal{A}$  and see whether it wins



## Reduction



So:

- ▶ If  $b'' = 0$ ,  $\Pr[D \rightarrow 1] \leq \frac{1}{2} + \frac{q(n)}{2^n}$ : idealized scheme
- ▶ If  $b'' = 1$ ,  $\Pr[D \rightarrow 1] = \frac{1}{2} + \eta(n)$ : normal security game
- ▶  $D$  distinguishes with  $\Pr \geq \eta(n) - \frac{q(n)}{2^n}$





# Distinguisher

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So,

$$\left| \Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1] \right| \geq \eta(n) - \frac{q(n)}{2^n}$$

And, since  $F$  is a PRF,  $\eta(n) - \frac{q(n)}{2^n}$  must be negligible,  
so,  $\eta(n)$  must be negligible



# Summary

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We showed that

- ▶ If we use a truly random function in our construction, then the scheme is secure
- ▶ If someone can break the scheme with a practical function  $F_k$ , then he can distinguish  $F_k$  from a truly random function

$\implies$  If  $F_k$  is a PRF, the scheme is secure



# Multiple-Message Security

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CPA security implies multiple-message security

- ▶ But focuses on the confidentiality of *one* message

(Reminder) Our central idea to randomize: encrypt each message as  $c := \langle r, F_k(r) \oplus m \rangle$  with a random  $r$

When we encrypt  $l$  messages, what is the probability that we draw the same  $r$  more than once?

- ▶ “How many messages can we encrypt before being at risk?”



## The birthday paradox

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“How many people do we need in a room to have a probability of at least one half that two of them have the same birth date?”

- ▶ Answer: 23
- ▶ For many people, this is surprisingly low
- ▶ This is known as the *birthday paradox*
- ▶ It can be roughly generalized as follows: If we take  $q$  random values from a space of size  $N$ , the probability of collision is about  $\frac{q^2}{2N}$
- ▶ So, more simply: If we take random values from a space of size  $N$ , we might expect a collision after about  $\sqrt{N}$  values



## Birthday paradox and security

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So, the risk to “draw” the same  $r$  twice is driven by the birthday paradox

Negligible if the number of messages is negligible compared to  $2^{n/2}$  (where  $n$  = block length)

- ▶ Number of blocks is polynomial,  $2^{n/2}$  is exponential

⇒ OK

*Remarks:*

- ▶ means that not only the key size, but also the *block size* is a security factor – DES had blocks of size 64 bits!
  - ▶ AES, the most widely used PRF today, has blocks of size 128 bits
- Independently of the key size!



# Efficiency

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OK, the construction

- ▶ choose random  $r \leftarrow \{0, 1\}^n$
- ▶  $c := \langle r, F_k(r) \oplus m \rangle$

is secure, but it is not very efficient (doubles length!)

Can we do better?



# Modes of Operation

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PRF (and PRPs, see next) are typically used in a *mode of operation*:

- ways to use a fixed length PRF/PRP in order to offer security for messages of arbitrary length.



## CTR Mode

Let  $F$  be a fixed-length PRF:  $\{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$

Define  $\Pi := \langle \text{Gen}, \text{Enc}, \text{Dec} \rangle$  as:

- ▶ Gen: choose random  $k \leftarrow \{0, 1\}^n$
- ▶ Enc: on input  $k \in \{0, 1\}^n$ ,  $m = \langle m_1, \dots, m_l \rangle \in (\{0, 1\}^n)^l$ 
  - ▶ choose random  $r \leftarrow \{0, 1\}^n$
  - ▶  $c := \langle r, c_1, \dots, c_l \rangle$  with  $c_i = F_k(r + i) \oplus m_i$   
(compute sums mod  $2^n$ )
- ▶ Dec: proceed in the natural way

Observe:

- ▶  $F_k(r + i)$  can be precomputed: encryption is then very fast
- ▶ Encryption can be made parallel: good for multi-core
- ▶ Easy to decrypt only block  $c_i$ : good for HDD encryption
- ▶ if  $m_l$  is not full length, just truncate the output of  $F_k(r + l)$





## CTR Mode Security

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If  $F$  is a PRF then CTR mode offers CPA security.

- ▶  $F_k(r+1) \parallel F_k(r+2) \parallel \dots \parallel F_k(r+l)$  is a pseudorandom stream, even if  $r$  is public
- ▶ Security proof follows previous single-block proof  
Main difference: bound on collision of  $r_1 + i_1$  and  $r_2 + i_2$

If  $q(n)$  queries of max length  $q(n)$  blocks

Proba that  $\text{Enc}_k(m_b)$  has  $r + i$  overlapping another block  
is  $\leq 2q(n)^2/2^n$

CTR is used to encrypt in TLS 1.2 and TLS 1.3 (with authentication)



# Pseudorandom permutations and block ciphers

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We have introduced the notion of pseudorandom function

Similarly, we can also define pseudorandom *permutations* (PRP), i.e. one-to-one functions

Most of the time PRPs are used for PRFs in practice!



# *Pseudorandom permutations and block ciphers*

---

A keyed permutation  $F$  is a function

$$F : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^* : (k, x) \rightarrow F(k, x)$$

such that,  $\forall k$ ,  $F_k(\cdot)$  is one-to-one

$F$  is said to be efficient if there is

- ▶ a deterministic polynomial-time algorithm to compute  $F_k(x)$  given  $k, x$
- ▶ *and* a deterministic polynomial-time algorithm to compute  $F_k^{-1}(x)$  given  $k, x$

We say that  $F$  is pseudorandom if, for random  $k$ ,  $F_k$  cannot be distinguished from a random permutation



## *How can we encrypt with a PRP?*

---

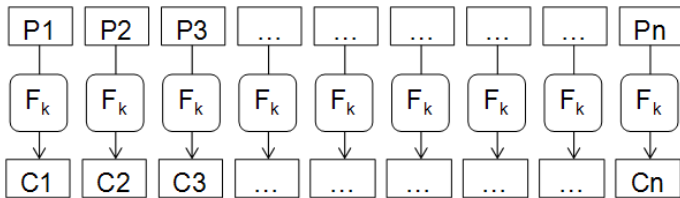
The bad idea (why?):  $c := F_k(m)$

This cannot be CPA-secure



## Electronic codebook (ECB)

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Basic mode: no mode of operation

Not CPA-secure

Not even secure against an eavesdropper



## *ECB is not secure*

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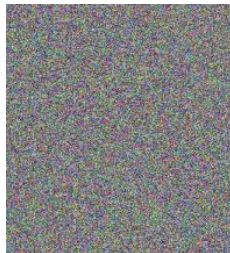
This



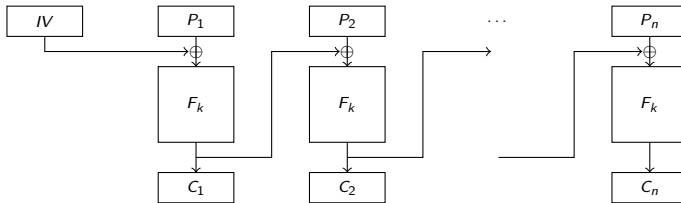
yields this



and not this



# Cipher block chaining (CBC)

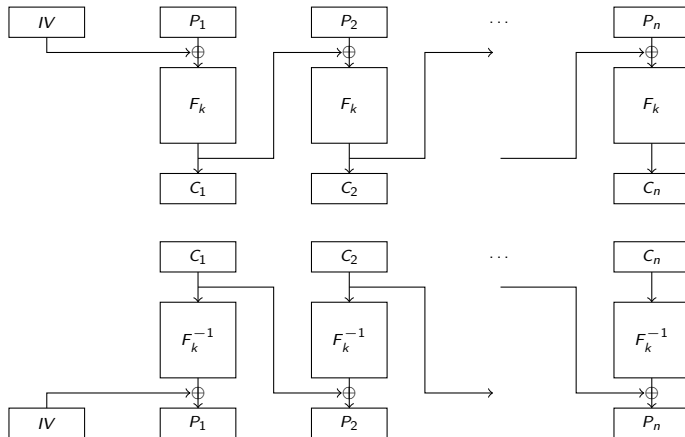


CPA-secure, provided  $F_k$  is a PRP

*Note:* in modes of operation, the random value  $r$  is traditionally called the Initialization Vector (IV)



## CBC : how do we decrypt?





## CBC-mode

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### Observations:

- ▶ Less efficient: no precomputation, not parallelizable
- ▶  $IV$  (near-)collisions less disastrous:  
different  $m_i$ 's cause divergence
- ▶ Can also serve for authentication (with fixed  $IV$ )  
(see next week)

### Uses:

- ▶ TLS 1.0, 1.1, 1.2



## *Modes of operation: summary*

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Constructions allowing to build secure encryption schemes based on PRPs (building blocks)

Can be proved secure (except ECB!) provided building block is secure

We only saw a few of them (also CFB, OFB. . .)

