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Devoir 3

Exercise (1).

Function succesion: $\{f_n\}$ where $f_n : [0,1] \to \mathbb{R}$ continue and converge puntually to $f : [0,1] \to \mathbb{R}$.

Puntually Convergence $\Rightarrow$ $f(x) = \lim_{n\to\infty} f_n(x)$, and it happens when: $\{f_n\}$,

(in our cases)

$$\forall x \in [0,1],\ \forall \varepsilon > 0\ \exists m \in \mathbb{N} : n \geq m \Rightarrow$$
$$\Rightarrow |f_n(x) - f(x)| < \varepsilon$$

i) If $\{f_n\}$ converge uniformly to $f \overset{?}{\Rightarrow} f$ continue.

The uniform converge happens when

$$\forall \varepsilon > 0\ \exists m \in \mathbb{N} : n \geq m \Rightarrow$$
$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in [0,1] \text{ (in our case)}$$

Then, we can fix $\varepsilon > 0$ and then the uniform convergence gives us a $m \in \mathbb{N}$:

$$|f_m(x) - f(x)| < \varepsilon/3 \quad \forall x \in [0,1]$$

Moreover, the continuity of f_n gives us that with a fixed $x_0 \in [0,1]$

$$|f_n(x) - f_n(x_0)| < \varepsilon/3 \quad \forall x \in [0,1]$$

Then fixed $x_0 \in [0,1]$, $\forall x \in [0,1]$:

$$|f(x) - f(x_0)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \leq$$
$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \Rightarrow x_0 \text{ is arbitrary in } [0,1]$$

$\Rightarrow f$ is continuous in $[0,1]$ $\square$

ii) We know ~~that~~ from (i) that if $\{f_n\}$ converge uniformly then f is continuous.

~~...~~ ~~...~~ Uniform convergence imply, in fact, puntual convergence but the reciprocal is not true. To prove this (what brings us to the conclusion that puntual convergence does not imply continuity) we will take a counter example:

$f_n(x) = x^n$. We know that $\forall x \in \mathbb{R} : |x| < 1$ $\{x^n\} \to 0$. For $\forall x \in \mathbb{R}$ $|x| > 1$ $\{x^n\} \to \infty$.

For $x = -1$ $\{-1^n\}$ does not converge and $x = 1$ $\{1^n\} \to 1$

Then we can use $]-1, 1]$ as the interval where $\{f_n\}$ converge puntually (to 0).

Lets prove that $\{f_n\}$ does not converge uniformly to 0 in $]-1, 1]$:

Fix $\varepsilon = 1/3 \Rightarrow \exists\, m \in \mathbb{N} : |x^m| < 1/3 \ \forall x \in]-1, 1]$

But if we take $x = \frac{1}{\sqrt[m]{2}}$ then $|x^m| = \frac{1}{2} > \frac{1}{3}$!!!

So puntual convergence does not imply uniform convergence. ~~...~~ But we can think that this does not even prove that puntual convergence $\not\Rightarrow$ $\not\Rightarrow$ continuity. Lets take another example:

$f_n(x) = \cos(\pi x)^{2n}$. The puntual limit is

$$f(x) = \lim_{n\to\infty} \cos(\pi x)^{2n} \quad \begin{cases} \cos(\pi x) = 1 \text{ if } x \in \mathbb{Z} \\ |\cos(\pi x)| < 1 \text{ if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

Then $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Z} \end{cases} \Rightarrow f$ not continuous

$$\lim_{n\to\infty} \underbrace{\cos(\pi x)}_{-1 < \ < 1}^{2n} = 0 \text{ if } x \in \mathbb{R}\setminus\mathbb{Z}$$

□

iii) First we will remember some definitions:

Rare sets: $\mathbb{X}$ endowed with d be a metric space. $A \subseteq \mathbb{X}$. Then A is rare if $\overline{\mathbb{X} \setminus \bar{A}} = \mathbb{X}$

Meagre sets: Let $\mathbb{X}$ endowed with d be a metric space. $A \subseteq \mathbb{X}$, then A is meagre if $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$ with $A_n \subseteq \mathbb{X}$ rare.

Cauchy's criterion: $\mathbb{X}$ a metric space. $\{x_n\}$ is a Cauchy sequence whenever $\forall \varepsilon > 0$ $\exists n_* \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m > n \geq n_*$ then $d(x_m, x_n) \leq \varepsilon$

We will use this definitions and conceps along the solution.

First, Fixed a $k \in \mathbb{N} \setminus \{0\}$, $p, q \in \mathbb{N}$ Telle que ~~we~~ we can define the following sets:

$$A(k, p, q) \doteq \{x \in [0,1] : |f_p(x) - f_q(x)| \leq 1/k\}$$

$$B(k, p) \doteq \bigcap_{q' \geq p} A(k, p, q')$$

$$A = \bigcap_{k \in \mathbb{N} \setminus \{0\}} \bigcup_{p \in \mathbb{N}} \operatorname{int}(B(k,p))$$

We know that for every $x \in A$ f is continuous. The set where f is not continuous $\subset [0,1] \setminus A$ is meagre, ~~and then prove it we have to prove~~ if ~~~~ $[0,1] \setminus A$ is meagre. We know that:

$$[0,1] \setminus A = [0,1] \setminus \bigcap_{k \in \mathbb{N} \setminus \{0\}} \bigcup_{p \in \mathbb{N}} \operatorname{int}(B(k,p)) = \bigcup_{k \in \mathbb{N} \setminus \{0\}} \left([0,1] \setminus \bigcup_{p \in \mathbb{N}} \operatorname{int}(B(k,p)) \right)$$

As the union is numerable, we just have to prove.

$$[0,1] \setminus \bigcup_{p \in \mathbb{N}} \operatorname{int}(B(k,p)) \text{ is meagre } \forall k \in \mathbb{N} \setminus \{0\}$$

We also, by definition, know that $[0,1] \setminus \bigcup_{p \in \mathbb{N}} \mathrm{int}(B(k,p))$ is meagre if it is content in $\bigcup_{n \in \mathbb{N}} A_n$ where each set A_n is rare. If we are able to prove that $[0,1] = \bigcup_{p \in \mathbb{N}} B(k,p)$ then

$$[0,1] \setminus \bigcup_{p \in \mathbb{N}} \mathrm{int}(B(k,p)) = \bigcup_{p \in \mathbb{N}} B(k,p) \setminus \bigcup_{p \in \mathbb{N}} \mathrm{int}(B(k,p)) \subset$$

$$\subset \bigcup_{p \in \mathbb{N}} \underbrace{\left(B(k,p) \setminus \mathrm{int}(B(k,p)) \right)}_{\text{This set is rare cause } B(k,p) \text{ is closed}}$$

Then we just have to prove that

$$[0,1] = \bigcup_{p \in \mathbb{N}} B(k,p)$$

$\supset$)

$[0,1] \supset \bigcup_{p \in \mathbb{N}} B(k,p)$ Trivially, as $B(k,p)$ is defined as:

$$B(k,p) = \bigcap_{q' \geq p} A(k,p,q') \text{ and } A(k,p,q') \subset [0,1]$$

$\subset$) As $\{f_n\}$ converge puntually ~~...~~ we know that for every $x \in [0,1]$ $\{f_n(x)\}$ converge.

And ~~as~~ Therefore, $\{f_n(x)\}$ satisfais Cauchy's criterion, that, as we defined before, says ~~that~~:

$$\forall x \in [0,1] \quad \exists n_* \in \mathbb{N} : \exists m,n \in \mathbb{N} : \quad m > n \geq n_* \text{ then}$$
$$|f_m(x) - f_n(x)| \leq \varepsilon.$$

On the other hand, if $x \in \bigcup_{p \in \mathbb{N}} B(k,p) \Rightarrow \exists p \in \mathbb{N}$:

$\forall q' \geq p \quad |f_p(x) - f_q(x)| \leq 1/k$ because of the definition of $B(k,p)$ and $A(k,p,q)$.

So, finally, if we take $\varepsilon = 1/k$, $n^* = n = p$ $m = q$. in the Cauchy's condition then:

$\forall x \in [0,1]$ $\exists p \in \mathbb{N}$: $\forall q \in \mathbb{N}$ such that $q > p$ then

$$|f_q(x) - f_p(x)| \leq 1/k$$ that is exactly

the definition of $\bigcup_{p \in \mathbb{N}} B(k,p)$.

So, then, $[0,1] \subset \bigcup_{p \in \mathbb{N}} B(k,p) \Rightarrow$

$\Rightarrow [0,1] = \bigcup_{p \in \mathbb{N}} B(k,p)$.

What means that $[0,1] \setminus \bigcup_{p \in \mathbb{N}} \operatorname{int}(B(k,p))$ is meagre

and what implies that its numerable union is meagre and what, finally, implies that $[0,1] \setminus A$ is meagre. Then, being D:

$$D = \{ x \in [0,1] : f \text{ is not continuous in } x \}$$

$D \subset [0,1] \setminus A$ is meagre

$\square$