Functional analysis

Lecture notes

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Contents

Part I

Metric and normed spaces

1 Metric, normed and inner product spaces

1.1 Abstract metric, normed and inner product spaces

1.1.1 Metrics and semi-metrics

Definition 1.1. Given a set X, a function $d : X \times X \to [0, \infty]$ is a *semi-metric* whenever

(i) (self-indiscernibility) for each $x \in X$, one has $d(x,x) = 0$,

(ii) (symmetry) for each $x, y \in X$, one has $d(x,y) = d(y,x)$,

(iii) (triangle inequality) for each $x, y, z \in X$, one has $d(x,z) \le d(x,y) + d(y,z)$.

Proposition 1.2. *If X is a set and d is a semi-metric on X, then for every $x, y, z, w \in X$, one has*

$$|d(x,y) - d(z,w)| \le d(x,z) + d(y,w). \tag{1.1}$$

Proof. By the triangle inequality (definition 1.1 (iii)), we have

$$\begin{aligned} d(x,y) &\le d(x,z) + d(z,y) \\ &\le d(x,z) + d(z,w) + d(w,y), \end{aligned} \tag{1.2}$$

and thus

$$d(x,y) - d(z,w) \le d(x,z) + d(w,y). \tag{1.3}$$

Similarly, we have

$$d(z,w) - d(x,y) \le d(z,x) + d(y,w). \tag{1.4}$$

The conclusion follows from (1.3), (1.4) and the symmetry property of the semi-metric (definition 1.1 (ii)). □

Definition 1.3. Given a set X endowed with a semi-metric d, the points $x \in X$ and $y \in X$ are *indiscernible* (with respect to d) whenever $d(x,y) = 0$.

Proposition 1.4. *If d is a semi-metric on X, then "x and y are indescernible with respect to d" is an equivalence relation on X.*

Proof. By definition 1.1 (i), for every point $x \in X$, we have $d(x,x) = 0$ and thus x and x are indescernible. The indiscernibility is thus reflexive.

If the points $x \in X$ and $y \in X$ are indescernible, then by definition we have $d(x,y) = 0$. By definition 1.1 (ii), we have $d(y,x) = d(x,y) = 0$ and thus the points y and x are indescernible. The indiscernibility is thus symmetric.

Finally, if $x \in X$ and $y \in X$ are indescernible and if y and $z \in X$ are indescernible, then by definition $d(x,y) = 0$ and $d(y,z) = 0$, by definition 1.1 (iii),

$$0 \le d(x,z) \le d(x,y) + d(y,z) = 0, \tag{1.5}$$

and thus x and z are indescernible. Indiscernibility is thus transitive.

Indescernibility, being reflexive, symmetric and transitive, satisfies thus the definition of an equivalence relation. □

Definition 1.5. Given a set X, a semi-metric d is *positive definite* whenever $x, y \in X$ are indiscernible with respect to d implies that $x = y$.

In other words, d is positive definite if and only if the indescenibility and equality equivalence relations on X are the same.

Definition 1.6. Given a set X, a function $d : X \times X \to [0, \infty]$ is a *metric* whenever d is a semi-metric and d is positive definite.

Proposition 1.7. *Let X be a set and d is semi-metric, then there exists a set X_* and a metric d_* and a mapping $i_* : X \to X_*$ such that i_* is surjective and for every $x, y \in X$,*

$$d_*(i_*(x), i_*(y)) = d(x,y). \tag{1.6}$$

The map i_*, the space X_* and the metric d_* are unique up to a bijective isometry: if for $j \in \{0,1\}$, the map $i_j : X \to X_j$ is surjective, the function d_j is a metric on X_j and for every $x, y \in X$, $d_j(i_j(x), i_j(y)) = d(x,y)$, then there exists a bijection $i : X_1 \to X_0$ such that $i_0 = i \circ i_1$ and for every $x, y \in X_1$, $d_1(x,y) = d_0(i(x), i(y))$.
Moreover, for every mapping $f : X \to Y$ such that for each $x, y \in X$, $d(x,y) = 0$ implies $f(x) = f(y)$, there exists a mapping $f_* : X_* \to Y$ such that $f = f_* \circ i_*$.

Proof. We define X_* as the set of equivalence classes of indescernible elements

$$X_* := \big\{\{z \in X \mid d(x,z) = 0\} \bigm| x \in X\big\} \tag{1.7}$$

and we set for each $x \in X$,

$$i_*(x) := \{z \in X \mid d(x,z) = 0\}. \tag{1.8}$$

By construction, the mapping $i : X \to X_*$ is surjective.

Given $x, y \in X$, we observe that $i_*(x) = i_*(y)$ if and only if $d(x,y) = 0$. Indeed, given $x, y \in X$ such that $d(x,y) = 0$, then if $z \in Z$ satisfies $d(x,z) = 0$, then $d(y,z) = 0$ by definition 1.1 (iii). Conversely, if $x, y \in X$ and $i_*(x) = i_*(y)$, then $x \in i_*(y)$ and thus $d(x,y) = 0$.

Given $x_*, y_* \in X_*$, and $x, y \in X$ such that $x_* = i_*(x)$ and $y_* = i_*(y)$, we define

$$d_*(x_*, y_*) := d(x, y). \tag{1.9}$$

The quantity $d_*(x_*, y_*)$ is well-defined since if $x_* = i_*(x')$ and $y_* = i_*(y')$, we have $d(x, x') = 0$ and $d(y, y') = 0$ and

$$|d(x, y) - d(x', y')| \le d(x, x') + d(y, y') = 0, \tag{1.10}$$

so that $d(x, y) = d(x', y')$.

The quantity d_* is a semi-metric. We clearly have for every $x \in X$,

$$d_*(i_*(x), i_*(x)) = d(x, x) = 0. \tag{1.11}$$

Next, we have for every $x, y \in X$,

$$d_*(i_*(x), i_*(y)) = d(x, y) = d(y, x) = d_*(i_*(x), i_*(y)). \tag{1.12}$$

For the triangle inequality, we have for every $x, y, z \in X$,

$$\begin{aligned} d_*(i_*(x), i_*(z)) &= d(x, z) \\ &\le d(x, y) + d(y, z) = d_*(i_*(x), i_*(y)) + d_*(i_*(y), i_*(z)). \end{aligned} \tag{1.13}$$

Finally, d_* is positive-definite: If $d_*(x_*, y_*) = 0$, then given $x, y \in X$ such that $x_* = i_*(x)$ and $y_* = i_*(y)$, we have

$$0 = d_*(x_*, y_*) = d(x, y), \tag{1.14}$$

and thus $x = y$ so that $x_* = i(x) = i(y) = y_*$.

Assume now that the mapping $f : X \to Y$ satisfies $f(x) = f(y)$ when $x, y \in X$ are indescernible. Then if $x_* \in X_*$, we define $f_*(x_*) = f(x)$. Since f is constant on equivalence classes for indescenibility, f_* is well-defined.

Finally, given for $j \in \{0, 1\}$, X_j, $i_j : X \to X_j$, and d_j, applying the previous property to $f = i_1$, we obtain f_* such that $i_1 = f_* \circ i_0$. We set $i_* := f_*$ and we have the conclusion. □

1.1.2 Norms and semi-norms

Definition 1.8. Given a vector space X, the function $\|\cdot\| : X \to [0, \infty)$ is a semi-norm on X whenever

(i) (homogeneity) for every $x \in X$ and $t \in \mathbb{R}$, one has $\|tx\| = |t|\|x\|$,

(ii) (subadditivity) for every $x, y \in X$, one has $\|x + y\| \le \|x\| + \|y\|$.

Proposition 1.9. *Let X be a vector space and $\|\cdot\| : X \to [0, \infty)$ be semi-norm, then for every $x, y \in X$,*

$$\big|\|x\| - \|y\|\big| \le \|x - y\|. \tag{1.15}$$

Proof. By definition 1.8 (ii), we have

$$\|x\| = \|(x-y)+y\| \le \|x-y\| + \|y\|, \tag{1.16}$$

and therefore

$$\|x\| - \|y\| \le \|x-y\|. \tag{1.17}$$

Similarly, we have

$$\|y\| - \|x\| \le \|y-x\|. \tag{1.18}$$

The conclusion follows from (1.16), (1.17) and definition 1.8 (ii). □

Definition 1.10. Given a vector space X and $\|\cdot\| : X \to [0,\infty)$ a semi-norm, the kernel of $\|\cdot\|$ is the set

$$\ker\|\cdot\| := \{x \in X \mid \|x\| = 0\}. \tag{1.19}$$

Proposition 1.11. *Given a vector space X and $\|\cdot\| : X \to [0,\infty)$ a semi-norm,* $\ker\|\cdot\|$ *is a linear subspace of X.*

Proof. For each $x \in \ker\|\cdot\|$ and $t \in \mathbb{R}$, we have by the homogeneity property of semi-norms (definition 1.8 (i)) and by definition of its kernel (definition 1.10)

$$\|tx\| = |t|\|x\| = 0, \tag{1.20}$$

and thus $tx \in \ker\|\cdot\|$.

For every $x, y \in \ker\|\cdot\|$, we have by definition 1.8 (ii) and by definition 1.10

$$\|x+y\| \le \|x\| + \|y\| = 0, \tag{1.21}$$

and thus $\|x+y\| = 0$ so that $x + y \in \ker\|\cdot\|$ by definition. □

Definition 1.12. Given a vector space X and $\|\cdot\| : X \to [0,\infty)$ a semi-norm, $\|\cdot\|$ is *positive definite* whenever $\ker\|\cdot\| = \{0\}$.

Definition 1.13. Given a vector space X, the function $\|\cdot\| : X \to [0,\infty)$ is a *norm*, whenever $\|\cdot\|$ is a semi-norm and $\|\cdot\|$ is positive definite.

Proposition 1.14. *Let X be a vector space. If $\|\cdot\| : X \to \mathbb{R}$ is a semi-norm on X, then the function $d : X \times X \to X$ defined for $x, y \in X$ by*

$$d(x,y) := \|x-y\| \tag{1.22}$$

is a semi-metric. Moreover d is a metric if and only if $\ker\|\cdot\| = \{0\}$.

Proof. We first have for every $x \in X$, by definition 1.8 (i)

$$d(x,x) = \|x - x\| = \|0x\| = |0|\|x\| = 0, \tag{1.23}$$

so that definition 1.1 (i) holds.

Next, we have for every $x, y \in X$, by definition 1.8 (i) again

$$d(x,y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\| = d(y,x), \tag{1.24}$$

so that definition 1.1 (ii) holds.

Finally, if $x, y, z \in X$, then by definition 1.8 (ii)

$$d(x,z) = \|x - z\| = \|x - y + y - z\| \le \|x - y\| + \|y - z\| = d(x,y) + d(y,z), \tag{1.25}$$

so that definition 1.1 (iii) holds.

Finally we observe that $d(x,y) = \|x - y\| = 0$ implies $x = y$ if and only if $\|\cdot\|$ is positive definite. □

Proposition 1.15. *Let X be a vector space and let $d : X \times X \to [0,\infty]$ be a semi-metric. There exists a semi-norm $\|\cdot\| : X \to [0,\infty)$ such that for every $x, y \in X$*

$$d(x,y) = \|x - y\|, \tag{1.26}$$

if and only if

(i) for every $x, y \in X$, one has $d(x,y) < \infty$,

(ii) for every $x, y, z \in X$, one has $d(x+z, y+z) = d(x,y)$,

(iii) for every $x \in X$ and $t \in \mathbb{R}$, one has $d(tx,0) = |t|d(x,0)$.

Proof. If (1.26) holds, then for every $x, y, z \in X$,

$$d(x+z, y+z) = \|(x+z) - (y+z)\| = \|x - y\| = d(x,y), \tag{1.27}$$

so that (ii) holds.

Next, by (1.26) and by definition 1.8 (i), we have for every $x \in X$ and $t \in \mathbb{R}$,

$$d(tx,0) = \|tx - 0\| = \|t(x - 0)\| = |t|\|x - 0\| = |t|d(x,0), \tag{1.28}$$

so that (iii) holds.

Conversely, if (i), (ii) and (iii) hold, we define for $x \in X$,

$$\|x\| := d(x,0). \tag{1.29}$$

By (i), $\|x\| < \infty$. For every $x \in X$ and $t \in \mathbb{R}$, we have by (iii) and (1.29),

$$\|tx\| = d(tx,0) = |t|d(x,0) = \|x\|, \tag{1.30}$$

so that (i) in definition 1.8 holds. Next, if $x, y \in X$, then by the triangle inequality and by (ii),

$$\|x + y\| = d(x+y,0) \le d(x+y,y) + d(y,0) = d(x,0) + d(y,0) = \|x\| + \|y\|, \tag{1.31}$$

so that (ii) in definition 1.8 holds. □

1.1.3 Semi-inner product and inner product

Definition 1.16. Let X be a vector space. The mapping $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is a *semi-inner product* whenever

(i) for every $x \in X$, one has $(x|x) \geq 0$,

(ii) for every $x, y \in X$, one has $(x|y) = (y|x)$,

(iii) for every $x, y \in X$ and $t \in \mathbb{R}$, one has $(tx|y) = t(x|y)$,

(iv) for every $x, y, z \in X$, one has $(x + y|z) = (x|z) + (y|z)$.

Proposition 1.17. *Let X be a vector space. If $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product, then*

(i) for every $x, y \in X$ and $t \in \mathbb{R}$, one has $(x|ty) = t(x|y)$,

(ii) for every $x, y, z \in X$, one has $(x|y + z) = (x|y) + (x|z)$.

Proof. For every $x, y \in X$ and $t \in \mathbb{R}$, we have by definition 1.16 (ii) and (iii),

$$(x|ty) = (ty|x) = t(y|x) = t(x|y). \tag{1.32}$$

For every $x, y, z \in X$, we have by definition 1.16 (ii) and (iv),

$$(x|y + z) = (y + z|x) = (y|x) + (z|x) = (x|y) + (x|z). \tag{1.33}$$ □

Proposition 1.18 (General Pythagorean identity)**.** *Let X be a vector space endowed with a semi-inner product $(\cdot|\cdot)$. For every $x, y \in X$,*

$$\|x + y\|^2 = \|x\|^2 + 2(x|y) + \|y\|^2. \tag{1.34}$$

Proof. This follows from definition 1.16. □

Definition 1.19. Let X be a vector space and let $(\cdot|\cdot) : X \times X \to \mathbb{R}$ be a semi-inner product, $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is *positive definite* whenever for every $x \in X \setminus \{0\}$, $(x|x) > 0$.

Definition 1.20. Let X be a vector space. The function $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is an *inner product* whenever $(\cdot|\cdot)$ is a semi-inner product and positive definite.

Proposition 1.21. *If X is a vector space and $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product, then for every $x, y \in X$,*

$$(x|y)^2 \leq (x|x)(y|y). \tag{1.35}$$

Proof. For every $s, t \in \mathbb{R}$, we have by proposition 1.18

$$\begin{aligned} 0 &\leq (sx - ty|sx - ty) \\ &= s^2(x|x) - 2ts(x|y) + t^2(y|y), \end{aligned} \tag{1.36}$$

so that

$$ts(x\,|\,y) \le \frac{s^2(x\,|\,x) + t^2(y\,|\,y)}{2}. \tag{1.37}$$

If $(x\,|\,x) > 0$ and $(y\,|\,y) > 0$, taking $t = \sqrt[4]{(x\,|\,x)/(y\,|\,y)}$ and $s = \sqrt[4]{(y\,|\,y)/(x\,|\,x)}$, we get

$$(x\,|\,y) \le \sqrt{(x\,|\,x)}\sqrt{(y\,|\,y)}, \tag{1.38}$$

while taking $t = \sqrt[4]{(x\,|\,x)/(y\,|\,y)}$ and $s = -\sqrt[4]{(y\,|\,y)/(x\,|\,x)}$,

$$-(x\,|\,y) \le \sqrt{(x\,|\,x)}\sqrt{(y\,|\,y)}, \tag{1.39}$$

and the conclusion follows.

If $(x\,|\,x) = 0$, we have, taking $t = 1$ in (1.37)

$$s(x\,|\,y) \le \frac{(y\,|\,y)}{2}, \tag{1.40}$$

which implies that $(x\,|\,y) = 0$. The case $(y\,|\,y) = 0$ is similar. □

Proposition 1.22. *If X is a vector space and $(\cdot\,|\,\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product, then the function $\|\cdot\| : X \to \mathbb{R}$ defined by*

$$\|x\| = \sqrt{(x\,|\,x)} \tag{1.41}$$

is a semi-norm. If moreover, $(\cdot\,|\,\cdot)$ is positive-definite, then $\|\cdot\|$ is a norm.

Proof. For every $x \in X$ and $t \in \mathbb{R}$, we have by definition 1.16 (iii) and proposition 1.17 (i)

$$\|tx\| = \sqrt{(tx\,|\,tx)} = \sqrt{t^2(x\,|\,x)} = |t|\sqrt{(x\,|\,x)} = |t|\|x\|. \tag{1.42}$$

For every $x, y \in X$, we have by definition 1.16 (iv), proposition 1.17 (ii) and proposition 1.21

$$\begin{aligned} \|x+y\| = \sqrt{(x+y\,|\,x+y)} &= \sqrt{(x\,|\,x) + 2(x\,|\,y) + (y\,|\,y)} \\ &\le \sqrt{\|x\|^2 + 2\|x\|\|y\| + \|y\|^2} = \sqrt{(\|x\| + \|y\|)^2} \\ &= \|x\| + \|y\|. \end{aligned} \tag{1.43}$$

□

Proposition 1.23 (Cauchy–Schwarz inequality)**.** *If X is a vector space and $(\cdot\,|\,\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product and $\|\cdot\|$ be the associated norm, then for every $x, y \in X$,*

$$|(x\,|\,y)| \le \|x\|\|y\|. \tag{1.44}$$

Proof. This follows from proposition 1.21. □

Proposition 1.24 (Parallelogram law)**.** *If X is a vector space and $(\cdot\,|\,\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product and $\|\cdot\|$ be the associated norm, then for every $x, y \in X$,*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.45}$$

Proof. We have by proposition 1.18

$$\|x + y\|^2 = \|x\|^2 + 2(x\,|\,y) + \|y\|^2 \tag{1.46}$$

and

$$\|x - y\|^2 = \|x\|^2 - 2(x\,|\,y) + \|y\|^2. \tag{1.47}$$ □

Adding (1.46) and (1.47) yields (1.45).

Proposition 1.25 (Polarisation identity)**.** *If X is a vector space and $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is a semi-inner product and $\|\cdot\|$ be the associated norm, then for every $x, y \in X$,*

$$(x\,|\,y) = \frac{\|x + y\|^2 - \|x - y\|^2}{4}. \tag{1.48}$$

Proof. This follows from (1.46) and (1.47). □

Proposition 1.26 (Jordan–von Neumann theorem)**.** *Let X be a vector space and $\|\cdot\|$ be a semi-norm on X. Then there exists a semi-inner product $(\cdot|\cdot) : X \times X \to \mathbb{R}$, such that for every $x \in X$,*

$$\|x\| = \sqrt{(x\,|\,x)} \tag{1.49}$$

if and only if for every $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.50}$$

Proof. If there exists a semi-inner product, then (1.50) holds by proposition 1.24.

Conversely, assuming that (1.50) holds, we define, as suggested by the polarisation identity (proposition 1.25),

$$(x\,|\,y) := \frac{\|x + y\|^2 - \|x - y\|^2}{4} \tag{1.51}$$

We check immediately that for every $x \in X$, by (1.50)

$$(x\,|\,x) = \frac{\|x + x\|^2 - \|x - x\|^2}{4} = \frac{\|2x\|^2}{4} = \|x\|^2. \tag{1.52}$$

We also have for every $x, y \in X$,

$$(x\,|\,y) = \frac{\|x + y\|^2 - \|x - y\|^2}{4} = \frac{\|y + x\|^2 - \|y - x\|^2}{4} = (y\,|\,x). \tag{1.53}$$

Next, if $x, y, z \in X$, we have by (1.51)

$$\left(\frac{x+y}{2}\,\middle|\, z\right) = \frac{\|x+y+2z\|^2 - \|x+y-2z\|^2}{16} \tag{1.54}$$

We have by (1.50)

$$\|x+y+2z\|^2 = \|x+z+y+z\|^2 = 2\|x+z\|^2 + 2\|y+z\|^2 - \|x-y\|^2, \tag{1.55}$$

and

$$\|x+y-2z\|^2 = 2\|x-z+y-z\|^2 = 2\|x-z\|^2 + 2\|y-z\|^2 - \|x-y\|^2, \tag{1.56}$$

and thus by (1.54), (1.55) and (1.56)

$$\begin{aligned} \left(\frac{x+y}{2}\,\middle|\, z\right) &= \frac{\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 + \|y-z\|^2}{8} \\ &= \frac{(x\,|\,z) + (y\,|\,z)}{2}. \end{aligned} \tag{1.57}$$

in view of (1.51). In particular, if $y = 0$, $(y\,|\,z) = 0$, and thus

$$\left(\frac{x}{2}\,\middle|\, z\right) = \frac{(x\,|\,z)}{2}, \tag{1.58}$$

and it follows thus from (1.57) that

$$(x+y\,|\,z) = (x\,|\,z) + (y\,|\,z). \tag{1.59}$$

We deduce then that for any $t \in \mathbb{Z}$,

$$(tx\,|\,z) = t(x\,|\,z); \tag{1.60}$$

it follows then that the same identity holds for $t \in \mathbb{Q}$. Finally, the identity holds for every $t \in \mathbb{R}$ by continuity. □

1.1.4 Complex semi-norms and semi-inner products

Definition 1.27. Given a complex vector space X, the function $\|\cdot\| : X \to [0, \infty)$ is a complex semi-norm on X whenever

(i) (homogeneity) for every $x \in X$ and $t \in \mathbb{C}$, one has $\|tx\| = |t|\|x\|$,

(ii) (subadditivity) for every $x, y \in X$, one has $\|x+y\| \le \|x\| + \|y\|$.

Proposition 1.28. *Let X be a complex vector space. The function $\|\cdot\| : X \to [0, \infty)$ be is a complex semi-norm on X if and only if $\|\cdot\| : X \to [0, \infty)$ is a real seminorm and for every $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, $\|\alpha x\| = \|x\|$.*

Proof. If $\|\cdot\| : X \to [0,\infty)$ is a real seminorm, and $t \in \mathbb{C}$, we write $t = |t|\alpha$, with $\alpha \in \mathbb{C}$ and $|\alpha| = 1$. By assumption we have then

$$\|tx\| = \|t\alpha x\| = |t|\|\alpha x\| = |t|\|x\|, \tag{1.61}$$

so that $\|\cdot\|$ is a complex norm. □

Definition 1.29. Let X be a complex vector space. The mapping $(\cdot|\cdot) : X \times X \to \mathbb{C}$ is a *complex semi-inner product* whenever

(i) for every $x \in X$, one has $(x|x) \in [0,\infty) \subseteq \mathbb{R} \subseteq \mathbb{C}$,

(ii) for every $x, y \in X$, one has $(x|y) = \overline{(y|x)}$,

(iii) for every $x, y \in X$ and $t \in \mathbb{C}$, one has $(tx|y) = t(x|y)$,

(iv) for every $x, y, z \in X$, one has $(x + y|z) = (x|z) + (y|z)$.

Proposition 1.30 (Right anti-linearity)**.** *Let X be a complex vector space. If $(\cdot|\cdot) : X \times X \to \mathbb{R}$ is a complex semi-inner product, then*

(i) for every $x, y \in X$ and $t \in \mathbb{C}$, one has $(x|ty) = \bar{t}(x|y)$,

(ii) for every $x, y, z \in X$, one has $(x|y + z) = (x|y) + (x|z)$.

Proof. For every $x, y \in X$ and $t \in \mathbb{C}$, we have by definition 1.29 (ii) and (iii),

$$(x|ty) = \overline{(ty|x)} = \overline{t(y|x)} = \bar{t}\,\overline{(y|x)} = \bar{t}(x|y), \tag{1.62}$$

and thus (i) holds.

For every $x, y, z \in X$, we have by definition 1.29 (ii) and (iv),

$$(x|y+z) = \overline{(y+z|x)} = \overline{(y|x)+(z|x)} = \overline{(y|x)} + \overline{(z|x)} = (x|y) + (x|z), \tag{1.63}$$

and thus (ii) holds. □

Proposition 1.31 (Real and imaginary parts of complex inner product)**.** *Let X endowed with $(\cdot|\cdot)$ be a semi-inner product space. Then for every $x, y \in X$,*

$$(x|y) = \operatorname{Re}(x|y) + i\operatorname{Re}(x|iy). \tag{1.64}$$

Proof. By proposition 1.30 (i), we have

$$(x|y) = i(x|iy), \tag{1.65}$$

and thus

$$\operatorname{Im}(x|y) = \operatorname{Re}(x|iy). \tag{1.66}$$

□

Proposition 1.32. *Let X be a vector space and $(\cdot|\cdot)_{\mathbb{R}} : X \times X \to \mathbb{R}$ be a semi-inner product. There exists a complex semi-inner product $(\cdot|\cdot)_{\mathbb{C}} : X \times X \to \mathbb{R}$ such that for every $x, y \in X$,*

$$(x\,|\,y)_{\mathbb{R}} = \operatorname{Re}(x\,|\,y)_{\mathbb{C}}, \tag{1.67}$$

if and only if for every $x \in X$,

$$(ix\,|\,ix)_{\mathbb{R}} = (x\,|\,x)_{\mathbb{R}}. \tag{1.68}$$

Proof. Assuming that for every $x \in X$, we have (1.68), then for every $x, y \in X$ by the polarisation identity (proposition 1.25) and by (1.68) we have

$$\begin{aligned}(ix\,|\,iy)_{\mathbb{R}} &= \frac{(ix+iy\,|\,ix+iy)_{\mathbb{R}} - (ix-iy\,|\,ix-iy)_{\mathbb{R}}}{4}\\ &= \frac{(x+y\,|\,x+y)_{\mathbb{R}} - (x-y\,|\,x-y)_{\mathbb{R}}}{4} = (x\,|\,y)_{\mathbb{R}}.\end{aligned} \tag{1.69}$$

Inspired by proposition 1.31, we define for each $x, y \in X$,

$$(x\,|\,y)_{\mathbb{C}} \coloneqq (x\,|\,y)_{\mathbb{R}} + i(x\,|\,iy)_{\mathbb{R}}. \tag{1.70}$$

We have by (1.70) and (1.69)

$$\begin{aligned}(y\,|\,x)_{\mathbb{C}} &= (y\,|\,x)_{\mathbb{R}} + i(y\,|\,ix)_{\mathbb{R}} = (x\,|\,y)_{\mathbb{R}} + i(ix\,|\,y)_{\mathbb{R}}\\ &= (x\,|\,y)_{\mathbb{R}} + i(i^2x\,|\,iy)_{\mathbb{R}} = (x\,|\,y)_{\mathbb{R}} - i(x\,|\,iy)_{\mathbb{R}} = \overline{(x\,|\,y)_{\mathbb{C}}}\,.\end{aligned} \tag{1.71}$$

We also have by (1.69)

$$\begin{aligned}(ix\,|\,y)_{\mathbb{C}} &= (ix\,|\,y)_{\mathbb{R}} + i(ix\,|\,iy)_{\mathbb{R}} = (i^2x\,|\,iy)_{\mathbb{R}} + i(x\,|\,y)_{\mathbb{R}}\\ &= -(x\,|\,iy)_{\mathbb{R}} + i(x\,|\,y)_{\mathbb{R}} = i\big((x\,|\,y)_{\mathbb{R}} + i(x\,|\,iy)_{\mathbb{R}}\big)\\ &= i(x\,|\,y)_{\mathbb{C}}.\end{aligned} \tag{1.72}$$

Conversely, if (1.67) holds, then by definition 1.29 (ii) and proposition 1.30 (i)

$$(ix\,|\,ix)_{\mathbb{R}} = \operatorname{Re}(ix\,|\,ix)_{\mathbb{C}} = \operatorname{Re}(x\,|\,x)_{\mathbb{C}} = (x\,|\,x)_{\mathbb{R}}. \tag{1.73}$$

□

Proposition 1.33 (Complex polarisation identity)**.** *If X endowed with $(\cdot|\cdot)$ is a complex semi-inner product space, then for every $x, y \in X$,*

$$\begin{aligned}(x\,|\,y) &= \frac{1}{4}\sum_{j=0}^{3} i^j \|x + i^j y\|^2\\ &= \frac{\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2}{4}.\end{aligned} \tag{1.74}$$

Proof. We have for every $j \in \{0, \dots, 3\}$, by definition 1.29 and proposition 1.30

$$\begin{aligned} \|x + i^j y\|^2 &= \|x\|^2 + (x \,|\, i^j y) + (i^j x \,|\, y) + \|y\|^2 \\ &= \|x\|^2 + i^{-j}(x \,|\, y) + i^j (x \,|\, y) + \|y\|^2. \end{aligned} \tag{1.75}$$

Hence,

$$\begin{aligned} \sum_{j=0}^{3} i^j \|x + i^j y\|^2 &= \sum_{j=0}^{3} i^j \|x\|^2 + \sum_{j=0}^{3} (x \,|\, y) + \sum_{j=0}^{3} i^{2j} (x \,|\, y) + \sum_{j=0}^{3} i^j \|y\|^2 \\ &= 4(x \,|\, y), \end{aligned} \tag{1.76}$$

since

$$\sum_{j=0}^{3} i^j = \sum_{j=0}^{3} i^{2j} = 0. \tag{1.77}$$

□

1.2 Concrete metric, normed and inner product spaces

1.2.1 Linear space of bounded linear mappings

Definition 1.34. Let $\|\cdot\|_X$ be a norm on X and $\|\cdot\|_Y$ be a norm on Y. The linear mapping $L : X \to Y$ is *bounded* (with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$) whenever

$$\|L\|_{\mathscr{L}(X,Y)} \coloneqq \sup\{\|L(x)\|_Y \mid x \in X \text{ and } \|x\|_X \le 1\} < \infty. \tag{1.78}$$

Proposition 1.35. *Let $\|\cdot\|_X$ be a norm on X and $\|\cdot\|_Y$ be a norm on Y. If the linear mapping $L : X \to Y$ is bounded, then for every $x \in X$,*

$$\|L(x)\|_Y \le \|L\|_{\mathscr{L}(X,Y)} \|x\|_X. \tag{1.79}$$

Proof. If $x = 0$, then (1.79) holds because $\|L(0)\|_Y = \|0\|_Y = 0$. Otherwise, letting $t \coloneqq \|x\|_X > 0$, we have by linearity

$$L(x) = L(t(x/t)) = tL(x/t), \tag{1.80}$$

and therefore, since $\|x/t\|_X = \|x\|_X / t = 1$, we have

$$\|L(x)\|_Y = |t| \|L(x/t)\|_Y \le \|L\|_{\mathscr{L}(X,Y)} \|x\|_X. \tag{1.81}$$

□

Definition 1.36. Let $\|\cdot\|_X$ be a norm on the vector space X and $\|\cdot\|_Y$ be a norm on the vector space Y. The space of bounded continuous linear maps from X to Y with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$ is

$$\mathscr{L}(X,Y) = \{L : X \to Y \mid L \text{ is linear and bounded}\}. \tag{1.82}$$

Proposition 1.37. *Let $\|\cdot\|_X$ be a norm on the vector space X and $\|\cdot\|_Y$ be a norm on the vector space Y, then the set $\mathscr{L}(X,Y)$ is a vector space and $\|\cdot\|_{\mathscr{L}(X,Y)}$ is a norm on $\mathscr{L}(X,Y)$.*

Proof. Let $L \in \mathscr{L}(X,Y)$ and assume that $\|L\|_{\mathscr{L}(X,Y)} = 0$. Then by proposition 1.35, for every $x \in X$, we have $\|L(x)\|_Y = 0$ and thus $L(x) = 0$, that is, $L = 0$.

Let $L \in \mathscr{L}(X,Y)$ and $t \in \mathbb{R}$. For every $x \in X$, we have

$$(tL)(x) = tL(x), \tag{1.83}$$

and thus if $\|x\|_X \le 1$, we have

$$\|(tL)(x)\|_Y = |t|\|L(x)\|_Y; \tag{1.84}$$

it follows thus that $\|tL\|_{\mathscr{L}(X,Y)} = |t|\|L\|_{\mathscr{L}(X,Y)}$ and $tL \in \mathscr{L}(X,Y)$.

Finally, if $L, M \in \mathscr{L}(X,Y)$, then for every $x \in X$,

$$(L+M)(x) = Lx + Mx, \tag{1.85}$$

and thus if $\|x\|_X \le 1$, then

$$\|(L+M)(x)\|_Y \le \|L(x)\|_Y + \|M(x)\|_Y \le \|L\|_{\mathscr{L}(X,Y)} + \|M\|_{\mathscr{L}(X,Y)}, \tag{1.86}$$

and thus

$$\|L+M\|_{\mathscr{L}(X,Y)} \le \|L\|_{\mathscr{L}(X,Y)} + \|M\|_{\mathscr{L}(X,Y)}, \tag{1.87}$$

and $L + M \in \mathscr{L}(X,Y)$. □

Proposition 1.38. *Let $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ be norms on the vector spaces X, Y, Z. If $L \in \mathscr{L}(X,Y)$ and $M \in \mathscr{L}(Y,Z)$, then $M \circ L \in \mathscr{L}(X,Z)$ and*

$$\|M \circ L\|_{\mathscr{L}(X,Z)} \le \|M\|_{\mathscr{L}(Y,Z)}\|L\|_{\mathscr{L}(X,Y)}. \tag{1.88}$$

Proof. Let $x \in X$. By proposition 1.35, we have

$$\|(M \circ L)(x)\|_Z \le \|M\|_{\mathscr{L}(Y,Z)}\|L(x)\|_Y. \tag{1.89}$$

Next, in view of (1.78), we have if $\|x\|_X \le 1$, $\|L(x)\|_Y \le \|L\|_{\mathscr{L}(X,Y)}$ and thus in view of (1.89)

$$\|(M \circ L)(x)\|_Z \le \|M\|_{Y,Z}\|L\|_{\mathscr{L}(X,Y)}, \tag{1.90}$$

so that $M \circ L \in \mathscr{L}(X,Z)$ with the announced estimate. □

1.2.2 Finite-dimensional spaces

Definition 1.39. Given $d \in \mathbb{N}$, we define $(\cdot \mid \cdot) : \mathbb{R}^d \to \mathbb{R}$ for $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d$ by

$$(x \mid y) \coloneqq \sum_{k=1}^{d} x_k y_k. \tag{1.91}$$

Proposition 1.40. *For every $d \in \mathbb{N}$, $(\cdot\,|\,\cdot)$ is an inner product on $\mathbb{R}^d$.*

Definition 1.41. Given $d \in \mathbb{N}$ and $p \in [1, \infty)$, we define $|\cdot|_p : \mathbb{R}^d \to [0, \infty)$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ by

$$|x|_p \coloneqq \Bigl(\sum_{k=1}^{d} |x_k|^p\Bigr)^{\frac{1}{p}}. \tag{1.92}$$

Proposition 1.42. *For every $x \in \mathbb{R}^d$, $|x|_2 = \sqrt{(x\,|\,x)}$.*

Proposition 1.43. *For every $d \in \mathbb{N}$ and $p \in [1, \infty)$, $|\cdot|_p$ is a norm on $\mathbb{R}^d$.*

Proof. For every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t \in \mathbb{R}$, we have

$$|tx|_p^p = \sum_{k=1}^{d} |tx_k|^p = \sum_{k=1}^{d} |t|^p |x_k|^p = |t|^p \sum_{k=1}^{d} |x_k|^p = |t|^p |x|_p^p, \tag{1.93}$$

and thus

$$|tx|_p = |t||x|_p. \tag{1.94}$$

Next given $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d \setminus \{0\}$, we set $t \coloneqq |y|_p/(|x|_p + |y|_p)$, and we have in view of lemma A.2

$$\begin{aligned}
|x+y|_p^p &= \Bigl|(1-t)\frac{x}{1-t} + t\frac{y}{t}\Bigr|_p^p = \sum_{k=1}^{d} \Bigl|(1-t)\frac{x_k}{1-t} + t\frac{y_k}{t}\Bigr|^p \\
&\le \sum_{k=1}^{d} \Bigl((1-t)\Bigl|\frac{x_k}{1-t}\Bigr| + t\Bigl|\frac{y_k}{t}\Bigr|\Bigr)^p \\
&\le \sum_{k=1}^{d} \Bigl((1-t)\Bigl|\frac{x_k}{1-t}\Bigr|^p + t\Bigl|\frac{y_k}{t}\Bigr|^p\Bigr) \\
&= \frac{|x|_p^p}{(1-t)^{p-1}} + \frac{|y|_p^p}{t^{p-1}} = \bigl(|x|_p + |y|_p\bigr)^p,
\end{aligned} \tag{1.95}$$

and it follows thus that

$$|x+y|_p \le |x|_p + |y|_p. \tag{1.96}$$

The inequality (1.96) holds trivially if either $x = 0$ or $y = 0$.

Finally, if $x = (x_1, \dots, x_k) \in \mathbb{R}^d$ and $|x|_p = 0$, then

$$0 = |x|_p^p = \sum_{k=1}^{d} |x_k|^p, \tag{1.97}$$

which implies that $x_1 = \dots = x_d = 0$ and thus $x = 0$. □

Definition 1.44. Given $d \in \mathbb{N}$, we define $|\cdot|_\infty : \mathbb{R}^d \to [0, \infty)$ for $x = (x_1, \dots, x_d)$ by

$$|x|_\infty := \max\{|x_k| \mid k \in \{1, \dots, d\}\}. \tag{1.98}$$

Proposition 1.45. *For every $d \in \mathbb{N}$, $|\cdot|_\infty$ is a norm on $\mathbb{R}^d$.*

Proof. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t \in \mathbb{R}$. We have

$$\begin{aligned} |tx|_\infty &= \max\{|tx_k| \mid k \in \{1, \dots, k\}\} \\ &= \max\{|t||x_k| \mid k \in \{1, \dots, k\}\} \\ &= |t| \max\{|x_k| \mid k \in \{1, \dots, k\}\} \\ &= |t||x|_\infty. \end{aligned} \tag{1.99}$$

Next, if $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$, we have for every $k \in \{1, \dots, d\}$, by the triangle inequality,

$$|x_k + y_k| \le |x_k| + |y_k| \le |x|_\infty + |y|_\infty, \tag{1.100}$$

and thus

$$|x + y|_\infty \le |x|_\infty + |y|_\infty. \tag{1.101}$$

Finally, if $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $|x|_\infty = 0$, we have for every $k \in \{1, \dots, d\}$, $x_k = 0$ and thus $x = 0$. ☐

1.2.3 Uniform norm

Definition 1.46. Given a set Γ, we define

$$\ell^\infty(\Gamma) := \{f : \Gamma \to \mathbb{R} \mid \sup\{|f(x)| \mid x \in \Gamma\} < \infty\} \tag{1.102}$$

and $\|\cdot\|_\infty : \ell^\infty(\Gamma) \to [0, \infty)$ for every $f : \Gamma \to \mathbb{R}$ by

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \Gamma\}. \tag{1.103}$$

Proposition 1.47. *Given any set Γ, $\ell^\infty(\Gamma)$ is a vector space and $\|\cdot\|_\infty : \ell^\infty(\Gamma) \to [0, \infty)$ is a norm.*

Proof. We first have $0 \in \ell^\infty(\Gamma)$.

Next, if $f \in \ell^\infty(\Gamma)$ and $t \in \mathbb{R}$, then for every $x \in \Gamma$,

$$|tf(x)| = |t||f(x)|, \tag{1.104}$$

and thus

$$\sup\{|tf(x)| \mid x \in \Gamma\} = |t| \sup\{|f(x)| \mid x \in \Gamma\}, \tag{1.105}$$

from which it follows that $tf \in \ell^\infty(\Gamma)$ and

$$\|tf\|_\infty = |t|\|f\|_\infty. \tag{1.106}$$

Next, if $f, g \in \ell^\infty(\Gamma)$, then for every $x \in \Gamma$,

$$|(f+g)(x)| = |f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty, \tag{1.107}$$

and thus $f + g \in \ell^\infty(\Gamma)$ and

$$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty. \tag{1.108}$$

Finally, if $f \in \ell^\infty(\Gamma)$ and $\|f\|_\infty = 0$, then for every $x \in \Gamma$, $|f(x)| = 0$ and thus $f = 0$. □

1.2.4 Sequence spaces

Definition 1.48. Given a set Γ and $p \in [1, \infty)$, we define

$$\ell^p(\Gamma) := \left\{ f : \Gamma \to \mathbb{R} \;\middle|\; \sum_{x \in \Gamma} |f(x)|^p < \infty \right\} \tag{1.109}$$

and $\|\cdot\|_p : \ell^p(\Gamma) \to [0, \infty)$ for every $f : \Gamma \to \mathbb{R}$ by

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p \right)^{\frac{1}{p}}. \tag{1.110}$$

The sum over an infinite set is defined in definition C.1.

Proposition 1.49. *Given any set Γ and $p \in [1, \infty)$, the set $\ell^p(\Gamma)$ is a vector space and $\|\cdot\|_p : \ell^p(\Gamma) \to [0, \infty)$ is a norm.*

Proof. If $f \in \ell^p(\Gamma)$ and $t \in \mathbb{R}$, then by proposition C.4

$$\sum_{x \in \Gamma} |tf(x)|^p = |t|^p \sum_{x \in \Gamma} |f(x)|^p, \tag{1.111}$$

so that $tf \in \ell^p(\Gamma)$ and

$$\|tf\|_p = |t|\|f\|_p. \tag{1.112}$$

If $f, g \in \ell^p(\Gamma) \setminus \{0\}$, setting $t := \|g\|_p / (\|f\|_p + \|g\|_p)$, by monotonicity and linearity of sums (proposition C.2, proposition C.5 and proposition C.4), we have

$$
\begin{aligned}
\sum_{x\in\Gamma} |f(x)+g(x)|^p &= \sum_{x\in\Gamma} \left| (1-t)\frac{f(x)}{1-t} + t\frac{g(x)}{t}\right|^p \\
&\le \sum_{x\in\Gamma} \left((1-t)\left|\frac{f(x)}{1-t}\right| + t\left|\frac{g(x)}{t}\right|\right)^p \\
&\le \sum_{x\in\Gamma} \left((1-t)\left|\frac{f(x)}{1-t}\right|^p + t\left|\frac{g(x)}{t}\right|^p\right) \\
&= \sum_{x\in\Gamma} (1-t)\left|\frac{f(x)}{1-t}\right|^p + \sum_{x\in\Gamma} t\left|\frac{g(x)}{t}\right|^p \\
&= \frac{\|f\|_p^p}{(1-t)^{p-1}} + \frac{\|g\|_p^p}{t^{p-1}} = \bigl(\|f\|_p + \|g\|_p\bigr)^p,
\end{aligned}
\tag{1.113}
$$

and thus

$$
\|f+g\|_p \le \|f\|_p + \|g\|_p. \tag{1.114}
$$

Finally, if $f \in \ell^p(\Gamma)$ and $\|f\|_p = 0$, then for every $x \in \Gamma$, $f(x) = 0$ and thus $f = 0$, so that $\|\cdot\|_p$ is positive-definite. ☐

Proposition 1.50. *If $f, g \in \ell^2(\Gamma)$, then $fg : \Gamma \to \mathbb{R}$ is unconditionnally summable.*

Proof. For every $x \in \Gamma$, we have

$$
|f(x)g(x)| \le \frac{|f(x)|^2 + |g(x)|^2}{2}. \tag{1.115}
$$

and therefore by proposition C.18, fg is unconditionnally summable. ☐

Definition 1.51. If Γ is a set, we define the function $(\cdot\,|\,\cdot) : \ell^2(\Gamma) \times \ell^2(\Gamma) \to \mathbb{R}$ for each $f, g \in \ell^2(\Gamma)$ by

$$
(f\,|\,g) = \sum_{x\in\Gamma} f(x)g(x). \tag{1.116}
$$

Proposition 1.52. *The function $(\cdot|\cdot)$ is an inner product on $\ell^2(\Gamma)$.*

Proof. This follows from the linearity properties of sums (proposition C.5 and proposition C.4) ☐

1.2.5 Space of integrable functions

Definition 1.53. Given a measure $\mu : \Sigma \to [0, \infty)$ on the set Ω, we define

$$
\mathscr{L}^p(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \int_\Omega |f|^p \,\mathrm{d}\mu < \infty \right\} \tag{1.117}
$$

and $\|\cdot\|_p : \mathscr{L}^p(\Omega,\mu) \to [0,\infty)$ for $f \in \mathscr{L}^p(\Omega,\mu)$

$$\|f\|_p := \Big(\int_\Omega |f|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}}. \tag{1.118}$$

Proposition 1.54. *Given a measure $\mu : \Sigma \to [0,\infty]$ on the set Ω, the set $\mathscr{L}^p(\Omega,\mu)$ is a vector space and $\|\cdot\|_p$ is a seminorm. Moreover, for every $f \in \mathscr{L}^p(\Omega,\mu)$, one has $\|f\|_p = 0$ if and only if $\mu(\{x \in \Omega \mid f(x) \neq 0\}) = 0$.*

Proof. If $f \in \mathscr{L}^p(\Omega,\mu)$ and $t \in \mathbb{R}$, then tf is measurable and

$$\int_\Omega |tf|^p \,\mathrm{d}\mu = |t|^p \int_\Omega |f|^p \,\mathrm{d}\mu < \infty; \tag{1.119}$$

hence $tf \in \mathscr{L}^p(\Omega,\mu)$ and

$$\|tf\|_p = |t|\|f\|_p. \tag{1.120}$$

If $f, g \in \mathscr{L}^p(\Omega,\mu)$, the function $f+g$ is measurable. Moreover, defining $t := \|g\|_p/(\|f\|_p + \|g\|_p)$, we have by lemma A.2, proposition D.26, proposition D.27 and proposition D.28

$$\begin{aligned}\int_\Omega |f+g|^p \,\mathrm{d}\mu &= \int_\Omega \Big|(1-t)\frac{f}{1-t} + t\frac{g}{t}\Big|^p \,\mathrm{d}\mu\\ &\le \int_\Omega \frac{|f|^p}{(1-t)^{p-1}} + \frac{|g|^p}{t^{p-1}} \,\mathrm{d}\mu\\ &= \frac{1}{(1-t)^{p-1}} \int_\Omega |f|^p \,\mathrm{d}\mu + \frac{1}{t^{p-1}} \int_\Omega |g|^p \,\mathrm{d}\mu\\ &= \big(\|f\|_p + \|g\|_p\big)^p,\end{aligned} \tag{1.121}$$

and therefore $f + g \in \mathscr{L}^p(\Omega,\mu)$ and

$$\|f+g\|_p \le \|f\|_p + \|g\|_p. \tag{1.122}$$

Finally, we assume that $\|f\|_p = 0$. By proposition D.32, we have

$$\mu(\{x \in \Omega \mid f(x) \neq 0\}) = \mu(\{x \in \Omega \mid |f(x)|^p \neq 0\}) = 0, \tag{1.123}$$

which is the conclusion. □

Proposition 1.55. *If $f, g \in \mathscr{L}^2(\Omega,\mu)$, then $fg : \Omega \to \mathbb{R}$ is integrable.*

Proof. For every $x \in \Omega$, we have

$$|f(x)g(x)| \le \frac{|f(x)|^2 + |g(x)|^2}{2}, \tag{1.124}$$

and the conclusion follows by the comparison criterion for integrability (proposition D.49). □

Definition 1.56. Given a measure $\mu : \Sigma \to [0, \infty]$ on the set Ω, we define $(\cdot|\cdot) : \mathscr{L}^2(\Omega, \mu) \times \mathscr{L}^2(\Omega, \mu)$ by

$$(f \,|\, g) = \int_\Omega f\, g \,\mathrm{d}\mu. \tag{1.125}$$

Proposition 1.57. *Given a measure $\mu : \Sigma \to [0, \infty]$ on the set Ω, $(\cdot|\cdot) : \mathscr{L}^2(\Omega, \mu) \times \mathscr{L}^2(\Omega, \mu)$ is a semi-inner product on $\mathscr{L}^2(\Omega, \mu)$ and for every $f \in \mathscr{L}^2(\Omega, \mu)$,*

$$\|f\|_2^2 = (f \,|\, f). \tag{1.126}$$

1.2.6 Essentially bounded functions

Definition 1.58. Given a measure $\mu : \Sigma \to [0, \infty)$ of the set Ω, a function $f : \Omega \to \mathbb{R}$ is essentially bounded whenever there exists $M \in \mathbb{R}$ such that

$$\mu(\{x \in \Omega \mid |f(x)| > M\}) = 0. \tag{1.127}$$

Definition 1.59. Given a measure $\mu : \Sigma \to [0, \infty)$ of the set Ω, we define

$$\mathscr{L}^\infty(\Omega, \mu) := \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable and essentially bounded}\} \tag{1.128}$$

and $\|\cdot\|_\infty : \mathscr{L}^\infty(\Omega, \mu) \to [0, \infty)$ for $f \in \mathscr{L}^\infty(\Omega, \mu)$

$$\|f\|_\infty := \inf\{M \in [0, \infty) \mid \mu(\{x \in \Omega \mid |f(x)| > M\}) = 0\}. \tag{1.129}$$

Proposition 1.60. *Given a measure $\mu : \Sigma \to [0, \infty]$ on the set Ω, the set $\mathscr{L}^\infty(\Omega, \mu)$ is a vector space and $\|\cdot\|_\infty$ is a seminorm. Moreover, for every $f \in \mathscr{L}^\infty(\Omega, \mu)$, then $\|f\|_\infty = 0$ if and only if $\mu(\{x \in \Omega \mid |f(x)| > 0\}) = 0$.*

Proof. Let $f \in \mathscr{L}^\infty(\Omega, \mu)$ and $t \in \mathbb{R}$. Assume that $M \in [0, \infty)$ and

$$\mu(\{x \in \Omega \mid |f(x)| > M\}) = 0. \tag{1.130}$$

Then

$$\mu(\{x \in \Omega \mid |tf(x)| > |t|M\}) = 0, \tag{1.131}$$

and thus $tf \in \mathscr{L}^\infty(\Omega, \mu)$ and

$$\|tf\|_\infty \le |t|\|f\|_\infty. \tag{1.132}$$

Similarly if $t \neq 0$, we have

$$|t|\|f\|_\infty = |t|\|tf/t\|_\infty \le \|tf\|_\infty. \tag{1.133}$$

Next, if $f, g \in \mathscr{L}^\infty(\Omega, \mu)$ and $M, N \in [0, \infty)$, then

$$\begin{aligned}\{x \in \Omega \mid |f(x) + g(x)| > M + N\} &\subseteq \{x \in \Omega \mid |f(x)| + |g(x)| > M + N\}\\ &\subseteq \{x \in \Omega \mid |f(x)| > M\} \cup \{x \in \Omega \mid |g(x)| > N\}.\end{aligned} \tag{1.134}$$

Hence if $\mu(\{x \in \Omega \mid |f(x)| > M\}) = 0$ and $\mu(\{x \in \Omega \mid |g(x)| > N\}) = 0$, then $\mu(\{x \in \Omega \mid |f(x) + g(x)| > M + N\}) = 0$. Therefore, $f + g \in \mathscr{L}^\infty(\Omega, \mu)$ and

$$\|f + g\|_\infty \le M + N. \tag{1.135}$$

Taking the infimum with respect to such M and N, we conclude that

$$\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty. \tag{1.136}$$

Finally, we assume that $\|f\|_\infty = 0$. This implies that for every $M \in (0, \infty)$,

$$\mu(\{x \in \Omega \mid |f(x)| > M\}) = 0. \tag{1.137}$$

Since

$$\{x \in \Omega \mid |f(x)| \ne 0\} = \bigcup_{n \in \mathbb{N}} \{x \in \Omega \mid |f(x)| \ge 2^{-n}\}, \tag{1.138}$$

we have by the measure of an nondecreasing sequence of sets (proposition D.14)

$$\begin{aligned} \mu(\{x \in \Omega \mid f(x) \ne 0\}) &= \mu(\{x \in \Omega \mid |f(x)| \ne 0\}) \\ &= \lim_{n \to \infty} \mu(\{x \in \Omega \mid |f(x)| \ge 2^{-n}\}) = 0. \end{aligned} \tag{1.139}$$

□

1.2.7 Distance in measure of sets

Definition 1.61. Given a measure $\mu : \Sigma \to [0, \infty]$ on Ω, we define $d_\mu : \Sigma \times \Sigma \to [0, \infty]$ for $A, B \in \Sigma$ by

$$d_\mu(A, B) = \mu(A \setminus B) + \mu(B \setminus A). \tag{1.140}$$

Proposition 1.62. *Given a measure $\mu : \Sigma \to [0, \infty]$ on Ω, the function $d_\mu : \Sigma \times \Sigma \to [0, \infty]$ is a semi-metric on Σ.*

Proof. If $A \in \Sigma$, then

$$d_\mu(A, A) = \mu(A \setminus A) + \mu(A \setminus A) = \mu(\emptyset) + \mu(\emptyset) = 0 + 0 = 0. \tag{1.141}$$

Next, if $A, B \in \Sigma$,

$$d_\mu(A, B) = \mu(A \setminus B) + \mu(B \setminus A) = d_\mu(B, A). \tag{1.142}$$

Finally, if $A, B, C \in \Sigma$, we have

$$A \setminus C \subseteq (A \setminus B) \cup (B \setminus C) \tag{1.143}$$

and

$$C \setminus A \subseteq (C \setminus B) \cup (B \setminus A), \tag{1.144}$$

and therefore, by subadditivity of measures

$$\begin{aligned} d_\mu(A, C) &= \mu(A \setminus C) + \mu(C \setminus A) \\ &\le \mu(A \setminus B) + \mu(B \setminus C) + \mu(C \setminus B) + \mu(B \setminus A) \\ &= d_\mu(A, B) + d_\mu(B, C). \end{aligned} \tag{1.145}$$

□

1.2.8 Distance in measure for functions

Definition 1.63. Given a measure $\mu : \Omega \to [0, \infty]$, we define the space

$$M(\Omega, \mu) := \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable}\} \tag{1.146}$$

and the function $d_\mu : M(\Omega, \mu) \times M(\Omega, \mu) \to [0, \infty]$ by

$$d_\mu(f, g) := \inf\bigl\{\delta \in (0, \infty) \bigm| \mu(\{x \in \Omega \mid |f(x) - g(x)| > \delta\} < \delta)\bigr\}. \tag{1.147}$$

Proposition 1.64. *The function $d_\mu : M(\Omega, \mu) \times M(\Omega, \mu) \to [0, \infty]$ is a metric on $M(\Omega, \mu)$.*

Proof. For every $f \in M(\Omega, \mu)$, we have for each $\delta > 0$,

$$\mu(\{x \in \Omega \mid |f(x) - f(x)| > \delta\}) = 0 < \delta, \tag{1.148}$$

and thus

$$d_\mu(f, f) = 0. \tag{1.149}$$

Next, for every $f, g \in M(\Omega, \mu)$, we have for each $\delta > 0$,

$$\mu(\{x \in \Omega \mid |f(x) - g(x)| > \delta\}) = \mu(\{x \in \Omega \mid |g(x) - f(x)| > \delta\}) \tag{1.150}$$

and thus

$$d_\mu(f, g) = d_\mu(g, f). \tag{1.151}$$

Finally, for every $f, g, h \in M(\Omega, \mu)$ and $\delta, \varepsilon > 0$, we have

$$\begin{aligned} \{x \in \Omega \mid |f(x) - h(x)| > \delta + \varepsilon\} \subseteq \{x \in \Omega \mid |f(x) - g(x)| > \delta\} \\ \cup \{x \in \Omega \mid |g(x) - h(x)| > \varepsilon\} \end{aligned} \tag{1.152}$$

and thus by monotonicity and subadditivity of measures (propositions D.12 and D.13)

$$\begin{aligned} \mu(\{x \in \Omega \mid |f(x) - h(x)| > \delta + \varepsilon\}) \le \mu(\{x \in \Omega \mid |f(x) - g(x)| > \delta\}) \\ + \mu(\{x \in \Omega \mid |g(x) - h(x)| > \varepsilon\}). \end{aligned} \tag{1.153}$$

It follows thus that if

$$\mu(\{x \in \Omega \mid |f(x) - g(x)| > \delta\}) < \delta \tag{1.154}$$

and

$$\mu(\{x \in \Omega \mid |g(x) - h(x)| > \varepsilon\}) < \varepsilon, \tag{1.155}$$

then

$$\mu(\{x \in \Omega \mid |f(x) - g(x)| > \delta + \varepsilon\}) < \delta + \varepsilon, \tag{1.156}$$

and thus

$$d_\mu(f, h) \le d_\mu(f, g) + d_\mu(g, h). \tag{1.157}$$

□

1.2.9 Local spaces

Definition 1.65. Let $\Omega \subseteq \mathbb{R}^d$ be open. We define

$$\mathscr{L}^p_{\mathrm{loc}}(\Omega, \mathscr{L}^d) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \text{for each compact set } K,\, f|_K \in \mathscr{L}^p(K, \mathscr{L}^d) \right\}; \tag{1.158}$$

given a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$, we define

$$d_{p,\mathrm{loc}}(f, g) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|(f-g)|_{K_n}\|_p}{1 + \|(f-g)|_{K_n}\|_p}. \tag{1.159}$$

Proposition 1.66. *$\mathscr{L}^p_{\mathrm{loc}}(\Omega, \mathscr{L}^d)$ is a vector space and $d_{p,\mathrm{loc}} : \mathscr{L}^p_{\mathrm{loc}}(\Omega, \mathscr{L}^d) \times \mathscr{L}^p_{\mathrm{loc}}(\Omega, \mathscr{L}^d) \to [0, \infty)$ is a semi-metric. Moreover, if $d_{p,\mathrm{loc}}(f, g) = 0$, then*

$$\mathscr{L}^d(\{x \in \Omega \mid |f(x) - g(x)| > 0\}) = 0. \tag{1.160}$$

1.2.10 Spaces of smooth functions

Definition 1.67. Let $\Omega \subseteq \mathbb{R}^n$ and let

$$C^k_b(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \in C^k(\Omega) \text{ and } f, Df, \dots, D^k f \text{ are bounded}\}, \tag{1.161}$$

and let $\|\cdot\|_{k,\infty} : C^k_b(\Omega) \times C^k_b(\Omega) \to [0, \infty]$ be defined for $f \in C^k_b(\Omega)$ by

$$\|f\|_{k,\infty} = \|D^k f\|_\infty. \tag{1.162}$$

1.2.11 Sobolev spaces

Definition 1.68. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The function $f \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, \mathscr{L}^d)$ is weakly differentiable, whenever there exists $\nabla u \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}))$ such that for every $\psi \in C^\infty_c(\Omega, \mathbb{R}^d)$, one has

$$\int_\Omega u \operatorname{div} \psi \,\mathrm{d}\mathscr{L}^d = -\int_\Omega \nabla u \cdot \psi \,\mathrm{d}\mathscr{L}^d. \tag{1.163}$$

Proposition 1.69. *The function $\nabla u \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}))$ in definition 1.68 is unique up to its values on a negiglible set for the Lebesgue measure $\mathscr{L}^d$.*

Proof. Assume that $g_0, g_1 \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}))$ satisfy

$$\int_\Omega u \operatorname{div} \psi \,\mathrm{d}\mathscr{L}^d = -\int_\Omega g_j \cdot \psi \,\mathrm{d}\mathscr{L}^d. \tag{1.164}$$

We have then

$$\int_\Omega (g_1 - g_0) \cdot \psi \,\mathrm{d}\mathscr{L}^d = 0. \tag{1.165}$$

Since $\psi \in C^\infty_c(\Omega, \mathbb{R}^d)$ is arbitrary, this implies that

$$\mathscr{L}^d(\{x \in \Omega \mid |g_1(x) - g_0(x)| > 0\}) = 0. \tag{1.166}$$

□

Definition 1.70. Given an open set $\Omega \subseteq \mathbb{R}^d$, we define

$$\dot{W}^{1,p}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is weakly differentiable and } \int_\Omega |\nabla u|^p \,\mathrm{d}\mathscr{L}^d < \infty \right\}. \tag{1.167}$$

and $\|\cdot\|_{1,p} : \dot{W}^{1,p}(\Omega) \times \dot{W}^{1,p}(\Omega)$ by

$$\|f\|_{1,p} = \|\nabla f\|_p. \tag{1.168}$$

Proposition 1.71. *The space $\dot{W}^{1,p}(\Omega)$ is vector space and the function $\|\cdot\|_{1,p} : \dot{W}^{1,p}(\Omega) \to [0, \infty)$ is a semi-norm on $\dot{W}^{1,p}(\Omega)$.*

1.3 Comments

The characterisation of norms induced by an inner product is due to Pascual Jordan and John von Neumann [JVN35].

The triangle inequality on $\ell^p(\Gamma)$ and $\mathscr{L}^p(\Omega)$ (Minkowski's inequality) can also be obtained as a consequence of Hölder's inequality (proposition 4.26) [Bre11, th. 4.7] or as a consequence of the homogeneity and concavity of the function $(s, t) \in [0, \infty) \times [0, \infty) \mapsto [0, \infty)$ [Wil13, th. 4.1.9; RW02].

2 Convergent sequences and topology

2.1 Convergence in metric spaces

Definition 2.1. Let d be a metric on the set X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_* \in X$ whenever for every $\varepsilon \in (0,\infty)$, there exists $n_* \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ satisfying $n \geq n_*$ one has $d(x_n, x_*) \leq \varepsilon$.

Proposition 2.2. *Let d be a metric on the set X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_* \in X$ if and only if the sequence $(d(x_n, x_*))_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{R}$.*

Proof. For every $n \in \mathbb{N}$, we have

$$|d(x_n, x_*) - 0| = d(x_n, x_*). \tag{2.1}$$

□

Definition 2.3. Let d be a metric on the set X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence $(x_n)_{n\in\mathbb{N}}$ is *bounded* whenever there exists $M \in \mathbb{R}$ such that for every $n, m \in \mathbb{N}$, $d(x_n, x_m) \leq M$.

Proposition 2.4. *Let d be a metric on the set X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ converges, then $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. We assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_*. For every $n, m \in \mathbb{N}$ we have by the triangle inequality (definition 1.1 (iii)) and by symmetry (definition 1.1 (ii))

$$d(x_n, x_m) \leq d(x_n, x_*) + d(x_*, x_m) \leq d(x_n, x_*) + d(x_m, x_*). \tag{2.2}$$

By definition of convergence (definition 2.1) with $\varepsilon = 1$, there exists some $n_* \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq n_*$, then $d(x_n, x_*) \leq 1$. Defining,

$$M := 2\max(\{1\} \cup \{d(x_n, x_*) \mid n \in \mathbb{N} \text{ and } n < n_*\}). \tag{2.3}$$ □

We have for each $n \in \mathbb{N}$,

$$d(x_n, x_*) \leq \frac{M}{2}, \tag{2.4}$$

and the conclusion then follows from (2.2).

Proposition 2.5. *Let X endowed with d be a metric space. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_*^0 and x_*^1, then $x_*^0 = x_*^1$.*

Proof. For every $n \in \mathbb{N}$, we have by the triangle inequality (definition 1.1 (iii)) and by symmetry (definition 1.1 (ii))

$$d(x_*^0, x_*^1) \le d(x_*^0, x_n) + d(x_n, x_*^1) \le d(x_n, x_*^0) + d(x_n, x_*^1). \tag{2.5}$$

Let $\varepsilon \in (0, \infty)$. By definition of convergence (definition 2.1), there exist $n_*^0, n_*^1 \in \mathbb{N}$, such that if $j \in \{0, 1\}$, $n \in \mathbb{N}$ and $n \ge n_*^j$, then

$$d(x_n, x_*^j) \le \frac{\varepsilon}{2}. \tag{2.6}$$

In particular, if $n \ge \max\{n_*^0, n_*^1\}$, then, in view of (2.5),

$$d(x_*^0, x_*^1) \le \varepsilon. \tag{2.7}$$

Since $\varepsilon \in (0, \infty)$ is arbitrary, it follows that $d(x_*^0, x_*^1) = 0$, and thus $x_*^0 = x_*^1$. □

Definition 2.6. Let X endowed with d be a metric space. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_* we define the *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$ to be

$$\lim_{n\to\infty} x_n := x_*. \tag{2.8}$$

As usual with such definition, the limit and the notation *only make sense when the sequence converges*. Writing limits for sequences that are not known to converge shall lead in many cases to false and contradictory consequences.

Proposition 2.7. *Let X endowed with d be a metric space. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_* and the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y_*, then*

$$\lim_{n\to\infty} d(x_n, y_n) = d(x_*, y_*). \tag{2.9}$$

Proof. By proposition 1.2, we have

$$|d(x_n, y_n) - d(x_*, y_*)| \le d(x_n, x_*) + d(y_n, y_*), \tag{2.10}$$

and the conclusion follows. □

Definition 2.8. Let X be a set and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. The sequence $(y_n)_{n\in\mathbb{N}}$ is a *subsequence* of the sequence $(x_n)_{n\in\mathbb{N}}$ whenever there exists a map $\sigma : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, one has $y_n = x_{\sigma(n)}$ and $\sigma(n+1) > \sigma(n)$.

By induction, the map $\sigma : \mathbb{N} \to \mathbb{N}$ given in definition 2.8 satisfies for every $n, m \in \mathbb{N}$ $\sigma(n+m) \ge \sigma(n) + m$ and in particular for every $n \in \mathbb{N}$, $\sigma(n) \ge n$.

Proposition 2.9. *Let X endowed with d be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges and if the sequnence $(y_n)_{n\in\mathbb{N}}$ is a subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$ then the sequence $(y_n)_{n\in\mathbb{N}}$ converges and*

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n. \tag{2.11}$$

Proof. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be given by the definition of subsequence (definition 2.8) and let $x_* := \lim_{n\to\infty} x_n$.

Let $\varepsilon \in (0, \infty)$. By definition of convergence (definition 2.1), there exists $n_* \in \mathbb{N}$ such that if $n \ge n_*$, then $d(x_n, x_*) \le \varepsilon$. If $n \ge n_*$, we also have $\sigma(n) \ge n \ge n_*$, and thus $d(y_n, x_*) = d(x_{\sigma(n)}, x_*) \le \varepsilon$, and the conclusion follows. □

2.1.1 Convergence in normed spaces

Proposition 2.10. *Let $\|\cdot\|$ be a norm on the vector space X, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. If both sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge in X, then the sequence $(x_n+y_n)_{n\in\mathbb{N}}$ converges in X and*

$$\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}x_n+\lim_{n\to\infty}y_n. \tag{2.12}$$

Proof. Let $x_*:=\lim_{n\to\infty}x_n$ and $y_*:=\lim_{n\to\infty}y_n$. We have for each $n\in\mathbb{N}$

$$(x_n+y_n)-(x_*+y_*)=(x_n-x_*)+(y_n-y_*), \tag{2.13}$$

and thus by the triangle inequality for norms (definition 1.8 (ii))

$$\|(x_n+y_n)-(x_*+y_*)\|\le\|x_n-x_*\|+\|y_n-y_*\|. \tag{2.14}$$

Given $\varepsilon\in(0,\infty)$, there exists n_*^1 such that for every $n\in\mathbb{N}$ satisfying $n\ge n_*^1$,

$$\|x_n-x_*\|\le\frac{\varepsilon}{2} \tag{2.15}$$

and there exists n_*^2 such that for every $n\in\mathbb{N}$ satisfying $n\ge n_*^2$,

$$\|y_n-y_*\|\le\frac{\varepsilon}{2}. \tag{2.16}$$

Letting $n_*=\max\{n_*^1,n_*^2\}$, if $n\ge n_*$ we have $n\ge n_*^1$ and $n\ge n_*^2$ and thus by (2.14), (2.15) and (2.16)

$$\|(x_n+y_n)-(x_*+y_*)\|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \tag{2.17}$$

and thus sequence $(x_n+y_n)_{n\in\mathbb{N}}$ converges to x_*+y_*. □

Proposition 2.11. *Let $\|\cdot\|$ be a norm on the vector space X, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X and if the sequence $(t_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, then the sequence $(t_nx_n)_{n\in\mathbb{N}}$ converges in X and*

$$\lim_{n\to\infty}t_nx_n=\Bigl(\lim_{n\to\infty}t_n\Bigr)\Bigl(\lim_{n\to\infty}x_n\Bigr). \tag{2.18}$$

Proof. Let $t_*:=\lim_{n\to\infty}t_n$ and $x_*=\lim_{n\to\infty}x_n$. For every $n\in\mathbb{N}$, we have

$$\begin{aligned}t_nx_n-t_*x_*&=((t_n-t_*)+t_*)((x_n-x_*)+x_*)-t_*x_*\\&=(t_n-t_*)x_*+t_*(x_n-x)+(t_n-t_*)(x_n-x),\end{aligned} \tag{2.19}$$

and thus by the properties of semi-norms (definition 1.8)

$$\begin{aligned}\|t_nx_n-t_*x_*\|&\le\|(t_n-t_*)x_*\|+\|t_*(x_n-x)\|+\|(t_n-t_*)(x_n-x)\|\\&=|t_n-t_*|\|x_*\|+|t_*|\|x_n-x_*\|+|t_n-t_*|\|x_n-x\|.\end{aligned} \tag{2.20}$$

□

Proposition 2.12. *Let $\|\cdot\|$ be a norm on the vector space X, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X, then the sequence $(\|x_n\|)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ and*

$$\lim_{n\to\infty} \|x_n\| = \Big\|\lim_{n\to\infty} x_n\Big\|. \tag{2.21}$$

Proof. Let $x_* := \lim_{n\to\infty} x_n$. By proposition 1.9, we have for every $n \in \mathbb{N}$,

$$\big|\|x_n\| - \|x_*\|\big| \le \|x_n - x_*\|. \tag{2.22}$$

□

2.1.2 Convergence and inner products

Proposition 2.13. *Let $(\cdot\,|\,\cdot)$ be an inner product on the vector space X, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge with respect to the metric induced by $(\cdot|\cdot)$, then the sequence $((x_n\,|\,y_n))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ and*

$$\lim_{n\to\infty} (x_n\,|\,y_n) = \Big(\lim_{n\to\infty} x_n \,\Big|\, \lim_{n\to\infty} y_n\Big). \tag{2.23}$$

Proof. Let $x_* := \lim_{n\to\infty} x_n$ and $y_* := \lim_{n\to\infty} y_n$. We have for each $n \in \mathbb{N}$,

$$\begin{aligned}(x_n\,|\,y_n) - (x_*\,|\,y_*) &= ((x_n - x_*) + x_*\,|\,(y_n - y) + y_*) - (x_*\,|\,y_*)\\ &= (x_n - x_*\,|\,y_*) + (x_*\,|\,y_n - y_*) + (x_n - x_*\,|\,y_n - y_*).\end{aligned} \tag{2.24}$$

and thus

$$|(x_n\,|\,y_n) - (x_*\,|\,y_*)| \le |(x_n - x_*\,|\,y_*)| + |(x_*\,|\,y_n - y_*)| + |(x_n - x_*\,|\,y_n - y_*)|. \tag{2.25}$$

□

2.2 Characterisations of concrete convergences

2.2.1 Convergence in sequence spaces

Proposition 2.14. *If the set Γ is finite, if $p \in [1,\infty]$, then the sequence $(f_n)_{n\in\mathbb{N}}$ in $\ell^p(\Gamma)$ converges in $\ell^p(\Gamma)$ to $f \in \ell^p(\Gamma)$ if and only if for every $x \in \Gamma$, $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ in $\mathbb{R}$.*

The space $\ell^p(\Gamma)$ was defined in definition 1.46 and definition 1.48.

Proof of proposition 2.14. This follows from the definitions. □

Example 2.15. If the set Γ is infinite, one can take a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of Γ. Defining for each $n \in \mathbb{N}$, the function $f_n := \mathbb{1}_{\{x_n\}}$, we have that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{R}$ whereas for each $n \in \mathbb{N}$ $\|f_n - 0\|_p = 1$, so that the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge to 0 in $\ell^p(\Gamma)$.

Proposition 2.16. *Let Γ be a set and let $p \in [1, \infty)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^p(\Gamma)$ and let $f_* \in \ell^p(\Gamma)$. The following are equivalent.*

(i) The sequence $(f_n)_{n\in\mathbb{N}}$ converges to f_* in $\ell^p(\Gamma)$.

(ii) for every $x \in \Gamma$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f_*(x)$ and the sequence $(\|f_n\|_{\ell^p(\Gamma)})_{n\in\mathbb{N}}$ converges to $\|f_*\|_{\ell^p(\Gamma)}$,

(iii) for every $x \in \Gamma$, $(f_n(x))_{n\in\mathbb{N}}$ converges to $f_*(x)$ and for every $\varepsilon > 0$ there exists a finite set $F \subset \Gamma$ such that for each $n \in \mathbb{N}$, $\sum_{x\in\Gamma\setminus F} |f_n(x)|^p \le \varepsilon$.

Proof. Assume that (i) holds. If the sequence $(f_n)_{x\in\mathbb{N}}$ converges in $\ell^p(\Gamma)$, then for every $x \in \mathbb{N}$,

$$|f_n(x) - f_*(x)| \le \|f_n - f_*\|_{\ell^p(\Gamma)}, \tag{2.26}$$

and thus the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f_*(x)$ in $\mathbb{R}$. The convergence of the norm follows from the general property in normed spaces (proposition 2.12), and thus assertion (ii) holds.

We assume now that (ii) holds and we fix $\varepsilon \in (0, \infty)$. For each $n \in \mathbb{N}$, since $f_n \in \ell^p(\mathbb{N})$, by definitions 1.48 and C.1, there exists a finite set $F_n \subseteq \Gamma$ such that

$$\sum_{x\in\Gamma\setminus F_n} |f_n(x)|^p \le \varepsilon. \tag{2.27}$$

Since $f_* \in \ell^p(\Gamma)$, there exists a finite set $F_* \subseteq \Gamma$ such that

$$\sum_{x\in\Gamma\setminus F_*} |f_*(x)|^p \le \frac{\varepsilon}{2}. \tag{2.28}$$

By the pointwise convergence and the assumption, there exists $n_* \in \mathbb{N}$ such that if $n \ge n_*$,

$$\sum_{x\in\Gamma} |f_n(x)|^p \le \sum_{x\in\Gamma} |f_*(x)|^p + \frac{\varepsilon}{2}. \tag{2.29}$$

and thus

$$\begin{aligned}\sum_{x\in\Gamma\setminus F_*} |f_n(x)|^p &= \sum_{x\in\Gamma} |f_n(x)|^p - \sum_{x\in F_*} |f_n(x)|^p\\ &\le \sum_{x\in\Gamma} |f_*(x)|^p - \sum_{x\in F_*} |f_n(x)|^p + \frac{\varepsilon}{2}\\ &= \sum_{x\in\Gamma\setminus F_*} |f_*(x)|^p + \frac{\varepsilon}{2} \le \varepsilon\end{aligned} \tag{2.30}$$

Defining $F = F_* \cup \bigcup_{n=0}^{n_*-1} F_n$, we get that for every $n \in \mathbb{N}$,

$$\sum_{\Gamma\setminus F} |f_n(x)|^p \le \varepsilon, \tag{2.31}$$

and thus (iii) holds.

Assume now that (iii) holds. By Fatou's lemma for sums (proposition C.7) for every finite set $F \subseteq \Gamma$,

$$\sum_{x \in \Gamma \setminus F} |f_*(x)|^p \le \liminf_{n \to \infty} \sum_{x \in \Gamma \setminus F} |f_n(x)|^p \le \sup_{n \in \mathbb{N}} \sum_{x \in \Gamma \setminus F} |f_n(x)|^p \tag{2.32}$$

Since by lemma A.2,

$$|f_n(x) - f_*(x)|^p \le \left(\frac{2|f_n(x)| + 2|f_*(x)|}{2} \right)^p \le 2^{p-1}|f_n(x)| + 2^{p-1}|f_*(x)| \tag{2.33}$$

we have

$$\begin{aligned} \sum_{x \in \Gamma \setminus F} |f_n(x) - f_*(x)|^p &\le 2^{p-1} \sum_{x \in \Gamma \setminus F} |f_*(x)|^p + 2^{p-1} \sum_{x \in \Gamma \setminus F} |f_n(x)|^p \\ &\le 2^p \sup_{m \in \mathbb{N}} \sum_{x \in \Gamma \setminus F} |f_m(x)|^p. \end{aligned} \tag{2.34}$$

By assumption, there exists a finite set $F \subseteq \Gamma$ such that for every $n \in \mathbb{N}$,

$$\sum_{x \in \Gamma \setminus F} |f_n(x)|^p \le \frac{\varepsilon^p}{2}. \tag{2.35}$$

On the other hand, by the pointwise convergence assumption, there exists $n_* \in \mathbb{N}$ such that if $n \ge n_*$,

$$\sum_{x \in F} |f_n(x) - f_*(x)|^p \le \frac{\varepsilon^p}{2}. \tag{2.36}$$

Combining (2.35) and (2.36), we get for $n \ge n_*$,

$$\sum_{x \in \Gamma} |f_n(x) - f_*(x)|^p \le \varepsilon^p. \tag{2.37}$$

The assertion (i) then follows. □

2.2.2 Convergence in $L^p(\Omega, \mu)$

Proposition 2.17. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $p \in [1, \infty)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L^p(\Omega, \mu)$ and let $f_* \in L^p(\Omega, \mu)$. The following are equivalent*

(i) the sequence $(f_n)_{n \in \mathbb{N}}$ converges to f_* in $L^p(\Omega)$,

(ii) for every $\varepsilon \in (0, \infty)$ and every set $A \in \Sigma$ such that $\mu(A) < \infty$,

$$\lim_{n \to \infty} \mu(\{x \in A \mid |f_n(x) - f_*(x)| > \varepsilon\}) = 0. \tag{2.38}$$

and the sequence $(\|f_n\|_p)_{n \in \mathbb{N}}$ converges to $\|f\|_p$, *(iii) for every $\varepsilon > 0$ and every set $A \in \Sigma$ such that $\mu(A) < \infty$,*

$$\lim_{n\to\infty} \mu(\{x \in A \mid |f_n(x) - f_*(x)| > \varepsilon\}) = 0, \tag{2.39}$$

there exists $\delta > 0$ such that if $A \in \Sigma$ and $\mu(A) \le \delta$, then for every $n \in \mathbb{N}$ $\int_A |f_n|^p \,\mathrm{d}\mu \le \varepsilon$ and there exists $F \in \Sigma$ such that $\mu(F) < \infty$ and for every $n \in \mathbb{N}$ $\int_{\Omega \setminus F} |f_n|^p \,\mathrm{d}\mu \le \varepsilon$

Moreover, one has then for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in \Omega \mid |f_n(x) - f_*(x)| > \varepsilon\}) = 0, \tag{2.40}$$

Proof. Assume that (i) holds. If the sequence $(f_n)_{j\in\mathbb{N}}$ converges to f_* in $L^p(\Omega,\mu)$, then for every $n \in \mathbb{N}$, we have by Chebyshev's inequality (proposition D.30)

$$\begin{aligned} \mu(\{x \in \Omega \mid |f_n(x) - f_*(x)| > \varepsilon\}) &= \mu(\{x \in \Omega \mid |f_n(x) - f_*(x)|^p > \varepsilon^p\}) \\ &\le \frac{1}{\varepsilon^p} \int_\Omega |f_n - f_*|^p \,\mathrm{d}\mu, \end{aligned} \tag{2.41}$$

and thus

$$\lim_{n\to\infty} \mu(\{x \in \Omega \mid |f_n(x) - f_*(x)| > \varepsilon\}) \le \frac{1}{\varepsilon^p} \lim_{n\to\infty} \int_\Omega |f_n - f_*|^p \,\mathrm{d}\mu = 0. \tag{2.42}$$

The convergence of the norms follows from the general property in normed spaces (proposition 2.12). Thus (ii) holds.

If (ii) holds, we have by lemma A.2, for every $x \in \Omega$ and $n \in \mathbb{N}$,

$$|f_n(x) - f_*(x)|^p \le 2^p\Big(\frac{|f_n(x)|^p}{2} + \frac{|f_*(x)|^p}{2}\Big) = 2^{p-1}(|f_n(x)|^p + |f_*(x)|^p). \tag{2.43}$$

Since

$$|f_n(x)|^p \le |f_*(x)|^p + p|f_*(x)|^{p-1}|f_n(x) - f(x)|, \tag{2.44}$$

for every $\varepsilon > 0$, we have for $A \in \Sigma$

$$\begin{aligned} &\mu\big(\{x \in A \mid 2^{p-1}(|f_*(x)|^p - |f_n(x)|^p) + |f_n(x) - f_*(x)|^p > \varepsilon\}\big) \\ &\qquad \le \mu\big(\{x \in A \mid 2^{p-1}p|f_*(x)|^{p-1}|f_n(x) - f_*(x)| + |f_n(x) - f_*(x)|^p > \varepsilon\}\big) \end{aligned} \tag{2.45}$$

and thus if $2^{p-1}pM^{p-1}\delta + \delta^p < \varepsilon$, we have

$$\begin{aligned} &\mu\big(\{x \in A \mid 2^{p-1}(|f_*(x)|^p - |f_n(x)|^p) + |f_n(x) - f_*(x)|^p > \varepsilon\}\big) \\ &\qquad \le \mu(\{x \in A \mid |f_n(x) - f_*(x)| > \delta\}) + \mu(\{x \in A \mid |f_n(x)| > M\}), \end{aligned} \tag{2.46}$$

where by Chebyshev's inequality (proposition D.30), we have

$$\mu(\{x \in A \mid |f_n(x)| > M\}) \le \frac{1}{M^p} \int_\Omega |f_n|^p \,\mathrm{d}\mu. \tag{2.47}$$

Taking M sufficiently large, and δ small enough, we obtain by our assumption if $A \in \Sigma$ and if $\mu(A) < \infty$,

$$\lim_{n\to\infty} \mu\bigl(\{x \in A \mid 2^{p-1}(|f_*(x)|^p - |f_n(x)|^p) + |f_n(x) - f_*(x)|^p > \varepsilon\}\bigr) = 0, \tag{2.48}$$

or equivalently

$$\lim_{n\to\infty} \mu\bigl(\{x \in A \mid 2^{p-1}(|f_*(x)|^p + |f_n(x)|^p) - |f_n(x) - f_*(x)|^p \le 2^p|f_*(x)|^p - \varepsilon\}\bigr) = 0, \tag{2.49}$$

Moreover if $\inf\{|f_*(x)| \mid x \in A\} > 0$, then since $f_* \in L^p(\Omega, \mu)$, we have $\mu(A) < \infty$. Hence by Fatou's lemma for convergence in measure (proposition D.37), we have

$$\begin{aligned}\int_\Omega 2^p |f_*|^p \,\mathrm{d}\mu &\le \liminf_{n\to\infty} \int_\Omega \bigl(2^{p-1}(|f_n|^p + |f_*|^p) - |f_n - f_*|^p\bigr)\,\mathrm{d}\mu\\ &= \liminf_{n\to\infty} \Bigl(\int_\Omega 2^{p-1}(|f_n|^p + |f_*|^p)\,\mathrm{d}\mu - \int_\Omega |f_n - f_*|^p\Bigr)\,\mathrm{d}\mu\\ &\le \limsup_{n\to\infty} \int_\Omega 2^{p-1}(|f_n|^p + |f_*|^p)\,\mathrm{d}\mu - \limsup_{n\to\infty} \int_\Omega |f_n - f_*|^p \,\mathrm{d}\mu\\ &= \int_\Omega 2^p |f_*|^p \,\mathrm{d}\mu - \limsup_{n\to\infty} \int_\Omega |f_n - f_*|^p \,\mathrm{d}\mu.\end{aligned} \tag{2.50}$$

It follows thus that

$$\limsup_{n\to\infty} \int_\Omega |f_n - f_*|^p \,\mathrm{d}\mu = 0, \tag{2.51}$$

and thus the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f_* in $L^p(\Omega, \mu)$.

If (i) holds, we deduce (iii) from the convergence in $L^p(\Omega, \mu)$ and the corresponding properties of $|f_*|^p$ and $|f_n|^p$ (proposition D.34).

Finally, if (iii) holds, we use the estimate

$$\int_\Omega |f_n|^p \,\mathrm{d}\mu \le \int_{\Omega\setminus F} |f_n|^p \,\mathrm{d}\mu + \int_{A_n^\eta \cap F} |f_n|^p \,\mathrm{d}\mu + \int_F (|f_*| + \eta)^p \,\mathrm{d}\mu. \tag{2.52}$$

with the set

$$A_n^\eta \coloneqq \{x \in \Omega \mid |f_n(x) - f_*(x)| > \eta\}, \tag{2.53}$$

Given $\varepsilon > 0$, there exists by our assumption a set $F \in \Sigma$ such that $\mu(F) < \infty$ and for each $n \in \mathbb{N}$

$$\int_{\Omega\setminus F} |f_n|^p \,\mathrm{d}\mu \le \frac{\varepsilon}{3}. \tag{2.54}$$

Moreover, since $\mu(F) < \infty$, by Lebesgue's dominated convergence theorem, there exists $\eta \in (0, \infty)$ such that

$$\int_F (|f_*| + \eta)^p \,\mathrm{d}\mu \le \int_F |f_*|^p \,\mathrm{d}\mu + \frac{\varepsilon}{3} \tag{2.55}$$

Finally, since $\mu(F) < \infty$, we have

$$\lim_{n\to\infty} \mu(A_n^\eta \cap F) = 0, \tag{2.56}$$

and there exists thus by assumption $n_* \in \mathbb{N}$ such that if $n \ge n_*$,

$$\int_{A_n^\eta \cap F} |f_n|^p \,\mathrm{d}\mu \le \frac{\varepsilon}{3}. \tag{2.57}$$

Hence, in view of (2.52), (2.54) and (2.57), we have

$$\int_\Omega |f_n|^p \,\mathrm{d}\mu \le \int_\Omega |f_*|^p \,\mathrm{d}\mu + \varepsilon. \tag{2.58}$$

and the conclusion then follows. □

2.3 Non-metrisable convergences

2.3.1 Convergence almost everywhere

Proposition 2.18. *There is no metric d on $\mathscr{M}([0,1],\mathscr{L}^1)$ such that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{M}([0,1],\mathscr{L}^1)$ converges with respect to d if and only if it converges almost everywhere in $[0,1]$ with respect to $\mathscr{L}^1$.*

Proof. We consider a sequence of sets $A_n \in \mathscr{B}([0,1])$ such that for every $k \in \mathbb{N}$, $\bigcup_{k\ge\mathbb{N}} A_n = [0,1]$ and $\lim_{n\to\infty} \mu(A_n) = 0$. (For example, we could take $A_n = [k2^{-\ell}, (k+1)2^{-\ell}]$ with $n = 2^\ell + k - 1$ and $k \in \{0, \dots, 2^\ell - 1\}$.) By assumption, the sequence $(\mathbb{1}_{A_n})_{n\in\mathbb{N}}$ does not converge almost everywhere to 0 in $[0,1]$. There exists thus an increasing sequence $(n_m)_{m\in\mathbb{N}}$ such that $\liminf_{m\to\infty} d(\mathbb{1}_{A_{n_m}}, 0) > 0$ and for every $m \in \mathbb{N}$, $\mathscr{L}^1(A_{n_m}) \le 2^{-m}$. The latter condition implies that

$$\mu\Bigl(\bigcup_{j\ge m} A_{n_m}\Bigr) \le \sum_{j\ge m} \mu\bigl(A_{n_m}\bigr) \le \sum_{j\ge m} 2^{-j} = 2^{-(m-1)};$$

and thus the sequence $(\mathbb{1}_{A_{n_m}})_{m\in\mathbb{N}}$ converges almost everywhere to 0 in $[0,1]$, which is a contradiction. □

2.3.2 Convergence everywhere

Proposition 2.19. *There is no metric on $C([0,1])$ such that a sequence $(f_n)_{n\in\mathbb{N}}$ in $C([0,1])$ converges with respect to d if and only if it converges everywhere in $[0,1]$.*

Proof. We fix a function $\varphi \in C(\mathbb{R},\mathbb{R})$ such that $\varphi(x) = 0$ if $x \in \mathbb{R} \setminus [0,1]$ and $\varphi(x) = 1$ on $[1/3, 2/3]$. We define $f_0^m(x) := \varphi(2^m x)$. The sequence $(f_0^m)_{m\in\mathbb{N}}$ converges to 0 everywhere, and thus for $m_0 \in \mathbb{N}$ large enough if we set $f_0 := f_0^{m_0}$ and $a_0 := 2^{-m_0}/3$ $b_0 := 2^{-(m_0-1)}/3$, we have $d(f_0^{m_0}, 0) \leq 1$ and $f_{m_0} = 1$ on $[a_0, b_0]$. Next, we assume that f_n has been defined for some $n \in \mathbb{N}$ so that $d(f_n, 0) \leq 2^{-n}$ and $f_n = 1$ on the interval $[a_n, b_n]$. We define

$$f_{n+1}^m(x) := \varphi\Big(2^m \frac{x - a_n}{b_n - a_n}\Big).$$

There exists $m_{n+1} \in \mathbb{N}$ such that if we set $f_{n+1} := f_{n+1}^{m_n}$, $a_{n+1} := a_n + 2^{-n_m}(b_n - a_n)/3$ and $b_{n+1} := a_n + 2^{-(n_m - 1)}(b_n - a_n)/3$, we have $d(f_{n+1}, 0) \leq 2^{-(n+1)}$ and $f_{n+1} = 1$ on $[a_{n+1}, b_{n+1}]$.

It follows that $\lim_{n\to\infty} d(f_n, 0) = 0$. Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, by the nested interval property, there exists a point $x_* \in \bigcap_{n\in\mathbb{N}} [a_n, b_n]$. We then have $f_n(x_*) = 1$ so that $(f_n(x_*))_{n\in\mathbb{N}}$ converges to 1, in contradiction with our assumption on the metric d. □

Proposition 2.20. *If the set* Γ *is uncountable, there is no metric on the set of functions from* Γ *to* $\mathbb{R}$ *such that a sequence of functions converge eveyrwhere if and only if it converges with respect to the metric.*

Proof. For every $x \in \Gamma$, we define the set

$$F_x := \{f : \Gamma \to \mathbb{R} \mid |f(x)| \geq 1\}. \tag{2.59}$$

We have for every $x \in (0, \infty)$,

$$\inf\{d(f, 0) \mid f \in F_x\} > 0. \tag{2.60}$$

Indeed, otherwise there would exist a sequence $(f_n)_{n\in\mathbb{N}}$ such that $(f_n(x))_{n\in\mathbb{N}}$ does not converge to 0 but $(d(f_n, 0))_{n\in\mathbb{N}}$ converges to 0.

Since the set Γ is uncountable, there exists $\delta \in (0, \infty)$ such that the set

$$A := \{x \in \Gamma \mid \{d(f, 0) \mid f \in F_x\} \geq \delta\} \tag{2.61}$$

is infinite. We take a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements in A and we define for every $n \in \mathbb{N}$ the function $f_n = \mathbb{1}_{\{x_n\}}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges everywhere to 0, but $\liminf_{n\to\infty} d(f_n, 0) \geq \delta > 0$, which is a contradiction. □

2.4 Comments

The proof of proposition 2.19 is due to Marion Kirkland Fort Jr. [For51].

For the proof of proposition 2.20, see [BBT97, §9.15].

3 Topology

3.1 Open and closed set in a metric space

3.1.1 Open sets

Definition 3.1. Let d be a metric on X. For every $x \in X$ and $\delta \in [0, \infty)$, the *closed ball* of radius δ centered at x with respect to d in X is the set

$$B[x, \delta] := \{y \in X \mid d(x, y) \leq \delta\}.$$

Definition 3.2. Let d be a metric on X. For every $x \in X$ and $\delta \in (0, \infty)$, the *open ball* of radius δ centered at x with respect to d in X is the set

$$B(x, \delta) := \{y \in X \mid d(x, y) < \delta\}.$$

Definition 3.3. Let d be a metric on X. The set $U \subseteq X$ is *open* (with respect to d) whenever for every $x \in U$, there exists $\delta \in (0, \infty)$ such that $B(x, \delta) \subseteq U$.

Proposition 3.4. *Let d be a metric on X. The sets $\emptyset$ and X are open.*

Proposition 3.5 (Finite intersection of open sets)**.** *Let d be a metric on X. If both sets $U \subseteq X$ and $V \subseteq X$ are open, then the set $U \cap V$ is open.*

Proof. If $x \in U \cap V$, then there exists $\delta \in (0, \infty)$ such that $B(x, \delta) \subseteq U$ and $\varepsilon \in (0, \infty)$ such that $B(x, \varepsilon) \subseteq V$. Setting $\eta := \min\{\delta, \varepsilon\}$, we have

$$B(x, \eta) = B(x, \delta) \cap B(x, \varepsilon) \subseteq U \cap V. \tag{3.1}$$ □

Proposition 3.6 (Arbitrary union of open sets)**.** *Let d be a metric on X and let J be a set. If for every $j \in J$, the set $U_j \subseteq X$ is open with respect to d, then the set $\bigcup_{j \in J} U_j$ is open.*

Proof. If $x \in \bigcup_{j \in J} U_j$, then there exists $j_x \in J$ such that $x \in U_{j_x}$. Since the set U_{j_x} is open, there exists $\delta \in (0, \infty)$ such that

$$B(x, \delta) \subseteq U_{j_x} \subseteq \bigcup_{j \in J} U_j. \tag{3.2}$$ □

Proposition 3.7. *Let d be a metric on X. For every $r > 0$, the set $B(x, r)$ is open.*

Proof. Let $y \in B(x, r)$. Choosing $\delta := r - d(x, y) > 0$, we have for every $z \in B(y, \delta)$, by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = r, \tag{3.3}$$

and thus

$$B(y, \delta) \subseteq B(x, r). \tag{3.4}$$ □

3.1.2 Closed sets

Definition 3.8. Let d be a metric on X. The set $F \subseteq X$ is *closed* with respect to d whenever the set $X \setminus F$ is open.

Proposition 3.9. *Let d be a metric on X. The sets $\emptyset$ and X are closed.*

Proof. Since X is open, $\emptyset = X \setminus X$ is closed; since $\emptyset$ is open, $X = X \setminus \emptyset$ is closed. □

Proposition 3.10 (Finite union of closed sets)**.** *Let d be a metric on X. If both sets $F \subseteq X$ and $G \subseteq X$ are closed, then the set $F \cup G$ is closed.*

Proof. This follows from the set identity

$$X \setminus (F \cup G) = (X \setminus F) \cap (X \setminus G), \tag{3.5}$$

the definition of closed sets as complements of open set (definition 3.8) and the finite intersection property for open sets (proposition 3.5). □

Proposition 3.11 (Arbitrary intersection of closed sets)**.** *Let d be a metric on X. If for every $j \in J$, $F_j \subseteq X$ is closed with respect to d, then $\bigcap_{j \in J} F_j$ is open.*

Proof. This follows from the set identity,

$$X \setminus \bigcap_{j \in J} F_j = \bigcup_{j \in J} \big(X \setminus F_j\big). \tag{3.6}$$

the definition of closed sets as complements of open set (definition 3.8) and the arbitrary union property for open sets (proposition 3.6). □

Proposition 3.12. *Let d be a metric on X. For every $x \in X$, the set $\{x\}$ is closed.*

Proof. Let $y \in X \setminus \{x\}$. We define $\delta := d(x, y)$. We have for every $z \in B(y, \delta)$, by the triangle inequality,

$$d(z, x) \ge d(y, x) - d(y, z) > \delta - \delta = 0, \tag{3.7}$$

and therefore $z \ne x$. This proves that $B(y, \delta) \subseteq X \setminus \{x\}$. Hence the set $X \setminus \{x\}$ is open and thus the set $\{x\}$ is closed by definition of closed sets as complements of open set (definition 3.8). □

Proposition 3.13. *For every $x \in X$ and for every $r > 0$, the set $B[x, r]$ is closed.*

Proof. Let $y \in X \setminus B[x, r]$. Choosing $\delta := d(x, y) - r > 0$, we have for every $z \in B(y, \delta)$, by the triangle inequality

$$d(x, z) \ge d(x, y) - d(y, z) < d(x, y) - \delta = r, \tag{3.8}$$

and thus

$$B(y, \delta) \subseteq X \setminus B[x, r]. \tag{3.9}$$

Hence the set $X \setminus B[x, r]$ is open and thus the ball $B[x, r]$ is closed. □

3.2 Convergence and topology

Proposition 3.14. *Let d be a metric on X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_* if and only if for every set $U \subseteq X$ which is open with respect to d and such that $x_* \in U$, there exists $n_* \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq n_*$, $x_n \in U$.*

Proof. Assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_* with respect to d, that the set $U \subseteq X$ is open with respect to d and that $x_* \in U$. By definition of open set (definition 3.3), there exists $\delta \in (0, \infty)$ such that $B(x_*, \delta) \subseteq U$. By definition of convergence (definition 2.1), there exists n_* such that if $n \in \mathbb{N}$ and $n \geq n_*$, $d(x_n, x_*) \leq \delta/2$ and thus

$$x_n \in B[x_*, \delta/2] \subseteq B(x_*, \delta) \subseteq U.$$

Conversely, we assume that for every open set $U \subseteq X$ containing x_* there exists n_* such that $x_n \in U$ when $n \geq n_*$. Given $\varepsilon \in (0, \infty)$, taking $U = B(x_*, \varepsilon)$ we obtain that for $n \in \mathbb{N}$ satisfying $n \geq n_*$, we have

$$d(x_n, x_*) < \varepsilon, \tag{3.10}$$

which implies the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ with respect to d. □

Proposition 3.15. *Let d be a metric on X. A set $F \subseteq X$ is closed if and only for every convergent sequence $(x_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ $x_n \in F$, one has $\lim_{n\to\infty} x_n \in F$.*

Proof. We first assume that the set F is closed. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_* \in X \setminus F$, then by proposition 3.14, one cannot have $x_n \in X$ for each $n \in \mathbb{N}$.

Conversely, we assume that the set F is not closed. There exists $x_* \in X \setminus F$ such that for every $\delta > 0$, $F \cap B(x_*, \delta) \neq \emptyset$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $x_n \in F$ and $(x_n)_{n\in\mathbb{N}}$ converges to $x_* \in X \setminus F$. This proves the reverse implication. □

3.3 Closed linear subspaces

Proposition 3.16. *Let X endowed with $\|\cdot\|$ be a normed space. If $V \subseteq X$ is a finite-dimensional linear subspace, then V is closed.*

Proposition 3.16 will be a particular case of the following proposition.

Proposition 3.17. *Let X endowed with $\|\cdot\|$ be a normed space. If $V \subseteq X$ is a closed linear subspace and if $W \subseteq X$ is finite-dimensional, then $V + W$ is closed.*

Lemma 3.18. *Let X endowed with $\|\cdot\|_X$ be a normed space and let $x_* \in X$. If $L : X \to \mathbb{R}$ is linear and $L(x_*) \neq 0$, then*

$$\|L\|_{\mathscr{L}(X,\mathbb{R})} = |L(x_*)| \sup\left\{ \frac{1}{\|v - x_*\|_X} \;\middle|\; v \in \ker(L) \right\}. \tag{3.11}$$

Proof. First we note, that if $x \in X$, then $x - L(x)x_*/L(x_*) \in \ker L$. Thus any vector $x \in X$ can be written as $x = v + tx_*$ with $v := x - L(x)x_*/L(x_*) \in \ker L$ and $t := L(x)/L(x_*) \in \mathbb{R}$. By linearity, we have for every $v \in \ker L$ and $t \in \mathbb{R}$.

$$L(v + tx_*) = tL(x_*). \tag{3.12}$$

If $t \neq 0$, we have

$$|L(v + tx_*)| = |t||L(x_*)| = \frac{\|v + tx_*\||L(x_*)|}{\|v/t + x_*\|}, \tag{3.13}$$

and thus

$$\begin{aligned} \sup\Bigl\{\frac{|L(v + tx_*)|}{\|v + tx_*\|} \Bigm| v \in \ker L, t \in \mathbb{R} \text{ and } v + tx_* \neq 0\Bigr\} \\ = \sup\Bigl\{\frac{|L(x_*)|}{\|v/t + x_*\|} \Bigm| v \in \ker L \text{ and } t \in \mathbb{R} \setminus \{0\}\Bigr\} \\ = \sup\Bigl\{\frac{|L(x_*)|}{\|x_* - v\|} \Bigm| v \in \ker L\Bigr\}. \end{aligned} \tag{3.14}$$

□

Proof of proposition 3.17. By an induction argument we can assume that $\dim W = 1$, and thus $W = \mathbb{R}w$ for some $w \in X \setminus V$. Let $L : V + W \to \mathbb{R}$ be the linear map defined by the conditions that $L|_V = 0$ and $L(w) = 1$. Since the set V is closed, there exists $\delta \in (0, \infty)$, such that $B(w, \delta) \subseteq X \setminus V$ and thus in view of lemma 3.18, we have $\|L\|_{\mathscr{L}(V+W,\mathbb{R})} \le |L(w)|/\delta$ and thus $L \in \mathscr{L}(V + W, \mathbb{R})$.

If $(z_n)_{n\in\mathbb{N}}$ is a sequence in $V + \mathbb{R}w$ converging to $z_* \in X$, we define for each $n \in \mathbb{N}$, $t_n := L(z_n)$ and $v_n := z_n - t_n w$. For every $m, n \in \mathbb{N}$ with $m > n$, we have

$$|t_n - t_m| = |L(z_n - z_m)| \le \|L\|_{\mathscr{L}(V+W,\mathbb{R})}\|z_n - z_m\|. \tag{3.15}$$

The sequence $(t_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$; by completeness of the real numbers, it converges in $\mathbb{R}$ to some $t_* \in \mathbb{R}$. For every $n \in \mathbb{N}$, $v_n \in V$ and $(v_n)_{n\in\mathbb{N}}$ converges to $v_* := z_* - t_* w \in X$. Since by assumption the set V is closed we have $v_* \in V$. Hence, $z_* = v_* + t_* w \in V + \mathbb{R}w$ and the space $V + \mathbb{R}w$ is closed. □

Proof of proposition 3.16. This follows from proposition 3.17, since $V = \{0\} + V$, and $\{0\}$ is closed. □

The assumption in proposition 3.16 that X is a real vector space plays a crucial role. Indeed, definining for $q = (q_1, \dots, q_d) \in \mathbb{Q}^d$, $\|q\| := |\alpha_1 q_1 + \cdots \alpha_d q_d|$, the function $\|\cdot\|$ is a norm if and only if $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ are linearly independent in $\mathbb{R}$ as a linear space over $\mathbb{Q}$. We claim that if $V \subseteq \mathbb{Q}^d$ is a closed linear subspace, then either $V = \{0\}$ or $V = \mathbb{Q}^d$. Indeed, assuming that $V \neq \{0\}$, if $q = (q_1, \dots, q_d) \in V \setminus \{0\}$ and $p = (p_1, \dots, p_d) \in \mathbb{Q}^d$, then for every $\varepsilon > 0$, there exists $t \in \mathbb{Q}$ such that $\|p - tq\| = |(\alpha_1 p_1 + \cdots + \alpha_d p_d) - t(\alpha_1 q_1 + \cdots + \alpha_d q_d)| \le \varepsilon$. Since $tq \in V$ and V is closed, this implies that $p \in V$. We have thus $V = \mathbb{Q}^d$.

3.4 Closure

Definition 3.19. Let d be a metric on X. The *closure* of a set $A \subseteq X$ with respect to d is the set

$$\bar{A} := \{x \in X \mid \text{ for every } \delta > 0,\ B(x,\delta) \cap A \neq \emptyset\}. \tag{3.16}$$

Proposition 3.20 (Closedness of the closure)**.** *Let d be a metric on X. For every $A \subseteq X$, the set $\bar{A}$ is closed.*

Proof. Let $x \in X \setminus \bar{A}$. By definition of the closure (definition 3.19), there exists $\delta \in (0,\infty)$ such that for every $z \in A$, we have $d(x,z) \geq \delta$. If $y \in \bar{A}$, then there exists $z \in A$, such that $d(y,z) \leq \delta/2$, and hence $d(x,y) \geq d(x,z) - d(y,z) \geq \delta/2$. Hence $B(x,\delta/2) \subseteq X \setminus \bar{A}$. The set $X \setminus \bar{A}$ is thus open (definition 3.3) and therefore $\bar{A}$ is closed as the complement of an open set (definition 3.8). □

Proposition 3.21. *Let d be a metric space. The set $A \subseteq X$ is closed if and only if $\bar{A} = A$.*

Proof. By proposition 3.20, the closure $\bar{A}$ is closed; hence if $A = \bar{A}$, the set A is closed.

Conversely, we assume that the set A is closed. One clearly has $A \subseteq \bar{A}$. Since the set $X \setminus A$ is open, if $x \in X \setminus A$, then by definition of open set, there exists $\delta > 0$ such that $B(x,\delta) \subseteq X \setminus A$ and thus $A \cap B(x,\delta) = \emptyset$, so that $x \in X \setminus \bar{A}$. We have thus proved that $X \setminus A \subseteq X \setminus \bar{A}$, and thus $\bar{A} \subseteq A$. □

Proposition 3.22 (Monotonicity of the closure)**.** *For every $A, B \subseteq X$, if $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Proof. For every $\delta \in (0,\infty)$ and $x \in X$, we have

$$B(x,\delta) \cap A \subseteq B(x,\delta) \cap B, \tag{3.17}$$

and the conclusion follows from the definition of closure (definition 3.19). □

Proposition 3.23. *If the set $F \subseteq X$ is closed and if $A \subseteq F$, then $\bar{A} \subseteq F$.*

Proof. By monotonicity of the closure (proposition 3.22) and the characterisation of closed sets by their closure (proposition 3.21), we have $\bar{A} \subseteq \bar{F} = F$. □

Proposition 3.24. *For every $A, B \subseteq X$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

Proof. By the proposition, we have $\bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$ and thus $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Conversely, we first have $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$ so that $A \cup B \subseteq \bar{A} \cup \bar{B}$. Since the sets $\bar{A}$ and $\bar{B}$ are closed (proposition 3.20), the set $\bar{A} \cup \bar{B}$ is closed (proposition 3.10), and thus by propositions 3.21 and 3.22 $\overline{A \cup B} \subseteq \overline{\bar{A} \cup \bar{B}} = \bar{A} \cup \bar{B}$. □

Proposition 3.25. *For every $A, B \subseteq X$, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.*

Proof. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$ by proposition 3.22, from which it follows that $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. □

Example 3.26. If $\{x\}$ is not open, then $\overline{X \setminus \{x\}}$ and $\overline{\{x\}} = \{x\}$, while $(X \setminus \{x\}) \cap \{x\} = \emptyset$, and thus

$$\overline{(X \setminus \{x\})} \cap \overline{\{x\}} \subsetneq \overline{(X \setminus \{x\}) \cap \{x\}}. \tag{3.18}$$

Proposition 3.27. *Let d be a metric on X. If $A \subseteq X$,*

$$\bar{A} = \Big\{ \lim_{n\to\infty} x_n \;\Big|\; (x_n)_{n\in\mathbb{N}} \textit{ converges and for every } n \in \mathbb{N}\ x_n \in A \Big\}, \tag{3.19}$$

where the closure and convergence are taken with respect to d.

Proof. Since $\bar{A}$ is closed in view of proposition 3.21, it can be characterised sequentially by proposition 3.15 as

$$\begin{aligned}\bar{A} &= \Big\{ \lim_{n\to\infty} x_n \;\Big|\; (x_n)_{n\in\mathbb{N}} \text{ converges and for every } n \in \mathbb{N}\ x_n \in \bar{A} \Big\}\\ &\supseteq \Big\{ \lim_{n\to\infty} x_n \;\Big|\; (x_n)_{n\in\mathbb{N}} \text{ converges and for every } n \in \mathbb{N}\ x_n \in A \Big\}.\end{aligned} \tag{3.20}$$

Conversely, if $x \in \bar{A}$, then by definition (definition 3.19), there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ one has $x_n \in A$ and $(x_n)_{n\in\mathbb{N}}$ converges to x_*. □

Proposition 3.28. *If X endowed with $\|\cdot\|$ is a normed space, then for every set $A \subseteq X$,* $\operatorname{span}\bar{A} \subseteq \overline{\operatorname{span} A}$*.*

Since $\operatorname{span} A \subseteq \operatorname{span}\bar{A}$, we have $\operatorname{span} A = \overline{\operatorname{span} A}$ if and only if $\operatorname{span}(A)$ is closed.

Proof of proposition 3.28. If $v \in \operatorname{span}\bar{A}$, then $v = \sum_{i=1}^{n} t_i v_i$, with $v_i \in \bar{A}$. Given $\varepsilon > 0$, there exist $w_i \in A$ such that $n t_i \|v_i - w_i\| \le \varepsilon$ and thus if $w := \sum_{i=1}^{d} t_i v_i$, we have $w \in \operatorname{span}(A)$ and

$$\|v - w\| \le \sum_{i=1}^{n} \|t_i(v_i - w_i)\| \le \sum_{i=1}^{n} |t_i| \|v_i - w_i\| \le \varepsilon. \tag{3.21}$$

This proves that $v \in \overline{\operatorname{span}(A)}$. □

3.5 Comments

For a similar metric approach to open sets, closed sets and closures, see [Sut09].

4 Continuity

4.1 Continuity in metric spaces

Definition 4.1. Let d_X be a metric on the set X, let d_Y be a metric on the set Y. The mapping $f : X \to Y$ is *continuous* at the point $a \in X$ (with respect to d_X and d_Y) whenever for every $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that for every $x \in X$, $d_X(x, a) \le \delta$ implies $d_Y(f(x), f(a)) \le \varepsilon$.

Definition 4.2. Let d_X be a metric on the set X and let d_Y be a metric on the set Y. The mapping $f : X \to Y$ is continuous on $A \subseteq X$ (with respect to d_X and d_Y) whenever, for every $a \in A$, f is continuous at a.

Proposition 4.3. *Let d_X be a metric on the set X and let d_Y be a metric on the set Y. If the mapping $f : X \to Y$ is continuous at $a \in X$ (with respect to d_X and d_Y) and if the mapping $g : Y \to Z$ is continuous at $f(a)$ (with respect to d_Y and d_Z) then the mapping $g \circ f$ is continuous at a (with respect to d_X and d_Z).*

Proof. Let $\varepsilon \in (0, \infty)$. By definition of continuity (definition 4.1), there exists $\eta \in (0, \infty)$ such that for every $y \in Y$, $d_Y(y, f(a)) \le \eta$ implies $d_Z(g(y), g(f(a))) \le \varepsilon$. Next, by definition of continuity (definition 4.1), there exists $\delta \in (0, \infty)$ such that for every $x \in X$, $d_X(x, a) \le \delta$ implies $d_Y(f(x), f(a)) \le \eta$, and thus $d_Z(g(f(x)), g(f(a))) \le \varepsilon$. □

Proposition 4.4. *Let d_X be a metric on X, d_Y be a metric on the set Y and let $f : X \to Y$. The function f is continuous at $a \in X$ with respect to d_X and d_Y if and only if for every sequence $(a_n)_{n\in\mathbb{N}}$ in X converging to a with respect to d_X, the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$ with respect to d_Y.*

Proof. Let f be continuous at the point a and let the sequence $(a_n)_{n\in\mathbb{N}}$ converge to a. By definition of continuity (definition 4.1), for every $\varepsilon \in (0, \infty)$, there exists $\delta > 0$ such that $d_X(x, a) \le \delta$ implies $d_Y(f(x), f(a)) \le \varepsilon$. By definition of convergence (definition 2.1), there exists n_* such that $n \ge n_*$ implies $d_X(a_n, a) \le \delta$. By the choice of δ, we deduce that $d_Y(f(a_n), f(a)) \le \varepsilon$ and thus in view of the definition of convergence (definition 2.1), the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$ with respect to d_Y.

For every $n \in \mathbb{N}$, we define

$$\varepsilon_n := \sup\bigl\{d_Y(f(x), f(a)) \bigm| x \in A \text{ and } d_X(x, a) \le 2^{-n}\bigr\} \in [0, \infty]. \tag{4.1}$$

By definition of supremum as a least upper bound, there exists $a_n \in X$ such that $d_X(a_n, a) \le 2^{-n}$ and

$$d_Y(f(a_n), f(a)) \ge \min\{1, \varepsilon_n - 2^{-n}\}. \tag{4.2}$$

By construction and definition of convergence (definition 2.1), the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a. Hence by assumption, the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$ with respect to d_Y, and thus $(\varepsilon_n)_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{R}$. In particular, for every $\varepsilon \in (0,\infty)$, there exists $n_* \in \mathbb{N}$ such that $\varepsilon_{n_*} \le \varepsilon$ and thus the definition of continuity at a is satisfied for ε with $\delta := 2^{-n_*}$. □

Proposition 4.5. *Let d_X be a metric on X, d_Y be a metric on the set Y and let $f : X \to Y$. The function f is continuous with respect to d_X and d_Y if and only if for every open set $U \subseteq Y$, the set $f^{-1}(U)$ is open.*

Proof. Let $U \subseteq Y$ be open and let $a \in f^{-1}(U)$. Since the set U is open, by definition (definition 3.3) there exists $\varepsilon \in (0,\infty)$ such that for every $y \in U$, $d_Y(y, f(a)) \le \varepsilon$ implies $y \in U$. By continuity, there exists $\delta \in (0,\infty)$ such that for every $x \in X$, $d_X(x,a) \le \delta$ implies $d_Y(f(x), f(a)) \le \varepsilon$. This shows that $B(a,\delta) \subseteq f^{-1}(U)$, and thus the set $f^{-1}(U)$ is open.

Conversely, let $\varepsilon \in (0,\infty)$. Since the set $U = B(f(a),\varepsilon)$ is open, the set $f^{-1}(U)$ is open. Since $a \in f^{-1}(U)$, there exists $\delta > 0$ such that for every $x \in X$, $d_X(x,a) \le \delta$ implies that $x \in f^{-1}(U)$ and thus $f(x) \in U$ and finally $d_Y(f(x), f(a)) \le \varepsilon$. □

4.2 Uniform continuity

Definition 4.6. Let d_X be a metric on the set X and let d_Y be a metric on the set Y. The mapping $f : X \to Y$ is *uniformly continuous* (with respect to d_X and d_Y) whenever for every $\varepsilon \in (0,\infty)$, there exists $\delta > 0$ such that for every $x, y \in X$, $d_X(x,y) \le \varepsilon$ implies $d_Y(f(x), f(y)) \le \varepsilon$.

Proposition 4.7. *Let d_X be a metric on the set X and let d_Y be a metric on the set Y. If the mapping $f : X \to Y$ is uniformly continuous, then for every $a \in X$, f is continuous at a.*

Proof. Given $\varepsilon \in (0,\infty)$, we take $\delta \in (0,\infty)$ given by the definition of uniform continuity (definition 4.6) and check directly that definition 4.1 is satisfied. □

Proposition 4.8. *Let d_X be a metric on the set X and let d_Y be a metric on the set Y. If the mapping $f : X \to Y$ is uniformly continuous (with respect to d_X and d_Y) and if the mapping $g : Y \to Z$ is uniformly continuous (with respect to d_Y and d_Z) then the mapping $g \circ f$ is uniformly continuous (with respect to d_X and d_Z).*

Proof. Let $\varepsilon \in (0,\infty)$. By definition of continuity (definition 4.1), there exists $\eta \in (0,\infty)$ such that for every $z, w \in Y$, $d_Y(z,w) \le \eta$ implies $d_Z(g(z), g(w)) \le \varepsilon$. Next, by definition of continuity (definition 4.1), there exists $\delta \in (0,\infty)$ such that for every $x, y \in X$, $d_X(x,y) \le \delta$ implies $d_Y(f(x), f(y)) \le \eta$, and thus $d_Z(g(f(x)), g(f(y))) \le \varepsilon$. □

4.3 Bounded linear mappings

4.3.1 Characterisation of continuous linear mappings

Proposition 4.9. *Let $\|\cdot\|_X$ be a norm on the vector space X, $\|\cdot\|_Y$ be a norm on the vector spaceY and let $L : X \to Y$ be linear. The following are equivalent:*

(i) $L \in \mathscr{L}(X, Y)$,

(ii) L is uniformly continuous,

(iii) L is continuous at 0.

The set of bounded linear mappings $\mathscr{L}(X, Y)$ was defined in definition 1.34.

Proof of proposition 4.9. If (i) holds, then for every $x, y \in X$, we have by linearity,

$$L(x) - L(y) = L(x - y); \tag{4.3}$$

hence by proposition 1.35, we have

$$\|L(x) - L(y)\|_Y \le \|L\|_{\mathscr{L}(X,Y)} \|x - y\|_X. \tag{4.4}$$

Given $\varepsilon \in (0, \infty)$, taking $\delta > 0$ such that $\|L\|_{\mathscr{L}(X,Y)}\delta \le \varepsilon$ gives the conclusion in view of the definition of uniform continuity (definition 4.6), and thus (ii) holds.

Next, if (ii) holds, then (iii) holds immediately by proposition 4.7.

Finally, if (iii) holds, then by definition of continuity at 0 (definition 4.1), there exists $\delta \in (0, \infty)$ such that if $x \in X$ and $\|x - 0\|_X = \|x\|_X \le \delta$, then $\|L(x) - L(0)\|_Y = \|L(x)\|_Y \le 1$. In view of definition 1.34, we have

$$\|L\|_{\mathscr{L}(X,Y)} \le \frac{1}{\delta} < \infty, \tag{4.5}$$

and thus the mapping L is bounded and (i) holds. □

Proposition 4.10. *Let X endowed with $\|\cdot\|_X$ and Y endowed with $\|\cdot\|_Y$ be normed spaces. If $L \in \mathscr{L}(X, \mathbb{R})$, then $\ker L$ is closed.*

Proof. If $L = 0$, then $\ker L = X$, which is open (see proposition 3.9).

Assume that $a \in X \setminus \ker L$. For every $x \in X$, we have

$$\|L(x)\|_Y \ge \|L(a)\|_Y - \|L(a - x)\|_Y \ge \|L(a)\|_Y - \|L\|_{\mathscr{L}(X,Y)} \|a - x\|_X, \tag{4.6}$$

and thus if $\|a - x\|_X < \|L(a)\|_Y / \|L\|_{\mathscr{L}(X,Y)}$, we have $x \in X \setminus \ker L$. This proves that the set $X \setminus \ker L$ is open, and thus

If $\|L\|_{\mathscr{L}(X,\mathbb{R})} = 0$, then $\ker L = X$. Otherwise, if $x_* \in X \setminus \ker L$, we have by (4.6), for every $x \in \ker L$, $\|x - x_*\| \ge |L(x_*)| / \|L\|_{\mathscr{L}(X,\mathbb{R})}$. Thus the set $X \setminus \ker L$ is open and $\ker L$ is closed. □

Example 4.11. In contrast with what happens for the kernel, the *range* of a bounded linear operator can be any subset V of a normed space Y endowed with a norm $\|\cdot\|_Y$. Indeed, one can take $X = V$ and L to be the identity.

4.3.2 Linear mappings on finite-dimensional spaces

Proposition 4.12. *Let X endowed with $\|\cdot\|_X$ and Y endowed with $\|\cdot\|_Y$ a normed space and $L : X \to Y$ be a linear mapping. If $\dim(L(X)) < \infty$, then $L \in \mathscr{L}(X,Y)$ if and only if $\ker L$ is closed.*

We first prove proposition 4.12 in the special case where $\dim Y = 1$, that is, $Y \simeq \mathbb{R}$.

Proposition 4.13. *Let X endowed with $\|\cdot\|_X$ be a normed space and $L : X \to \mathbb{R}$ be a linear mapping. Then $L \in \mathscr{L}(X,\mathbb{R})$ if and only if $\ker L$ is closed.*

Proof. We assume without loss of generality that $L \neq 0$; there exists thus $x_* \in X$ such that $L(x_*) \neq 0$. We have by lemma 3.18,

$$\|L\|_{\mathscr{L}(X,\mathbb{R})} = |L(x_*)| \sup\left\{ \frac{1}{\|v - x_*\|_X y} \;\middle|\; v \in \ker(L) \right\}. \tag{4.7}$$

If $\ker L$ is closed, then $X \setminus \ker L$ is open and there exists $\delta \in (0,\infty)$ such that $B(x_*,\delta) \cap \ker L = \emptyset$, and thus by (4.7)

$$\|L\|_{\mathscr{L}(X,\mathbb{R})} \le \frac{|L(x_*)|}{\delta} < \infty. \tag{4.8}$$

and thus $L \in \mathscr{L}(X,\mathbb{R})$.

The necessity of the closedness of the range follows from proposition 4.10. $\square$

Proof of proposition 4.12. The necessity of the closedness of the range follows from proposition 4.10.

Since the range $L(X) \subseteq Y$ is a finite-dimensional subspace, there exist $e_1,\dots,e_d \in X$ such that $L(e_1),\dots,L(e_d)$ is a basis for $L(X)$. Let $\xi_1,\dots,\xi_d : Y \to \mathbb{R}$ be linear mappings such that for each $i \in \{1,\dots,d\}$, $\xi_i(L(e_i)) = 1$ and for each $i,j \in \{1,\dots,d\}$, $\xi_i(L(e_j)) = 0$ when $i \neq j$. In particular, we have by linearity for each $y \in L(X)$,

$$y = \sum_{i=1}^d \xi_i(y) L(e_i). \tag{4.9}$$

and thus for every $x \in X$,

$$L(x) = \sum_{i=1}^d \xi_i(L(x)) L(e_i); \tag{4.10}$$

which can be rewritten by linearity as

$$L\left(x - \sum_{i=1}^d \xi_i(L(x)) e_i \right) = 0, \tag{4.11}$$

or equivalently

$$x - \sum_{i=1}^{d} \xi_i(L(x))e_i \in \ker L. \tag{4.12}$$

If E denotes the linear space spanned by $e_1, \dots, e_d$, we have thus $X = \ker L + E$. Since $\ker L \subseteq \ker(\xi_i \circ L)$, we have $\ker(\xi_i \circ L) = \ker L + E_i$ with $E_i := E \cap \ker(\xi_i \circ L)$, with $\dim E_i \le \dim E < \infty$. By proposition 3.17, the linear subspace $\ker L + E_i$ is closed, and thus by proposition 4.13, the linear mapping $\xi_i \circ L \in \mathscr{L}(X, \mathbb{R})$ is bounded. By (4.10), the operator $L = \sum_{i=1}^{d} L(e_i)(\xi_i \circ L)$ is bounded. □

Proposition 4.14. *Let X be endowed with the norm $\|\cdot\|_X$ and Y endowed with the norm $\|\cdot\|_Y$. If $L : X \to Y$ is linear and if $\dim(X) < \infty$, then $L \in \mathscr{L}(X, Y)$.*

Proof. This follows from proposition 4.13, since $\dim(L(X)) \le \dim(X) < \infty$ and $\dim(\ker L) \le \dim X < \infty$ so that $\ker L$ is closed by proposition 3.16. □

Proposition 4.15. *Let X be a finite-dimensional space. If $\|\cdot\|$ and $\|\cdot\|_\sharp$ are norms on X, then there exist constants $c, C \in (0, \infty)$ such that for every $x \in X$,*

$$c\|x\|_\sharp \le \|x\| \le C\|x\|_\sharp. \tag{4.13}$$

Proof. We let $X_\sharp = X$, and we let $\mathscr{L}(X_\sharp, X)$ be the space of bounded linear operators from $X_\sharp$ endowed with $\|\cdot\|_\sharp$ to X endowed with $\|\cdot\|$. We define $L : X_\sharp \to X$ by $L(x) := x$. By proposition 4.14, we have $C := \|\mathrm{L}_X\|_{\mathscr{L}(X_\sharp, X)} < \infty$. Thus for every $x \in X$,

$$\|x\| = \|L(x)\| \le C\|x\|_\sharp. \tag{4.14}$$

The proof of the second inequality is obtained by interchanging $\|\cdot\|$ and $\|\cdot\|_\sharp$. □

4.4 Concrete linear mappings

4.4.1 Bounded linear mappings on inner product spaces

Proposition 4.16. *Let X be an inner product space. If $y \in X$ and if one defines $\ell : X \to \mathbb{R}$ for every $x \in X$ by*

$$\ell(x) := (x \,|\, y), \tag{4.15}$$

then $\ell \in \mathscr{L}(X, \mathbb{R})$ and

$$\|\ell\|^2_{\mathscr{L}(X,\mathbb{R})} = \ell(y) = \|y\|^2_X. \tag{4.16}$$

Proof. For every $x \in X$, we have by the Cauchy–Schwarz inequality,

$$|\ell(x)| = |(x \,|\, y)| \le \|y\|_X \|x\|_X, \tag{4.17}$$

and thus $\ell \in \mathscr{L}(X,\mathbb{R})$

$$\|\ell\|_{\mathscr{L}(X,\mathbb{R})} \le \|y\|_X. \tag{4.18}$$

Finally, we have

$$\|y\|_X^2 = |(y\,|\,y)| = |\ell(y)| \le \|\ell\|_{\mathscr{L}(X,\mathbb{R})}\|y\|_X, \tag{4.19}$$

from which it follows that

$$\|\ell\|_{\mathscr{L}(X,\mathbb{R})} \ge \|y\|_X. \tag{4.20}$$

□

4.4.2 Bounded linear mappings on the space of polynomials

Proposition 4.17. *Let $I \subset \mathbb{R}$ be an open bounded nonempty interval. Let $P(I)$ be the set of polynomial functions on I endowed with the norm $\|\cdot\|_\infty$. If $L : P(I) \to P(I)$ is defined for every $f \in P(I)$ by*

$$L(f) := f', \tag{4.21}$$

then

$$\|L\|_{\mathscr{L}(P(I),P(I))} = \infty.$$

Proof. Let

$$\rho := \sup\{|x| \mid x \in I\}.$$

For every $n \in \mathbb{N}_*$, we define $f_n \in P(I)$ for $x \in I$ by $f_n(x) := x^n/\rho^n$. For every $n \in \mathbb{N}$ and $x \in I$, we have have $(Lf_n)(x) = nx^{n-1}/\rho^n$. Moreover

$$\|f_n\|_\infty = \sup\{|x^n/\rho^n| \mid x \in I\} = 1 \tag{4.22}$$

and

$$\|Lf_n\|_\infty = \sup\{|nx^{n-1}/\rho^n| \mid x \in I\} = n/\rho. \tag{4.23}$$

Therefore, we have for every $n \in \mathbb{N}$,

$$\|L\|_{\mathscr{L}(P(I),P(I))} \ge \|Lf_n\|_{P(I)} = n/\rho, \tag{4.24}$$

and we conclude that

$$\|L\|_{\mathscr{L}(P(I),P(I))} = \infty. \tag{4.25}$$

□

Proposition 4.18. *Let $P(I)$ be the set of polynomial functions on I endowed with $\|\cdot\|_\infty$ and let $a \in I$. If $L : P(I) \to P(I)$ is defined for every $f \in P(I)$ and $x \in I$ by*

$$L(f)(x) := \int_a^x f(t)\,\mathrm{d}t, \tag{4.26}$$

then

$$\|L\|_{\mathscr{L}(P(I),P(I))} = \sup\{|x-a| \mid x \in I\}.$$

Proof. We define

$$\rho := \sup\{|x-a| \mid x \in I\}. \tag{4.27}$$

For every $f \in P(I)$ and every $x \in I$, we have

$$|f(x)| \le \|f\|_\infty, \tag{4.28}$$

and hence

$$|Lf(x)| = \left|\int_0^x f(t)\,\mathrm{d}t\right| \le \int_{[a,x]} |f(t)|\,\mathrm{d}t \le \int_{[a,x]} \|f\|_\infty \,\mathrm{d}t = |x-a|\|f\|_\infty, \tag{4.29}$$

and thus

$$\|Lf\|_\infty \le \rho\|f\|_\infty, \tag{4.30}$$

so that

$$\|L\|_{\mathscr{L}(P(I),P(I))} \le \rho. \tag{4.31}$$

Moreover, if we consider now the particular case where $f(x) = 1$, we then have $\|f\|_\infty = 1$ and for every $x \in I$,

$$Lf(x) = x - a, \tag{4.32}$$

and thus

$$\|L(f)\|_\infty = \sup\{|x-a| \mid x \in I\} = \rho, \tag{4.33}$$

and therefore

$$\|L\|_{\mathscr{L}(P(I),P(I))} \ge \|f\|_\infty = \rho. \tag{4.34}$$

We conclude from (4.31) and (4.34) that $\|L\|_{\mathscr{L}(P(I),P(I))} = \rho$. □

Proposition 4.19. *Let $P(I)$ be the set of polynomial functions on I endowed with $\|\cdot\|_\infty$ and let $a \in I$. If $L : P(I) \to P(I)$ is defined for every $f \in P(I)$ and $x \in I$ by*

$$L(f)(x) := f(a), \tag{4.35}$$

then

$$\|L\|_{\mathscr{L}(P(I),P(I))} = 1.$$

Proposition 4.20. *Let I be a bounded interval. Let $P(I)$ be the set of polynomial functions on I endowed with $\|\cdot\|_{L^1(I)}$ and let $a \in I$. If $L : P(I) \to P(I)$ is defined for every $f \in P(I)$ and $x \in I$ by*

$$L(f)(x) := f(a), \tag{4.36}$$

then

$$\|L\|_{\mathscr{L}(P(I),P(I))} = \infty.$$

Proof. We take $\rho \in (0, \infty)$ such that $I \subseteq (a - \rho, a + \rho)$. For every $n \in \mathbb{N}_*$, we define $f_n \in P(I)$ for $x \in I$ by $f_n(x) := (1 - (x - a)^2/\rho^2)^n$. We have

$$Lf_n = 1, \tag{4.37}$$

whereas

$$\begin{aligned} \int_I |f_n| &\leq \int_{a-\rho}^{a+\rho} f_n = 2\rho \int_0^1 (1 - t^2)^n \,\mathrm{d}t \\ &\leq 2\rho \biggl(\int_0^\varepsilon 1 + \int_\varepsilon^1 (1 - \varepsilon^2)^n \biggr) \leq 2\rho(\varepsilon + (1 - \varepsilon^2)^n), \end{aligned} \tag{4.38}$$

which implies by taking small ε and then large n that

$$\lim_{n \to \infty} \int_I |f_n| = 0. \tag{4.39}$$

□

4.4.3 Bounded linear mappings on sequence spaces

Linear forms

Definition 4.21. The *Hölder conjugate exponent* of $p \in [1, \infty]$ is

$$p' := \frac{p}{p - 1} \in [1, \infty], \tag{4.40}$$

with the convention that $1' = \infty$ and $\infty' = 1$.

The Hölder conjugate exponent satisfies the identity

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{4.41}$$

Proposition 4.22. *Let Γ be a set. Let $p \in [1, \infty]$. For every $g \in \ell^{p'}(\Gamma)$, the function $\ell : \ell^p(\Gamma) \to \mathbb{R}$ defined for $f \in \ell^p(\Gamma)$ by*

$$\ell(f) = \sum_{x \in \Gamma} f(x) g(x) \tag{4.42}$$

is well-defined, $\ell \in \mathscr{L}(\ell^p(\Gamma), \mathbb{R})$ and,

$$\|\ell\|_{\mathscr{L}(\ell^p(\Gamma), \mathbb{R})} = \|g\|_{\ell^{p'}(\Gamma)}. \tag{4.43}$$

Proof of proposition 4.22 when $p = \infty$. We have $p' = 1$. Let $f \in \ell^\infty(\Gamma)$. By definition, we have for every $x \in \Gamma$, $|f(x)| \leq \|f\|_{\ell^\infty(\Gamma)}$. Hence for every $x \in \Gamma$, $|f(x)g(x)| \leq \|f\|_{\ell^\infty(\Gamma)} |g(x)|$ and thus

$$\sum_{x \in \Gamma} |f(x)g(x)| \leq \|f\|_{\ell^\infty} \sum_{x \in \Gamma} |g(x)|. \tag{4.44}$$

By proposition C.21, it follows that the function gf is unconditionally summable on Γ and that

$$\Bigl|\sum_{x\in\Gamma} f(x)g(x)\Bigr| \le \|g\|_{\ell^1(\Gamma)}\|f\|_{\ell^\infty(\Gamma)}, \tag{4.45}$$

hence ℓ is well-defined and

$$\|\ell\|_{\mathscr{L}(\ell^\infty(\Gamma),\mathbb{R})} \le \|g\|_{\ell^1(\Gamma)}. \tag{4.46}$$

Finally, we let $f = \operatorname{sgn}(g)$. We have $\|f\|_{\ell^\infty(\Omega)} \le 1$ and

$$\ell(f) = \sum_{x\in\Gamma} f(x)g(x) = \sum_{x\in\Gamma} |g(x)|. \tag{4.47}$$

It follows thus that

$$\|\ell\|_{\mathscr{L}(\ell^\infty(\Gamma),\mathbb{R})} \ge \|g\|_{\ell^1(\Gamma)}. \tag{4.48}$$

The conclusion follows then from (4.46) and (4.48). □

Proposition 4.23 (Hölder's inequality for sums)**.** *Let Γ be a set and let $f, g : \Gamma \to [0,\infty]$. Then for every $p \in (1,\infty)$,*

$$\sum_{x\in\Gamma} f(x)g(x) \le \Bigl(\sum_{x\in\Gamma} f(x)^p\Bigr)^{\frac{1}{p}} \Bigl(\sum_{x\in\Gamma} g^{p'}(x)\Bigr)^{\frac{1}{p'}}. \tag{4.49}$$

Proof. Let $\lambda \in (0,\infty)$. By Young's inequality (proposition A.3), for every $x \in \Gamma$, we have

$$f(x)g(x) \le \frac{\lambda^p f(x)^p}{p} + \frac{g(x)^{p'}}{\lambda^{p'} p'}, \tag{4.50}$$

and thus

$$\sum_{x\in\Gamma} f(x)g(x) \le \frac{\lambda^p}{p}\sum_{x\in\Gamma} f(x)^p + \frac{1}{\lambda^{p'}p'}\sum_{x\in\Gamma} g(x)^{p'}. \tag{4.51}$$

Choosing λ such that

$$\lambda^p \sum_{x\in\Gamma} f(x)^p = \frac{1}{\lambda^{p'}} \sum_{x\in\Gamma} g(x)^{p'}, \tag{4.52}$$

we reach the conclusion. □

Proof of proposition 4.22 when $1 < p < \infty$. By Hölder's inequality for sums (proposition 4.23), we have

$$\sum_{x\in\Gamma} |f(x)g(x)| = \sum_{x\in\Gamma} |g(x)||f(x)| \le \Bigl(\sum_{x\in\Gamma} |g(x)|^{p'}\Bigr)^{\frac{1}{p'}} \Bigl(\sum_{x\in\Gamma} |f(x)|^p\Bigr)^{\frac{1}{p}} \tag{4.53}$$

Hence, by proposition C.21 the function gf is unconditionally summable on Γ, $\ell(f)$ is well-defined and

$$(4.54)\qquad |\ell(f)| = \Bigl|\sum_{x\in\Gamma} f(x)g(x)\Bigr| \le \sum_{x\in\Gamma} |f(x)||g(x)| \le \|g\|_{\ell^{p'}(\Gamma)}\|f\|_{\ell^p(\Gamma)}.$$

Hence, we have

$$(4.55)\qquad \|\ell\|_{\mathscr{L}(\ell^p(\Gamma),\mathbb{R})} \le \|g\|_{\ell^{p'}(\Gamma)}.$$

Next, taking $f := |g|^{p'-2}g$, we have

$$(4.56)\qquad \|f\|_{\ell^p(\Gamma)} = \Bigl(\sum_{x\in\Gamma} |g(x)|^{(p'-1)p}\Bigr)^{\frac{1}{p}} = \Bigl(\sum_{x\in\Gamma} |g(x)|^{p'}\Bigr)^{\frac{1}{p}} = \|g\|_{\ell^{p'}(\Gamma)}^{p'-1}$$

and

$$(4.57)\qquad \ell(f) = \sum_{x\in\Gamma} |g(x)|^{p'-2}g(x)g(x) = \sum_{x\in\Gamma} |g(x)|^{p'} = \|g\|_{\ell^{p'}(\Gamma)}^{p'},$$

so that

$$(4.58)\qquad \|\ell\|_{\mathscr{L}(\ell^p(\Gamma),\mathbb{R})} \ge \frac{|\ell(f)|}{\|f\|_{\ell^p(\Gamma)}} = \|g\|_{\ell^{p'}(\Gamma)};$$

the conclusion follows. □

Proof of proposition 4.22 when $p = 1$. In this case we have $p' = \infty$. By definition, we have for every $x \in \Gamma$, $|g(x)| \le \|g\|_{\ell^\infty(\Gamma)}$ and hence $|f(x)g(x)| \le |f(x)|\|g\|_{\ell^\infty(\Gamma)}$

$$(4.59)\qquad \sum_{x\in\Gamma} |f(x)g(x)| \le \|g\|_{\ell^\infty(\Gamma)} \sum_{x\in\Gamma} |f(x)|.$$

It follows by proposition C.21 that fg is summable on Γ and that

$$(4.60)\qquad |\ell(g)| = \Bigl|\sum_{x\in\Gamma} g(x)f(x)\Bigr| \le \|g\|_{\ell^\infty(\Gamma)} \sum_{x\in\Gamma} |f(x)|;$$

hence ℓ is well-defined and

$$(4.61)\qquad \|\ell\|_{\mathscr{L}(\ell^1(\Gamma),\mathbb{R})} \le \|g\|_{\ell^\infty(\Gamma)}.$$

In order to prove equality in (4.43), for every $x \in \Gamma$, we define $f_x := \mathbb{1}_{\{x\}} \operatorname{sgn}(g)$, and we have

$$(4.62)\qquad \|f_x\|_{\ell^1(\Gamma)} = 1 < \infty$$

and

$$\ell(f_x) = \sum_{y \in \Gamma} f_x(y) g(y) = |g(x)|, \tag{4.63}$$

so that

$$|g(x)| \le \|\ell\|_{\mathscr{L}(\ell^1(\Gamma),\mathbb{R})} \tag{4.64}$$

Since $x \in \Gamma$ is arbitrary, this implies that

$$\|g\|_{\ell^\infty(\Gamma)} \le \|\ell\|_{\mathscr{L}(\ell^1(\Gamma),\mathbb{R})}. \tag{4.65}$$

The conclusion then follows. □

Multiplication operators

Proposition 4.24. *Let Γ be a set. Let $p, q, r \in [1, \infty]$. If*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \tag{4.66}$$

then for every $g \in \ell^q(\Gamma)$, the function $L : \ell^p(\Gamma) \to \ell^r(\Gamma)$ defined for each $f \in \ell^p(\Gamma)$ and $x \in \Gamma$ by

$$L(f)(x) = f(x)g(x) \tag{4.67}$$

is well-defined, $L \in \mathscr{L}(\ell^p(\Gamma), \ell^r(\Gamma))$ and,

$$\|L\|_{\mathscr{L}(\ell^p(\Gamma),\ell^r(\Gamma))} = \|g\|_{\ell^q(\Gamma)}. \tag{4.68}$$

4.4.4 Bounded linear mappings on Lebesgue spaces

Linear functionals

Proposition 4.25. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. Let $p \in [1, \infty]$. For every $g \in L^{p'}(\Omega, \mu)$, the function $\ell : L^p(\Omega, \mu) \to \mathbb{R}$ defined for $f \in L^p(\Omega, \mu)$ by*

$$\ell(f) = \int_\Omega f\, g \,\mathrm{d}\mu \tag{4.69}$$

is well-defined, $\ell \in \mathscr{L}(L^p(\Omega, \mu), \mathbb{R})$ and,

$$\|\ell\|_{\mathscr{L}(L^p(\Omega,\mu),\mathbb{R})} \le \|g\|_{L^{p'}(\Omega,\mu)}. \tag{4.70}$$

Equality holds in (4.70) *if either $p > 1$ or the measure μ is semi-finite.*

Semi-finite measures were defined in definition D.17 as measures for which every measurable set of positive measure has a subset of positive and finite measure.

Proof of proposition 4.25 when $p = \infty$. We have $p' = 1$. Let $f \in L^\infty(\Omega, \mu)$. By definition, we have $|f| \le \|f\|_{L^\infty(\Omega,\mu)}$ almost everywhere on Ω with respect to μ. Hence, $|fg| \le \|f\|_{L^\infty(\Omega,\mu)}|g|$ almost everywhere and

$$\int_\Omega |fg| \,\mathrm{d}\mu \le \|f\|_{L^\infty(\Omega,\mu)} \int_\Omega |g| \,\mathrm{d}\mu. \tag{4.71}$$

By proposition D.49, it follows that fg is integrable on Ω with respect to μ and

$$\left|\int_\Omega f g \,\mathrm{d}\mu\right| \le \|g\|_{L^1(\Omega,\mu)} \|f\|_{L^\infty(\Omega,\mu)}, \tag{4.72}$$

hence ℓ is well-defined and

$$\|\ell\|_{\mathscr{L}(L^\infty(\Omega,\mu),\mathbb{R})} \le \|g\|_{L^1(\Omega,\mu)}. \tag{4.73}$$

Finally, we let $f = \operatorname{sgn}(g)$. The function f is measurable, $\|f\|_{L^\infty(\Omega,\mu)} \le 1$ and

$$\ell(f) = \int_\Omega f g \,\mathrm{d}\mu = \int_\Omega |g| \,\mathrm{d}\mu. \tag{4.74}$$

It follows thus that

$$\|\ell\|_{\mathscr{L}(L^\infty(\Omega,\mu),\mathbb{R})} \ge \|g\|_{L^1(\Omega,\mu)}. \tag{4.75}$$

The conclusion follows then from (4.73) and (4.75). □

Proposition 4.26 (Hölder's inequality). *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $f, g : \Omega \to [0, \infty]$ be measurable. Then for every $p \in (1, \infty)$,*

$$\int_\Omega f g \,\mathrm{d}\mu \le \left(\int_\Omega f^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_\Omega g^{p'} \,\mathrm{d}\mu\right)^{\frac{1}{p'}}. \tag{4.76}$$

Proof. Let $\lambda \in (0, \infty)$. By Young's inequality (proposition A.3), for every $x \in \Omega$, we have

$$f(x)g(x) \le \frac{\lambda^p f(x)^p}{p} + \frac{g(x)^{p'}}{\lambda^{p'} p'}, \tag{4.77}$$

and thus

$$\int_\Omega f g \,\mathrm{d}\mu \le \frac{\lambda^p}{p} \int_\Omega f^p \,\mathrm{d}\mu + \frac{1}{\lambda^{p'} p'} \int_\Omega g^{p'} \,\mathrm{d}\mu. \tag{4.78}$$

Choosing λ such that

$$\lambda^p \int_\Omega f^p \,\mathrm{d}\mu = \frac{1}{\lambda^{p'}} \int_\Omega g^{p'} \,\mathrm{d}\mu, \tag{4.79}$$

we reach the conclusion. □

Proof of proposition 4.25 when $1 < p < \infty$. By Hölder's inequality (proposition 4.26), we have

$$\int_\Omega |f||g| \, \mathrm{d}\mu \le \Big(\int_\Omega |f|^p \, \mathrm{d}\mu\Big)^{\frac{1}{p}} \Big(\int_\Omega |g|^{p'} \, \mathrm{d}\mu\Big)^{\frac{1}{p'}}. \tag{4.80}$$

Hence, by proposition D.49 the function $f g$ is integrable on Ω, $\ell(f)$ is well-defined and

$$|\ell(f)| = \Big|\int_\Omega f g \, \mathrm{d}\mu\Big| \le \int_\Omega |f||g| \, \mathrm{d}\mu \le \|g\|_{L^{p'}(\Omega,\mu)} \|f\|_{L^p(\Omega,\mu)}. \tag{4.81}$$

Hence, we have

$$\|\ell\|_{\mathscr{L}(L^p(\Omega,\mu),\mathbb{R})} \le \|g\|_{L^{p'}(\Omega,\mu)}. \tag{4.82}$$

Next, taking $f := |g|^{p'-2} g$, we have

$$\|f\|_{L^p(\Omega,\mu)} = \Big(\int_\Omega |g|^{(p'-1)p} \, \mathrm{d}\mu\Big)^{\frac{1}{p}} = \Big(\int_\Omega |g|^{p'} \, \mathrm{d}\mu\Big)^{\frac{1}{p}} = \|g\|^{p'-1}_{L^{p'}(\Omega,\mu)} \tag{4.83}$$

and

$$\ell(f) = \int_\Omega |g|^{p'-2} g g \, \mathrm{d}\mu = \int_\Omega |g|^{p'} \, \mathrm{d}\mu = \|g\|^{p'}_{L^{p'}(\Omega,\mu)}, \tag{4.84}$$

so that

$$\|\ell\|_{\mathscr{L}(L^p(\Omega,\mu),\mathbb{R})} \ge \frac{|\ell(f)|}{\|f\|_{L^p(\Omega,\mu)}} = \|g\|_{L^{p'}(\Omega,\mu)}; \tag{4.85}$$

the conclusion follows. □

Proof of proposition 4.25 when $p = 1$. In this case we have $p' = \infty$. By definition, we have $|g| \le \|g\|_{L^\infty(\Omega,\mu)}$ almost everywhere on Ω with respect to μ. Hence, $|f g| \le \|g\|_{L^\infty(\Omega,\mu)} |f|$ almost everywhere and

$$\int_\Omega |f g| \, \mathrm{d}\mu \le \|g\|_{L^\infty(\Omega,\mu)} \int_\Omega |f| \, \mathrm{d}\mu. \tag{4.86}$$

It follows by proposition D.49 that $f g$ is integrable on Ω with respect to μ and

$$|\ell(g)| = \Big|\int_\Omega f g \, \mathrm{d}\mu\Big| \le \|g\|_{L^\infty(\Omega,\mu)} \int_\Omega |f| \, \mathrm{d}\mu; \tag{4.87}$$

hence ℓ is well-defined and

$$\|\ell\|_{\mathscr{L}(L^1(\Omega,\mu),\mathbb{R})} \le \|g\|_{L^\infty(\Omega,\mu)}. \tag{4.88}$$

We now prove equality in (4.70). If $\eta \in (0,1)$, we consider the set

$$A_\eta := \{x \in X \mid |g(x)| \ge (1-\eta)\|g\|_{L^\infty(\Omega,\mu)}\}. \tag{4.89}$$

By definition of $\|g\|_{L^\infty(\Omega,\mu)}$, we have $\mu(A_\eta) > 0$. Since by assumption μ is semi-finite, there exists a set $B_\eta \in \Sigma$ such that $B_\eta \subseteq A_\eta$ and $0 < \mu(B_\eta) < \infty$ (see definition D.17). We define $f_\eta \coloneqq \mathbb{1}_{B_\eta} \operatorname{sgn}(g)$, and we have

$$\|f_\eta\|_{L^1(\Omega,\mu)} = \mu(B_\eta) < \infty \tag{4.90}$$

and

$$\ell(f_\eta) = \int_\Omega f_\eta g \,\mathrm{d}\mu \ge (1-\eta)\|g\|_{L^\infty(\Omega,\mu)}\mu(B_\eta), \tag{4.91}$$

so that

$$\|\ell\|_{\mathscr{L}(L^1(\Omega,\mu),\mathbb{R})} \ge (1-\eta)\|g\|_{L^\infty(\Omega,\mu)}. \tag{4.92}$$

Since $\eta \in (0,1)$ is arbitrary, we conclude that (4.70) holds. □

Multliplication operators

Proposition 4.27. *Let $\mu : \Sigma \to [0,\infty]$ be a measure on Ω. Let $p,q,r \in [1,\infty]$. If*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \tag{4.93}$$

then for every $g \in L^q(\Omega,\mu)$, the function $L : L^p(\Omega,\mu) \to L^r(\Omega,\mu)$ defined for each $f \in L^p(\Omega,\mu)$ and $x \in \Omega$ by

$$L(f)(x) = g(x)f(x) \tag{4.94}$$

is well-defined, $L \in \mathscr{L}(L^p(\Omega,\mu), L^r(\Omega,\mu))$ and,

$$\|\ell\|_{\mathscr{L}(L^p(\Omega,\mu),L^r(\Omega,\mu))} \le \|g\|_{L^q(\Omega,\mu)}. \tag{4.95}$$

Equality holds in (4.95) *if either $p > 1$ or the measure μ is semi-finite.*

Translation operator

Definition 4.28. Let $f : \mathbb{R}^d \to \mathbb{R}$ and $h \in \mathbb{R}^d$. The translation $\tau_h f : \mathbb{R}^d \to \mathbb{R}$ is defined for each $x \in \mathbb{R}^d$ by

$$\tau_h f(x) = f(x-h). \tag{4.96}$$

Graphically, the graph of $\tau_h f$ is obtained by a horizontal translation of the graph of f.

Proposition 4.29. *For every $d \in \mathbb{N}$, $p \in [1,\infty]$ and $h \in \mathbb{R}^d$, one has $\tau_h \in \mathscr{L}\big(L^p(\mathbb{R}^d), L^p(\mathbb{R}^d)\big)$. Moreover, for every $f \in L^p(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |\tau_h f|^p = \int_{\mathbb{R}^d} |f|^p. \tag{4.97}$$

In particular

$$\|\tau_h\|_{\mathscr{L}(L^p(\mathbb{R}^d),L^p(\mathbb{R}^d))} = 1. \tag{4.98}$$

Proof. This follows from the invariance under translations of the Lebesgue measure. □

Convolution

Definition 4.30. The convolution product of $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that for almost every $x \in \mathbb{R}^d$, the function

$$y \in \mathbb{R}^d \mapsto f(y)g(x-y) \tag{4.99}$$

is integrable, is a function $f * g : \mathbb{R}^d \to \mathbb{R}$ such that for almost every $y \in \mathbb{R}^d$,

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y. \tag{4.100}$$

Proposition 4.31.

$$\tau_h(f * g) = (\tau_h f) * g = f * (\tau_h g).$$

Proof. For every $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\tau_h(f * g)(x) &= (f * g)(x-h) = \int_{\mathbb{R}^d} f(y)g(x-h-y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(z-h)g(x-z)\,\mathrm{d}y = \int_{\mathbb{R}^d} \tau_h f(z)g(x-z)\,\mathrm{d}y \\ &= ((\tau_h f) * g)(x)\end{aligned} \tag{4.101}$$

and

$$\begin{aligned}\tau_h(f * g)(x) &= (f * g)(x-h) = \int_{\mathbb{R}^d} f(y)g(x-h-y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y)\tau_h g(x-y)\,\mathrm{d}y = (f * (\tau_h g))(x).\end{aligned} \tag{4.102}$$

□

Proposition 4.32 (Young's convolution inequality)**.** *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. If $d \in \mathbb{N}$ and $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}y < \infty, \tag{4.103}$$

and

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}, \tag{4.104}$$

with $r \in [1, \infty]$ satisfying

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{4.105}$$

Proof in the case $r < \infty$. We assume that the functions $f, g, h : \mathbb{R}^d \to \mathbb{R}$ are integrable. We have for every $x, y \in \mathbb{R}^d$,

$$|f(y)g(x-y)h(x)| = (|f(y)|^p |g(x-y)|^q)^{\frac{1}{r}} (|f(y)|^{\frac{r-p}{r-1}} |g(x-y)|^{\frac{r-q}{r-1}} |h(x)|^{\frac{r}{r-1}})^{1-\frac{1}{r}}. \tag{4.106}$$

By Hölder's inequality (proposition 4.26)

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)h(x)| \,\mathrm{d}x\,\mathrm{d}y &\le \Bigl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)|^p |g(x-y)|^q \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{r}}\\ &\qquad \Bigl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)|^{\frac{r-p}{r-1}} |g(x-y)|^{\frac{r-q}{r-1}} |h(x)|^{\frac{r}{r-1}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{1-\frac{1}{r}}.\end{aligned} \tag{4.107}$$

Defining

$$s := \frac{r-1}{\frac{r}{p}-1} \qquad \text{and} \qquad t := \frac{r-1}{\frac{r}{q}-1}, \tag{4.108}$$

we have in view of (4.105),

$$\frac{1}{t} + \frac{1}{s} = 1, \tag{4.109}$$

and thus

$$|f(y)|^{\frac{r-p}{r-1}} |g(x-y)|^{\frac{r-q}{r-1}} |h(x)|^{\frac{r}{r-1}} = \Bigl(|f(y)|^p |h(x)|^{\frac{r}{r-1}}\Bigr)^{\frac{1}{s}} \Bigl(|g(x-y)|^q |h(x)|^{\frac{r}{r-1}}\Bigr)^{\frac{1}{t}}. \tag{4.110}$$

Hence, by Hölder's inequality (proposition 4.26) again, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)|^{\frac{r-p}{r-1}} |g(x-y)|^{\frac{r-q}{r-1}} |h(x)|^{\frac{r}{r-1}} \,\mathrm{d}x\,\mathrm{d}y &\\ \le \Bigl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)|^p |h(x)|^{\frac{r}{r-1}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{s}} &\Bigl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(x-y)|^q |h(x)|^{\frac{r}{r-1}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{t}}.\end{aligned} \tag{4.111}$$

To conclude, we insert (4.111) into (4.107) and note that

$$\frac{1}{r} + \Bigl(1 - \frac{1}{r}\Bigr)\frac{1}{s} = \frac{1}{p} \qquad \text{and} \qquad \frac{1}{r} + \Bigl(1 - \frac{1}{r}\Bigr)\frac{1}{t} = \frac{1}{q},$$

so that in view of (4.109), we get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)h(y)| \,\mathrm{d}x\,\mathrm{d}y \le \Bigl(\int_{\mathbb{R}^d} |f|^p\Bigr)^{\frac{1}{p}} \Bigl(\int_{\mathbb{R}^d} |g|^q\Bigr)^{\frac{1}{q}} \Bigl(\int_{\mathbb{R}^d} |h|^{\frac{r}{r-1}}\Bigr)^{\frac{r-1}{r}}. \tag{4.112}$$

Taking $h : \mathbb{R}^d \to \mathbb{R}$ such that $h > 0$ and $\int_{\mathbb{R}^d} |h|^{\frac{r}{r-1}} < \infty$, we get (4.103) for almost every $x \in \mathbb{R}^d$. The estimate then follows from (4.112) by proposition 4.25. □

4.5 Comments

One can define the following variant of $\|\cdot\|_\infty$ and $L^\infty(\Omega,\mu)$:

(4.113)
$$\|f\|_\propto = \inf\{M \in \mathbb{R} \mid \text{for every } A \in \Sigma \text{ with } \mu(A) < \infty,\ \mu(\{x \in A \mid |f(x)| > M\}) = 0\},$$

and set

$$L^\propto(\Omega,\mu) = \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \|f\|_\propto < \infty\}. \tag{4.114}$$

then we have, with the notations of proposition 4.25, if $g \in L^\propto(\Omega,\mu)$, then we have $\ell \in \mathscr{L}(L^1(\Omega,\mu),\mathbb{R})$ and

$$\|\ell\|_{\mathscr{L}(L^1(\Omega,\mu),\mathbb{R})} = \|g\|_\propto \tag{4.115}$$

(see [Fol99, ch. 6 ex. 23–24]). A similar remark holds for proposition 4.27.

Part II

Completeness

5 Complete spaces

5.1 Definition of completeness

Definition 5.1. Let X be a space endowed with a metric d. A sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* (with respect to d) whenever for every $\varepsilon \in (0,\infty)$ there exists $n_* \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m > n \geq n_*$, then $d(x_n, x_m) \leq \varepsilon$.

Proposition 5.2. *Let X be a space endowed with a metric d and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ converges to $x_* \in X$ with respect to d, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.*

Proof. For every $m, n \in \mathbb{N}$, we have by the triangle inequality (definition 1.1 (iii)) and by symmetry (definition 1.1 (ii))

$$d(x_n, x_m) \leq d(x_n, x_*) + d(x_*, x_m) = d(x_n, x_*) + d(x_m, x_*). \tag{5.1}$$

Let $\varepsilon \in (0,\infty)$. By definition of convergence of sequences (definition 2.1), there exists $n_* \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq n_*$, we have

$$d(x_n, x_*) \leq \frac{\varepsilon}{2}. \tag{5.2}$$

Hence, if $m \in \mathbb{N}$ and $m > n$, in view of (5.1),

$$d(x_n, x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{5.3}$$

and thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence by definition (definition 5.1). □

Proposition 5.3. *Let X be a space endowed with a metric d, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $(n_m)_{m\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and if $(x_{n_m})_{m\in\mathbb{N}}$ converges to $x_* \in X$, then $(x_n)_{n\in\mathbb{N}}$ converges to x_*.*

Proof. For every $m, n \in \mathbb{N}$, we have by the triangle inequality

$$d(x_n, x_*) \leq d(x_n, x_{n_m}) + d(x_{n_m}, x_*). \tag{5.4}$$

Let $\varepsilon \in (0,\infty)$. By definition of convergence (definition 2.1), there exists $m_* \in \mathbb{N}$ such that if $m \geq m_*$, then

$$d(x_{n_m}, x_*) \leq \frac{\varepsilon}{2}. \tag{5.5}$$

By definition of Cauchy sequence (definition 5.1), there exists $n_* \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m > n \geq n_*$, then $n_m \geq m > n$ and thus

$$d(x_n, x_{n_m}) \leq \frac{\varepsilon}{2}. \tag{5.6}$$

Hence, $n > n_*$, taking $m = \max\{n, m_*\}$, we have by (5.4), (5.5) and (5.6),

$$d(x_n, x_*) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{5.7}$$

and thus the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_* by definition (definition 2.1). □

Proposition 5.4. *Let d_X be a metric on X, d_Y be a metric on the set Y and let $f : X \to Y$. The function f is uniformly continuous on X and if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $f(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. By definition of uniform continuity (definition 4.6), for every $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that for every $x, y \in X$, $d_X(x, y) \leq \delta$ implies $d_Y(f(x), f(y)) \leq \varepsilon$. By definition of Cauchy sequence (definition 5.1), there exists n_* such that $n, m \geq n_*$ implies $d_X(x_n, x_m) \leq \delta$. By the choice of δ, we deduce that $d_Y(f(x_n), f(x_m)) \leq \varepsilon$ and thus in view of the definition of convergence (definition 5.1), the sequence $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Definition 5.5. Let X be a space endowed with a metric d. The set $A \subseteq X$ is *complete* (with respect to d) whenever for every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in A (with respect to d) there exists $x_* \in A$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x_* (with respect to d).

Proposition 5.6. *Let X be a space endowed with a metric d. If A is complete, then A is closed.*

Proof. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in A that converges to some $x_* \in X$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A (by proposition 5.2) and converges to some $a_* \in A$ by assumption. By uniqueness of the limit, $a_* = x_*$ and thus $x_* \in A$, and thus by the sequential characterisation of closed sets (proposition 3.15), the set A is closed. □

Proposition 5.7. *Assume that the space X endowed with a metric d is complete. If $A \subseteq X$ is closed, then A is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A with respect to d. Since the space X is complete and $A \subseteq X$, by definition of completeness of X (proposition 5.7) there exists $x_* \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_*. Since the set A is closed, by the sequential characterisation of closed sets (proposition 3.15), we have $x_* \in A$. □

Proposition 5.8. *Let X endowed with $\|\cdot\|$ be a normed space. The space X is complete if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that*

$$\sum_{n\in\mathbb{N}} \|x_n\| < \infty, \tag{5.8}$$

there exists $y \in X$ such that the sequence $\left(\sum_{n=0}^{m} x_n\right)_{m\in\mathbb{N}}$ converges to y.

Proof. If X is complete and $(x_n)_{n\in\mathbb{N}}$ is a sequence such that (5.8), then for every $m, \ell \in \mathbb{N}$ such that $m < \ell$, by the triangle inequality,

$$\|y_m - y_\ell\| \le \sum_{n=m}^{\ell-1} \|y_{n+1} - y_n\| = \sum_{n=m}^{\ell-1} \|x_n\|. \tag{5.9}$$

The sequence $(y_m)_{m\in\mathbb{N}}$ is thus a Cauchy sequence. By definition of completeness, there exists $y \in X$ such that $(y_m)_{m\in\mathbb{N}}$ converges to y.

Conversely, we assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We define $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and $n \ge n_0$,

$$\|x_n - x_{n_0}\| \le 1. \tag{5.10}$$

Next, if n_m is defined for $m \in \mathbb{N}$, we choose $n_{m+1} \in \mathbb{N}$ such that $n_{m+1} > n_m$ and for every $n \in \mathbb{N}$ and $n \ge n_{m+1}$,

$$\|x_n - x_{n_m}\| \le 2^{-m}. \tag{5.11}$$

In particular, it follows that

$$\|x_{n_{m+1}} - x_n\| \le 2^{-m}, \tag{5.12}$$

and thus

$$\sum_{m\in\mathbb{N}} \|x_{n_{m+1}} - x_n\| \le \sum_{m\in\mathbb{N}} 2^{-m} \le 2. \tag{5.13}$$

By assumption, there exists $x \in X$ such that the sequence $(x_{n_m})_{m\in\mathbb{N}}$ converges to x. By proposition 5.3, the sequence $(x_n)_{n\in\mathbb{N}}$ also converges to x. This proves that the space X is complete. □

5.2 Finite-dimensional spaces

Proposition 5.9. *Let X endowed with $\|\cdot\|$ be a normed space. If* $\dim X < \infty$*, then X is complete.*

Proposition 5.10. *Let X endowed with $\|\cdot\|$ be a normed space. If $V \subseteq X$ is complete and if* $\dim W < \infty$ *then $V + W$ is complete.*

Proof. By an induction argument we can assume that $\dim W = 1$, with $W = \mathbb{R} w$ for some $w \in X \setminus V$. Let $L : V + W \to \mathbb{R}$ be the linear map defined by $L|_V = 0$ and $L(w) = 1$. Since V is complete, by proposition 5.6 V is closed, there exists $\delta \in (0, \infty)$, such that $B(w, \delta) \subseteq X \setminus V$ and thus in view of lemma 3.18, we have $\|L\|_{\mathscr{L}(V+W,\mathbb{R})} \le |L(w)|/\delta$ and thus $L \in \mathscr{L}(V + W, \mathbb{R})$.

If $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $V + \mathbb{R} w$, we define for each $n \in \mathbb{N}$, $t_n := L(z_n)$ and $v_n := z_n - t_n w$. For every $m, n \in \mathbb{N}$ with $m > n$, we have

$$|t_n - t_m| = |L(z_n - z_m)| \le \|L\|_{\mathscr{L}(V+W,\mathbb{R})} \|z_n - z_n\|. \tag{5.14}$$

The sequence $(t_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$; by completeness of the real numbers, it converges in $\mathbb{R}$ to some $t_* \in \mathbb{R}$. Similarly, for every $m, n \in \mathbb{N}$ with $m > n$, we have

$$\|v_n - v_m\| = \|z_n - z_m - (t_n - t_m)w\| \le \|z_n - z_m\| + \|t_n - t_m\|, \tag{5.15}$$

so that the sequence $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. By our assumption that V is complete, it converges to some $v_* \in V$. We then have that the sequence $(z_n)_{n\in\mathbb{N}}$ converges to $z_* = v_* + t_* w$, and thus $V + W$ is complete. □

Proof of proposition 5.9. This follows from proposition 5.10. □

5.3 Concrete complete spaces

5.3.1 Uniform convergence

Proposition 5.11. *Let Γ be a set. The space $\ell^\infty(\Gamma)$ endowed with $\|\cdot\|_{\ell^\infty(\Gamma)}$ is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^\infty(\Gamma)$. For every $x \in \Gamma$, we have by definition of $\|\cdot\|_{\ell^\infty(\Gamma)}$ (definition 1.46),

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{\ell^\infty(\Gamma)}. \tag{5.16}$$

The sequence $(f_n(x))_{n\in\mathbb{N}}$ is thus a Cauchy sequence in $\mathbb{R}$. By completeness of the set $\mathbb{R}$ of real numbers, there exists $f_*(x) \in \mathbb{R}$ such that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f_*(x)$ in $\mathbb{R}$.

By definition of Cauchy sequence (definition 5.1), for every $\varepsilon > 0$, there exists $n_* \in \mathbb{N}$, such that if $m, n \in \mathbb{N}$ satisfy $m > n \ge n_*$, then

$$\|f_n - f_m\|_{\ell^\infty(\Gamma)} \le \varepsilon. \tag{5.17}$$

By (5.16), we have thus for every $x \in \Gamma$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{\ell^\infty(\Gamma)} \le \varepsilon. \tag{5.18}$$

Letting $m \to \infty$ in (5.18), we obtain then for every $x \in \Gamma$,

$$|f_n(x) - f_*(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \varepsilon. \tag{5.19}$$

By definition of $\|\cdot\|_{\ell^\infty(\Gamma)}$ (definition 1.46), this implies that

$$\|f_n - f_*\|_{\ell^\infty(\Gamma)} \le \varepsilon \tag{5.20}$$

By definition of convergence (definition 2.1), the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f_* with respect to $\|\cdot\|_{\ell^\infty(\Gamma)}$. □

Definition 5.12. Given a set Γ and $p \in [1, \infty)$, we define

$$c_0(\Gamma) := \{f : \Gamma \to \mathbb{R} \mid \text{for every } \varepsilon > 0, \text{ the set } \{x \in \Gamma \mid |f(x)| \ge \varepsilon\} \text{is finite}\}. \tag{5.21}$$

Proposition 5.13. *Let Γ be a set. The space $c_0(\Gamma)$ endowed with $\|\cdot\|_{\ell^\infty(\Gamma)}$ is complete.*

Lemma 5.14. *Let Γ be a set. The set $c_0(\Gamma)$ is a closed subset of $\ell^\infty(\Gamma)$.*

Proof. Let $f \in \ell^\infty(\Gamma) \setminus c_0(\Gamma)$. For every $g \in \ell^\infty(\Gamma)$ and ε, we have

$$\{x \in \Gamma \mid |f(x)| \ge \varepsilon\} \subseteq \{x \in \Gamma \mid |g(x)| \ge \varepsilon - \|f - g\|_{\ell^\infty(\Gamma)}\}. \tag{5.22}$$

Since $f \in \ell^\infty(\Gamma)\setminus c_0(\Gamma)$, by definition 5.12, there exists $\varepsilon \in (0,\infty)$ such that the set $\{x \in \Gamma \mid |f(x)| \ge \varepsilon\}$ is infinite. If $g \in \ell^\infty(\Gamma)$ and $\|f - g\|_{\ell^\infty(\Gamma)} < \varepsilon$ and $\eta := \varepsilon - \|f - g\|_{\ell^\infty(\Gamma)} > 0$, in view of (5.22), the set $\{x \in \Gamma \mid |g(x)| \ge \eta\}$ is infinite, and thus $g \notin c_0(\Gamma)$. □

Proof of proposition 5.13. In view of proposition 5.7, this follows from proposition 5.11 and lemma 5.14. □

Definition 5.15. Let Γ be a set. One defines

$$c_c(\Gamma) := \{f : \Gamma \to \mathbb{R} \mid \text{the set } \{x \in \Gamma \mid |f(x)| > 0\} \text{ is finite}\} \tag{5.23}$$

Proposition 5.16. *Let Γ be a set. The set $c_c(\Gamma)$ endowed with $\|\cdot\|_{\ell^p(\Gamma)}$ is complete if and only if the set Γ is finite.*

Proof. If the set Γ is infinite, then there exists an injective map $g : \mathbb{N} \to \Gamma$. We define for every $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 2^{-g^{-1}(x)} & \text{if } x \in g(\{0,\dots,n\}),\\ 0 & \text{otherwise.}\end{cases} \tag{5.24}$$

If $p < \infty$, we have for $m, n \in \mathbb{N}$ with $m > n$,

$$\|f_m - f_n\|^p_{\ell^p(\Gamma)} = \sum_{\ell=n+1}^{m} 2^{-\ell p} = \frac{2^{-(n+1)p} - 2^{-(m+1)p}}{1 - 2^{-p}} \le \frac{2^{-(n+1)p}}{1 - 2^{-p}}, \tag{5.25}$$

so that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which cannot converge.

The proof for $p = \infty$ is similar. □

Proposition 5.17. *Let Γ be a metric space. The space $C(\Gamma, \mathbb{R})$ endowed with $\|\cdot\|_{\ell^\infty(\Gamma)}$ is complete.*

Lemma 5.18. *If Γ endowed with d is a metric space, then $C(\Gamma,\mathbb{R}) \cap \ell^\infty(\Gamma)$ is a closed subset of space $\ell^\infty(\Gamma)$ endowed with $\|\cdot\|_{\ell^\infty(\Gamma)}$.*

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $C(\Gamma,\mathbb{R})$ converging to $f_* \in \ell^\infty(\Gamma)$ with respect to $\|\cdot\|_{\ell^\infty(\Gamma)}$. We need to prove that f_* is continuous. Let $a \in \Gamma$. By the triangle inequality and by definition of $\|\cdot\|_{\ell^\infty(\Gamma)}$ (definition 1.46), we have for every $x \in \Gamma$ and $n \in \mathbb{N}$,

$$\begin{aligned} |f_*(x) - f_*(a)| &\le |f_*(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_*(a) - f_n(a)|\\ &\le 2\|f_* - f_n\|_{\ell^\infty(\Gamma)} + |f_n(x) - f_n(a)|. \end{aligned} \tag{5.26}$$

Let $\varepsilon \in (0, \infty)$. By definition of convergence (definition 2.1), there exists $n \in \mathbb{N}$ such that

$$(5.27) \qquad \|f_* - f_n\|_{\ell^\infty(\Gamma)} \le \frac{\varepsilon}{4}.$$

Since the function f_n is continuous at a, there exists $\delta > 0$ such that if $x \in \Gamma$ and $d(x, a) \le \delta$, then

$$(5.28) \qquad |f_n(x) - f_n(a)| \le \frac{\varepsilon}{2}.$$

Hence, by (5.26) and (5.28),

$$(5.29) \qquad |f_*(x) - f_*(a)| \le 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,$$

and thus the function f_* is continuous at a. □

Proof of proposition 5.17. In view of proposition 5.7, this follows from proposition 5.11 and lemma 5.18. □

5.3.2 Bounded linear operators

Proposition 5.19. *Let $\|\cdot\|_X$ be a norm on the vector space X and $\|\cdot\|_Y$ be a norm on the vector space Y. If Y is complete, then the vector space $\mathscr{L}(X, Y)$ endowed with $\|\cdot\|_{\mathscr{L}(X,Y)}$ is complete.*

Proof. Let $(L_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$. For every $x \in X$, we have by definition of $\|\cdot\|_{\mathscr{L}(X,Y)}$ (definition 1.34)

$$(5.30) \qquad \|L_n(x) - L_m(x)\|_Y = \|(L_n - L_m)(x)\|_Y \le \|L_n - L_m\|_{\mathscr{L}(X,Y)}\|x\|_X,$$

and therefore $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. By assumption Y is complete, therefore there exists $L_*(x) \in Y$ such that $(L_n(x))_{n\in\mathbb{N}}$ converges to $L_*(x)$ in Y.

By proposition 2.10 and proposition 2.11, $L_* : X \to Y$ is linear.

Let $\varepsilon > 0$. Since $(L_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m \ge n \ge N$,

$$(5.31) \qquad \|L_n(x) - L_m(x)\|_Y \le \varepsilon\|x\|_X.$$

Letting $m \to \infty$ in (5.31), we have

$$(5.32) \qquad \|L_n(x) - L_*(x)\|_Y \le \varepsilon\|x\|_X,$$

from which it follows that

$$(5.33) \qquad \|L_n - L_*\|_{\mathscr{L}(X,Y)} \le \varepsilon.$$

We have thus proved that $L_* \in \mathscr{L}(X, Y)$ and that the sequence $(L_n)_{n\in\mathbb{N}}$ converges to L_*. □

5.3.3 Lebesgue spaces

Proposition 5.20. *If $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. The space $L^\infty(\Omega, \mu)$ endowed with $\|\cdot\|_{L^\infty(\Omega,\mu)}$ is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable bounded functions such that there equivalence classes form a Cauchy sequence in $L^\infty(\Omega, \mu)$. By definition of $\|\cdot\|_{L^\infty(\Omega,\mu)}$, for every $m, n \in \mathbb{N}$, there exists a set $A_{m,n} \in \Sigma$ such that $\mu(A_{m,n} = 0$ and for every $x \in \Omega \setminus A_{m,n}$, $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{L^\infty(\Omega,\mu)}$. Letting $A = \bigcup_{m,n\in\mathbb{N}} A$, we have $\mu(A) = \emptyset$ and $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{L^\infty(\Omega,\mu)}$. By proposition 5.11, there exists $f : \Omega \setminus A \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in \Omega \setminus A\} = 0. \tag{5.34}$$

Extending f by 0 to Ω, we get that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^\infty(\Omega, \mu)$. □

Proposition 5.21. *If $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $p \in [1, \infty)$. The space $L^p(\Omega, \mu)$ endowed with $\|\cdot\|_{L^p(\Omega,\mu)}$ is complete.*

Lemma 5.22. *If $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $p \in [1, \infty)$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions from Ω to $\mathbb{R}$ such that*

$$\sum_{n\in\mathbb{N}} \Big(\int_\Omega |f_n|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} < \infty \tag{5.35}$$

then there exist a measurable function $f_* : \Omega \to \mathbb{R}$ and a function $g : \Omega \to [0, \infty]$ such that

$$\int_\Omega g \, \mathrm{d}\mu \le \sum_{n\in\mathbb{N}} \Big(\int_\Omega |f_n|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} \tag{5.36}$$

and for every $x \in \Omega$ such that $g(x) < \infty$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f_*(x)$ and for every $m \in \mathbb{N}$, $|\sum_{n=0}^m f_n(x)|^p \le g(x)$.

Proof. We define $m \in \mathbb{N}$ the function $g_m : \Omega \to [0, \infty)$ for each $x \in \Omega$ by

$$g_m(x) := \sum_{n=0}^m |f_n|^p \in [0, \infty). \tag{5.37}$$

Since for every $m \in \mathbb{N}$ and $x \in \Omega$, $g_m(x) \le g_{m+1}(x)$, we can define for each $x \in \Omega$

$$g(x) := \lim_{m\to\infty} g_m(x) = \sup\{g_m(x) \mid m \in \mathbb{N}\} \in [0, \infty]. \tag{5.38}$$

By the triangle inequality, we have for every $n \in \mathbb{N}$

$$\Big| \sum_{n=0}^m f_n(x) \Big|^p \le g_n(x) \le g(x). \tag{5.39}$$

Moreover, by the triangle inequality again if $m > \ell$,

$$\left|\sum_{n=0}^{m} f_n(x) - \sum_{n=0}^{\ell} f_m(x)\right| \le g_m(x)^{1/p} - g_\ell(x)^{1/p} \le g(x)^{1/p} - g_n(x)^{1/p} \tag{5.40}$$

In particular, if $g(x) < \infty$, then $(\sum_{n=0}^m f_n(x))_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. We define

$$f_*(x) = \begin{cases} \lim_{m\to\infty} \sum_{n=0}^m f_m(x) & \text{if } g(x) < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{5.41}$$

By the monotone convergence theorem for integrals (proposition D.38), we have

$$\int_\Omega g \,\mathrm{d}\mu = \lim_{m\to\infty} \int_\Omega g_m \,\mathrm{d}\mu. \tag{5.42}$$

By the triangle inequality in $L^p(\Omega,\mu)$ (see proposition 1.54), we can estimate

$$\begin{aligned} \Big(\int_\Omega g_m \,\mathrm{d}\mu\Big)^{\frac{1}{p}} &= \Big(\int_\Omega \big(g_m^{1/p}\big)^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}} \\ &\le \sum_{n=0}^m \Big(\int_\Omega |f_n|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}}. \end{aligned} \tag{5.43}$$

Combining (5.42) and (5.43), we get

$$\Big(\int_\Omega g \,\mathrm{d}\mu\Big)^{\frac{1}{p}} \le \sum_{n\in\mathbb{N}} \Big(\int_\Omega |f_n|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}}. \tag{5.44}$$

This concludes the proof. □

Proof of proposition 5.21. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions such that their equivalence classes in $L^p(\Omega,\mu)$ satisfy

$$\sum_{n\in\mathbb{N}} \|f_n\|_{L^p(\Omega,\mu)} < \infty. \tag{5.45}$$

By lemma 5.22, there exists $f_* : \Omega \to \mathbb{R}$ and $g : \Omega \to [0,\infty]$ such that

$$\int_\Omega g \,\mathrm{d}\mu < \infty, \tag{5.46}$$

for every $m \in \mathbb{N}$, $|\sum_{n=0}^m f_n(x)|^p \le g(x)$ and if $g(x) < \infty$, then $f_*(x) = \lim_{m\to\infty} \sum_{n=0}^m f_n(x)$. In view of (5.46) and of the finiteness property of integrable functions (proposition D.31), we have $g(x) < \infty$ almost everywhere on Ω, and thus $f_*(x) = \lim_{m\to\infty} \sum_{n=0}^m f_n(x)$ for almost every $x \in \Omega$. For every $m \in \mathbb{N}$, we have by convexity,

$$\left|\sum_{n=0}^m f_n(x) - f_*(x)\right|^p \le 2^{p-1}\Big(|\sum_{n=0}^m f_n(x)|^p + |f_*(x)|^p\Big) \le 2^p g(x). \tag{5.47}$$

Hence by Lebesgue's dominated convergence theorem (proposition D.50),

$$\lim_{m\to\infty} \int_\Omega \left| \sum_{n=0}^{m} f_m - f_* \right|^p \mathrm{d}\mu = 0. \tag{5.48}$$

We conclude by proposition 5.8 that $L^p(\Omega,\mu)$ is complete. □

The space of polynomials on an interval endowed with the uniform norm is not complete:

Proposition 5.23. *There exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials such that*

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} |P_n(x) - |x|| = 0. \tag{5.49}$$

Proof. We set $P_0(x) := 0$ and define inductively

$$P_{n+1}(x) := P_n(x) + \frac{x^2 - P_n(x)^2}{2}. \tag{5.50}$$

Since,

$$P_{n+1}(x)^2 = \left(1 - \frac{x^2}{2}\right) P_n(x)^2 + \frac{P_n(x)^4 + x^4}{4}. \tag{5.51}$$

by induction, we have for every $n \in \mathbb{N}$ and $x \in [-1,1]$, $0 \le P_n(x) \le |x|$.

We have for each $n \in \mathbb{N}$ and $x \in [-1,1]$,

$$|x| - P_{n+1}(x) = (|x| - P_n(x)) \frac{2 - |x| - P_n(x)}{2} \le \left(1 - \frac{|x|}{2}\right)(|x| - P_n(x)). \tag{5.52}$$

By induction, this implies that

$$0 \le |x| - P_n(x) \le \left(1 - \frac{|x|}{2}\right)^n |x|. \tag{5.53}$$

This shows the uniform convergence. □

6 Contractive mapping theorem

6.1 Fixed point theorem for contractive mappings

Theorem 6.1 (Banach fixed-point theorem)**.** *Let X endowed with d be a metric space and let $f : X \to X$. If there exists $\kappa \in [0,1)$ such that for every $x, y \in X$,*

$$d(f(x), f(y)) \le \kappa d(x, y), \tag{6.1}$$

if X is complete and if $X \ne \emptyset$, then there exists a unique $x_* \in X$ such that $f(x_*) = x_*$. Moreover, for every $n \in \mathbb{N}$ and every $x_0 \in X$,

$$d(f^n(x), x_*) \le \frac{\kappa^n}{1-\kappa} d(x_0, f(x_0)). \tag{6.2}$$

Here we have set $f^0(x) := x$ and $f^{n+1}(x) := f(f^n(x))$.

Proof. Let $x_0 \in X$. For every $n \in \mathbb{N}$, we have by our assumption (6.1)

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \le \kappa d(f^n(x_0), f^{n+1}(x_0)), \tag{6.3}$$

and thus by induction

$$d(f^n(x_0), f^{n+1}(x_0)) \le \kappa^n d(x_0, f(x_0)). \tag{6.4}$$

If $m \in \mathbb{N}$ and $m > n$, we have thus by the triangle inequality and by (6.4),

$$d(f^n(x_0), f^m(x_0)) \le \sum_{\ell=n}^{m-1} d(f^\ell(x_0), f^{\ell+1}(x_0)) \le \sum_{\ell=n}^{m-1} \kappa^\ell d(x_0, f(x_0)). \tag{6.5}$$

Since $\kappa \in [0,1)$, we have

$$\sum_{\ell=n}^{m-1} \kappa^\ell = \frac{\kappa^n - \kappa^m}{1-\kappa} \le \frac{\kappa^n}{1-\kappa} \tag{6.6}$$

and thus

$$d(f^n(x_0), f^m(x_0)) \le \frac{\kappa^n}{1-\kappa} d(x_0, f(x_0)). \tag{6.7}$$

The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is thus a Cauchy sequence. Since X is complete, it converges to some $x_* \in X$ with respect to d.

By (6.7), we have for every $n \in \mathbb{N}$,

$$d(f^n(x_0), x_*) \leq \frac{\kappa^n}{1-\kappa} d(x_0, f(x_0)). \tag{6.8}$$

By the triangle inequality, we also have

$$\begin{aligned} d(x_*, f(x_*)) &\leq d(x_*, f^{n+1}(x_0)) + d(f^{n+1}(x_0), f(x_*)) \\ &\leq d(x_*, f^{n+1}(x_0)) + \kappa d(x_*, f^n(x_0)) \leq \frac{2\kappa^{n+1}}{1-\kappa} d(x_0, f(x_0)), \end{aligned} \tag{6.9}$$

and thus $d(x_*, f(x_*)) = 0$ and hence $f(x_*) = x_*$.

Finally, if $x \in X$ and $f(x) = x$, then

$$d(x, x_*) = d(f(x), f(x_*)) \leq \kappa d(x, x_*), \tag{6.10}$$

and thus since $\kappa < 1$, we have $d(x, x_*) = 0$ and thus $x = x_*$. □

6.2 Inverting linear operators

Proposition 6.2. *Let X endowed with $\|\cdot\|_X$ be a normed space and let $L \in \mathscr{L}(X, X)$. If X is complete and if $\|I - L\|_{\mathscr{L}(X,X)} < 1$, then L is invertible and*

$$\|L^{-1}\|_{\mathscr{L}(X,X)} \leq \frac{1}{1 - \|I - L\|_{\mathscr{L}(X,X)}}. \tag{6.11}$$

Proof. For a given $z \in X$, we define the function $f : X \to X$ for each $x \in X$ by

$$f(x) := x + z - L(x). \tag{6.12}$$

We have for every $x, y \in X$, by linearity of L

$$\begin{aligned} f(x) - f(y) &= x + z - L(x) - (y + z - L(y)) \\ &= x - y - L(x - y) = (I - L)(x - y), \end{aligned} \tag{6.13}$$

and thus,

$$\|f(x) - f(y)\|_X \leq \kappa \|x - y\|_X, \tag{6.14}$$

with

$$\kappa := \|I - L\|_{\mathscr{L}(X,X)} \tag{6.15}$$

By theorem 6.1 with $x_0 = 0$, there exists $x_* \in X$ such that $f(x_*) = x_*$, and thus $L(x_*) = z$. Moreover, we have

$$\begin{aligned} \|x_*\| = \|f(x_0) - x_*\| + \|f(x_0)\| &\leq \frac{\kappa}{1-\kappa} \|x_0 - f(x_0)\| + \|f(x_0)\| \\ &\leq \frac{\kappa}{1-\kappa} \|z\| + \|z\| = \frac{\|z\|}{1-\kappa}. \end{aligned} \tag{6.16}$$

We set $L^{-1}(z) := x_*$ and (6.11) follows from (6.16). □

6.3 Solving ordinary differential equations

Proposition 6.3. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Assume that there exists $L \in [0, \infty)$ such that for every $t \in I$, $x, y \in \mathbb{R}^d$,*

$$|f(t,x) - f(t,y)| \le L|x - y|. \tag{6.17}$$

Then for every $t_0 \in I$ and $x_0 \in \mathbb{R}^d$, there exists a unique function $u : I \to \mathbb{R}^d$ such that for every $t \in I$,

$$u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s. \tag{6.18}$$

Proof. We define the mapping $\Phi : C(I, \mathbb{R}^d) \to C(I, \mathbb{R}^d)$ for $u \in C(I, \mathbb{R}^d)$ and $t \in I$ by

$$\Phi(u)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s. \tag{6.19}$$

For $u, v \in C(I, \mathbb{R}^d)$, we have for each $t \in I$,

$$\Phi(u)(t) - \Phi(v)(t) = \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))\,\mathrm{d}s, \tag{6.20}$$

and thus for each $t \in I$

$$\begin{aligned} |\Phi(u)(t) - \Phi(v)(t)| &\le \int_{[t_0,t]} |f(s, u(s)) - f(s, v(s))|\,\mathrm{d}s \\ &\le L \int_{[t_0,t]} |u(s) - v(s))|\,\mathrm{d}s \\ &\le L \int_{[t_0,t]} e^{\lambda|s-s_0|}\,\mathrm{d}s \sup_{s\in[t_0,t]} e^{-\lambda|t_0-s|}|u(s) - v(s))| \\ &\le L \frac{e^{\lambda|t_0-t|}}{\lambda} \sup_{s\in I} e^{-\lambda|t_0-s|}|u(s) - v(s))|. \end{aligned} \tag{6.21}$$

If we define

$$\|u\|_\lambda := \sup_{s\in I} e^{-\lambda|t_0-s|}|u(s)|, \tag{6.22}$$

then (6.21) reads as

$$\|\Phi(u) - \Phi(v)\|_\lambda \le \frac{L}{\lambda}\|u - v\|_\lambda. \tag{6.23}$$

On the other hand, if I is bounded, we have

$$\nu\|u\|_{\ell^\infty(I)} \le \|u\|_\lambda \le \|u\|_{\ell^\infty(I)}, \tag{6.24}$$

with

$$\nu := \inf\{e^{-\lambda|t-t_0|} \mid t \in I\}. \tag{6.25}$$

The inequalities (6.24) imply that the Cauchy sequences in $C(I, \mathbb{R}^d)$ with respect to $\|\cdot\|_{\ell^\infty(I)}$ and with respect to $\|\cdot\|_\lambda$ are the same. Similarly, convergent sequences in $C(I, \mathbb{R}^d)$ with respect to $\|\cdot\|_{\ell^\infty(I)}$ and with respect to $\|\cdot\|_\lambda$ are the same. Since $C(I, \mathbb{R}^d)$ is complete with respect to $\|\cdot\|_{\ell^\infty(I)}$, it is thus also complete with respect to $\|\cdot\|_\lambda$.

If we take $\lambda > L$, by theorem 6.1, there exists a unique $u \in C(I, \mathbb{R}^d)$ such that $\Phi(u) = u$, that is, (6.18) holds.

In the case where I is unbounded, the proof can be used to construct u on any bounded subinterval of I containing t_0. By uniqueness all these functions coincide on their common domain define a solution u on the whole interval I. □

7 Density and completion

7.1 Density

Definition 7.1. Let X endowed with d be a metric space. A set $D \subseteq A$ is *dense* in A whenever $\bar{D} \supseteq A$.

Proposition 7.2. *Let X endowed with d be a metric space, and let $D \subseteq E \subseteq A \subseteq X$. If D is dense in E and E is dense in A, then D is dense in A.*

Proof. By proposition 3.20 and proposition 3.21, we have $\bar{D} = \bar{\bar{D}} \supseteq \bar{E} \supseteq A$. □

Proposition 7.3. *Let X endowed with d_X and let Y endowed with d_Y be metric spaces. If $A \subseteq X$ is dense in X, if $f : A \to Y$ is uniformly continuous and if Y is complete, then there exists a unique function $\bar{f} : \bar{A} \to Y$ such that $\bar{f}$ is uniformly continuous and $\bar{f}|_A = f$.*

Proof. For every $x \in X$, since the set A is dense in X, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A converging to x with respect to d_X. In particular, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X (proposition 5.2). By proposition 5.4, the $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since the space Y is complete, there exists $y \in Y$ such that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to y with respect to d_Y. We set $\bar{f}(x) := y$.

We prove that $\bar{f}|_A = f$. Indeed, if $x \in A$, by continuity of f, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$, and thus by uniqueness of the limit (proposition 2.5) we have $\bar{f}(x) = f(x)$.

For the uniform continuity of $\bar{f}$, we observe that if $x, y \in X$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in A converging respectively to x and y, we have by the triangle inequality, for every $n \in \mathbb{N}$,

$$d_Y(\bar{f}(x), \bar{f}(y)) \le d_Y(\bar{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), \bar{f}(y)) \tag{7.1}$$

and

$$d_X(x_n, y_n) \le d_X(x_n, x) + d_X(x, y) + d_X(y_n, y). \tag{7.2}$$

Given $\varepsilon > 0$, by uniform continuity of f, there exists $\delta > 0$ such that $z, w \in A$ and $d_X(z, w) \le 2\delta$ implies $d_Y(f(z), f(w)) \le \varepsilon/2$. In view of (7.2), if $d_X(x, y) \le \delta$ and if $n \in \mathbb{N}$ is large enough, then $d_X(x_n, y_n) \le 2\delta$; hence $d_Y(f(x_n), f(y_n)) \le \varepsilon/2$; and thus if $n \in \mathbb{N}$ was taken large enough, $d_Y(\bar{f}(x), \bar{f}(y)) \le \varepsilon$ in view of (7.1).

Finally, for the uniqueness, assume that $\bar{f}_0, \bar{f}_1 : X \to Y$ are continuous and $\bar{f}_0|_A = \bar{f}_1|_A$. For every $x \in X$, there exists by density of A in X a sequence $(x_n)_{n\in\mathbb{N}}$ of points in A converging to x with respect to d_X. One has then by continuity of $\bar{f}_0$ and of $\bar{f}_1$

$$\bar{f}_0(x) = \lim_{n\to\infty} \bar{f}_0(x_n) = \lim_{n\to\infty} \bar{f}_1(x_n) = \bar{f}_1(x), \tag{7.3}$$

and thus $\bar{f}_0 = \bar{f}_1$ on X, which is the announced uniqueness property. □

Proposition 7.4. *Let X endowed with $\|\cdot\|_X$ and Y endowed with $\|\cdot\|_Y$ be normed spaces. If $V \subseteq X$ is a dense linear subspace, if $L \in \mathscr{L}(V,Y)$ and if Y is complete, then there exists a unique $\bar{L} \in \mathscr{L}(X,Y)$ such that $\bar{L}|_V = L$. Moreover, $\|\bar{L}\|_{\mathscr{L}(X,Y)} = \|L\|_{\mathscr{L}(V,Y)}$.*

Proof. Let $\bar{L} : X \to Y$ be given by proposition 7.3.

We claim that $\bar{L}$ is linear. Indeed, if $x \in X$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V converging to x and $t \in \mathbb{R}$. One has then $tx_n \in V$ and $(tx_n)_{n\in\mathbb{N}}$ converges to tx. Hence by continuity of $\bar{L}$,

$$\begin{aligned}\bar{L}(tx) &= \lim_{n\to\infty} \bar{L}(tx_n) = \lim_{n\to\infty} L(tx_n) = \lim_{n\to\infty} tL(x_n)\\ &= t\lim_{n\to\infty} L(x_n) = t\lim_{n\to\infty} \bar{L}(x_n) = t\bar{L}(x).\end{aligned} \tag{7.4}$$

Next, if $x, y \in X$, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in V converging respectively to x and y. Then $(x_n + y_n)_{n\in\mathbb{N}}$ is a sequence in V converging to $x + y$ and

$$\begin{aligned}\bar{L}(x+y) &= \lim_{n\to\infty} \bar{L}(x_n+y_n) = \lim_{n\to\infty} L(x_n+y_n)\\ &= \lim_{n\to\infty} (L(x_n)+L(y_n)) = \lim_{n\to\infty} L(x_n) + \lim_{n\to\infty} L(y_n)\\ &= \bar{L}(x) + \bar{L}(y).\end{aligned} \tag{7.5}$$

Finally, one has immediately, since $\bar{L}|_V = L$,

$$\|L\|_{\mathscr{L}(V,Y)} = \|\bar{L}|_V\|_{\mathscr{L}(V,Y)} \le \|\bar{L}\|_{\mathscr{L}(X,Y)}. \tag{7.6}$$

Moreover, if $x \in X$, letting $(x_n)_{n\in\mathbb{N}}$ be a sequence in V converging to x, one has

$$\|\bar{L}(x)\|_Y = \lim_{n\to\infty} \|\bar{L}(x_n)\|_Y = \lim_{n\to\infty} \|L(x_n)\|_Y \le \|L\|_{\mathscr{L}(V,Y)} \lim_{n\to\infty} \|x_n\|_Y = \|L\|_{\mathscr{L}(V,Y)} \|x\|_X, \tag{7.7}$$

and thus

$$\|\bar{L}\|_{\mathscr{L}(X,Y)} \le \|L\|_{\mathscr{L}(V,Y)}. \tag{7.8}$$

The last conclusion follows from (7.6) and (7.8). □

Proposition 7.5. *Let X endowed with d be a metric space. If the set $U \subseteq X$ is open and if the set $D \subseteq U$ is dense in U, then for every $x \in U$, there exists $y \in D$ and $r \in (0,\infty)$ such that*

$$x \in B(y,r) \subseteq U. \tag{7.9}$$

Proof. Since the set U is open, by definition of open set there exists $r \in (0,\infty)$ such that $B(x,2r) \subseteq U$. Since D is dense in U, there exists $y \in D$ such that $d(y,x) < r$. We then have $x \in B(y,r)$. Moreover, for every $z \in B(y,r)$, we have by the triangle inequality

$$d(z,x) \le d(z,y) + d(y,x) < 2r, \tag{7.10}$$

and thus

$$B(y,r) \subseteq B(x,2r) \subseteq U, \tag{7.11}$$

and the conclusion follows. □

7.2 Examples of dense spaces

7.2.1 Rational linear subspaces

Proposition 7.6. *Let X be a normed space and let $V \subseteq X$. If V is a vector space over $\mathbb{Q}$, then V is dense in $\overline{\operatorname{span}(V)}$.*

Proof. Let $v \in \operatorname{span}(V)$. There exists then $\ell \in \mathbb{N}$, $t_1, \dots, t_\ell \in \mathbb{R}$ and $v_1, \dots, v_\ell \in V$ such that $v = \sum_{i=1}^{\ell} t_i v_i$.

If $\tilde{t}_1, \dots, \tilde{t}_\ell \in \mathbb{R}$, then

$$\left\| v - \sum_{i=1}^{\ell} \tilde{t}_i v_i \right\| \le \sum_{i=1}^{\ell} |\tilde{t}_i - t_i| \|v_i\|. \tag{7.12}$$

For every $\varepsilon \in (0, \infty)$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $\tilde{t}_1, \dots, \tilde{t}_\ell \in \mathbb{Q}$, such that

$$\sum_{i=1}^{\ell} |\tilde{t}_i - t_i| \|v_i\| \le \varepsilon. \tag{7.13}$$

Since by assumption the set V is a linear space over $\mathbb{Q}$, we have $\sum_{i=1}^{\ell} \tilde{t}_i v_i \in V$, and thus we have proved that $v \in \bar{V}$. □

7.2.2 Sequence spaces

Proposition 7.7. *Let Γ be a set, and let $p \in [1, \infty)$. The set $c_c(\Gamma)$ is dense in $\ell^p(\Gamma)$ if and only if either $p < \infty$ or the set Γ is finite.*

The set $c_c(\Gamma)$ was defined in definition 5.15.

Proof. When Γ is finite, then $c_c(\Gamma) = \ell^p(\Gamma)$ and the density is trivial.

We assume that $p < \infty$. Let $f \in \ell^p(\Gamma)$. Given $\delta > 0$, we define the set

$$F_\delta := \{ x \in \Gamma \mid |f(x)| \ge \delta \}. \tag{7.14}$$

By construction, we have

$$\# F_\delta \le \frac{1}{\delta^p} \sum_{x \in F_\delta} |f(x)|^p \le \frac{1}{\delta^p} \sum_{x \in \Gamma} |f(x)|^p = \frac{\|f\|_p^p}{\delta^p}, \tag{7.15}$$

Moreover, for every finite set $F \subseteq \Gamma$, there exists $\delta > 0$ such that

$$\{x \in F \mid |f(x)| \ne 0\} = F \cap F_\delta, \tag{7.16}$$

and thus

$$\sum_{x \in F} |f(x)|^p = \sum_{x \in F \cap F_\delta} |f(x)|^p, \tag{7.17}$$

so that

$$\lim_{\delta\to 0} \sum_{x\in F_\delta} |f(x)|^p = \sum_{x\in\Gamma} |f(x)|^p \tag{7.18}$$

Next, we define $f_\delta : \Gamma \to \mathbb{R}$ for each $x \in \Gamma$ by

$$f_\delta(x) := \begin{cases} f(x) & \text{if } x \in F_\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

By (7.15), we have $f_\delta \in c_c(\Gamma)$. Moreover, we have

$$\|f_\delta - f\|_p^p = \sum_{\Gamma\setminus F_\delta} |f(x)|^p = \sum_{\Gamma} |f(x)|^p - \sum_{x\in F_\delta} |f(x)|^p, \tag{7.20}$$

and thus by (7.18)

$$\lim_{\delta\to 0} \|f_\delta - f\|_p^p = 0. \tag{7.21}$$

This implies that $f \in \overline{c_c(\Gamma)}$.

Finally, let us assume that $p = \infty$ and that $c_c(\Gamma)$ is dense in $\ell^\infty(\Gamma)$. Defining $f : \Gamma \to \mathbb{R}$ by $f(x) = 1$, there exists $g \in c_c(\Gamma)$ such that $\|f - g\|_\infty \le 1/2$. By definition of $\|\cdot\|$, we have for every $x \in \Gamma$,

$$|g(x)| \ge |f(x)| - |f(x) - g(x)| \ge 1 - \|f - g\|_\infty \ge \frac{1}{2} > 0, \tag{7.22}$$

and thus $\Gamma = \{x \in \Gamma \mid g(x) \ne 0\}$ is finite by definition of $c_c(\Gamma)$. □

Proposition 7.8. *Let Γ be a set. The set $c_c(\Gamma)$ is dense in $c_0(\Gamma)$. endowed with $\|\cdot\|_{\ell^\infty(\Gamma)}$.*

Proof. Let $f \in c_0(\Gamma)$. Given $\delta > 0$, we define the set

$$F_\delta := \{x \in \Gamma \mid |f(x)| \ge \delta\}. \tag{7.23}$$

By definition of $c_0(\Gamma)$ (definition 5.12), the set F_δ is finite. Next, we define $f_\delta : \Gamma \to \mathbb{R}$ for each $x \in \Gamma$ by

$$f_\delta(x) := \begin{cases} f(x) & \text{if } x \in F_\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{7.24}$$

We have $f_\delta \in c_c(\Gamma)$ and

$$\|f_\delta - f\|_{\ell^\infty(\Gamma)} = \sup(\{0\} \cup \{|f(x)| \mid x \in F_\delta\}) \le \delta. \tag{7.25}$$

□

7.2.3 Lebesgue spaces

Proposition 7.9. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. For every $p \in [1, \infty)$, the set $L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ is dense in $L^p(\Omega, \mu)$.*

Proof. For every $n \in \mathbb{N}$, we define the function

$$f_n \coloneqq f \mathbb{1}_{A_n}, \tag{7.26}$$

where the set $A_n \subseteq \Omega$ is defined by

$$A_n \coloneqq \bigl\{x \in \Omega \bigm| 2^{-n} \le |f_n(x)| \le 2^n\bigr\}. \tag{7.27}$$

The function f_n is measurable because f is measurable. Since for every $x \in \Omega$, $|f_n(x)| \le 2^n$, we have $f_n \in L^\infty(\Omega, \mu)$. Moreover, by Hölder's inequality and Chebyshev's inequality, we have

$$\begin{aligned}\int_\Omega |f_n| \,\mathrm{d}\mu &\le \Bigl(\int_\Omega |f_n|^p \,\mathrm{d}\mu\Bigr)^{1/p} \mu(A_n)^{1-1/p}\\ &\le \Bigl(\int_\Omega |f_n|^p \,\mathrm{d}\mu\Bigr)^{1/p} \mu\bigl(\{x \in \Omega \mid |f_n(x)|^p \ge 2^{-np}\}\bigr)^{1-1/p}\\ &\le 2^{n(p-1)} \int_\Omega |f_n|^p \,\mathrm{d}\mu < \infty,\end{aligned} \tag{7.28}$$

and thus $f_n \in L^1(\Omega, \mu)$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \|f_n - f\|^p_{L^p(\Omega,\mu)} = \lim_{n\to\infty} \int_{\Omega\setminus A_n} |f|^p \,\mathrm{d}\mu = 0. \tag{7.29}$$

□

Proposition 7.10. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $p \in [1, \infty)$. Let $T \subseteq \mathbb{R}$. and $\mathscr{H} \subseteq \{A \in \Sigma \mid \mu(A) < \infty\}$. If for every $A \in \Sigma$ and ε, there exists $B \in \mathscr{H}$ such that $\mu(A \triangle B) \le \varepsilon$ and if T is dense in $\mathbb{R}$ and countable, then the set*

$$\Bigl\{\sum_{i=1}^n t_i \mathbb{1}_{A_i} \Bigm| n \in \mathbb{N}, t_i \in T \text{ and } A_i \in \mathscr{H}\Bigr\} \tag{7.30}$$

is dense in $L^p(\Omega, \mu)$.

Proof. Since the set T is countable, we can write $T = \{t_n \mid n \in \mathbb{N}\}$, with $(t_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$. For every $\delta > 0$ and $n \in \mathbb{N}$, we have

$$\delta|t| \le \delta|t - t_n| + \delta|t_n|, \tag{7.31}$$

so that if $\delta < 1$ and $|t - t_n| \le \delta|t|/(1-\delta)$, we have $|t - t|_n \le \delta|t_n|$. We have thus,

$$\mathbb{R} = \{0\} \cup \bigcup_{n\in\mathbb{N}} B[t_n, \delta|t_n|]. \tag{7.32}$$

Moreover, for every $n \in \mathbb{N}$ and $y \in B[t_n, \delta|t_n|]$, we have by the triangle inequality

$$(1-\delta)|y-t_n| \le \delta(|t_n|-|y-t_n|) \le \delta|y|, \tag{7.33}$$

Setting $C_n^\delta := B[t_n, \delta|t_n|] \setminus \bigcup_{k=0}^{n-1} C_k^\delta$, we have

$$\mathbb{R} \setminus \{0\} = \bigcup_{n \in \mathbb{N}} C_n^\delta. \tag{7.34}$$

Defining

$$f_m^\delta := \sum_{n=0}^{m} t_n \mathbb{1}_{f^{-1}(C_n^\delta)} \tag{7.35}$$

we have if $\delta < 1$, in view of (7.33),

$$\begin{aligned}\int_\Omega |f - f_m^\delta|^p \,\mathrm{d}\mu &= \sum_{n=0}^{m} \int_{f^{-1}(C_n^\delta)} |f - t_n|^p \,\mathrm{d}\mu + \int_{\Omega \setminus \bigcup_{n=m+1}^\infty f^{-1}(C_n^\delta)} |f| \,\mathrm{d}\mu \\ &= \frac{\delta}{1-\delta} \sum_{n=0}^{m} \int_{f^{-1}(C_n^\delta)} |f|^p \,\mathrm{d}\mu + \int_{\Omega \setminus \bigcup_{n=m+1}^\infty f^{-1}(C_n^\delta)} |f|^p \,\mathrm{d}\mu \\ &\le \frac{\delta}{1-\delta} \int_\Omega |f|^p \,\mathrm{d}\mu + \int_{\Omega \setminus \bigcup_{n=m+1}^\infty f^{-1}(C_n^\delta)} |f|^p \,\mathrm{d}\mu.\end{aligned} \tag{7.36}$$

Given $\varepsilon \in (0, \infty)$, we choose $\delta \in (0, 1)$ such that

$$\frac{\delta}{1-\delta} \int_\Omega |f|^p \,\mathrm{d}\mu \le \frac{\varepsilon}{2}. \tag{7.37}$$

Moreover, by Lebesgue's dominated convergence theorem, there exists $m \in \mathbb{N}$, so that

$$\int_{\Omega \setminus \bigcup_{n=m+1}^\infty f^{-1}(C_n^\delta)} |f| \,\mathrm{d}\mu \le \frac{\varepsilon}{2}. \tag{7.38}$$

Thus by (7.36), we have

$$\int_\Omega |f - f_m^\delta|^p \,\mathrm{d}\mu \le \varepsilon. \tag{7.39}$$

Finally, given a set $B \in \mathscr{H}$, we have

$$\int_\Omega |t_n \mathbb{1}_{f^{-1}(C_n^\delta)} - t_n \mathbb{1}_B|^p \,\mathrm{d}\mu \le |t_n| \,\mu\big(f^{-1}(C_n^\delta) \triangle B\big), \tag{7.40}$$

and we conclude the proof by the triangle inequality. □

Proposition 7.11. *Let $d \in \mathbb{N} \setminus \{0\}$ and let $\mu : \mathscr{B}(\mathbb{R}^d) \to [0, \infty]$ is the Lebesgue measure on $\mathbb{R}^d$ Let $p \in [1, \infty)$. The set*

$$\left\{ \sum_{i=1}^{n} h_i \mathbb{1}_{R_i} \;\middle|\; h_i \in \mathbb{R} \text{ and } R_i \subseteq \mathbb{R}^d \text{ is an open rectangle} \right\} \tag{7.41}$$

is dense in $L^p(\mathbb{R}^d, \mu)$.

Proof. By definition of Lebesgue measure, if $A \subseteq \mathbb{R}^d$, there exist $n \in \mathbb{N}$ and open rectangles $R_1, \dots, R_n \subseteq \mathbb{R}^d$, such that

$$\mu\Big(A \triangle \bigcup_{i=1}^{n} R_i\Big) \le \varepsilon. \tag{7.42}$$

We reach the conclusion by proposition 7.10, taking $\mathscr{H}$ to be the set of open rectangles in $\mathbb{R}^d$. □

Lemma 7.12. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases} \tag{7.43}$$

is smooth on $\mathbb{R}$.

Proof. We define teh function $g := f|_{(0,\infty)}$. We claim that for every $k \in \mathbb{N} \setminus \{0\}$, the function g is k times differentiable on $(0, \infty)$ and that for every $x \in (0, \infty)$,

$$g^{(k)}(x) = P_k(x) e^{-1/x}/x^{2k}, \tag{7.44}$$

where P_k is a polynomial of degree at most $k - 1$. Indeed for $k = 1$, we have

$$g'(x) = \frac{e^{-1/x}}{x^2} \tag{7.45}$$

so that (7.44) holds with $P_1 = 1$. Assuming that the claim holds for $k \in \mathbb{N}$, we have for every $k \in \mathbb{N}$,

$$g^{(k+1)}(x) = (g^{(k)})'(x) = \big(P_k'(x)x^2 + P_k(x) - 2kxP_k(x)\big)\frac{e^{-1/x}}{x^{2k+2}}, \tag{7.46}$$

where the degree of the first factor is at most the degree of P_k increased by one. The claim is thus proved with $P_{k+1}(x) = P_k'(x)x^2 + P_k(x) - 2kxP_k(x)$.

In order to conclude, we note that for every $k \in \mathbb{N}$, $\lim_{x \to 0} g^{(k)}(x) = 0$. □

Proposition 7.13. *Let $d \in \mathbb{N} \setminus \{0\}$, let $\mu : \mathscr{B}(\mathbb{R}^d) \to [0, \infty]$ is the Lebesgue measure on $\mathbb{R}$ and $p \in [1, \infty)$, then $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R})$.*

Here $C^\infty(\mathbb{R}^d)$ is the set of smooth functions $f \in C^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} f = \overline{\mathbb{R}^d \setminus f^{-1}(\{0\})}$ is compact.

Proof of proposition 7.13. In view of proposition 7.11 and of proposition 3.28, it is sufficient to approximate the function $\mathbb{1}_R$ with $R = \prod_{i=1}^d (a_i, b_i)$. In order to do this we define for every $\varepsilon \in (0, \infty)$ the function $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ for each $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ by

$$f_\varepsilon(x) := \begin{cases} \prod_{i=1}^d e^{-\varepsilon/((x_i - a_i)(b_i - x_i))} & \text{if } x \in R, \\ 0 & \text{otherwise} \end{cases} \tag{7.47}$$

In view of lemma 7.12, We have $f_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ and $0 \le f_\varepsilon \le \mathbb{1}_R$, so that by Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |f_\varepsilon - \mathbb{1}_R|^p \,\mathrm{d}\mu = \int_{\mathbb{R}^d} 0 \,\mathrm{d}\mu = 0. \tag{7.48}$$ □

7.3 Separable space

Definition 7.14. Let X endowed with d be a metric space. A set $A \subseteq X$ is separable whenever there exists a countable set $D \subseteq A$ such that D is dense in A.

Proposition 7.15. *Let X endowed with d be a metric space. If $A \subseteq X$ is separable and if $B \subseteq A$, then B is separable.*

Proof. Let $D \subseteq X$ be a dense and countable subset of A. For every $n \in \mathbb{N} \setminus \{0\}$, we have

$$A \subseteq \bigcup_{y \in D} B[y, 1/n]. \tag{7.49}$$

If we consider the set

$$D_n := \{y \in D \mid B[y, 1/n] \cap B \neq \emptyset\}, \tag{7.50}$$

we have

$$B \subseteq \bigcup_{y \in D_n} B[y, 1/n]. \tag{7.51}$$

There exists a countable set $\tilde{D}_n \subseteq B$ such that

$$D_n \subseteq \bigcup_{y \in \tilde{D}_n} B[y, 1/n], \tag{7.52}$$

and thus

$$B \subseteq \bigcup_{y \in \tilde{D}_n} B[y, 2/n]. \tag{7.53}$$

Taking

$$\tilde{D} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \tilde{D}_n, \tag{7.54}$$

we have for every $n \in \mathbb{N}$, by (7.53)

$$B \subseteq \bigcup_{y \in \tilde{D}} B[y, 2/n] \tag{7.55}$$

we conclude that $\tilde{D} \subseteq B$ and $\tilde{D}$ is dense in B; hence the set B is separable. □

Proposition 7.16. *Let X endowed with $\|\cdot\|$ be a normed space. If $E \subseteq X$ is countable, then $\overline{\operatorname{span}(E)}$ is separable.*

Proof. This follows from proposition 7.6 applied to the set of rational finite linear combinations of elements of E. □

Proposition 7.17. *Let Γ be a set. The space $c_0(\Gamma)$ is separable if and only if Γ is countable.*

Proof. Assume that the set Γ is a set. Then $c_c(\Gamma) = \operatorname{span}\{\mathbb{1}_{\{x\}} \mid x \in \Gamma\}$. If Γ is countable, it follows by proposition 7.16 and proposition 7.8 that $c_0(\Gamma)$ is separable.

Conversely, if $c_0(\Gamma)$ is separable, let $D \subseteq c_0(\Gamma)$ be a countable dense set. For every $x \in \Gamma$, there exists $f_x \in D$ such that $\|f_x - \mathbb{1}_{\{x\}}\|_{\ell^\infty(\Gamma)} \le 1/3$. If $x, y \in \Gamma$ and $y \ne x$, then

$$\|f_x - f_y\|_{\ell^\infty(\Gamma)} \ge \|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\|_{\ell^\infty(\Gamma)} - \|f_x - \mathbb{1}_{\{x\}}\|_{\ell^\infty(\Gamma)} - \|f_y - \mathbb{1}_{\{y\}}\|_{\ell^\infty(\Gamma)} \ge \frac{1}{3}. \tag{7.56}$$

This implies that $x \in \Gamma \mapsto f_x \in D$ is injective and therefore Γ is countable. □

Proposition 7.18. *Let Γ be a set and let $p \in [1, \infty)$. The space $\ell^p(\Gamma)$ is separable if and only if Γ is countable.*

Proof. Assume that the set Γ is a set. Then $c_c(\Gamma) = \operatorname{span}\{\mathbb{1}_{\{x\}} \mid x \in \Gamma\}$. If Γ is countable, it follows by proposition 7.16 and proposition 7.7 that $\ell^p(\Gamma)$ is separable.

Conversely, if $\ell^p(\Gamma)$ is separable, let $D \subseteq \ell^p(\Gamma)$ be a countable dense set. For every $x \in \Gamma$, there exists $f_x \in D$ such that $\|f_x - \mathbb{1}_{\{x\}}\|_{\ell^p(\Gamma)} \le 1/3$. If $x, y \in \Gamma$ and $y \ne x$, then

$$\|f_x - f_y\|_{\ell^p(\Gamma)} \ge \|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\|_{\ell^p(\Gamma)} - \|f_x - \mathbb{1}_{\{x\}}\|_{\ell^p(\Gamma)} - \|f_y - \mathbb{1}_{\{y\}}\|_{\ell^p(\Gamma)} \ge \frac{1}{3}. \tag{7.57}$$

This implies that $x \in \Gamma \mapsto f_x \in D$ is injective and therefore Γ is countable. □

Proposition 7.19. *Let Γ be a set. The space $\ell^\infty(\Gamma)$ is separable if and only if Γ is finite.*

Proof. If $\ell^\infty(\Gamma)$ is separable, let $D \subseteq \ell^\infty(\Gamma)$ be a countable dense set. For every $A \subseteq X$, there exists $f_A \in D$ such that $\|f_x - \mathbb{1}_A\|_{\ell^\infty(\Gamma)} \le 1/3$. If $A, B \subseteq \Gamma$ and $A \ne B$, then

$$\|f_A - f_B\|_{\ell^p(\Gamma)} \ge \|\mathbb{1}_A - \mathbb{1}_B\|_{\ell^\infty(\Gamma)} - \|f_A - \mathbb{1}_A\|_{\ell^\infty(\Gamma)} - \|f_B - \mathbb{1}_B\|_{\ell^\infty(\Gamma)} \ge \frac{1}{3}. \tag{7.58}$$

This implies that the map $A \subseteq \Gamma \mapsto f_A \in D$ is injective and therefore Γ is finite. □

Definition 7.20. A measure $\mu : \Sigma \to [0, \infty]$ on X is locally separable whenever there exists a countable set $\mathscr{D} \subseteq \Sigma$ such that for every $A \in X$ with $\mu(A) < \infty$,

$$\inf\{\mu(A \triangle D) \mid D \in \mathscr{D}\} = 0. \tag{7.59}$$

Proposition 7.21. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. For every $p \in [1, \infty)$, then $L^p(\Omega, \mu)$ is separable if and only if μ is locally separable.*

Proof. This follows from proposition 7.10 and definition 7.20.

The converse follows from □

7.4 Completion

Proposition 7.22. *Let X endowed with d be a metric space. There exists a metric space $\bar{X}$ endowed with a metric $\bar{d}$ and a mapping $i : X \to \bar{X}$ such that*

(i) $\bar{X}$ is complete,

(ii) i is an isometry: for every $x, y \in X$, $\bar{d}(i(x), i(y)) = d(x, y)$,

(iii) $i(X)$ is dense in $\bar{X}$.

Moreover $\bar{X}$, $\bar{d}$ and i are unique in the following sense: if $\bar{X}_j$, $\bar{d}_j$ and i_j satisfy for $j \in \{0, 1\}$ the conclusion of the proposition, then there exists $\Phi : X_0 \to X_1$ such that $\bar{d}_1(\Phi(x), \Phi(y)) = \bar{d}_0(x, y)$, $\Phi(X_0) = X_1$ and $i_0 = \Phi \circ i_1$.

Proof. We define

$$\tilde{X} := \{(x_n)_{n\in\mathbb{N}} \mid \text{for each } n \in \mathbb{N},\ x_n \in X \text{ and } (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\}. \tag{7.60}$$

Given $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \tilde{X}$, we have for every $n, m \in \mathbb{N}$,

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m); \tag{7.61}$$

thus by definition of $\tilde{X}$, $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. We set

$$\tilde{d}((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) := \lim_{n\to\infty} d(x_n, y_n). \tag{7.62}$$

We say $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ whenever $\tilde{d}((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0$. By proposition 1.7, we let $\bar{X}$ be the quotient of $\tilde{X}$ by the equivalence relation $\sim$ and let $\bar{d}$ be then defined by $\tilde{d}$. The map i is defined by taking $i(x)$ to be the equivalence class of the constant sequence $(x)_{n\in\mathbb{N}}$; the density of $i(X)$ follows by showing that for every $(x_n)_{n\in\mathbb{N}}$, the sequence $(i(x_n))_{n\in\mathbb{N}}$ approximates the equivalence class of $(x_n)_{n\in\mathbb{N}}$ with respect to $\bar{d}$.

Finally, if $\bar{X}_j$, $\bar{d}_j$ and i_j satisfy for $j \in \{0, 1\}$ the conclusion of the proposition, we observe that $i_1 \circ i_0^{-1} : i_0(X) \to \bar{X}_1$ and $i_0 \circ i_1^{-1} : i_1(X) \to \bar{X}_0$ are uniformly continuous. By proposition 7.3, they have uniformly continuous extensions $\Phi : \bar{X}_0 \to \bar{X}_1$ and $\Psi : \bar{X}_1 \to \bar{X}_0$. $\square$

8 Meagre sets

8.1 Definitions

Definition 8.1. Let X endowed with d be a metric space. A set $A \subseteq X$ is *rare* in X (with respect to d) whenever $\overline{X \setminus \bar{A}} = X$.

In view of the definition of closure of a set, the set $A \subseteq X$ is rare in X if and only if for every $x \in X$ and $\varepsilon \in (0, \infty)$, there exists $y \in B(x, \varepsilon)$ and $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \varepsilon) \setminus A$. Informally, any ball of X contains a ball contained in $X \setminus A$.

Proposition 8.2. *Let X endowed with d be a metric space. A set $A \subseteq X$ is rare in X if and only $\bar{A}$ is rare.*

Proof. Since the closure $\bar{A}$ is closed (proposition 3.20), we have $\bar{\bar{A}} = \bar{A}$ (proposition 3.21). Hence $\overline{X \setminus \bar{A}} = \overline{X \setminus \bar{\bar{A}}}$. The conclusion then follows from definition 8.1. □

Proposition 8.3. *Let X endowed with d be a metric space. If $A \subseteq B \subseteq X$ and if B is rare, then A is rare.*

Proof. Since $A \subseteq B$, we have successively $\bar{A} \subseteq \bar{B}$, $X \setminus \bar{A} \supseteq X \setminus \bar{B}$ and $\overline{X \setminus \bar{A}} \supseteq \overline{X \setminus \bar{B}}$ in view of the monotonicity of the closure (proposition 3.22). □

Proposition 8.4. *Let X endowed with d be a metric space. If the sets $A \subseteq X$ and $B \subseteq X$ are rare, then $A \cup B$ is rare.*

Proof. Assume now that $x \in X$ and $\varepsilon > 0$. Since A is rare, there exists $a \in X$ and $\delta > 0$ such that $B(a, \delta) \subseteq B(x, \varepsilon) \setminus A$. Since B is rare, there exists $b \in X$ and $\eta \in (0, \infty)$ such that $B(b, \eta) \subseteq B(a, \delta) \setminus B \subseteq (B(x, \varepsilon) \setminus A) \setminus B = B(x, \varepsilon) \setminus (A \cup B)$. □

Proposition 8.5. *Let X endowed with d be a metric space. The set $\{x\}$ is rare if and only if the set $\{x\}$ is not open.*

Proof. By proposition 3.12, the set $\{x\}$ is closed, and thus $\overline{\{x\}} = \{x\}$. The $\{x\}$ is open if and only if $X \setminus \{x\}$ is closed. This is in turn equivalent to having $\overline{X \setminus \{x\}} = X \setminus \{x\}$. Finally, having $X \setminus \{x\} \subsetneq \overline{X \setminus \{x\}}$ is equivalent to having $\overline{X \setminus \{x\}} = X$. □

Proposition 8.6. *Let X endowed with $\|\cdot\|$ be a normed space and let $V \subseteq X$ be a closed linear subspace. The set V is rare if and only if $V \neq X$.*

Proof. If V is rare, then $\overline{X \setminus V} = X$ and thus $X \setminus V \neq \emptyset$.

Conversely, assume that $V \neq X$ and let $x \in X \setminus V$. For every $t \in \mathbb{R} \setminus \{0\}$ and $v \in V$, $v + tx \notin V$ and $\|v + tx - v\| = |t|\|x\|$ so that $v + tx \in B[v, |t|\|x\|]$; therefore $\overline{X \setminus V} = X$ so that V is rare. □

Definition 8.7. Let X endowed with d be a metric space. A set $A \subseteq X$ is *meagre* whenever $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$, with $A_n \subseteq X$ rare.

Proposition 8.8. *Let X endowed with d be a metric space and let $B \subseteq X$. If $A \subseteq B$ and if B is meagre, then A is meagre.*

Proposition 8.9. *Let X endowed with d be a metric space and let for every $n \in \mathbb{N}$, the set $A_n \subseteq X$. If for every $n \in \mathbb{N}$, A_n is meagre, then $\bigcup_{n \in \mathbb{N}} A_n$ is meagre.*

Proposition 8.10. *Let X endowed with d be a metric space. A set $A \subseteq X$ is meagre if and only if $A = \bigcup_{n \in \mathbb{N}} A_n$, with $A_n \subseteq X$ rare and disjoint.*

Proof. We assume that A is meagre. By definition 8.7, we have $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$, with $A_n \subseteq X$ rare. Defining

$$\tilde{A}_n := A \cap A_n \setminus \bigcup_{\ell=0}^{n-1} A_k, \tag{8.1}$$

we have $A = \bigcup_{n \in \mathbb{N}} \tilde{A}_n$, with $\tilde{A}_n \subseteq X$ rare and disjoint.

The proof of the converse implication is immediate. □

Theorem 8.11 (Baire category theorem). *Let X endowed with d be a metric space. If $A \subseteq X$ is meagre and if X is complete, then $\overline{X \setminus A} = X$.*

Proof. Let $x \in X$. We are going to prove that $x \in \overline{X \setminus A}$ by proving that $B(x, \varepsilon) \setminus A \neq \emptyset$ for every $\varepsilon \in (0, \infty)$.

We fix $\varepsilon \in (0, \infty)$. By definition of meagre set (definition 8.7), we have $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$, with $A_n \subseteq X$ rare. Since A_0 is rare, in view of definition 8.1 there exists $x_0 \in X$ and ε_0 such that $B[x_0, \varepsilon_0] \subseteq B[x, \varepsilon] \setminus A_0$. Next, we define inductively $x_n \in X$ and $\varepsilon_n \in (0, \infty)$ so that $\varepsilon_{n+1} \leq \varepsilon_n/2$ and $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B[x_n, \varepsilon_n] \setminus A_{n+1}$.

It follows then that if $m \geq n$, $d(x_n, x_m) \leq \varepsilon_n$. The sequence $(x_n)_{n \in \mathbb{N}}$, being a Cauchy sequence, converges to some $x_* \in X$ by our completeness assumption. Moreover, for every $n \in \mathbb{N}$, we have

$$x_* \in B[x_n, \varepsilon_n] \subseteq B[x, \varepsilon] \setminus A_n, \tag{8.2}$$

and thus

$$x_* \in B[x, \varepsilon] \setminus \bigcup_{n \in \mathbb{N}} A_n \subseteq B[x, \varepsilon] \setminus A. \tag{8.3}$$

Since $\varepsilon \in (0, \infty)$ is arbitrary, we have $x \in \overline{X \setminus A}$. □

Proposition 8.12. *Let X be a metric space. If X is countable and not empty, and if for every $x \in X$, the set $\{x\}$ is not open, then X is not complete.*

Proof. Since for every $x \in X$, the set $\{x\}$ is not open, by proposition 8.5, the set $\{x\}$ is rare and thus the set X is meagre. Since the set X is not empty, by Baire's category theorem (theorem 8.11), this implies that the space X is not complete. □

In particular, proposition 8.12 implies that the set $\mathbb{R}$ of real numbers is uncountable.

Proposition 8.13. *Let X endowed with $\|\cdot\|$ be a normed space. If for each $n \in \mathbb{N}$, $X_n \subsetneq X$ is a closed linear subspace and if $X = \bigcup_{n\in\mathbb{N}} X_n$, then X is not complete.*

Proof. For every $n \in \mathbb{N}$, by proposition 8.6 the set $X_n \subsetneq X$ is rare, and thus $X = \bigcup_{n\in\mathbb{N}} X_n$ is meagre. By Baire's category theorem (theorem 8.11), this implies that X is not complete. □

Since finite dimensional spaces are closed (proposition 3.16) and the space of polynomials is the countable union of sets of polynomials with a given bound on the degree, proposition 8.13 implies that there is no norm such that the space of polynomials is complete under that norm.

8.2 Comments

Rare sets are also known as *nowhere dense sets* or *sets of the first category*.

9 Convergence of linear mappings

9.1 Pointwise limit of a bounded sequence

Proposition 9.1. *Let X and Y be normed spaces. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear mappings from X to Y and let $L_* : X \to Y$. If for every $x \in X$, $(L_n(x))_{n\in\mathbb{N}}$ converges to $L_*(x)$ in Y, then L_* is linear and*

$$\|L_*\|_{\mathscr{L}(X,Y)} \le \liminf_{n\to\infty} \|L_n\|_{\mathscr{L}(X,Y)}. \tag{9.1}$$

Proof. The linearity of L_* follows from the linearity of L_n for each $n \in \mathbb{N}$ and in view of proposition 2.10 and proposition 2.11.

Next, by proposition 2.12, for every $x \in \mathbb{R}$, we have

$$\|L_*(x)\|_Y = \lim_{n\to\infty} \|L_n(x)\|_Y = \liminf_{n\to\infty} \|L_n(x)\|_Y \le \liminf_{n\to\infty} \|L_n\|_{\mathscr{L}(X,Y)} \|x\|_X, \tag{9.2}$$

and (9.1) follows. □

9.2 Convergence on a dense set

Proposition 9.2. *Let X and Y be normed spaces. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear mappings from X to Y, let $Z \subseteq X$. If $\operatorname{span} Z$ is dense in X, if for every $z \in Z$, $(L_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence and if $(\|L_n\|_{\mathscr{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded, for every $x \in X$, $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. By linearity, for every $z \in \operatorname{span}(Z)$, $(L_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence. We fix $x \in X$. For every $x \in X$, $z \in \operatorname{span}(Z)$ and $n, m \in \mathbb{N}$, we have

$$L_n(x) - L_m(x) = L_n(x - z) + L_n(z) - L_m(z) + L_m(z - x), \tag{9.3}$$

and therefore, by the triangle inequality

$$\begin{aligned} \|L_n(x) - L_m(x)\|_Y &\le \|L_n(x - z)\|_Y + \|L_n(z) - L_m(z)\|_Y + \|L_m(z - x)\|_Y \\ &\le \big(\|L_n\|_{\mathscr{L}(X,Y)} + \|L_m\|_{\mathscr{L}(X,Y)}\big)\|x - z\|_X + \|L_n(z) - L_m(z)\|_Y. \end{aligned} \tag{9.4}$$

Let $\varepsilon > 0$, since the sequence $(\|L_n\|_{\mathscr{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded, there exists $\delta > 0$ such that for every $n \in \mathbb{N}$,

$$\|L_n\|_{\mathscr{L}(X,Y)}\delta \le \frac{\varepsilon}{4}. \tag{9.5}$$

Since Z is dense in X, there exists $z \in Z$ such that $\|x - z\|_X \le \delta$. Next, since $(L_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence, we have for $m, n \in \mathbb{N}$ large enough

$$\|L_n(z) - L_m(z)\|_Y \le \frac{\varepsilon}{2}, \tag{9.6}$$

and it follows from (9.4), (9.5) and (9.6) thus that $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. □

9.3 Uniform boundedness principle

9.3.1 Uniform boundedness of families of operators

Theorem 9.3. *Let X endowed with $\|\cdot\|_X$ and Y endowed with $\|\cdot\|_Y$ be normed spaces and let $\mathscr{F} \subseteq \mathscr{L}(X,Y)$. If*

$$\sup\{\|L\|_{\mathscr{L}(X,Y)} \mid L \in \mathscr{F}\} = \infty. \tag{9.7}$$

then the set

$$\{x \in X \mid \sup\{\|L(x)\|_Y \mid L \in \mathscr{F}\} < \infty\} \tag{9.8}$$

is meagre.

Proof. We write

$$\left\{x \in X \,\middle|\, \sup\left\{\|L(x)\|_Y \,\middle|\, L \in \mathscr{F}\right\} < \infty\right\} = \bigcup_{n \in \mathbb{N}} A_n, \tag{9.9}$$

where for every $n \in \mathbb{N}$, we have defined the set

$$A_n \coloneqq \{x \in X \mid \text{for every } L \in \mathscr{F}, \|L(x)\|_Y \le n\}. \tag{9.10}$$

Since every $L \in \mathscr{F}$ is continuous, the set A_n is closed.

We claim that the set A_n is rare. Since A_n is closed, it is sufficient to prove that $X \setminus A_n$ is dense in X. By assumption (9.7), there exists $L \in \mathscr{F}$ and $v \in B[0,\varepsilon]$ such that

$$\|L(v)\|_Y > n, \tag{9.11}$$

Given $x \in X$, we have for every $v \in B[0,\varepsilon]$, by linearity of L,

$$L(v) = \frac{L(v-x) + L(v+x)}{2}, \tag{9.12}$$

and thus

$$\|L(v)\|_Y \le \frac{\|L(x-v) + L(x+v)\|_Y}{2}. \tag{9.13}$$

and thus by (9.11), either $x + v \notin A_n$ or $x - v \notin A_n$, so that $B[x,\varepsilon] \setminus A_n \ne \emptyset$. This implies that the set A_n is rare.

The set $\bigcup_{n \in \mathbb{N}} A_n$ is thus meager by definition, which is the conclusion. □

Theorem 9.4. *Let X endowed with $\|\cdot\|_X$ and Y endowed with $\|\cdot\|_Y$ be normed space and let $\mathscr{F} \subseteq \mathscr{L}(X,Y)$. If X is complete and if*

$$\sup_{L \in \mathscr{F}} \|L\|_{\mathscr{L}(X,Y)} = \infty. \tag{9.14}$$

then there exists $x \in X$ such that

$$\sup\{\|L(x)\|_Y \mid L \in \mathscr{F}\} = \infty. \tag{9.15}$$

Proof of theorem 9.4. We let

$$A \coloneqq \{x \in X \mid \sup\{\|L(x)\|_Y \mid L \in \mathscr{F}\} < \infty\}. \tag{9.16}$$

By theorem 9.3, A is meagre. Since X is complete, we have by theorem 8.11 $A \ne X$; we take then $x \in X \setminus A$. □

Direct proof of theorem 9.4. We consider a sequence $(L_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$. We fix $\kappa \in (0,1)$ and $\varepsilon \in (0,1)$. We let $x_0 = 0$. Assuming that x_n is defined for some $n \in \mathbb{N}$, there exists $v_n \in X$ such that $\|v_n\|_X = \kappa^n$ and $\|L_n(v_n)\|_Y \ge (1-\varepsilon)\|L_n\|_{\mathscr{L}(X,Y)}\|v_n\|_X$. We then have

$$\|L_n(v_n)\|_Y \le \frac{\|L_n(x_n + v_n)\|_Y + \|L_n(x_n - v_n)\|_Y}{2}. \tag{9.17}$$

Taking either $x_{n+1} \coloneqq x_n + v_n$ or $x_{n+1} \coloneqq x_n - v_n$, can ensure that

$$\|L_n(x_{n+1})\|_Y \ge (1-\varepsilon)\|L_n\|_{\mathscr{L}(X,Y)}\kappa^n; \tag{9.18}$$

we also have

$$\|x_{n+1} - x_n\|_X = \kappa^n. \tag{9.19}$$

By the triangle inequality, we have thus for $m, n \in \mathbb{N}$ satisfying $m > n$,

$$\|x_m - x_n\|_X \le \sum_{\ell=n}^{m-1} \|x_{\ell+1} - x_\ell\|_X = \frac{\kappa^n - \kappa^m}{1-\kappa} \tag{9.20}$$

Since $\sum_{n\in\mathbb{N}} \kappa^n < \infty$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence; by completeness of X, $(x_n)_{n\in\mathbb{N}}$ converges to some $x_* \in X$. We then have

$$\begin{aligned} \|L_n(x_*)\|_Y &\ge \|L_n(x_{n+1})\|_Y - \|L_n(x_{n+1} - x_*)\|_Y \\ &\ge \|L_n\|_{\mathscr{L}(X,Y)}\Bigl((1-\varepsilon)\kappa^n - \sum_{\ell \ge n+1} \kappa^\ell\Bigr) \\ &= \|L_n\|_{\mathscr{L}(X,Y)}\kappa^n\Bigl(\frac{1-2\kappa}{1-\kappa} - \varepsilon\Bigr), \end{aligned} \tag{9.21}$$

since

$$\sum_{\ell \ge n+1} \kappa^k = \frac{\kappa^{n+1}}{1-\kappa}. \tag{9.22}$$

Choosing $\kappa \in (0, 1/2)$ and ε such that $\varepsilon < \frac{1-2\kappa}{1-\kappa}$ and $(L_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|L_n\|_{\mathscr{L}(X,Y)}\kappa^n = \infty, \tag{9.23}$$

we reach the conclusion. □

9.3.2 Uniform boundedness of sequences

Proposition 9.5. *Let X and Y be normed spaces. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear mappings from X to Y. If X is complete and if for every $x \in X$, $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, then the sequence $(\|L_n\|_{\mathscr{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded.*

Proof. Since $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$, $(L_n(x))_{n\in\mathbb{N}}$ is a bounded sequence, and thus by the uniform boundedness principle (theorem 9.4), the sequence $(\|L_n\|_{\mathscr{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded. □

9.3.3 Application to Fourier series

Proposition 9.6. *Define for $f \in C_{\mathrm{per}}([0,1])$ and $n \in \mathbb{N}$*

$$S_n(f)(x) = \sum_{k=-n}^{n} e^{2\pi i k x} \int_0^1 f(t) e^{-2\pi i k t} \,\mathrm{d}t, \tag{9.24}$$

then the set

$$\big\{ f \in C_{\mathrm{per}}([0,1]) \;\big|\; (S_n f(0))_{n\in\mathbb{N}} \text{ is bounded} \big\} \tag{9.25}$$

is meagre in $C_{\mathrm{per}}([0,1])$ endowed with $\|\cdot\|_\infty$.

Proof. We define

$$\ell_n(f) = S_n f(0). \tag{9.26}$$

We compute

$$\ell_n(f) = \int_0^1 D_n(x) f(x) \,\mathrm{d}x, \tag{9.27}$$

where $D_n : \mathbb{R} \to \mathbb{R}$ is the Dirichlet kernel defined as

$$D_n(f)(x) = \begin{cases} \frac{\sin((2n+1)\pi x)}{\sin(\pi x)} & \text{if } x \notin \mathbb{Z}, \\ 2n+1 & \text{otherwise.} \end{cases} \tag{9.28}$$

We have

$$\|\ell_n\|_{\mathscr{L}(C_{\mathrm{per}}([0,1]),\mathbb{R})} = \int_0^1 |D_n(x)| \,\mathrm{d}x. \tag{9.29}$$

We estimate

$$\int_0^1 |D_n(x)| \,\mathrm{d}x \ge \int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} |D_n(x)| \,\mathrm{d}x = \frac{1}{\pi} \int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} \frac{1}{\sin(\pi x)} \,\mathrm{d}x + \int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} \frac{h'((2n+1)\pi x)}{\sin(\pi x)} \,\mathrm{d}x, \tag{9.30}$$

with

$$h(x) := \int_0^x |\sin(t)| - \frac{2}{\pi}\, \mathrm{d}t. \tag{9.31}$$

We compute directly

$$\int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} \frac{1}{\sin(\pi x)}\, \mathrm{d}x = \frac{2}{\pi} \ln \frac{1}{\tan \frac{\pi}{2(2n+1)}}. \tag{9.32}$$

Next, since

$$\frac{h(2n\pi)}{\sin(\pi 2n/(2n+1))} = \frac{h(\pi)}{\sin(\pi/(2n+1)}. \tag{9.33}$$

by integration by parts we have

$$\int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} \frac{h'((2n+1)\pi x)}{\sin(\pi x)}\, \mathrm{d}x = \frac{1}{(2n+1)\pi} \int_{\frac{1}{2n+1}}^{\frac{2n}{2n+1}} \frac{h((2n+1)\pi x)}{\sin(\pi x)^2} \cos(\pi x)\, \mathrm{d}x, \tag{9.34}$$

which remains bounded as $n \to \infty$.

□

Part III

Approximation and compactness

10 Bases in Hilbert spaces

10.1 Orthonormal sets and bases in inner product spaces

Definition 10.1. Let X be endowed with the inner product $(\cdot\,|\,\cdot)$. A set $E \subset X$, is an *orthonormal set* (with respect to $(\cdot|\cdot)$) whenever for every $x \in E$, $(x\,|\,x) = 1$ and for every $y \in E \setminus \{x\}$, one has $(x\,|\,y) = 0$ if $x \neq y$.

In particular, $\emptyset$ is always an orthonormal set and $\{x\}$ is an orthonormal set if and only if $\|x\| = 1$.

Proposition 10.2. *Let X be endowed with the inner product $(\cdot\,|\,\cdot)$. If $E \subset X$ is a finite orthonormal set then*

(i) for every $x \in X$,

$$\Big\|\sum_{e\in E}(x\,|\,e)e\Big\|^2 = \sum_{e\in E}|(x\,|\,e)|^2, \tag{10.1}$$

(ii) for every $x \in X$ and $y \in \operatorname{span} E$,

$$\Big(x - \sum_{e\in E}(x\,|\,e)e\,\Big|\,y\Big) = 0, \tag{10.2}$$

(iii) for every $x \in X$

$$\|x\|^2 = \Big\|x - \sum_{e\in E}(x\,|\,e)e\Big\|^2 + \sum_{e\in E}|(x\,|\,e)|^2, \tag{10.3}$$

(iv) for every $x \in X$ and $y \in \operatorname{span} E$,

$$\Big\|x - \sum_{e\in E}(x\,|\,e)e\Big\| \le \|x - y\|. \tag{10.4}$$

Proof. First, since E is an orthonoral set, we have by the Pythagorean identity (proposition 1.18),

$$\Big\|\sum_{e\in E}(x\,|\,e)e\Big\|^2 = \sum_{e\in E}\|(x\,|\,e)e\|^2 = \sum_{e\in E}|(x\,|\,e)|^2, \tag{10.5}$$

so that (i) holds.

If $y \in E$, by linearity of the inner product

$$\left(x - \sum_{e \in E} (x \,|\, e) e \,\middle|\, y\right) = (x \,|\, y) - \sum_{e \in E} (x \,|\, e)(e \,|\, y) = (x \,|\, y) - (x \,|\, y) = 0. \tag{10.6}$$

In particular (10.2) holds for $y \in E$. By anti-linearity, of the inner product, (10.2) is valid for every $y \in \operatorname{span} E$ and thus (ii) holds.

Next, we have in view of (ii), of the Pythagorean identity (proposition 1.18) and of (i),

$$\begin{aligned} \|x\|^2 &= \left\| \left(x - \sum_{e \in E} (x \,|\, e) e\right) + \sum_{e \in E} (x \,|\, e) e \right\|^2 \\ &= \left\| x - \sum_{e \in E} (x \,|\, e) e \right\|^2 + \left\| \sum_{e \in E} (x \,|\, e) e \right\|^2 \\ &= \left\| x - \sum_{e \in E} (x \,|\, e) e \right\|^2 + \sum_{e \in E} |(x \,|\, e)|^2, \end{aligned} \tag{10.7}$$

from which (iii) follows.

Finally, if $y \in \operatorname{span} E$, then $y = \sum_{e \in E} t_e e$, and

$$\begin{aligned} \left\| x - \sum_{e \in E} t_e e \right\|^2 &= \left\| \left(x - \sum_{e \in E} (x \,|\, e) e\right) + \sum_{e \in E} (t_e - (x \,|\, e)) e \right\|^2 \\ &= \left\| x - \sum_{e \in E} (x \,|\, e) e \right\| + \sum_{e \in E} |t_e - (x \,|\, e)|^2, \end{aligned} \tag{10.8}$$

from which (iv) follows. □

Proposition 10.3 (Bessel's inequality)**.** *Let X be endowed with the inner product $(\cdot|\cdot)$. If $E \subset X$ is an orthonormal set then for every $x \in X$,*

$$\sum_{e \in E} |(x \,|\, e)|^2 \le \|x\|^2, \tag{10.9}$$

Proof. For every finite set $F \subseteq E$, F is an orthonormal set in view of definition 10.1. We have thus by proposition 10.2 (iii),

$$\sum_{e \in F} |(x \,|\, e)|^2 \le \|x\|^2. \tag{10.10}$$

Hence, in view of the definition of infinite sum (definition C.1), we have

$$\sum_{e \in E} |(x \,|\, e)|^2 = \sup \left\{ \sum_{e \in F} |(x \,|\, e)|^2 \,\middle|\, F \subseteq E \text{ is finite} \right\} \le \|x\|^2. \tag{10.11}$$

□

Definition 10.4. An orthonormal set $E \subset X$ is an *orthonormal basis*, whenever for every $x \in X$, $e \in E \mapsto (x \,|\, e)e$ is unconditionnally summable and

$$x = \sum_{e \in E} (x \,|\, e) e. \tag{10.12}$$

Proposition 10.5. *Let X be endowed with the inner product $(\cdot \,|\, \cdot)$. If $E \subseteq X$ is an orthonormal set, then the following are equivalent*

(i) E is an orthonormal basis,

(ii) $\overline{\operatorname{span}(E)} = X$,

(iii) (Parseval identity) for every $x \in X$,

$$\sum_{e \in E} |(x \,|\, e)|^2 = \|x\|^2. \tag{10.13}$$

Moreover, the previous equivalent assertions imply

(iv) $\{x \in X \mid \text{for every } e \in E, (x \,|\, e) = 0\} = \{0\}$.

As a consequence of item (ii), if $E \subset Y \subset X$ is an orthonormal set and Y is a linear space, E is an orthonormal basis of X if and only if E is an orthonormal basis of Y and Y is dense in X.

Proof of proposition 10.5. Assuming that (i) hold and let $x \in X$. By definition of orthonormal basis (definition 10.4) and of unconditional summability (definition C.9), for every $\varepsilon > 0$, there exists a finite set $F \subseteq E$ such that

$$\Big\| x - \sum_{e \in F} (x \,|\, e) e \Big\| \le \varepsilon. \tag{10.14}$$

Letting

$$y := \sum_{e \in F} (x \,|\, e) e, \tag{10.15}$$

we have $y \in B[x, \varepsilon] \cap \operatorname{span} E$, and thus $B[x, \varepsilon] \cap \operatorname{span} E \ne \emptyset$. Since $\varepsilon > 0$ is arbitrary, this implies that $x \in \overline{\operatorname{span}(E)}$, and thus (ii) holds.

If (ii) holds, then for every $\varepsilon > 0$, there exists $y \in \operatorname{span}(E)$ such that

$$\|x - y\| \le \varepsilon. \tag{10.16}$$

By proposition 10.2 (iv), if $F \subseteq E$ is finite and $y \in \operatorname{span}(F)$, we have

$$\Big\| x - \sum_{e \in F} (x \,|\, e) e \Big\| \le \|x - y\| \le \varepsilon, \tag{10.17}$$

and thus (i) holds.

Assume now again that (i) holds. If the set $F \subseteq E$ is finite then by orthogonality we get

$$\Big\|\sum_{e\in F}(x\,|\,e)e\Big\|^2 = \sum_{e\in F}|(x\,|\,e)|^2. \tag{10.18}$$

On the other hand, by the triangle inequality, we have

$$\Big\|\sum_{e\in F}(x\,|\,e)e\Big\| \ge \|x\| - \Big\|\sum_{e\in F}(x\,|\,e)e - x\Big\|. \tag{10.19}$$

By assumption, the second term in the right-hand side of (10.19) can be made arbitrarily small by taking the set F to be large enough, and it follows thus from (10.18) and (10.19) that

$$\sum_{e\in E}|(x\,|\,e)|^2 \ge \|x\|^2. \tag{10.20}$$

In view of proposition 10.3 and of (10.20), (iii) holds.

Assume that (iii) holds. Let $x \in X$. Given a finite set $F \subset E$, we let $y := x - \sum_{e\in F}(x\,|\,e)e$,

$$\Big\|x - \sum_{e\in F}(x\,|\,e)e\Big\|^2 = \|y\|^2 = \sum_{e\in E}|(y\,|\,e)|^2 = \sum_{e\in E}|(x\,|\,e)|^2 - \sum_{e\in F}|(x\,|\,e)|^2, \tag{10.21}$$

from which it follows by definition of unconditional convergence (definition C.1) that $e \in\mapsto (x\,|\,e)e$ is unconditionally summable and that

$$x = \sum_{e\in E}(x\,|\,e)e, \tag{10.22}$$

and thus (i) holds in view of definition 10.4.

Finally, assuming that (i) holds, if $x \in X$ and for every $e \in E$, $(x \mid e) = 0$, then by definition of orthonormal basis, we have

$$x = \sum_{e\in E}(x\,|\,e)e = 0, \tag{10.23}$$

and thus $x = 0$, that is, (iv) holds. □

Proposition 10.6. *Let X be endowed with the inner product $(\cdot\,|\,\cdot)$ and let $A \subseteq X$. If $E \subseteq X$ is an orthonormal set. If* span(A) *is dense in X, then the following are equivalent*

(i) E is an orthonormal basis,

(ii) for every $x \in A$, $e \in E \mapsto (x\,|\,e)e$ is unconditionnally summable and

$$x = \sum_{e\in E}(x\,|\,e)e. \tag{10.24}$$

(iii) (Parseval identity) for every $x \in A$,

$$\sum_{e\in E} |(x\,|\,e)|^2 = \|x\|^2. \tag{10.25}$$

Proof. We prove that (ii) implies (i). By linearity, we know that for every $y \in \operatorname{span}(A)$, $e \in E \mapsto (y\,|\,e)e$ is unconditionnally summable and

$$y = \sum_{e\in E} (y\,|\,e)e. \tag{10.26}$$

Given $x \in X$ and $y \in A$, and a finite set $F \subseteq E$, we have

$$\begin{aligned} \Bigl\|\sum_{e\in F}(x\,|\,e)e - x\Bigr\| &\le \Bigl\|\sum_{e\in F}(x\,|\,e)e - \sum_{e\in F}(y\,|\,e)e\Bigr\| + \Bigl\|\sum_{e\in F}(y\,|\,e)e - y\Bigr\| + \|y-x\| \\ &\le 2\|y-x\| + \Bigl\|\sum_{e\in F}(y\,|\,e)e - y\Bigr\|, \end{aligned} \tag{10.27}$$

which can be made arbitrarily small by our assumptions on A. □

Proposition 10.7. *If X endowed with an inner product $(\cdot\,|\,\cdot)$ is separable, then there exists a set $E \subseteq X$ such that X is an orthonormal basis and E is countable.*

The proof of proposition 10.7 follows the classical Gram–Schmidt orthonormalisation algorithm.

Proof of proposition 10.7. Let $A = \{a_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X given by definition of separable space (definition 7.14). We set $E_{-1} := \emptyset$, so that $\operatorname{span}(E_{-1}) = \{0\}$. For every $n \in \mathbb{N} \cup \{-1\}$, we assume that we have a finite set $E_n \subseteq X$ such that $\operatorname{span}(\{a_1, \dots, a_n\}) = \operatorname{span}(E_n)$. If $a_{n+1} \in \operatorname{span}(E_n)$, we set $E_{n+1} := E_n$; otherwise we set

$$E_{n+1} := E_n \cup \left\{ \frac{a_{n+1} - \sum_{e\in E_n}(x\,|\,e_n)e_n}{\|a_{n+1} - \sum_{e\in E_n}(x\,|\,e_n)e_n\|} \right\}. \tag{10.28}$$

In order to conclude, we define

$$E := \bigcup_{n\in\mathbb{N}} E_n. \tag{10.29}$$

Since $E_n \subset E_{n+1}$ is orthonormal, E is orthonormal. Moreover, we have

$$\operatorname{span}(E) = \operatorname{span}(A) \supseteq A, \tag{10.30}$$

so that

$$X = \bar{A} \subseteq \overline{\operatorname{span}(E)} \subseteq X, \tag{10.31}$$

and hence E is an orthonormal basis in view of proposition 10.5. □

10.2 Examples of Fourier bases

Proposition 10.8. *Define for $k \in \mathbb{Z}$, $e_k : [0,1] \to \mathbb{C}$ by*

$$e_k(x) := e^{2\pi i k x}. \tag{10.32}$$

The set $\{e_k\}_{k\in\mathbb{Z}}$ is an othonormal basis of $L^2((0,1),\mathbb{C})$.

Proof. We first consider the case where $f = \mathbb{1}_{(a,b)}$. We have

$$|(e_k\,|\,f)|^2 = \int_{(0,1)}\int_{(0,1)} \overline{f(y)} f(x) e^{2\pi i k(y-x)}\,\mathrm{d}x\,\mathrm{d}y, \tag{10.33}$$

and thus

$$\begin{aligned}\sum_{k=-n}^{n}\Bigl(1-\frac{|k|}{n}\Bigr)|(e_k\,|\,f)|^2 &= \int_{(0,1)}\int_{(0,1)} \overline{f(y)} f(x) F_n(y-x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{(a,b)}\int_{(a,b)} F_n(y-x)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{10.34}$$

where the Fejér kernel $F_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_n(z) := \sum_{k=-n}^{n}\Bigl(1-\frac{|k|}{n}\Bigr)e^{2\pi i k z} = \frac{1}{n}\Bigl(\frac{\sin(\pi n z)}{\sin(\pi z)}\Bigr)^2. \tag{10.35}$$

It follows then that

$$\int_{(a,b)} F_n(y-x)\,\mathrm{d}y \le \int_{(0,1)} F_n(y-x)\,\mathrm{d}y = 1, \tag{10.36}$$

and that for every $x \in (0,1)\setminus[a,b]$,

$$\begin{aligned}0 \le \int_{(a,b)} F_n(y-x)\,\mathrm{d}y &\le \int_{(a,b)} \frac{1}{n\sin(\pi(x-y))^2}\,\mathrm{d}y\\ &= \frac{1}{n\pi}\Bigl(\frac{\cos(\pi(x-b))}{\sin(\pi(x-b))} - \frac{\cos(\pi(x-a))}{\sin(\pi(x-a))}\Bigr)\\ &= \frac{\sin(\pi(b-a))}{n\pi\sin(\pi(x-b))\sin(\pi(x-a))},\end{aligned} \tag{10.37}$$

and thus

$$\lim_{n\to\infty}\int_{(a,b)} F_n(y-x)\,\mathrm{d}y = 0. \tag{10.38}$$

It follows then also that for every $x \in (a,b)$,

$$\begin{aligned}&\lim_{n\to\infty}\int_{(a,b)} F_n(y-x)\,\mathrm{d}y\\ &\quad= \int_{(0,1)} F_n(y-x)\,\mathrm{d}y\\ &\qquad - \lim_{n\to\infty}\int_{(0,a)} F_n(y-x)\,\mathrm{d}y - \lim_{n\to\infty}\int_{(b,1)} F_n(y-x)\,\mathrm{d}y\\ &\quad= 1,\end{aligned} \tag{10.39}$$

and thus by Lebesgue's dominated convergence theorem, and by the monotone convergence theorem for sums

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|(e_k\,|\,f)|^2 &= \lim_{n\to\infty}\sum_{k=-n}^{n}\Bigl(1-\frac{|k|}{n}\Bigr)|(e_k\,|\,f)|^2\\ &= \lim_{n\to\infty}\int_{(a,b)}\int_{(a,b)}F_n(y-x)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b |f|^2.\end{aligned} \tag{10.40}$$

We conclude by proposition 7.11 and by proposition 10.6. □

10.3 Orthonormal sets and bases in complete spaces

Proposition 10.9. *Let X be endowed with the inner product $(\cdot\,|\,\cdot)$. If $E\subseteq X$ is an orthonormal set and if X is complete, then for every $f\in\ell^2(E)$, $e\in E\mapsto f(e)e$ is unconditionally summable and*

$$\Bigl\|\sum_{e\in E}f(e)e\Bigr\|^2 = \sum_{e\in E}|f(e)|^2, \tag{10.41}$$

and for every $d\in E$,

$$\Bigl(d\,\Big|\,\sum_{e\in E}f(e)e\Bigr) = f(d). \tag{10.42}$$

Proof. For every finite set $F\subseteq E$, we have by orthogonality

$$\Bigl\|\sum_{e\in F}f(e)e\Bigr\|^2 = \sum_{e\in F}|f(e)|^2. \tag{10.43}$$

In particular, if $F\subseteq E$ and $F\cap F_0=\emptyset$, we have

$$\Bigl\|\sum_{e\in F}f(e)e\Bigr\|^2 \le \sum_{e\in E\setminus F_0}|f(e)|^2 = \sum_{e\in E}|f(e)|^2 - \sum_{e\in F_0}|f(e)|^2. \tag{10.44}$$

By definition of infinite sums (definition C.1), for every $\varepsilon>0$, there exists a finite set such that $F_0\subseteq E$ and

$$\sum_{e\in F_0}|f(e)|^2 \ge \sum_{e\in E}|f(e)|^2 - \varepsilon^2, \tag{10.45}$$

and thus if the set $F\subseteq E\setminus F_0$ is finite, we have by (10.44) and (10.45),

$$\Bigl\|\sum_{e\in F}f(e)e\Bigr\| \le \varepsilon. \tag{10.46}$$

Since X is complete, this implies by proposition C.18 that $e \in E \mapsto f(e)e$ is unconditionnally summable.

By proposition 10.2 10.1

$$
\Bigl|\sum_{e\in E}|f(e)|^2 - \Bigl\|\sum_{e\in E} f(e)e\Bigr\|^2\Bigr| \le \Bigl|\Bigl(\sum_{e\in E}|f(e)|^2\Bigr) - \Bigl(\sum_{e\in F}|f(e)|^2\Bigr)\Bigr| + \Bigl|\Bigl\|\sum_{e\in F} f(e)e\Bigr\|^2 - \Bigl\|\sum_{e\in E} f(e)e\Bigr\|^2\Bigr|, \tag{10.47}
$$

from which (10.41) follows. □

Theorem 10.10. *If X is endowed with an inner product $(\cdot\,|\,\cdot)$, if X is complete and if E is an orthogonal basis, then the mapping $T : X \to \ell^2(E)$ defined for $x \in X$ and $e \in E$ by $(T(x))(e) = (x\,|\,e)$ is a surjective isometry.*

Proof. This follows from the definition of orthonormal basis (definition 10.4) and from proposition 10.9. □

Proposition 10.11. *Let X be endowed with the inner product $(\cdot\,|\,\cdot)$ and let $E \subseteq X$ be an othonormal set. If X is complete then*

(i) for every $x \in X$, $e \in E \mapsto (x\,|\,e)e$ is unconditionally summable,

(ii) for every $x \in X$,

$$
\Bigl\|\sum_{e\in E}(x\,|\,e)e\Bigr\|^2 = \sum_{e\in E}|(x\,|\,e)|^2, \tag{10.48}
$$

(iii) for every $x \in X$ and $y \in \overline{\operatorname{span} E}$,

$$
\Bigl(x - \sum_{e\in E}(x\,|\,e)e \Bigm| y\Bigr) = 0, \tag{10.49}
$$

(iv) for every $x \in X$,

$$
\Bigl\|x - \sum_{e\in E}(x\,|\,e)e\Bigr\|^2 + \sum_{e\in E}|(x\,|\,e)|^2 \le \|x\|^2, \tag{10.50}
$$

(v) for every $x \in X$ and $y \in \overline{\operatorname{span} E}$,

$$
\Bigl\|x - \sum_{e\in E}(x\,|\,e)e\Bigr\| \le \|x - y\|. \tag{10.51}
$$

Proof. In view of proposition 10.3, we have

$$\sum_{e \in E} |(x \,|\, e)|^2 < \infty, \tag{10.52}$$

and thus by proposition 10.9 $e \in E \mapsto (x \,|\, e)e$ is unconditionnally summable. Assertion (ii) then follows.

By proposition 10.2 (ii), we have if $y \in E$ and if $F \subseteq E$ is a finite set such that $y \in F$,

$$\Big(x - \sum_{e \in E} (x \,|\, e)e \,\Big|\, y\Big) = \Big(\sum_{e \in E} (x \,|\, e)e - \sum_{e \in F} (x \,|\, e)e \,\Big|\, y\Big), \tag{10.53}$$

and thus by the Cauchy–Schwarz inequality

$$\Big|\Big(x - \sum_{e \in E} (x \,|\, e)e \,\Big|\, y\Big)\Big| \le \Big\|\sum_{e \in E} (x \,|\, e)e - \sum_{e \in F} (x \,|\, e)e\Big\| \|y\|. \tag{10.54}$$

By the unconditionnal convergence, of $e \mapsto (x \,|\, e)e$, the right-hand side of (10.54) can be made arbitrarily small, and thus (10.49) for every $y \in E$. By linearity and by continuity of the inner product, (10.49) holds for every $y \in \overline{\operatorname{span}(E)}$. We have thus proved (iii).

The assertion (iv) follows from (iii), proposition 1.18 and proposition C.19.

Finally, given $y \in \operatorname{span}(E)$. If $F \subseteq E$ is finite and if $y \in \operatorname{span}(E)$, then by the triangle inequality and by proposition 10.2 (v)

$$\begin{aligned} \Big\|x - \sum_{e \in E} (x \,|\, e)e\Big\| &\le \Big\|x - \sum_{e \in F} (x \,|\, e)e\Big\| + \Big\|\sum_{e \in F} (x \,|\, e)e - \sum_{e \in E} (x \,|\, e)e\Big\| \\ &\le \|x - y\| + \Big\|\sum_{e \in F} (x \,|\, e)e - \sum_{e \in E} (x \,|\, e)e\Big\|. \end{aligned} \tag{10.55}$$

By the unconditionnal convergence, of $e \mapsto (x \,|\, e)e$, the second term of the right-hand side of (10.55) can be made arbitrarily small, and thus (10.51) holds for every $y \in \operatorname{span}(E)$. By continuity of the norm, (10.51) also holds for every $y \in \overline{\operatorname{span}(E)}$ and thus (v) holds. □

Proposition 10.12. *Let X be endowed with the inner product $(\cdot \,|\, \cdot)$. If X is complete, $E \subseteq X$ is an orthonormal set and if*

$$\{x \in X \,|\, \textit{for every } e \in E, (x \,|\, e) = 0\} = \{0\}, \tag{10.56}$$

then E is an orthonormal basis.

Proof. If (10.56) holds, then by proposition 10.11, $e \in E \mapsto (x \,|\, e)e$ is unconditionnally summable and we have

$$\Big(x - \sum_{e \in E} (x \,|\, e)e \,\Big|\, e\Big) = 0. \tag{10.57}$$

By assumption, this implies that $x = \sum_{e \in E} (x \,|\, e)e$. Hence E is an orthonormal basis. □

Theorem 10.13. *Let X be endowed with an inner product. If X is complete, then X has an othornormal basis.*

Proof. We consider

$$\mathscr{E} := \{E \subset X \mid E \text{ is an othonormal set}\}, \tag{10.58}$$

and we define $E_1 \preccurlyeq E_2$ whenever $E_1 \subseteq E_2$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered, then $E := \bigcup_{F\in\mathscr{F}} F \in \mathscr{E}$ and E is an upper bound for $\mathscr{F}$.

The set $\mathscr{E}$ is not empty since $\emptyset \in \mathscr{E}$. By Zorn's lemma (theorem E.5), $\mathscr{E}$ has a maximal element E. We claim that E is an othonormal basis. If this is not the case, since X is complete, by proposition 10.12, there exists $x \in X \setminus \{0\}$ such that for every $e \in E$ and $(x \mid e) = 0$. Then $E \cup \{x/\|x\|\}$ is linearly independent, and thus $E \cup \{x/\|x\|\} \in \mathscr{E}$, in contradiction with the maximality of E. □

Theorem 10.14. *Let X be endowed with an inner product. If E and F are orthonormal bases, then E and E have the same cardinality.*

Proof. If E or F is finite, this is a standard result of linear algebra.

For every $f \in F$, we have

$$1 = \|f\|^2 = \sum_{e\in E} |(e|f)|^2 < \infty, \tag{10.59}$$

and thus

$$E \setminus f^{\perp} = \{e \in E \mid (e|f) \neq 0\} \tag{10.60}$$

is countable and non-empty. Hence, there exists an injective map from $\bigcup_{f\in F} E \setminus f^{\perp}$ into $F \times \mathbb{N}$, and thus by proposition F.14, into F.

Since we have $\bigcup_{f\in F} E \setminus f^{\perp} = E$. This implies that there is an injective mapping from E into F. We conclude by symmetry and by proposition F.7. □

11 Approximation in L^p

11.1 Approximation by averaging

Proposition 11.1. *Let $p \in [1, \infty)$. Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. Let $\mathscr{H} \subseteq \Sigma$ be a countable set of disjoints sets of finite positive measure. Then the operator $P : L^p(\Omega, \mu) \to L^p(\Omega, \mu)$ defined for $f \in L^p(\Omega, \mu)$ by*

$$Pf := \sum_{H \in \mathscr{H}} \frac{\mathbb{1}_H}{\mu(H)} \int_H f \,\mathrm{d}\mu \tag{11.1}$$

is well-defined and

$$\|P\|_{\mathscr{L}(L^p(\Omega,\mu),L^p(\Omega,\mu))} \le 1. \tag{11.2}$$

Moreover, for every $f \in L^p(\Omega, \mu)$, we have

$$\begin{aligned}\int_\Omega |Pf - f|^p &\\ \le \sum_{H \in \mathscr{H}} \frac{1}{\mu(H)} \int_H \int_H |f(x) - f(y)|^p \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) &+ \int_{\Omega \setminus \bigcup_{H \in \mathscr{H}} H} |f|^p \,\mathrm{d}\mu\\ &\le 2^p \int_\Omega |Pf - f|^p \,\mathrm{d}\mu.\end{aligned} \tag{11.3}$$

Proof. If $f \in L^p(\Omega, \mu)$, since $p \ge 1$, we have by Hölder's inequality (proposition 4.26), for every $H \in \mathscr{H}$,

$$\left|\int_H f \,\mathrm{d}\mu\right| \le \left(\int_H |f|^p\right)^{\frac{1}{p}} \mu(H)^{1-\frac{1}{p}}, \tag{11.4}$$

and thus

$$\begin{aligned}\int_\Omega |Pf|^p \,\mathrm{d}\mu &= \sum_{H \in \mathscr{H}} \int_H \left|\frac{1}{\mu(H)} \int_H f \,\mathrm{d}\mu\right|^p \mathrm{d}\mu\\ &\le \sum_{H \in \mathscr{H}} \int_H \frac{1}{\mu(H)} \int_H |f|^p \,\mathrm{d}\mu \,\mathrm{d}\mu\\ &= \sum_{H \in \mathscr{H}} \int_H |f|^p \,\mathrm{d}\mu \le \int_\Omega |f|^p.\end{aligned} \tag{11.5}$$

Hence, $Pf \in L^p(\Omega, \mu)$,

$$\|Pf\|_p \le \|f\|_p, \tag{11.6}$$

and (11.2) follows.

In order to continue, since $p \ge 1$, we observe that by Hölder's inequality (proposition 4.26) again, for every $H \in \mathscr{H}$ and every $x \in H$,

$$\left|\int_H f(y) - f(x) \,\mathrm{d}\mu(y)\right| \le \left(\int_H |f(y) - f(x)|^p \,\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \mu(H)^{1-\frac{1}{p}} \tag{11.7}$$

and therefore for every $H \in \mathscr{H}$,

$$\begin{aligned} \int_H |Pf - f|^p \,\mathrm{d}\mu &= \frac{1}{\mu(H)^p} \int_H \left|\int_H f(y) - f(x) \,\mathrm{d}\mu(y)\right|^p \mathrm{d}\mu(x) \\ &\le \frac{1}{\mu(H)} \int_H \int_H |f(y) - f(x)|^p \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y). \end{aligned} \tag{11.8}$$

Conversely, since $p \ge 1$, we have by convexity, since Pf is constant on H,

$$\begin{aligned} &\frac{1}{\mu(H)} \int_H \int_H |f(y) - f(x)|^p \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\ &\qquad = \frac{1}{\mu(H)} \int_H \int_H |f(y) - Pf(y) + Pf(x) - f(x)|^p \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\ &\qquad = \frac{2^p}{\mu(H)} \int_H \int_H \left|\frac{f(y) - Pf(y)}{2} + \frac{Pf(x) - f(x)}{2}\right|^p \mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\ &\qquad \le \frac{2^{p-1}}{\mu(H)} \int_H \int_H |f(y) - Pf(y)|^p + |Pf(x) - f(x)|^p \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\ &\qquad = 2^p \int_H |f - Pf|^p \,\mathrm{d}\mu. \end{aligned} \tag{11.9}$$

Finally, we have $Pf = 0$ on $\Omega \setminus \bigcup_{H \in \mathscr{H}} H$ and therefore

$$\int_{\Omega \setminus \bigcup_{H \in \mathscr{H}} H} |Pf - f|^p \,\mathrm{d}\mu = \int_{\Omega \setminus \bigcup_{H \in \mathscr{H}} H} |f|^p \,\mathrm{d}\mu. \tag{11.10}$$

Combining (11.8), (11.9) and (11.10), we get (11.3). □

Proposition 11.2. *Let $p \in [1, \infty)$. Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. Assume that for every $n \in \mathbb{N}$, $\mathscr{H}_n \subseteq \Sigma$ is a countable set of disjoints sets of finite positive measure such that for every set $A \in \Sigma$ with $\mu(A) < \infty$, one has*

$$\lim_{n \to \infty} \sum_{H \in \mathscr{H}_n} \min\bigl\{\mu(H \cap A), \mu(H \setminus A)\bigr\} + \mu\Bigl(A \setminus \bigcup_{H \in \mathscr{H}_n} H\Bigr) = 0, \tag{11.11}$$

then if $P_n : L^p(\Omega,\mu) \to L^p(\Omega,\mu)$ is defined for each $f \in L^p(\Omega,\mu)$ by

$$P_n(f) = \sum_{H\in\mathscr{H}_n} \frac{\mathbb{1}_H}{\mu(H)} \int_H f \,\mathrm{d}\mu, \tag{11.12}$$

for every $f \in L^p(\Omega,\mu)$,

$$\lim_{n\to\infty} \|P_n(f) - f\|_{L^p(\Omega,\mu)} = 0 \tag{11.13}$$

Moreover, for every $f \in L^p(\Omega,\mu)$

$$\lim_{n\to\infty} \sum_{H\in\mathscr{H}_n} \frac{1}{\mu(H)} \int_H \int_H |f(x) - f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_{\Omega\setminus\bigcup_{H\in\mathscr{H}_n}} |f|^p \,\mathrm{d}\mu = 0. \tag{11.14}$$

The condition (11.11) is equivalent to

$$\lim_{n\to\infty} \inf\Bigl\{\mu\Bigl(A \triangle \bigcup_{H\in\mathscr{J}} H\Bigr) \Bigm| \mathscr{J} \subseteq \mathscr{H}_n\Bigr\} = 0. \tag{11.15}$$

Proof of proposition 11.2. Let $A \in \Sigma$. Computing

$$P_n(\mathbb{1}_{A_i}) = \sum_{h\in\mathscr{H}_n} \frac{\mu(A_i \cap H)}{\mu(H)} \mathbb{1}_H, \tag{11.16}$$

we have

(11.17)

$$\begin{aligned}
\|P_n(\mathbb{1}_A) - \mathbb{1}_A\|^p_{L^p(\Omega,\mu)} &= \int_\Omega |P_n(\mathbb{1}_A) - \mathbb{1}_A|^p \,\mathrm{d}\mu \\
&= \sum_{H\in\mathscr{H}_n} \int_H |P_n(\mathbb{1}_A) - \mathbb{1}_A|^p \,\mathrm{d}\mu + \int_{\Omega\setminus\bigcup_{H\in\mathscr{H}_n} H} |P_n(\mathbb{1}_A) - \mathbb{1}_A|^p \,\mathrm{d}\mu \\
&= \sum_{H\in\mathscr{H}_n} \int_H \Bigl(\mathbb{1}_A - \frac{\mu(A\cap H)}{\mu(H)}\Bigr)^p \,\mathrm{d}\mu + \int_{\Omega\setminus\bigcup_{H\in\mathscr{H}_n} H} \mathbb{1}_A \,\mathrm{d}\mu \\
&= \sum_{H\in\mathscr{H}_n} \frac{\mu(A\cap H)\mu(H\setminus A)^p + \mu(A\setminus H)\mu(A\cap H)^p}{\mu(H)^p} + \mu\Bigl(A \setminus \bigcup_{H\in\mathscr{H}_n} H\Bigr) \\
&\le 2 \sum_{H\in\mathscr{H}_n} \min\bigl\{\mu(A\cap H), \mu(H\setminus A)\bigr\} + \mu\Bigl(A \setminus \bigcup_{H\in\mathscr{H}_n} H\Bigr),
\end{aligned}$$

and thus in view of (11.11),

$$\lim_{n\to\infty} \|P_n(\mathbb{1}_A) - \mathbb{1}_A\|^p_{L^p(\Omega,\mu)} = 0. \tag{11.18}$$

Next, we have

$$\|P_n(f) - P_n(g)\|_{L^p(\Omega,\mu)} \le \|f - g\|_{L^p(\Omega,\mu)}. \tag{11.19}$$

Therefore, taking g sufficiently close to f in view of proposition 7.10 applying proposition 9.2 with $Z = \{\mathbb{1}_A \mid A \in \Sigma\}$, we reach the conclusion. □

11.2 Approximation by convolution

Definition 11.3. Given a function $f : \mathbb{R}^d \to \mathbb{R}$ and a vector $h \in \mathbb{R}^d$, we define the function $\tau_h f : \mathbb{R}^d \to \mathbb{R}$ by setting for every $x \in \mathbb{R}^d$,

$$\tau_h f(x) \coloneqq f(x-h). \tag{11.20}$$

Proposition 11.4. *For every $d \in \mathbb{N} \setminus \{0\}$, for every $p \in (1,\infty)$ and every $h \in \mathbb{R}^d$, $\tau_h \in \mathscr{L}(L^p(\mathbb{R}^d), L^p(\mathbb{R}^d))$ and for every $f \in L^p(\mathbb{R}^d)$,*

$$\|\tau_h f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}. \tag{11.21}$$

Proposition 11.5. *Let $d \in \mathbb{N} \setminus \{0\}$, let $\mu : \mathfrak{B}(\mathbb{R}^d) \to [0,\infty]$ be the Lebesgue measure and let $p \in [1,\infty)$. For every $f \in L^p(\mathbb{R}^d)$,*

$$\lim_{h \to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} = 0. \tag{11.22}$$

Proof. Let $R \subseteq \mathbb{R}^d$ be an open rectangle. One computes immediately that

$$\lim_{h \to 0} \|\tau_h \mathbb{1}_R - \mathbb{1}_R\|_{L^p(\mathbb{R}^d)} = 0 \tag{11.23}$$

and one observes that for every $f \in L^p(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$,

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^d)} \le \|\tau_h f\|_{L^p(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} = 2\|f\|_{L^p(\mathbb{R}^d)}, \tag{11.24}$$

in view of proposition 11.4. We conclude by proposition 7.11 and proposition 9.2. □

Proposition 11.6. *Let $d \in \mathbb{N} \setminus \{0\}$, let $\mu : \mathfrak{B}(\mathbb{R}^d) \to [0,\infty]$ be the Lebesgue measure and let $p \in [1,\infty)$. Assume that for every $n \in \mathbb{N}$, $g_n \in L^1(\mathbb{R}^d, \mu)$. If*

(i) $\displaystyle\int_{\mathbb{R}^d} g_n \,\mathrm{d}\mu = 1$,

(ii) $\displaystyle\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |g_n| < \infty$,

(iii) for every $r > 0$, $\displaystyle\lim_{n \to \infty} \int_{\mathbb{R}^d \setminus B(0,r)} |g_n| \,\mathrm{d}\mu = 0$,

then for every $f \in L^p(\mathbb{R}^d, \mu)$,

$$\lim_{n \to \infty} \|g_n * f - f\|_{L^p(\mathbb{R}^d,\mu)} = 0. \tag{11.25}$$

Proof. We have for every $x \in \mathbb{R}^d$,

$$(g_n * f)(x) - f(x) = \int_{\mathbb{R}^d} g_n(h)(f(x-h) - f(x)) \,\mathrm{d}h, \tag{11.26}$$

and thus

$$|(g_n * f)(x) - f(x)| \le \int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)| \,\mathrm{d}h. \tag{11.27}$$

Hence, by Hölder's inequality,

$$|(g_n * f)(x) - f(x)| \le \Bigl(\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}h\Bigr)^{\frac{1}{p}} \Bigl(\int_{\mathbb{R}^d} |g_n|\Bigr)^{1-\frac{1}{p}}, \tag{11.28}$$

and thus

$$\int_{\mathbb{R}^d} |(g_n * f)(x) - f(x)|^p \,\mathrm{d}x \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h \Bigl(\int_{\mathbb{R}^d} |g_n|\Bigr)^{p-1}. \tag{11.29}$$

We note that for every $r \in (0, \infty)$

$$\int_{B(0,r)}\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h \le \int_{\mathbb{R}^d} |g_n(h)| \sup_{|k| \le r} \int_{\mathbb{R}^d} |f(x-k) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h, \tag{11.30}$$

and thus by proposition 11.5,

$$\lim_{r \to 0} \sup_{n \in \mathbb{N}} \int_{B(0,r)}\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h = 0. \tag{11.31}$$

We next note that for every $r \in (0, \infty)$, by proposition 11.4,

$$\int_{\mathbb{R}^d \setminus B(0,r)}\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h \le 2^p \int_{\mathbb{R}^d \setminus B(0,r)} |g_n| \int_{\mathbb{R}^d} |f|^p, \tag{11.32}$$

and thus

$$\lim_{n \to \infty} \int_{\mathbb{R}^d \setminus B(0,r)}\int_{\mathbb{R}^d} |g_n(h)||f(x-h) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}h = 0. \tag{11.33}$$

The conclusion follows from (11.29), (11.32) and (11.33). □

11.2.1 Application to Fourier analysis

If $f \in L^1([0,1], \mathbb{C})$ and $k \in \mathbb{Z}$, we define the Fourier coefficient

$$c_k(f) := \int_0^1 f(t) e^{-2\pi i k t} \,\mathrm{d}t. \tag{11.34}$$

The Dirichlet sum is defined by

$$\mathfrak{D}_n(f)(x) := \sum_{k=-n}^{n} c_k(f) e^{2\pi i k x}. \tag{11.35}$$

In view of the definition of Fourier coefficients (11.34), we have

$$\mathfrak{D}_n(f)(x) = \int_0^1 D_n(x-t) f(t)\,\mathrm{d}t \tag{11.36}$$

with the *Dirichlet kernel* defined by

$$D_n(t) = \sum_{k=-n}^{n} e^{2\pi i k x} = \frac{e^{2\pi i(n+1)x} - e^{-2\pi i n x}}{e^{2\pi i x} - 1} = \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}. \tag{11.37}$$

We define the Féjer summation operator by

$$\mathfrak{F}_n(f)(x) \coloneqq \frac{1}{n+1} \sum_{\ell=0}^{n} \mathfrak{D}_n(f). \tag{11.38}$$

In view of the definition of the Dirichlet kernel

$$\mathfrak{F}_n(f)(x) = \int_0^1 F_n(x-t) f(t)\,\mathrm{d}t, \tag{11.39}$$

where

$$\begin{aligned}
F_n(x) &\coloneqq \frac{1}{n+1} \sum_{\ell=0}^{n} D_n(x)\\
&= \frac{1}{(n+1)\sin(\pi x)} \sum_{\ell=0}^{n} \sin((2\ell+1)\pi x)\\
&= \frac{1}{2i(n+1)\sin(\pi x)} \sum_{\ell=0}^{n} \bigl(e^{i(2\ell+1)\pi x} - e^{-i(2\ell+1)\pi x}\bigr)\\
&= \frac{1}{2i(n+1)\sin(\pi x)} \biggl(\frac{e^{i(2n+3)\pi x} - e^{i\pi x}}{e^{i2\pi x} - 1} - \frac{e^{-i(2n+3)\pi x} - e^{-i\pi x}}{e^{-i2\pi x} - 1}\biggr)\\
&= \frac{1}{2i(n+1)\sin(\pi x)} \frac{e^{i2(n+1)\pi x} - 2 + e^{-i2(n+1)\pi x}}{e^{i\pi x} - e^{-i\pi x}}\\
&= \frac{1}{n+1} \biggl(\frac{\sin((n+1)\pi x)}{\sin(\pi x)}\biggr)^2.
\end{aligned} \tag{11.40}$$

If we define $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n \coloneqq \mathbb{1}_{(-1/2,1/2)} F_n, \tag{11.41}$$

and $\bar{f} : \mathbb{R} \to \mathbb{C}$ by

$$\bar{f}(x) \coloneqq \begin{cases} 0 & \text{if } x \le -1 \text{ or } x \ge 2,\\ f(x+1) & \text{if } -1 < x \le 0,\\ f(x) & \text{if } 0 < x \le 1,\\ f(x-1) & \text{if } 1 < x < 2. \end{cases} \tag{11.42}$$

We have

$$\mathfrak{F}_n(f)(x) = \int_0^1 F_n(x-t)f(t)\,\mathrm{d}t, = \int_{x-1/2}^{x+1/2} F_n(x-t)\bar{f}(t)\,\mathrm{d}t = g_n * \bar{f}(x). \tag{11.43}$$

We now observe that

$$\int_{\mathbb{R}} g_n(x)\,\mathrm{d}x = \int_{-1/2}^{1/2} F_n(x)\,\mathrm{d}x = \frac{1}{n+1}\sum_{\ell=0}^{n}\int_{-1/2}^{1/2} D_\ell(x)\,\mathrm{d}x = 1 \tag{11.44}$$

Since $g_n \geq 0$, we also have

$$\int_{\mathbb{R}} |g_n(x)|\,\mathrm{d}x = 1. \tag{11.45}$$

Finally, we have if $r \in (0, 1/2)$,

$$\int_{\mathbb{R}\setminus[-r,r]} |g_n(x)|\,\mathrm{d}x \leq \frac{1}{(n+1)|\sin(\pi r)|^2}\,\mathrm{d}x, \tag{11.46}$$

and thus

$$\lim_{n\to\infty} \int_{\mathbb{R}\setminus[-r,r]} |g_n(x)|\,\mathrm{d}x = 0. \tag{11.47}$$

It follows from proposition 11.6 that $g_n * \bar{f}$ converges to $\bar{f}$ in $L^p(\mathbb{R})$, and thus by restriction, we have $(\mathfrak{F}_n f)_{n\in\mathbb{N}}$ converges to f in $L^p([0,1])$. Hence the set

$$\left\{ f : [0,1] \to \mathbb{C} \;\middle|\; f(x) = \sum_{k=-n}^{n} t_i e^{2\pi i n x}, n \in \mathbb{N}, t_{-n}, \dots, t_n \in \mathbb{C} \right\} \tag{11.48}$$

is dense in $L^p([0,1], \mathbb{C})$.

In particular, the set

$$\{e_n \mid e_n(x) = e^{2\pi i n x}, n \in \mathbb{N}\} \tag{11.49}$$

is an orthonormal basis of $L^2([0,1], \mathbb{C})$.

12 Compactness

12.1 Compactness in metric spaces

Definition 12.1. Let X endowed with d be a metric space. The set $A \subseteq X$ is *(sequentially) compact* whenever for every sequence $(x_n)_{n\in\mathbb{N}}$ in A, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and $x_* \in A$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges to x_* in A.

Proposition 12.2. *Let X endowed with d be a metric space and let $A \subseteq B \subseteq X$. If the set B is compact, then the set A is compact if and only if A is closed.*

Proposition 12.3. *Let X endowed with d_X be a metric space and let Y endowed with d_Y be a metric space. If $A \subseteq X$ is compact and if $f : A \to Y$ is continuous, then $f(A)$ is compact.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $f(A)$. By definition of the image of a set, for every $n \in \mathbb{N}$, there exists $x_n \in A$ such that $y_n = f(x_n)$. By compactness of A, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. By continuity of f, the image of this subsequence converges and thus the set $f(A)$ is compact. $\square$

Definition 12.4. Let X endowed with d be a metric space. The set $A \subseteq X$ is *(sequentially) precompact* whenever for every sequence $(x_n)_{n\in\mathbb{N}}$ in A, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence.

Proposition 12.5. *Let X endowed with d be a metric space and let $A \subseteq B \subseteq X$. If B is precompact, then A is precompact.*

Proposition 12.6. *Let X endowed with d_X be a metric space and let Y endowed with d_Y be a metric space. If $A \subseteq X$ is precompact and if $f : A \to Y$ is uniformly continuous, then $f(A)$ is precompact.*

Proposition 12.7. *Let X endowed with d be a metric space. A set $A \subseteq X$ is compact if and only if A is complete and precompact.*

Proof. Assume that A is compact. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in A, then by definition 12.1, has a convergent subsequence, which is a Cauchy sequence in view of proposition 5.2; A is thus precompact. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, then by definition 12.1 it has a convergent subsequence; by proposition 5.3 the full sequence $(x_n)_{n\in\mathbb{N}}$ converges; A is thus complete.

Conversely, if A is precompact and complete. Let $(x_n)_{n\in\mathbb{N}}$ is a sequence in A, then by definition 12.4, $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence; by definition 5.5 this sequence converges and thus A is compact. $\square$

Definition 12.8. Let X endowed with d be a metric space. A set $A \subseteq X$ is *totally bounded* whenever for every $\varepsilon \in (0, \infty)$, there exists a finite set $F \subseteq A$ such that $A \subseteq \bigcup_{a \in F} B[a, \varepsilon]$.

Proposition 12.9. *Let X endowed with d be a metric space. If $A \subseteq X$ is totally bounded, then A is bounded.*

Proof. By definition 12.8, there exists a finite set $F \subseteq A$ such that $A \subseteq \bigcup_{a \in F} B[a, 1]$. For every $x, y \in A$, we then have

$$d(x, y) \le 2 + \max\{d(a, b) \mid a, b \in F\}. \tag{12.1}$$

□

Proposition 12.10. *Let X endowed with d be a metric space and let $A \subseteq X$. If for every $\varepsilon \in (0, \infty)$, there exists a set $\tilde{A} \subseteq X$ such that $\tilde{A}$ is totally bounded and $A \subseteq \bigcup_{a \in \tilde{A}} B[a, \varepsilon]$, then A is totally bounded.*

In particular, any subset of a totally bounded set is itself totally bounded.

Proof. Let $\varepsilon \in (0, \infty)$. By assumption, there exists a totally bounded set $\tilde{A} \subseteq X$ such that $A \subseteq \bigcup_{a \in \tilde{A}} B[a, \varepsilon/4]$. By definition of total boundedness (definition 12.8) there exists a finite set $\tilde{F}$ such that $\tilde{A} \subseteq \bigcup_{a \in \tilde{F}} B[a, \varepsilon/4]$. By the triangle inequality, we have $A \subseteq \bigcup_{a \in \tilde{F}} B[a, \varepsilon/2]$. We define

$$\tilde{F}_* := \{a \in \tilde{F} \mid B[a, \varepsilon/2] \cap A \neq \emptyset\}, \tag{12.2}$$

so that

$$A \subseteq \bigcup_{a \in \tilde{F}_*} B[a, \varepsilon/2] \tag{12.3}$$

We now take $F \subseteq A$ to be a finite set such that for every $a \in \tilde{F}_*$, $B[a, \varepsilon/2] \cap F \neq \emptyset$. We then have

$$A \subseteq \bigcup_{a \in \tilde{F}_*} B[a, \varepsilon/2] \subseteq \bigcup_{a \in F} B[a, \varepsilon]. \tag{12.4}$$

This proves that the set A is totally bounded by definition of total boundedness (definition 12.8). □

Proposition 12.11. *Let X endowed with d be a metric space. A set $A \subseteq X$ is precompact if and only if it is totally bounded.*

Proof. Assume that the set A is totally bounded. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in A. We are going to prove by induction that for every $k \in \mathbb{N}$, there exists an infinite set $N_k \subseteq \mathbb{N}$ such that for every $n, m \in N_k$, we have $d(x_n, x_m) \le 2^{-k}$. By definition of total boundedness, there exists a finite set $F_0 \subseteq A$ such that $A \subseteq \bigcup_{a \in F_0} B[a, 1/2]$. Since F_0 is finite, there exists $a_0 \in F_0$ and an infinite set $N_0 \subseteq \mathbb{N}$ such that for every $n \in N_0$, $x_n \in B[a_0, 1/2]$ and thus for every $n, m \in N_0$, $d(x_n, x_m) \le 1$. Next assuming that N_k has been constructed. By definition of total boundedness (definition 12.8), there exists a finite set $F_{k+1} \subseteq A$ such that $A \subseteq \bigcup_{a \in F_{k+1}} B[a, 1/2^{k+2}]$. Since F_k is finite, there exists $a_{k+1} \in F_{k+1}$ and an infinite set $N_{k+1} \subseteq N_k$ such that for every $n \in N_{k+1}$, $x_n \in B[a_{k+1}, 1/2^{k+2}]$ and thus for every $n, m \in N_{k+1}$, $d(x_n, x_m) \le 1/2^{k+1}$. In order to conclude, we choose $n_0 \in N_0$ and $n_{k+1} \in N_{k+1}$ such that $n_{k+1} > n_k$. The sequence $(x_{n_k})_{k \in \mathbb{N}}$ is then a Cauchy sequence and A is thus precompact.

Conversely, assume that A is not totally bounded. Then for some $\varepsilon \in (0, \infty)$ and for every finite set $F \subseteq A$, we have $A \setminus \bigcup_{a \in F} B[a, \varepsilon] \ne \emptyset$. This implies that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that for every $n, m \in \mathbb{N}$, $d(x_n, x_m) > \varepsilon$. This sequence cannot have a Cauchy subsequence and thus the set A is not precompact. □

12.2 Compactness in Banach spaces

Proposition 12.12. *Let X endowed with $\|\cdot\|$ be a normed space and let $V \subseteq X$ be a finite-dimensional space. A set $A \subseteq V$ is totally bounded if and only if A is bounded.*

Proof. We prove this by induction on the dimension of V. If $\dim V = 0$, then $V = \{0\}$ which is compact.

Otherwise $\dim V \ge 1$ and we assume that $V = W \oplus \mathbb{R}z$, with $z \in V \setminus W$ and $\dim W = \dim V - 1$. We define the linear mapping $\xi : V \to \mathbb{R}$ such that $\xi(z) = 1$ and $\xi|_W = \{0\}$. By proposition 3.16 and lemma 3.18, we have $\xi \in \mathscr{L}(X, \mathbb{R})$. For every $x \in V$, we can write $x = w + tz$ with $w \in W$ and $t \in \mathbb{R}$, we have then $x - \xi(x)z = w \in W$.

If $A \subseteq V$ is bounded, there exists $R \in \mathbb{R}$ such that for every $x \in A$, $\|x\| \le R$. Hence, for every $x \in A$, we have

$$|\xi(x)| \le \|\xi\|_{\mathscr{L}(V,\mathbb{R})} \|x\| \le \|\xi\|_{\mathscr{L}(V,\mathbb{R})} R. \tag{12.5}$$

Given $\varepsilon \in (0, \infty)$, we let $T_\varepsilon \subseteq \mathbb{R}$ be a finite set such that

$$[-\|\xi\|_{\mathscr{L}(V,\mathbb{R})} R, \|\xi\|_{\mathscr{L}(V,\mathbb{R})} R] \subseteq \bigcup_{\tau \in T_\varepsilon} [\tau - \varepsilon/\|z\|, \tau + \varepsilon/\|z\|]. \tag{12.6}$$

For every $t \in T_\varepsilon$ and $x \in A$ such that $|\xi(t) - t| \le \varepsilon/\|z\|$, we have

$$\|x - (x - \xi(x)z + tz)\| = |\xi(x) - t|\|z\| \le \varepsilon. \tag{12.7}$$

Hence we have

$$A \subseteq \bigcup_{a \in A_\varepsilon} B[a, \varepsilon], \tag{12.8}$$

with

$$A_\varepsilon := \bigcup_{t \in T_\varepsilon} A_\varepsilon^t \tag{12.9}$$

and

$$A_\varepsilon^t := \{x - (\xi(x)z + tz) \mid x \in A \text{ and } |\xi(x) - t| \le \varepsilon/\|z\|\}. \tag{12.10}$$

We have $A_\varepsilon^t + tz \subseteq W$ and for every $y \in A_\varepsilon^t + tz$, we have for some $x \in A$, $y = x - (\xi(x)z + tz) + tz = x - \xi(x)z \in W$ and

$$\|y\| \le \|x\| + |\xi(x) - t|\|z\| \le R + \varepsilon. \tag{12.11}$$

By our induction assumption, the sets $A_\varepsilon^t + tz \subseteq W$ are totally bounded, and thus A_ε is totally bounded. In view of (12.8) and proposition 12.10, the set A is totally bounded. □

Proposition 12.13. *Let X endowed with $\|\cdot\|$ be a normed space. The set $A \subseteq X$ is totally bounded if and only if A is bounded and for every $\varepsilon \in (0, \infty)$ there exists a finite dimensional space $V \subseteq X$ such that*

$$A \subseteq \bigcup_{v \in V} B[v, \varepsilon]. \tag{12.12}$$

Proof. If A is totally bounded, then A is bounded. Moreover, there exists a finite set $F \subseteq A$ such that $A \subseteq \bigcup_{a \in F} B[a, \varepsilon]$. Letting $V := \operatorname{span} F \supseteq F$ we reach the conclusion.

Conversely, if the set A is bounded, then

$$A \subseteq B[0, R] \tag{12.13}$$

for some $R \in (0, \infty)$. By assumption, for every $\varepsilon \in (0, \infty)$ there exists a finite-dimensional space $V \subseteq X$ such that

$$A \subseteq \bigcup_{a \in V} B[a, \varepsilon]. \tag{12.14}$$

By (12.13) and (12.14), we have

$$A \subseteq B[0, R] \cap \bigcup_{a \in V} B[a, \varepsilon] \subseteq \bigcup_{a \in V \cap B[a, R + \varepsilon]} B[a, \varepsilon]. \tag{12.15}$$

By proposition 12.12, the set $V \cap B[a, R + \varepsilon]$ is totally bounded as a bounded subset of a finite-dimensional subspace, and thus by proposition 12.10, the set A is totally bounded. □

Proposition 12.14. *Let X endowed with $\|\cdot\|$ be a normed space. The unit ball $B[0, 1]$ is totally bounded if and only if $\dim X < \infty$.*

Lemma 12.15 (Riesz lemma). *Let X endowed with $\|\cdot\|$ be a normed space and let $V \subseteq X$ be a linear subspace. If V is not dense in X, then for every $\kappa \in [0, 1)$ there exists $x \in X$ such that $\|x\| = 1$ and for every $v \in V$, $\|v - x\| \ge \kappa$.*

Proof. By assumption $\bar{V} \subsetneq X$ and there exists thus $y \in X \setminus \bar{V}$. In particular,

$$\eta := \inf\big\{\|v - y\| \;\big|\; v \in V\big\} > 0. \tag{12.16}$$

By definition of infimum as a greatest lower bound, there exists $w \in V$ such that

$$\|w - y\| \le \frac{\eta}{\kappa}. \tag{12.17}$$

We set

$$x := \frac{y - w}{\|y - w\|}. \tag{12.18}$$

For every $v \in V$, we have since $\|w - y\|v + w \in V$ by linearity

$$\|v - x\| = \frac{\|(\|w - y\|v + w) - y\|}{\|y - w\|} \ge \frac{\eta}{\eta/\kappa} = \kappa, \tag{12.19}$$

which ends the proof. □

Proof of proposition 12.14. If $\dim X$ is totally bounded, then $B[0,1]$ is totally bounded by proposition 12.12.

If $B[0,1]$ is totally bounded, then by proposition 12.13 there exists a finite dimension subspace $V \subseteq X$ such that $B[0,1] \subseteq \bigcup_{v\in V} B[v, 1/2]$. In view of lemma 12.15, we have $\bar{V} = X$ and thus the space X has finite dimension. □

12.3 Compactness in Hilbert spaces

Proposition 12.16. *Let X be endowed with the inner product $(\cdot\,|\,\cdot)$ and let $E \subseteq X$ be an orthonormal basis. A set $A \subseteq X$ is totally bounded if and only if A is bounded and*

$$\inf\Big\{\sup\Big\{\sum_{e\in E\setminus F} |(x\,|\,e)|^2 \;\Big|\; x \in A\Big\} \;\Big|\; F \subseteq E \text{ is finite}\Big\} = 0. \tag{12.20}$$

Proof. First assume that the set A is totally bounded. By proposition 12.9, the set A is bounded. Given $\varepsilon \in (0,\infty)$, there exists $a_1, \dots, a_n \in A$ such that $A \subseteq \bigcup_{j=1}^n B[a_i, \varepsilon/2]$. By the Parseval identity (proposition 10.5 (iii))for every $j \in \{1, \dots, m\}$,

$$\sum_{e\in E} |(a_j\,|\,e)|^2 = \|a_j\|^2 < \infty \tag{12.21}$$

and by definition of infinite sums (definition C.1), there exists a finite set $F \subseteq E$ such that for every $j \in \{1, \dots, m\}$,

$$\sum_{e\in E\setminus F} |(a_j\,|\,e)|^2 \le \frac{\varepsilon^p}{2^p}. \tag{12.22}$$

If now $x \in A$, there exists $j \in \{1, \dots, n\}$ such that $x \in B[a_j, \varepsilon/2]$ and by the triangle inequality and Parseval identity (proposition 10.5 (iii))

$$
\begin{aligned}
\Big(\sum_{e\in E\setminus F} |(x\,|\,e)|^2\Big)^{1/2} &\le \Big(\sum_{e\in E\setminus F} |(x-a_j\,|\,e)|^2\Big)^{1/2} + \Big(\sum_{e\in E\setminus F} |(a_j\,|\,e)|^2\Big)^{1/2}\\
&\le \Big(\sum_{e\in E} |(x-a_j\,|\,e)|^2\Big)^{1/2} + \Big(\sum_{e\in E\setminus F} |(a_j\,|\,e)|^2\Big)^{1/2}\\
&\le \|x-a\| + \Big(\sum_{e\in E\setminus F} |(a_j\,|\,e)|^2\Big)^{1/2} \le \varepsilon,
\end{aligned}
\tag{12.23}
$$

and the claim is proved.

Conversely, for every $\varepsilon \in (0, \infty)$, let $F \subseteq E$ be a finite set such that for every $x \in A$,

$$
\sum_{e\in E\setminus F} |(x\,|\,e)|^2 \le \varepsilon^2. \tag{12.24}
$$

Since E is a basis, we have by proposition 10.2 (iii) and Parseval's identity (proposition 10.5 (iii))

$$
\begin{aligned}
\Big\|x - \sum_{e\in F}(x\,|\,e)e\Big\|^2 &= \|x\|^2 - \sum_{e\in F}|(x\,|\,e)|^2\\
&= \sum_{e\in E}|(x\,|\,e)|^2 - \sum_{e\in F}|(x\,|\,e)|^2\\
&= \sum_{e\in E\setminus F} |(x\,|\,e)|^2 \le \varepsilon^2.
\end{aligned}
\tag{12.25}
$$

Setting

$$
V := \operatorname{span}(F), \tag{12.26}
$$

we have that V is finite-dimensional and that $A \subseteq V + B[0, \varepsilon]$ and we conclude by proposition 12.13. □

12.4 Compactness in sequence spaces

Proposition 12.17. *Let Γ be a set and let $p \in [1, \infty)$. A set $A \subseteq \ell^p(\Gamma)$ is totally bounded if and only if Γ is bounded and*

$$
\inf\Big\{\sup\Big\{\sum_{x\in\Gamma\setminus F} |f(x)|^p \,\Big|\, f \in A\Big\} \,\Big|\, F \subseteq \Gamma \textit{ is finite}\Big\} = 0. \tag{12.27}
$$

Proof. First assume that A is totally bounded. By proposition 12.9, the set A is bounded. Moreover, by definition of total boundedness (definition 12.8), given $\varepsilon \in (0,\infty)$, there exists $f_1,\dots,f_n \in A$ such that $A \subseteq \bigcup_{j=1}^n B[f_j,\varepsilon/2]$. By definition of sums (definition C.1), there exists a finite set $F \subseteq \Gamma$ such that for every $j \in \{1,\dots,m\}$,

$$\sum_{x\in\Gamma\setminus F} |f_j(x)|^p \le \frac{\varepsilon^p}{2^p}. \tag{12.28}$$

If now $f \in A$, there exists $j \in \{1,\dots,m\}$ such that $f \in B[f_j,\varepsilon/2]$ and by the triangle inequality

$$\Bigl(\sum_{x\in\Gamma\setminus F} |f(x)|^p\Bigr)^{1/p} \le \Bigl(\sum_{x\in\Gamma\setminus F} |f(x)-f_j(x)|^p\Bigr)^{1/p} + \Bigl(\sum_{x\in\Gamma\setminus F} |f_j(x)|^p\Bigr)^{1/p} \le \varepsilon, \tag{12.29}$$

and the claim is proved.

Conversely, for every $\varepsilon \in (0,\infty)$, let $F \subseteq \Gamma$ be a finite set such that for every $f \in A$,

$$\sum_{x\in\Gamma\setminus F} |f(x)|^p \le \varepsilon^p. \tag{12.30}$$

Setting

$$V := \bigl\{f : \Gamma \to \mathbb{R} \bigm| \text{for each } x \in \Gamma \setminus F,\ f(x) = 0\bigr\}, \tag{12.31}$$

we have $A \subseteq V + B[0,\varepsilon]$ and we conclude by proposition 12.13. □

Proposition 12.18. *Let Γ be a set. A set $A \subseteq c_0(\Gamma)$ is totally bounded if and only if Γ is bounded and*

$$\inf\bigl\{\sup\bigl\{|f(x)| \bigm| f \in A \textit{ and } x \in \Gamma \setminus F\bigr\} \bigm| F \subseteq \Gamma \textit{ is finite}\bigr\} = 0. \tag{12.32}$$

Proof. First assume that A is totally bounded. By proposition 12.9, the set A is bounded. Moreover, by definition of total boundedness (definition 12.8), given $\varepsilon \in (0,\infty)$, there exists $f_1,\dots,f_n \in A$ such that $A \subseteq \bigcup_{j=1}^n B[f_j,\varepsilon/2]$. By definition of the space $c_0(\Gamma)$ (definition 5.12), there exists a finite set $F \subseteq \Gamma$ such that for every $j \in \{1,\dots,m\}$ and every $x \in \Gamma \setminus F$

$$|f_j(x)| \le \frac{\varepsilon}{2}. \tag{12.33}$$

If now $f \in A$, there exists $j \in \{1,\dots,m\}$ such that $f \in B[f_j,\varepsilon/2]$ and by the triangle inequality for every $x \in \Gamma \setminus F$

$$|f(x)|^p \le |f(x)-f_j(x)| + |f_j(x)| \le \varepsilon, \tag{12.34}$$

and the claim is proved.

Conversely, for every $\varepsilon \in (0,\infty)$, let $F \subseteq \Gamma$ be a finite set such that for every $f \in A$ and $x \in \Gamma \setminus F$,

$$|f(x)| \le \varepsilon. \tag{12.35}$$

Setting

$$V := \{f : \Gamma \to \mathbb{R} \mid \text{for each } x \in \Gamma \setminus F, f(x) = 0\}, \tag{12.36}$$

we have $A \subseteq V + B[0,\varepsilon]$ and we conclude by proposition 12.13. $\square$

12.5 Compactness in $L^p(\Omega,\mu)$

12.5.1 Measure spaces

Proposition 12.19. *Let $\mathscr{F} \subseteq L^p(\Omega,\mu)$. Assume that for every $n \in \mathbb{N}$, $\mathscr{H}_n \subseteq \Sigma$ is a finite set of disjoints sets of finite measure such that for every $A \in \Sigma$ with $\mu(A) < \infty$, one has*

$$\lim_{n\to\infty} \sum_{H \in \mathscr{H}_n} \min\{\mu(H \cap A), \mu(H \setminus A)\} + \mu\Big(A \setminus \bigcup_{H \in \mathscr{H}_n} H\Big) = 0, \tag{12.37}$$

The following are equivalent

(i) $\mathscr{F}$ is totally bounded,

(ii) $\mathscr{F}$ is bounded and for every $f \in L^p(\Omega,\mu)$,

$$\lim_{n\to\infty} \sup\{\|P_n f - f\|_{L^p(\Omega,\mu)} \mid f \in \mathscr{F}\} = 0, \tag{12.38}$$

where $P_n : L^p(\Omega,\mu) \to L^p(\Omega,\mu)$ is defined for $f \in L^p(\Omega,\mu)$ by

$$P_n f = \sum_{H \in \mathscr{H}_n} \frac{\mathbb{1}_H}{\mu(H)} \int_H f \,\mathrm{d}\mu, \tag{12.39}$$

(iii) $\mathscr{F}$ is bounded and for every $f \in L^p(\Omega,\mu)$

$$\begin{aligned}\lim_{n\to\infty} \sup\Big\{ \sum_{H \in \mathscr{H}_n} \frac{1}{\mu(H)} \int_H \int_H |f(x) - f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)& \\ + \int_{X \setminus \bigcup_{H \in \mathscr{H}_n}} |f|^p \,\mathrm{d}\mu \,\Big|\, f \in \mathscr{F}\Big\} &= 0.\end{aligned} \tag{12.40}$$

Proof. Assume that (i) holds. Since $\mathscr{F}$ is totally bounded, there exists $f_1, \dots, f_m \in \mathscr{F}$ such that $\mathscr{F} \subseteq \bigcup_{j=1}^m B[f_j, \varepsilon/3]$. By proposition 11.2, there exists $n \in \mathbb{N}$ such that for every $j \in \{1, \dots, m\}$, we have

$$\|P_n f_j - f\|_{L^p(\Omega,\mu)} \le \varepsilon/3. \tag{12.41}$$

Given $f \in \mathscr{F}$, there exists $j \in \{1,\dots,m\}$ such that $\|f_j - f\|_{L^p(\Omega,\mu)} \le \varepsilon/3$. We have then by the triangle inequality and by proposition 11.1

$$\begin{aligned}\|P_n f - f\|_{L^p(\Omega,\mu)} &\le \|P_n f - P_n f_j\|_{L^p(\Omega,\mu)}\|P_n f_j - f_j\|_{L^p(\Omega,\mu)}\|f_j - f\|_{L^p(\Omega,\mu)}\\ &\le \|P_n f_j - f_j\|_{L^p(\Omega,\mu)} + 2\|f_j - f\|_{L^p(\Omega,\mu)}\\ &\le \varepsilon.\end{aligned} \tag{12.42}$$

Hence (ii) holds.

Conversely if (ii) holds, then there exists $n \in \mathbb{N}$,

$$\mathscr{F} \subseteq P_n(L^p(\Omega,\mu)) + B[0,\varepsilon]; \tag{12.43}$$

since $P_n(L^p(\Omega,\mu))$ is finite-dimensional, (i) follows from proposition 12.13.

The equivalence between (ii) and (iii) follows from proposition 11.1. □

12.5.2 The Lebesgue measure

Proposition 12.20. *Let $\mu : \mathscr{B}(\mathbb{R}^d) \to [0,\infty]$ be the Lebesgue measure on $\mathbb{R}^d$. The set $\mathscr{F} \subseteq L^p((0,1)^d,\mu)$ is totally bounded if and only if $\mathscr{F}$ is bounded and*

$$\lim_{\rho\to 0} \sup\left\{\frac{1}{\rho^d}\iint_{(0,1)^d\times(0,1)^d} \mathbb{1}_{B(0,\rho)}(|x-y|)|f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \;\middle|\; f \in \mathscr{F}\right\} = 0. \tag{12.44}$$

Proof. First assume that $\mathscr{F}$ is bounded and that (12.44) holds.

We define for $n \in \mathbb{N}$ the family

$$\mathscr{H}_n := \left\{(0,2^{-n})^d + 2^{-n}k \;\middle|\; k \in \{0,\dots,2^n-1\}\right\}. \tag{12.45}$$

For every $H \in \mathscr{H}_n$ and $x,y \in H$, we have $|x-y| \le \rho_n := 2^{-n}\sqrt{d}$ and $\mu(H) = 2^{-nd}$, and thus for each $f \in \mathscr{F}$,

$$\begin{aligned}\sum_{H\in\mathscr{H}_n} \frac{1}{\mu(H)}\int_H\int_H |f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)&\\ \le \frac{d^{d/2}}{\rho_n^d}\iint_{(0,1)^d\times(0,1)^d} \mathbb{1}_{B(0,\rho_n)}(|x-y|)|f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).&\end{aligned} \tag{12.46}$$

By proposition 12.19, the set $\mathscr{F}$ is totally bounded.

Conversely, we assume that the set $\mathscr{F}$ is totally bounded. By proposition 12.9, $\mathscr{F}$ is bounded. We define for $n \in \mathbb{N}$ and for $h \in \{0,1\}^d$ the family

$$\mathscr{H}_n^h := \left\{((0,2^{-n})^d + 2^{-n}k + 2^{-(n+1)}h) \cap (-1,1)^d \;\middle|\; k \in \mathbb{Z}^d\right\} \setminus \emptyset. \tag{12.47}$$

For every $H \in \mathscr{H}_n^h$, we have $\mu(H) \le 2^{-nd}$. Moreover, if $\rho \le 2^{-(n+2)}$, then for every $x \in (0,1)^d$, there exists $h \in \{0,1\}^d$ and $H \in \mathscr{H}_n^h$ such that $B(x,\rho) \subseteq H$. Hence, we have

$$\begin{aligned}\frac{1}{\rho^d}\iint_{(0,1)^d\times(0,1)^d} \mathbb{1}_{B(0,\rho)}(|x-y|)|f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)&\\ \le \frac{1}{2^{nd}\rho^d}\sum_{h\in\{0,1\}^d}\sum_{H\in\mathscr{H}_n^h} \frac{1}{\mu(H)}\int_H\int_H |f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).&\end{aligned} \tag{12.48}$$

The conclusion then follows from proposition 12.19. □

Proposition 12.21. *Let $\Omega \subseteq \mathbb{R}^d$ be open, and let μ be the Lebesgue measure on $\mathbb{R}^d$. The set $\mathscr{F} \subseteq L^p(\Omega, \mu)$ is totally bounded if and only if*

(i) *$\mathscr{F}$ is bounded,*

(ii) $\displaystyle \lim_{\rho \to 0} \sup \left\{ \frac{1}{\rho^d} \iint_{\Omega \times \Omega} \mathbb{1}_{B(0,\rho)}(|x-y|)|f(x)-f(y)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \,\middle|\, f \in \mathscr{F} \right\} = 0.$

(iii) $\displaystyle \inf \left\{ \sup \left\{ \int_{\Omega \setminus K} |f|^p = 0 \,\middle|\, f \in \mathscr{F} \right\} \,\middle|\, K \subseteq \Omega \textit{ is compact} \right\} = 0.$

Proof. We assume that the set $\mathscr{F}$ is totally bounded. By proposition 12.9, $\mathscr{F}$ is bounded. Given $\varepsilon \in (0, \infty)$, by definition of total boundedness (definition 12.8), there exist $f_1, \dots, f_m \in \mathscr{F}$ such that $\mathscr{F} \subseteq \bigcup_{j=1}^m B(f_j, \varepsilon)$. Since Ω is open, by regularity of Lebesgue's measure, there exists a compact set $K \subseteq \Omega$ such that for every $j \in \{1, \dots, m\}$,

$$\int_{\Omega \setminus K} |f_j|^p \le \varepsilon^p. \tag{12.49}$$

Given $f \in \mathscr{F}$, we take $j \in \{1, \dots, m\}$ such that $\|f - f_j\|_{L^p(\Omega,\mu)} \le \varepsilon$, and we have by the triangle inequality

$$\left(\int_{\Omega \setminus K} |f|^p \right)^{\frac{1}{p}} \le \left(\int_{\Omega \setminus K} |f_j|^p \right)^{\frac{1}{p}} + \left(\int_{\Omega \setminus K} |f - f_j|^p \right)^{\frac{1}{p}} \le 2\varepsilon, \tag{12.50}$$

which proves (iii). In order to prove (ii), we take a suitably large compact set K so that (iii) holds. Without loss of generality, we can assume that K is a cube. Combining (iii) outside the cube with proposition 12.20 inside the cube, we get (ii).

Conversely, in view of (iii), we can treat through proposition 12.20 the compactness on a compact subset $K \subseteq \Omega$, that can be assumed without loss of generality to be a union of the closure of disjoint open cubes. □

Comments

Proposition 12.17 with $\Gamma = \mathbb{N}$ and proposition 12.16 with $p = 2$ are due to Maurice Fréchet [MF08].

Proposition 12.19 is due to Ralph S. Phillips [Phi40, §3].

13 Spectral theory

13.1 Spectrum of a linear operator

Definition 13.1. Let X be a vector space over the field $\mathbb{K}$ and $L : X \to X$ be a linear map, $\lambda \in \mathbb{K}$ is an *eigenvalue of* L whenever the linear operator $L - \lambda\,\mathrm{id} : X \to X$ is not injective; An associate *eigenvector of* L is an element of $\ker(L - \lambda\,\mathrm{id}) \setminus \{0\}$.

Definition 13.2. Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$. The spectrum of $L \in \mathscr{L}(X,X)$ is the set

$$\sigma(L) := \{\lambda \in \mathbb{K} \mid (L - \lambda\,\mathrm{id}) \text{ has no inverse in } \mathscr{L}(X,X)\}. \tag{13.1}$$

Proposition 13.3. *Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$ and let $L \in \mathscr{L}(X,X)$. If $\lambda \in \mathbb{K}$ is an eigenvalue of L, then $\lambda \in \sigma(L)$.*

Proof. By definition of eigenvalue (definition 13.1), there exists $x \in X \setminus \{0\}$ such that $L(x) = \lambda x$, and thus $(L - \lambda\,\mathrm{id})x = 0$. Hence the mapping $(L - \lambda\,\mathrm{id})$ is not injective and has thus no inverse. □

Proposition 13.4. *Let X endowed with $\|\cdot\|$ be a normed space. If $\dim X < \infty$, then $L \in \mathscr{L}(X,X)$ has an inverse in $\mathscr{L}(X,X)$ if and only if L is injective.*

Proof. This follows from the characterisation in linear algebra of invertible operators as injective operators and the fact that any linear mapping on a finite-dimensional space is bounded (proposition 4.14). □

Proposition 13.5. *Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$. If $\dim X < \infty$, then $\lambda \in \sigma(L)$ if and only if λ is an eigenvalue of L.*

Proof. This follows from proposition 13.4 and the definitions of eigenvalue (definition 13.1) and of spectrum (definition 13.2). □

Definition 13.6. Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$. The number $\lambda \in \mathbb{K}$ is an *approximate eigenvalue* of L whenever

$$\inf\left\{\|L(x) - \lambda x\| \;\middle|\; x \in X \text{ and } \|x\| = 1\right\} = 0. \tag{13.2}$$

Proposition 13.7. *Let X be a vector space over the field $\mathbb{K}$ and $L \in \mathscr{L}(X,X)$ be a linear map. If $\lambda \in \mathbb{K}$ is an eigenvalue of L, then λ is an approximate eigenvalue.*

Proposition 13.8. *Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$ and let $\lambda \in \mathbb{K}$. The number $\lambda \in \mathbb{K}$ is not an approximate eigenvalue if and only if $(L-\lambda\,\mathrm{id}) : X \to (L-\lambda\,\mathrm{id})(X)$ has an inverse $(L-\lambda\,\mathrm{id})^{-1} \in \mathscr{L}((L-\lambda\,\mathrm{id})(X), X)$. Moreover,*

$$\|(L-\lambda\,\mathrm{id})^{-1}\|_{\mathscr{L}((L-\lambda\,\mathrm{id})(X),X)} = \frac{1}{\inf\{\|L(x)-\lambda x\| \mid x \in X \text{ and } \|x\| = 1\}}. \tag{13.3}$$

In particular, X is complete if and only if $(L-\lambda\,\mathrm{id})(X)$ is complete.

Proof. Since λ is not an approximate eigenvalue, λ is not an eigenvalue (proposition 13.7) and thus $L-\lambda\,\mathrm{id}$ is injective. There exists thus a linear mapping $(L-\lambda\,\mathrm{id})^{-1} : (L-\lambda\,\mathrm{id})(X) \to X$.

We set, in view of definition 13.6,

$$\nu := \inf\{\|L(x)-\lambda x\| \mid x \in X \text{ and } \|x\| = 1\} > 0. \tag{13.4}$$

For every $x \in (L-\lambda\,\mathrm{id})X \setminus \{0\}$, we set $y := x/\|(L-\lambda\,\mathrm{id})^{-1}(x)\|$. Hence $\|(L-\lambda\,\mathrm{id})^{-1}y\| = 1$, and thus

$$\nu \le \|L(y)-\lambda y\| = \frac{\|x\|}{\|(L-\lambda\,\mathrm{id})^{-1}(x)\|}, \tag{13.5}$$

and thus

$$\|(L-\lambda\,\mathrm{id})^{-1}(x)\| \le \frac{\|x\|}{\nu}, \tag{13.6}$$

so that $(L-\lambda\,\mathrm{id})^{-1} \in \mathscr{L}((L-\lambda\,\mathrm{id})(X,)X)$ and

$$\|(L-\lambda\,\mathrm{id})^{-1}\|_{\mathscr{L}((L-\lambda\,\mathrm{id})(X),X)} \le \frac{1}{\nu}. \tag{13.7}$$

For the converse inequality, if $x \in X$ and $\|x\| = 1$, then

$$1 = \|x\| = \|(L-\lambda\,\mathrm{id})^{-1}((L-\lambda\,\mathrm{id})(x))\| \le \|(L-\lambda)^{-1}\|_{\mathscr{L}((L-\lambda\,\mathrm{id})(X),X)}\|L(x)-\lambda x\|, \tag{13.8}$$

and thus

$$\|L(x)-\lambda x\| \ge \frac{1}{\|(L-\lambda)^{-1}\|_{\mathscr{L}((L-\lambda\,\mathrm{id})(X),X)}} \tag{13.9}$$

and then

$$\nu \ge \frac{1}{\|(L-\lambda\,\mathrm{id})^{-1}\|_{\mathscr{L}((L-\lambda\,\mathrm{id})(X),X)}} > 0, \tag{13.10}$$

and the conclusion follows. □

Proposition 13.9. *Let X endowed with $\|\cdot\|$ be a normed space over the field $\mathbb{K}$ and let $\lambda \in \mathbb{K}$. One has $\lambda \notin \sigma(L)$ if and only if*

(i) $(L-\lambda\,\mathrm{id})$ is surjective,

(ii) and λ is not an approximate eigenvalue.

Proof. If $\lambda \in \mathbb{K} \setminus \sigma(L)$, then $(L-\lambda\,\mathrm{id})$ is invertible and thus surjective and by proposition 13.8, λ is not an approximate eigenvalue.

Conversely, if neither (i) nor (ii) hold, then by proposition 13.8, $(L-\lambda\,\mathrm{id})$ is invertible. □

13.2 Compact operators

Definition 13.10. Let X, Y endowed with $\|\cdot\|_X$ and $\|\cdot\|_Y$ be *normed spaces*. The linear mapping $L : X \to Y$ is *compact* whenever the set $L(B(0,1)) \subseteq Y$ is precompact.

The set of linear compact mappings is denoted by $\mathscr{L}_c(X, Y)$.

Proposition 13.11. *Let X, Y endowed with $\|\cdot\|_X$ and $\|\cdot\|_Y$ be normed spaces. If the linear mapping $L : X \to Y$ is compact, then L is bounded.*

Proof. This follows from the fact that any precompact set is totally bounded (proposition 12.11) and any totally bounded set is bounded (proposition 12.9). □

Proposition 13.12. *Let X, Y, Z endowed with $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ be normed spaces, let $L \in \mathscr{L}(X, Y)$ and let $K \in \mathscr{L}(Y, Z)$. If either L or K is compact, then $K \circ L$ is compact.*

Proof. If L is compact this follows from the definition of bounded linear operator (definition 1.34) and from the definition of compact operator (definition 13.10).

When K is compact, this follows from the definition of compact operators (definition 13.10), the uniform continuity of bounded linear operators proposition 4.9 and the invariance of precompact sets under uniformly continuous mappings (proposition 12.6). □

Proposition 13.13. *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is complete, if $L \in \mathscr{L}_c(X, X)$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then λ is an eigenvalue of L if and only if λ is an approximate eigenvalue.*

Proof. Since λ is an approximate eigenvalue, we have by definition 13.6

$$\inf\Bigl\{\|L(x) - \lambda x\| \Bigm| x \in X \text{ and } \|x\| = 1\Bigr\} = 0. \tag{13.11}$$

By definition of infimum as a greatest lower bound, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\|x_n\| = 1$ and such that $\lim_{n\to\infty}\|L(x_n) - \lambda x_n\| = 0$. In particular, we have for each $n \in \mathbb{N}$, $L(x_n) \in L(B[0,1])$. Since L is a compact operator, $(L(x_n))_{n\in\mathbb{N}}$ has a Cauchy subsequence $(L(x_{n_k}))_{k\in\mathbb{N}}$ with an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ (see definition 13.10 and definition 12.4) and since the space X is complete, there exists $y \in X$ such that $(L(x_{n_k}))_{k\in\mathbb{N}}$ converges to y. Since

$$\|\lambda x_{n_k} - y\| \le \|L(x_{n_k}) - y\| + \|L(x_{n_k}) - \lambda x_{n_k}\|, \tag{13.12}$$

the sequence $(\lambda x_{n_k})_{k\in\mathbb{N}}$ converges to y in X; by continuity of the norm, we have $\|y\| = \lim_{k\to\infty}\|\lambda x_{n_k}\| = |\lambda| \ne 0$ since $\lambda \ne 0$. Moreover, we have

$$L(y) - \lambda y = (L - \lambda\,\mathrm{id})(y - L(x_n)) + L(L(x)_n - \lambda x_n) \tag{13.13}$$

and thus

$$\|L(y) - \lambda y\| \le \|L - \lambda\,\mathrm{id}\|_{\mathscr{L}(X,X)}\|y - L(x_n)\| + \|L\|_{\mathscr{L}(X,X)}\|L(x_n) - \lambda x_n\|_{\mathscr{L}(X,X)}. \tag{13.14}$$

Letting $n \to \infty$, we have $L(y) = \lambda y$, which proves that λ is an eigenvalue of L and y is an associated eigenvector. □

Proposition 13.14. *Let X be a normed space. If $L \in \mathscr{L}_c(X, X)$ and $\dim X = \infty$, then 0 is an approximate eigenvalue of L.*

Proof. We prove that if 0 is not an approximate value, then $\dim X < \infty$.

Let $(x_n)_{n\in\mathbb{N}}$ be sequence in X such that for each $n \in \mathbb{N}$, $\|x_n\| \le 1$. Since L is compact, $L(B(0,1))$, up to a subsequence we can assume that $(L(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

If 0 is not an approximate eigenvalue, setting

$$\nu := \Big\{\|Lx\| \;\Big|\; x \in X \text{ and } \|x\| = 1\Big\} > 0, \tag{13.15}$$

we have for every $m, n \in \mathbb{N}$,

$$\|x_m - x_n\| \le \frac{\|L(x_m) - L(x_n)\|}{\nu} = \frac{\|L(x_m - x_n)\|}{\nu}, \tag{13.16}$$

so that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This proves then that $B(0,1)$ is precompact. By proposition 12.14, this implies that $\dim X < \infty$. □

13.3 Self-adjoint operator

Definition 13.15. Let X endowed with $(\cdot|\cdot)$ be an inner product space. A linear operator $L \in \mathscr{L}(X, X)$ is self-adjoint whenever for every $x, y \in X$, $(L(x)|y) = (x|L(y))$.

Proposition 13.16. *Let X endowed with $(\cdot|\cdot)$ be an inner-product space. If $L \in \mathscr{L}(X, X)$ is self-adjoint, then any of*

$$\lambda_0^+ := \sup\{(L(x)|x) \mid x \in X \textit{ and } \|x\| = 1\} \tag{13.17}$$

and

$$\lambda_0^- := \inf\{(L(x)|x) \mid x \in X \textit{ and } \|x\| = 1\} \tag{13.18}$$

is an approximate eigenvalue.

Proof. We write the proof for λ_0^+. By definition, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for every $n \in \mathbb{N}$, $\|x_n\| = 1$ and $\lim_{n\to\infty}(L(x_n) \mid x_n) = \lambda_0^+$. Since the map $(x, y) \mapsto ((\lambda_0^+ \operatorname{id} - L)x \,|\, y)$ is a semi-inner product, we have by the Cauchy–Schwarz inequality

$$\begin{aligned} \|(\lambda_0^+ \operatorname{id} - L)x_n\|^4 &= ((\lambda_0^+ \operatorname{id} - L)x_n \,|\, (\lambda_0^+ \operatorname{id} - L)x_n)^2 \\ &\le ((\lambda_0^+ \operatorname{id} - L)x_n \,|\, x_n)((\lambda_0^+ \operatorname{id} - L)^2 x_n \,|\, (\lambda_0^+ \operatorname{id} - L)x) \\ &\le \|(\lambda_0^+ \operatorname{id} - L)\|^3_{\mathscr{L}(X,X)}((\lambda_0^+ \operatorname{id} - L)x_n \,|\, x_n), \end{aligned} \tag{13.19}$$

and thus

$$\lim_{n\to\infty} \|(\lambda_0^+ \operatorname{id} - L)x_n\| = 0, \tag{13.20}$$

so that λ_0^+ is an approximate eigenvalue. □

Proposition 13.17. *Let X endowed with $(\cdot|\cdot)$ be an inner-product space. If X is complete and if $L \in \mathscr{L}_c(X,X)$ is self-adjoint, then there exists a set N which is at most countable, families $(e_n)_{n\in N}$ in X and $(\lambda_n)_{n\in N}$ in $\mathbb{R}$ such that*

(i) $Le_n = \lambda_n e_n$,

(ii) $(e_n)_{n\in N}$ *is an orthonormal family,*

(iii) *for every* $\varepsilon > 0$, $\{n \in N \mid |\lambda_n| \ge \varepsilon\}$ *is finite.*

(iv) *for every* $x \in X$,

$$L(x) = \sum_{n\in N} \lambda_n (e_n|x) e_n \tag{13.21}$$

and

$$\ker L = \{x \in X \mid \textit{for every } n \in N,\ (e_n|x) = 0\}. \tag{13.22}$$

Proof. We proceed by induction, assuming that for some $n \in \{-1\} \cup \mathbb{N}$ there are $e_0, \dots, e_n \in X$ and $\lambda_0, \dots, \lambda_n \in \mathbb{R}$ such that for $j \in \{1, \dots, n\}$, $L(e_j) = \lambda_j e_j$ and $(e_j)_{j\in\mathbb{N}}$ is orthonormal. We define

$$X_{n+1} := \Big\{x \in X \;\Big|\; \text{for every } j \in \{1, \dots, n\}\ (e_j|x) = 0\Big\}. \tag{13.23}$$

Since X is complete and $X_n \subseteq X$ is closed, X_n is complete (see proposition 5.7). Moreover, for every $j \in \{1, \dots, n\}$ and $x \in X_{n+1}$, then

$$(e_j|L(x)) = (L(e_j)|x) = (\lambda_j e_j|x) = \lambda_j (e_j|x) = 0. \tag{13.24}$$

Defining $L_n := A|_{X_n}$, we have $A \in \mathscr{L}_c(X_n, X_n)$. By proposition 13.16, there exists $\lambda_{n+1} \in \mathbb{R}$ such that

$$|\lambda_{n+1}| = \sup\Big\{|(L_{n+1}(x)|x)| \;\Big|\; x \in X_{n+1} \text{ and } \|x\| \le 1\Big\} \tag{13.25}$$

is an approximate eigenvalue. If $|\lambda_{n+1}| = 0$ we stop the process, setting $N := \{1, \dots, n\}$. By proposition 13.13, λ_{n+1} is an eigenvalue. We let $e_{n+1} \in X_{n+1}$ be a corresponding eigenvector satisfying $\|e_{n+1}\| = 1$.

If the process does not finish in a finite number of steps, we set $N = \mathbb{N}$. For every $n, m \in \mathbb{N}$, we have

$$\|L(e_n) - L(e_m)\|^2 = \|\lambda_n e_n - \lambda_m e_m\|^2 = \lambda_n^2 + \lambda_m^2. \tag{13.26}$$

Since the sequence $(|\lambda_n|)_{n\in\mathbb{N}}$ is non-increasing and since the mapping L is compact, this implies that $(\lambda_n)_{n\in\mathbb{N}}$ converges to 0.

Finally, we set

$$X_* := \Big\{x \in X \;\Big|\; \text{for every } n \in N,\ (e_n|x) = 0\Big\}. \tag{13.27}$$

If $x \in X_*$, then for every $n \in N$,

$$(L(x)\,|\,e_n) = (x\,|\,L(e_n)) = \lambda_n (x\,|\,e) = 0, \tag{13.28}$$

so that $L(x) \in X_*$. By (13.25), we have $(L(x)\,|\,x) = 0$. Hence, we have for every $x, y \in X_*$, since L is self-adjoint

$$(L(x)\,|\,y) = \frac{(L(x+y)\,|\,x+y) - (L(x-y)\,|\,x-y)}{4} = 0, \tag{13.29}$$

so that $L(x) = 0$. This proves that $X_* \subseteq \ker L$. On the other hand, if $x \in \ker L$ then for every $n \in N$,

$$(e_n\,|\,x) = \frac{(L(e_n)\,|\,x)}{\lambda_n} = \frac{(e_n\,|\,L(x))}{\lambda_n} = 0, \tag{13.30}$$

and thus $\ker L \subseteq X_*$.

Since $(e_n)_{n\in N}$ is an orthonormal family, $\sum_{n\in N} (x\,|\,e_n) e_n$ converges unconditionally in X (proposition 10.11) and, by continuity of L, we have

$$0 = L\Big(x - \sum_{n\in N} (x\,|\,e_n) e_n\Big) = L(x) - \sum_{n\in N} (x\,|\,e_n) \lambda_n e_n, \tag{13.31}$$

which concludes the proof. □

13.4 Complements

Proposition 13.18. *Let X and Y be normed spaces. The set $\mathscr{L}_c(X,Y)$ is closed in $\mathscr{L}(X,Y)$.*

Proof. Let $L \in \mathscr{L}(X,Y)$ and assume that for every $\varepsilon \in (0,\infty)$, there exists $K \in \mathscr{L}_c(X,Y)$ such that $\|K - L\|_{\mathscr{L}(X,Y)} \le \varepsilon$. It follows that

$$L(B[0,1]) \subseteq \bigcup_{x\in B[0,1]} B[L(x),\varepsilon] = \bigcup_{y\in K(B[0,1])} B[y,\varepsilon]. \tag{13.32}$$

By assumption, the set $K(B[0,1])$ is precompact and thus by proposition 12.11. Since $\varepsilon \in (0,\infty)$ is arbitrary, by proposition 12.10 the set $L(B[0,1])$ is totally bounded and thus by proposition 12.11 again precompact. This proves that the operator L is compact. □

Proposition 13.19. *Let X be a complete inner product space. If $L \in \mathscr{L}(X,X)$ is self-adjoint and $\lambda \in \sigma(L) \cap \mathbb{R}$, then λ is an approximate eigenvalue.*

Proof. In view of proposition 13.9, either $\lambda \in \sigma(L)$ is not an approximate eigenvalue or$(L - \lambda\,\mathrm{id})$ is not surjective. We assume the latter, that is $V := (L - \lambda\,\mathrm{id})(X) \subsetneq X$. By proposition 13.8, V is complete and thus closed. There exists thus $e \in X \setminus \{0\}$ such that for every $v \in V$, $(e\,|\,v) = 0$. Hence for every $x \in X$,

$$(e\,|\,(L - \lambda\,\mathrm{id})x) = (L(e) - \lambda e\,|\,x). \tag{13.33}$$

Since x is arbitrary, e is an eigenvector of L, in contradiction with the fact that λ is not an approximate eigenvalue. □

Proposition 13.20. *Let X be a complete inner product space. If $L \in \mathscr{L}(X,X)$ is self-adjoint then $\sigma(L) \subseteq \mathbb{R}$.*

Proof. In view of the definition of the spectrum $\sigma(L)$, we need to prove that for every $\lambda = \mu + i\nu \in \mathbb{C} \setminus \mathbb{R}$, the operator $(L - \lambda \operatorname{id})$ is invertible. Since for every $\mu \in \mathbb{R}$, the operator $L - \mu \operatorname{id}$ is self-adjoint, it will be sufficient to prove that $L - i\nu \operatorname{id}$ is invertible for every $\nu \in \mathbb{R} \setminus \{0\}$. Since L is self-adjoint, we have for every $x \in X$,

$$\begin{aligned} \|L(x)\|^2 + \nu^2\|x\|^2 &= (Lx \,|\, Lx) + \nu^2(x \,|\, x) = (L^2 x \,|\, x) + \nu^2(x \,|\, x) \\ &= ((L^2 + \nu^2 \operatorname{id})x \,|\, x) \le \|(L^2 + \nu^2)x\|x. \end{aligned} \tag{13.34}$$

By the Cauchy–Schwarz inequality, this implies that

$$\nu^2\|x\|^2 \le \|(L^2 + \nu^2 \operatorname{id})x\|\|x\|, \tag{13.35}$$

and thus

$$\|x\| \le \frac{\|(L^2 + \nu^2 \operatorname{id})x\|}{\nu^2}. \tag{13.36}$$

In view of definition 13.6, $-\nu^2 \in \mathbb{R}$ is not an approximate eigenvalue of L^2. Since L is self-adjoint, for every $x, y \in X$, we have $(L^2(x) \mid y) = (L(x) \mid L(y)) = (x \mid L^2(y))$ and thus also the operator L^2 is self-adjoint. Since $-\nu^2$ is not an eigenvalue of the self-adjoint operator $(L^2 + \nu^2 \operatorname{id})$, this operator $(L^2 + \nu^2 \operatorname{id})$ is invertible. We conclude by setting $K_\nu := (L + i\nu \operatorname{id})(L^2 + \nu^2 \operatorname{id})^{-1} = (L^2 + \nu^2 \operatorname{id})^{-1}(L + i\nu \operatorname{id})$, and observe that $K_\nu \circ L = L \circ K_\nu = \operatorname{id}$. □

The spectrum of a bounded linear operator is bounded.

Proposition 13.21. *Let X be a complete normed space over the field $\mathbb{K}$ and let $L \in \mathscr{L}(X,X)$. For every $\lambda \in \sigma(L)$, one has $|\lambda| \le \|L\|_{\mathscr{L}(X,X)}$.*

Proof. If $\lambda \in \mathbb{K}$ and $|\lambda| > \|L\|_{\mathscr{L}(X,X)}$, then $\|L/\lambda\|_{\mathscr{L}(X,X)}$. By proposition 6.2, $\operatorname{id} - L/\lambda$ is invertible, and then $\lambda^{-1}(\operatorname{id} - L/\lambda)^{-1}$ is the inverse of $(\lambda \operatorname{id} - L)$. □

Proposition 13.22. *Let X be a complete normed space over the field $\mathbb{K}$ and let $L \in \mathscr{L}(X,X)$. The set $\sigma(L)$ is closed.*

Proof. Assume that $\lambda_* \in \mathbb{K} \setminus \sigma(L)$, so that in particular $(L - \lambda \operatorname{id})^{-1} \in \mathscr{L}(X,X)$. For every $\lambda \in \mathbb{K} \setminus \sigma(L)$, we have

$$L - \lambda \operatorname{id} = (L - \lambda_* \operatorname{id}) + (\lambda_* - \lambda) \operatorname{id} = \big(\operatorname{id} - (\lambda - \lambda_*)(L - \lambda_* \operatorname{id})^{-1}\big)(L - \lambda_* \operatorname{id}). \tag{13.37}$$

If

$$|\lambda - \lambda_*| \|L - \lambda_* \operatorname{id}^{-1}\|_{\mathscr{L}(X,X)} < 1, \tag{13.38}$$

then $(\operatorname{id} - (\lambda - \lambda_*))(L - \lambda_* \operatorname{id})^{-1}$ is invertible. Hence $(L - \lambda \operatorname{id})$ is invertible, with

$$(L - \lambda \operatorname{id})^{-1} = (L - \lambda_* \operatorname{id})^{-1}\big(\operatorname{id} - (\lambda - \lambda_*)(L - \lambda_* \operatorname{id})^{-1}\big)^{-1}. \tag{13.39}$$

□

Proposition 13.23. *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is complete, if $L \in \mathscr{L}_c(X,X)$ and $\lambda \in \mathbb{K} \setminus \{0\}$ and if $\lambda \in \sigma(L)$, then λ is an eigenvalue of L.*

Proof. By proposition 13.3 and proposition 13.8, we need to prove that if $\lambda \in \sigma(L) \setminus \{0\}$, then λ is an approximate eigenvalue.

We first prove that λ is an approximate eigenvalue. We define for every $n \in \mathbb{N}$ the space

$$V_n := (L - \lambda \,\mathrm{id})^n(X). \tag{13.40}$$

If λ is not an approximate eigenvalue, by proposition 13.8 for every $n \in \mathbb{N}$, the mapping $(L - \lambda \,\mathrm{id})|_{V_n} \in \mathscr{L}(V_n, V_{n+1})$ has a continuous inverse. In particular, the spaces V_n are complete, and thus closed. Moreover, if $V_1 \neq X$, then $V_n \supsetneq V_{n+1}$ and thus by lemma 12.15 there exists $x_n \in V_n$ such that for every $x \in V_{n+1}$, $\|x_n - x\| \geq 1/2$. Hence, we have if $m > n$,

$$\|L(x_m) - L(x_n)\| = \|(L - \lambda \,\mathrm{id})(x_m - x_n) + \lambda x_m - \lambda x_n\| \geq \frac{|\lambda|}{2}, \tag{13.41}$$

since $(L - \lambda \,\mathrm{id})(x_m - x_n) + \lambda x_m \in V_{n+1}$. Since $\lambda > 0$, the inequality (13.41) contradicts the fact that L is a compact operator. □

13.5 Comments

The definition of compact operator is due to Friedrich Riesz [18].

In general, compact operators are a closed subset of bounded linear operators that contains finite-rank operators. Finite-rank operators into Y are dense in compact operators with respect to the norm topology for bounded linear operators if and only if Y has the *approximation property* that the identity operator on Y can be approximated uniformly on compact subsets of Y by finite rank operators [Meg98, §3.4].

The notion of spectrum can be generalised into ε–pseudospectrum, defined by

$$\sigma_\varepsilon(L) = \{\lambda \in \mathbb{K} \mid \|(L - \lambda \,\mathrm{id})^{-1}\|_{\mathscr{L}(X,X)} > \varepsilon^{-1}\}; \tag{13.42}$$

see [TE05].

Part IV

Duality

14 Dual space

14.1 Definitions

Definition 14.1. Given a real linear space X endowed with a norm $\|\cdot\|_X$, the dual space of X is the linear space

$$X^* := \mathscr{L}(X, \mathbb{R}), \tag{14.1}$$

endowed with the norm defined for $\ell \in X^*$ by

$$\|\ell\|_{X^*} := \|\ell\|_{\mathscr{L}(X,\mathbb{R})} = \sup\big\{|\langle \ell, x\rangle| \;\big|\; x \in X \text{ and } \|x\|_X \le 1\big\}. \tag{14.2}$$

The *duality product* of $\ell \in X^*$ and $x \in X$, is defined as

$$\langle \ell, x\rangle := \ell(x). \tag{14.3}$$

Proposition 14.2 (Bilinearity of the duality product)**.** *Let X be a linear space endowed with the norm $\|\cdot\|_X$, then*

(i) for every $\ell \in X^*$, $x, y \in X$ and $s, t \in \mathbb{R}$,

$$\langle \ell, tx + sy\rangle = t\langle \ell, x\rangle + s\langle \ell, y\rangle, \tag{14.4}$$

(ii) for every $k, \ell \in X^*$, $x \in X$ and $s, t \in \mathbb{R}$,

$$\langle tk + s\ell, x\rangle = t\langle k, x\rangle + s\langle \ell, x\rangle, \tag{14.5}$$

Proof. This follows from the linearity of elements on X^* and the definition of the linear structure on $X^* = \mathscr{L}(X, \mathbb{R})$. □

Proposition 14.3. *If the real linear space X endowed with a norm $\|\cdot\|_X$, then X^* endowed with $\|\cdot\|_{X^*}$ is complete.*

Proof. This follows from the definition of dual space X^* (definition 14.1) and from the completeness of $\mathscr{L}(X, \mathbb{R})$ (proposition 5.19). □

Remarkably, the completeness of the dual space X^* is independent on the completeness of X. This results from the fact that the dual space of any dense linear subspace of X is naturally isomorphic to X^*:

Proposition 14.4. *Let X be a normed space and $V \subseteq X$ be a dense linear subspace. The operator $\ell \in X^* \mapsto \ell|_V \in V^*$ is a linear isometry.*

Proof. This follows from proposition 7.4. □

14.2 Representation of dual spaces

14.2.1 The dual of ℓ^p and c_0 spaces

Proposition 14.5. *Let $p \in [1, \infty)$ and let Γ be set. For every $\ell \in \ell^p(\Gamma)^*$, there exists $g \in \ell^{p'}(\Gamma)$ such that for every $f \in \ell^p(\Gamma)$,*

$$\langle \ell, f \rangle = \sum_{x \in \Gamma} g(x) f(x). \tag{14.6}$$

Moreover, one has

$$\|g\|_{\ell^{p'}(\Gamma)} = \|\ell\|_{\ell^p(\Gamma)^*}. \tag{14.7}$$

Proof. For every $x \in \Gamma$, we define $\delta_x : \Gamma \to \mathbb{R}$ for $y \in \Gamma$ by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases} \tag{14.8}$$

We define then $g : \Gamma \to \mathbb{R}$ for each $x \in \Gamma$ by

$$g(x) := \langle \ell, \delta_x \rangle. \tag{14.9}$$

If $f \in c_c(\Gamma)$, then by definition (see (5.23)) there exists a finite set such that $f|_{\Gamma \setminus F} = 0$ and then

$$\langle \ell, f \rangle = \left\langle \ell, \sum_{x \in F} \delta_x f(x) \right\rangle = \sum_{x \in F} g(x) f(x). \tag{14.10}$$

We claim that $g \in \ell^{p'}(\Gamma)$. When $p = 1$, we then have for each $x \in \Gamma$

$$|g(x)| \le \|\ell\|_{\ell^1(\Gamma)^*} \|\delta_x\|_{\ell^1(\Gamma)} = \|\ell\|_{\ell^1(\Gamma)^*}, \tag{14.11}$$

and thus

$$\|g\|_{\ell^\infty(\Gamma)} = \sup_{x \in \Gamma} |g(x)| \le \|\ell\|_{\ell^1(\Gamma)} \tag{14.12}$$

When $p > 1$, given a finite set $F \subseteq \Gamma$, we define $h_F : \Gamma \to \mathbb{R}$ for $x \in \Gamma$ by

$$h_F(x) = \begin{cases} \operatorname{sgn}(g(x)) |g(x)|^{p'-1} & \text{if } x \in F, \\ 0 & \text{otherwise.} \end{cases} \tag{14.13}$$

Since the set F is finite, $h_F \in c_c(\Gamma)$, we have

$$\sum_{x \in F} |g(x)|^{p'} = \langle \ell, h_F \rangle \le \|\ell\|_{\ell^p(\Gamma)'} \|h_F\|_{\ell^p(\Gamma)} = \|\ell\|_{\ell^p(\Gamma)^*} \left(\sum_{x \in F} |g(x)|^{p'} \right)^{\frac{1}{p}}, \tag{14.14}$$

from which it follows that

$$\sum_{x\in F} |g(x)|^{p'} \le \|\ell\|^{p'}_{\ell^p(\Gamma)^*} \tag{14.15}$$

and thus since the finite set $F \subseteq \Gamma$ is arbitrary, by definition of infinite sum (definition C.1)

$$\|g\|^{p'}_{\ell^{p'}(\Gamma)} = \sum_{x\in\Gamma} |g(x)|^{p'} \le \|\ell\|_{\ell^p(\Gamma)'}. \tag{14.16}$$

By Hölder's inequality, one can define $k \in \ell^p(\Gamma)^*$ for $f \in \ell^p(\Gamma)$ by

$$\langle k, f\rangle = \sum_{x\in E} g(x)f(x). \tag{14.17}$$

By (14.10), the functionals k and ℓ coincide on $c_c(\Gamma)$. Since $c_c(\Gamma)$ is dense in $\ell^p(\Gamma)$ (proposition 7.7), we have $k = \ell$.

The converse inequality follows from proposition 4.22. □

Proposition 14.6. *Let Γ be set. For every $\ell \in c_0(\Gamma)^*$, there exists $g \in \ell^1(\Gamma)$ such that for every $f \in c_0(\Gamma)$,*

$$\langle \ell, f\rangle = \sum_{x\in\Gamma} g(x)f(x). \tag{14.18}$$

Moreover, one has

$$\|g\|_{\ell^1(\Gamma)} = \|\ell\|_{c_0(\Gamma)^*}. \tag{14.19}$$

Proof. For every $x \in \Gamma$, defining $\delta_x : \Gamma \to \mathbb{R}$ for $y \in \Gamma$ by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x,\\ 0 & \text{otherwise.}\end{cases} \tag{14.20}$$

we define the function $g : \Gamma \to \mathbb{R}$ for each $x \in \Gamma$ by

$$g(x) := \langle \ell, \delta_x\rangle. \tag{14.21}$$

If $f \in c_c(\Gamma)$, then by definition (see (5.23)) there exists a finite set such that $f|_{\Gamma\setminus F} = 0$ and then

$$\langle \ell, f\rangle = \left\langle \ell, \sum_{x\in F} \delta_x f(x) \right\rangle = \sum_{x\in F} g(x)f(x). \tag{14.22}$$

We claim that $g \in \ell^1(\Gamma)$. Given a finite set $F \subseteq \Gamma$, we define the function $h_F : \Gamma \to \mathbb{R}$ for $x \in \Gamma$ by

$$h_F(x) = \begin{cases} \operatorname{sgn}(g(x)) & \text{if } x \in F,\\ 0 & \text{otherwise.}\end{cases} \tag{14.23}$$

Since the set F is finite, $h_F \in c_c(\Gamma)$, we have

$$\sum_{x \in F} |g(x)| = \langle \ell, h_F \rangle \le \|\ell\|_{c_0(\Gamma)'} \|h_F\|_{c_0(\Gamma)} = \|\ell\|_{c_0(\Gamma)^*}. \tag{14.24}$$

Since the finite set $F \subseteq \Gamma$ is arbitrary, by definition of infinite sum (definition C.1)

$$\|g\|_{\ell^1(\Gamma)} = \sum_{x \in \Gamma} |g(x)| \le \|\ell\|_{c_0(\Gamma)'}. \tag{14.25}$$

One can define $k \in c_0(\Gamma)^*$ for $f \in c_0(\Gamma)$ by

$$\langle k, f \rangle = \sum_{x \in E} g(x) f(x). \tag{14.26}$$

By (14.22), the functionals k and ℓ coincide on $c_c(\Gamma)$. Since $c_c(\Gamma)$ is dense in $\ell^p(\Gamma)$ (proposition 7.8), we have $k = \ell$.

The converse inequality follows from proposition 4.22. □

14.2.2 The dual of Hilbert spaces

Proposition 14.7. *Let X endowed by $(\cdot \,|\, \cdot)$ be an inner product space. If X is complete, then for every $\ell \in X^*$, there exists a unique $y \in X$ such that for every $x \in X$,*

$$\langle \ell, x \rangle = (y \,|\, x). \tag{14.27}$$

Moreover, one has

$$\|y\|_X = \|\ell\|_{X^*} \tag{14.28}$$

Proof. We define the function $\varphi : X \to \mathbb{R}$ for $x \in X$ by

$$\varphi(x) := \frac{\|x\|_X^2}{2} - \langle \ell, x \rangle. \tag{14.29}$$

For every $x \in X$, we have

$$\varphi(x) \ge \frac{\|x\|_X^2}{2} - \|\ell\|_{X^*} \|x\|_X = \frac{(\|x\|_X - \|\ell\|_{X^*})^2}{2} - \frac{\|\ell\|_{X^*}^2}{2} \ge -\frac{\|\ell\|_{X^*}^2}{2}. \tag{14.30}$$

Defining

$$\nu := \inf\{\varphi(x) \mid x \in X\}, \tag{14.31}$$

we have $\nu \in [-\|\ell\|_{X^*}^2/2, \infty)$.

By definition of infimum as a greatest lower bound, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $(\varphi(x_n))_{n \in \mathbb{N}}$ converges to ν.

Moreover we have for every $m, n \in \mathbb{N}$, by the parallelogram law (proposition 1.24)

$$
\begin{aligned}
\varphi(x_n) + \varphi(x_m) &= \frac{\|x_n\|_X^2}{2} - \langle \ell, x_n \rangle + \frac{\|x_m\|_X^2}{2} - \langle \ell, x_m \rangle \\
&= \frac{\|x_n + x_m\|_X^2}{4} + \frac{\|x_n - x_m\|_X^2}{4} - \langle \ell, x_n + x_m \rangle \\
&= 2\varphi\Big(\frac{x_n + x_m}{2}\Big) + \frac{\|x_n - x_m\|^2}{4} \\
&\geq 2\nu + \frac{\|x_n - x_m\|_X^2}{4}.
\end{aligned} \tag{14.32}
$$

It follows thus that

$$
\lim_{m,n \to \infty} \|x_n - x_m\|_X^2 = \lim_{m,n \to \infty} 4\varphi(x_n) + 4\varphi(x_m) - 8\nu = 0, \tag{14.33}
$$

and thus $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X. Since the space X is complete by assumption, there exists $y \in X$ such that $(x_n)_{n \in \mathbb{N}}$ converges to y in X. We have by continuity of the norm and of ℓ,

$$
\varphi(y) = \frac{\|y\|_X^2}{2} - \langle \ell, y \rangle = \lim_{n \to \infty} \frac{\|x_n\|_X^2}{2} - \langle \ell, x_n \rangle = \lim_{n \to \infty} \varphi(x_n) = \nu. \tag{14.34}
$$

Hence, for every $t \in \mathbb{R}$ and $x \in X$, we have

$$
\begin{aligned}
0 \leq \varphi(y + tx) - \varphi(y) &= \frac{\|y + tx\|_X^2}{2} - \langle \ell, y + tx \rangle - \frac{\|y\|_X^2}{2} + \langle \ell, y \rangle \\
&= t(y \,|\, x) + \frac{t^2 \|x\|_X^2}{2} - t\langle \ell, x \rangle.
\end{aligned} \tag{14.35}
$$

Therefore for every $t \in (0, +\infty)$, dividing (14.35) by t we have

$$
\langle \ell, x \rangle \leq (y \,|\, x) + \frac{t\|x\|_X^2}{2}; \tag{14.36}
$$

letting $t \to 0$, this implies that

$$
\langle \ell, x \rangle \leq (y \,|\, x). \tag{14.37}
$$

By linearity, (14.37) implies that for every $x \in X$, we have

$$
\langle \ell, x \rangle = (y \,|\, x). \tag{14.38}
$$

Therefore, in view of proposition 4.16,

$$
\|\ell\|_{X^*} = \|y\|_X. \tag{14.39}
$$

Finally, assume that $z \in X$ and that for every $x \in X$,

$$
(z \,|\, x) = \langle \ell, x \rangle. \tag{14.40}
$$

Then, taking $x = y - z$, we get

$$
\|y - z\|^2 = (y \,|\, y - z) - (z \,|\, y - z) = \langle \ell, y - z \rangle - \langle \ell, y - z \rangle = 0, \tag{14.41}
$$

and therefore $y = z$. □

14.2.3 The dual of smooth uniformly convex spaces

Definition 14.8. Let X endowed with $\|\cdot\|_X$ be a normed space. The space X is uniformly convex whenever for every $\varepsilon > 0$, there exists $\delta > 0$ such that if for every $x, y \in X$ satisfying $\|x\|_X \le 1$, $\|y\|_X \le 1$ and $\|x + y\|_X \ge 2 - \delta$ implies $\|x - y\| \le \varepsilon$.

Proposition 14.9. *Let X endowed with $\|\cdot\|_X$ be a normed space. The space X is uniformly convex if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if for every $R \in [0, \infty)$, $x, y \in X$ satisfying $\|x\|_X \le R$, $\|y\|_X \le R$ and $\|x + y\|_X \ge (2 - \delta)R$ implies $\|x - y\| \le \varepsilon R$.*

Proof. If X is uniformly convex, then given $\varepsilon > 0$ let $\delta > 0$ be given by definition of uniform convexity (definition 14.8) if $R \in (0, \infty)$ and $\|x\|_X \le R$, $\|y\|_X \le R$ and $\|x + y\|_X \ge (2-\delta)R$ implies that $\|x/R\|_X \le 1$, $\|y/R\|_X \le 1$ and $\|x/R + y/R\|_X \ge (2 - \delta)$ and thus $\|x - y\|_X = R\|x/R - y/R\|_X \le \varepsilon R$ When $R = 0$, $\|x\|_X \le R$, $\|y\|_X \le R$ implies that $\|x - y\|_X \le 0$.

The converse implication is straightforward. □

Proposition 14.10. *If the space X is endowed with an inner product, then X is uniformly convex.*

Proof. Let $\|x\|_X$ be the norm associated to the inner product. For every $x, y \in X$ such that $\|x\|_X \le 1$ and $\|y\|_X \le 1$, we have by the parallelogram identity (proposition 1.24)

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 \le 4, \tag{14.42}$$

and thus

$$\|x - y\|^2 \le 4 - \|x + y\|^2. \tag{14.43}$$

Hence, for every $\varepsilon \in [0, 2)$ taking $\delta = 2 - \sqrt{4 - \varepsilon^2}$ we check that if $\|x + y\| \ge 2 - \delta$, then

$$\|x - y\|^2 \le 4 - (2 - \delta)^2 = \varepsilon. \tag{14.44}$$

The space X is thus uniformly convex in view of definition 14.8. □

Proposition 14.11. *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is uniformly convex and complete, then for every $\ell \in X^*$, there exists a unique $x \in X$ such that*

$$\|x\|_X^2 = \langle \ell, x\rangle = \|\ell\|_{X^*}^2. \tag{14.45}$$

Proof. We first have by definition of the dual norm (definition 14.1),

$$\|\ell\|_{X^*}^2 = \sup\bigl\{\langle \ell, x\rangle \bigm| x \in X \text{ and } \|x\|_X \le \|\ell\|_{X^*}\bigr\}. \tag{14.46}$$

By definition of supremum there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for every $n \in \mathbb{N}$, $\|x_n\|_X \le \|\ell\|_{X^*}$ and the sequence $(\langle \ell, x_n\rangle)_{n\in\mathbb{N}}$ converges to $\|\ell\|_{X^*}^2$ in $\mathbb{R}$.

Moreover, for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \langle \ell, x_n\rangle + \langle \ell, x_m\rangle &= \langle \ell, x_n + x_m\rangle \\ &\le \|\ell\|_{X^*}\|x_n + x_m\|_X \le \|\ell\|_{X^*}(\|x_n\|_X + \|x_m\|_X) \le 2\|\ell\|_{X^*}^2. \end{aligned} \tag{14.47}$$

By the sandwich property for the convergence of sequences, we have

$$\lim_{m,n\to\infty} \|x_n + x_m\|_X = 2\|\ell\|_{X^*}, \tag{14.48}$$

and thus by proposition 14.9, we have

$$\lim_{m,n\to\infty} \|x_m - x_n\|_X = 0. \tag{14.49}$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is thus a Cauchy sequence in X. By completeness of X, there exists $x \in X$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x in X. By construction, we have $\|x\|_X = \lim_{n\to\infty} \|x_n\|_X \le \|\ell\|_{X^*}$ and

$$\langle \ell, x\rangle = \lim_{n\to\infty} \langle \ell, x_n\rangle = \|\ell\|_{X^*}^2. \tag{14.50}$$

Finally, assume that for some $y \in X$, we have $\|y\|_X^2 = \langle \ell, y\rangle = \|\ell\|_{X^*}^2$. If $\ell = 0$, we have $x = y = 0$. Otherwise, we have

$$2\|\ell\|_{X^*}^2 = \langle \ell, x\rangle + \langle \ell, y\rangle \le \|\ell\|_{X^*}\|x + y\|_X, \tag{14.51}$$

from which it follows by uniform convexity (see proposition 14.9) that $x = y$. □

Definition 14.12. Let X endowed with $\|\cdot\|_X$ be a normed space. The space X is *smooth* whenever there exists a mapping $F' : X \to X^*$ such that if the function $F : X \to \mathbb{R}$ is defined for each $x \in X$ by $F(x) := \|x\|_X^2/2$, then for every $x \in X$,

$$\langle F'(x), y\rangle = \lim_{t\to 0} \frac{F(x + ty) - F(x)}{t}. \tag{14.52}$$

Proposition 14.13. *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is smooth, then for every $x \in X$ and $\ell \in X^*$,*

$$\|\ell\|_{X^*}^2 = \langle \ell, x\rangle = \|x\|_X^2. \tag{14.53}$$

if and only if $\ell = F'(x)$.

Proof. First given $x \in X$, we have by definition of smooth space (definition 14.12)

$$\begin{aligned}\langle F'(x), x\rangle &= \lim_{t\to 0} \frac{\|x + tx\|_X^2 - \|x\|_X^2}{2t}\\ &= \|x\|_X^2 \lim_{t\to 0} \frac{(1+t)^2 - 1}{2t} = \|x\|_X^2.\end{aligned} \tag{14.54}$$

The identity (14.54) also implies that

$$\|F'(x)\|_{X^*} \ge \|x\|_X. \tag{14.55}$$

Next, for every $t \in \mathbb{R}$ and $y \in X$, we have

$$\begin{aligned}\left|\|x + ty\|_X^2 - \|y\|_X^2\right| &= \bigl(\|x + ty\|_X + \|x\|_X\bigr)\left|\|x + ty\|_X - \|y\|_X\right|\\ &\le (\|x + ty\|_X + \|x\|_X)|t|\|y\|_X,\end{aligned} \tag{14.56}$$

and thus for every $t \in \mathbb{R} \setminus \{0\}$,

$$\left|\frac{\|x + ty\|_X^2 - \|y\|_X^2}{2t}\right| \le \frac{\|x + ty\|_X + \|x\|_X}{2}\|y\|_X, \tag{14.57}$$

so that

$$|\langle F'(x), y\rangle| \le \lim_{t\to 0} \frac{\|x + tv\|_X + \|x\|_X}{2}\|y\|_X = \|x\|_X\|y\|_X, \tag{14.58}$$

that is,

$$\|F'(x)\|_{X^*} \le \|x\|_X. \tag{14.59}$$

Hence we have proved that (14.53) holds when $\ell = F'(x)$.

Conversely, if (14.53) holds, then by definition of the dual norm (definition 14.1), for every $y \in X$,

$$\frac{\|y\|_X^2}{2} - \langle \ell, y\rangle \ge -\frac{\|\ell\|_{X^*}^2}{2} = \frac{\|x\|_X^2}{2} - \langle \ell, x\rangle. \tag{14.60}$$

It follows then from the smoothness of X (definition 14.12) that $\ell = F'(x)$. □

Proposition 14.14. *If the space X is endowed with an inner product, then X is smooth.*

Proof. For every $x, y \in X$ and $t \in \mathbb{R}$, we have, with the notations of definition 14.12,

$$\frac{F(x + ty) - F(x)}{t} = (x \,|\, y) + t\frac{\|y\|_X^2}{2}, \tag{14.61}$$

and thus the conclusion follows with

$$\langle F'(x), y\rangle = (x \,|\, y). \tag{14.62}$$

□

Theorem 14.15 (James representation theorem)**.** *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is smooth, uniformly convex and complete, then for every $\ell \in X^*$, there exists a unique $x \in X$ such that*

$$\ell = F'(x). \tag{14.63}$$

Moreover, one has

$$\|x\|_X^2 = \langle \ell, x\rangle = \|\ell\|_{X^*}^2. \tag{14.64}$$

Proof. This follows from proposition 14.11 and proposition 14.13. □

14.2.4 The dual of L^p spaces

Proposition 14.16. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If $1 < p < \infty$, then the space $L^p(\Omega, \mu)$ is uniformly convex.*

Proposition 14.17 (Hanner inequalities)**.** *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $p \in (1, \infty)$.*

(i) if $1 < p \le 2$, then for every $f, g \in L^p(\Omega, \mu)$,

$$\begin{aligned}\|f+g\|^p_{L^p(\Omega,\mu)} &+ \|f-g\|^p_{L^p(\Omega,\mu)}\\ &\ge \bigl|\|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)}\bigr|^p + \bigl|\|f\|_{L^p(\Omega,\mu)} - \|g\|_{L^p(\Omega,\mu)}\bigr|^p\end{aligned} \tag{14.65}$$

(ii) if $2 \le p < \infty$, then for every $f, g \in L^p(\Omega, \mu)$,

$$\begin{aligned}\|f+g\|^p_{L^p(\Omega,\mu)} &+ \|f-g\|^p_{L^p(\Omega,\mu)}\\ &\le \bigl|\|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)}\bigr|^p + \bigl|\|f\|_{L^p(\Omega,\mu)} - \|g\|_{L^p(\Omega,\mu)}\bigr|^p\end{aligned} \tag{14.66}$$

Lemma 14.18. *Let $a, b \in [0, \infty)$ and $s, t \in (0, \infty)$.*

(i) If $p \le 2$,

$$\begin{aligned}(s+t)^{p-1}\Bigl(\frac{a^p}{s^{p-1}} + \frac{b^p}{t^{p-1}}\Bigr) + \operatorname{sgn}(s-t)|s-t|^{p-1}\Bigl(\frac{a^p}{s^{p-1}} - \frac{b^p}{t^{p-1}}\Bigr)\\ \le (a+b)^p + |a-b|^p.\end{aligned} \tag{14.67}$$

(ii) If $p \ge 2$, then

$$\begin{aligned}(s+t)^{p-1}\Bigl(\frac{a^p}{s^{p-1}} + \frac{b^p}{t^{p-1}}\Bigr) + \operatorname{sgn}(s-t)|s-t|^{p-1}\Bigl(\frac{a^p}{s^{p-1}} - \frac{b^p}{t^{p-1}}\Bigr)\\ \ge (a+b)^p + |a-b|^p.\end{aligned} \tag{14.68}$$

Proof. Let the function $h : (0, \infty) \to \mathbb{R}$ be defined for $r \in (0, \infty)$ by

$$\begin{aligned}h(r) :=&\bigl((1+r)^{p-1} + \operatorname{sgn}(1-r)|1-r|^{p-1}\bigr)a^p\\ &+ \bigl((1+1/r)^{p-1} - \operatorname{sgn}(1-r)|1-1/r|^{p-1}\bigr)b^p,\end{aligned} \tag{14.69}$$

so that if $r = t/s$ with $s, t \in (0, \infty)$, we have

$$\begin{aligned}h(t/s) &= \bigl((s+t)^{p-1} + \operatorname{sgn}(s-t)|s-t|^{p-1}\bigr)\frac{a^p}{s^{p-1}}\\ &\qquad + \bigl((s+t)^{p-1} - \operatorname{sgn}(s-t)|s-t|^{p-1}\bigr)\frac{b^p}{t^{p-1}}\\ &= (s+t)^{p-1}\Bigl(\frac{a^p}{s^{p-1}} + \frac{b^p}{t^{p-1}}\Bigr) + \operatorname{sgn}(s-t)|s-t|^{p-1}\Bigl(\frac{a^p}{s^{p-1}} - \frac{b^p}{t^{p-1}}\Bigr).\end{aligned} \tag{14.70}$$

We have if $r \in (0,1) \cup (1,\infty)$,

$$h'(r) = (p-1)\big((1+r)^{p-2} - |1-r|^{p-2}\big)\Big(a^p - \frac{b^p}{r^p}\Big). \tag{14.71}$$

Since for every $r \in (0,\infty)$

$$|1-r| \le 1+r, \tag{14.72}$$

When $p \le 2$, the function h is nondecreasing when $r \le b/a$ and nonincreasing when $r \ge b/a$, and achieves thus its maximum at $r = b/a$. By our optimality result above, we have if $s,t \in (0,\infty)$,

$$h(t/s) \le h(b/a). \tag{14.73}$$

which is the desired inequality.

When $p \ge 2$, the function h is nonincreasing when $r \le b/a$ and nondecreasing when $r \ge b/a$, and achieves thus its minimum at $r = b/a$. We then have $s,t \in (0,\infty)$,

$$h(t/s) \ge h(b/a), \tag{14.74}$$

which is the conclusion. □

Proof of proposition 14.17. We consider the case $p \le 2$. We assume without loss of generaly that $\|f\|_{L^p(\Omega,\mu)} > 0$ and $\|g\|_{L^p(\Omega,\mu)} > 0$,

By lemma 14.18, we have for every $s,t \in (0,\infty)$,

$$\begin{aligned}(s+t)^{p-1}\Big(\frac{|f(x)|^p}{s^{p-1}} + \frac{|g(x)|^p}{t^{p-1}}\Big) + \operatorname{sgn}(s-t)|s-t|^{p-1}\Big(\frac{|f(x)|^p}{s^{p-1}} - \frac{|g(x)|^p}{t^{p-1}}\Big)\\ \le \big(|f(x)|+|g(x)|\big)^p + \big||f(x)|-|g(x)|\big|^p = |f(x)+g(x)|^p + |f(x)-g(x)|^p.\end{aligned} \tag{14.75}$$

Integrating (14.75), we get

$$\begin{aligned}(s+t)^{p-1}\left(\frac{\|f\|^p_{L^p(\Omega,\mu)}}{s^{p-1}} + \frac{\|g\|^p_{L^p(\Omega,\mu)}}{t^{p-1}}\right) + \operatorname{sgn}(s-t)|s-t|^{p-1}\left(\frac{\|f\|^p_{L^p(\Omega,\mu)}}{s^{p-1}} - \frac{\|g\|^p_{L^p(\Omega,\mu)}}{t^{p-1}}\right)\\ \le \|f+g\|^p_{L^p(\Omega,\mu)} + \|f-g\|^p_{L^p(\Omega)}.\end{aligned} \tag{14.76}$$

Taking $s = \|f\|_{L^p(\Omega,\mu)}$ and $t = \|g\|_{L^p(\Omega,\mu)}$, we conclude. □

Proof of proposition 14.16. For each $f,g \in L^p(\Omega,\mu)$ satisfying $\|f\|_{L^p(\Omega,\mu)} \le 1$, $\|g\|_{L^p(\Omega,\mu)} \le 1$ and $\|f+g\|_{L^p(\Omega,\mu)} \ge 2-\delta$, we have by the triangle inequality

$$\|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)} \ge \|f+g\|_{L^p(\Omega,\mu)} \ge 2-\delta, \tag{14.77}$$

and thus

$$\|f\|_{L^p(\Omega,\mu)} - \|g\|_{L^p(\Omega,\mu)} = \|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega)} - 2\|g\|_{L^p(\Omega,\mu)} \le \delta \tag{14.78}$$

and

$$\|g\|_{L^p(\Omega,\mu)} - \|f\|_{L^p(\Omega,\mu)} = \|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega)} - 2\|f\|_{L^p(\Omega,\mu)} \le \delta, \tag{14.79}$$

so that

$$\big|\|f\|_{L^p(\Omega,\mu)} - \|g\|_{L^p(\Omega,\mu)}\big| \le \delta. \tag{14.80}$$

If $p \ge 2$, by Hanner's inequality, we deduce that proposition 14.17,

$$\|f - g\|^p_{L^p(\Omega,\mu)} \le 2^p + \delta^p - (2-\delta)^p. \tag{14.81}$$

Taking $\delta > 0$ small enough, we can make the right-hand side smaller than any given $\varepsilon > 0$. The space $L^p(\Omega,\mu)$ satisfies thus the definition of uniformly convex space (definition 14.8) when $2 \le p < \infty$.

When $1 < p \le 2$, we have as a consequence of Hanner's inequality (proposition 14.17) that

$$H\big(\|f+g\|_{L^p(\Omega,\mu)}, \|f-g\|_{L^p(\Omega,\mu)}\big) \le 2^p\Big(\|f\|^p_{L^p(\Omega,\mu)} + \|g\|^p_{L^p(\Omega,\mu)}\Big). \tag{14.82}$$

where the function $H : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is defined for $(s,t) \in [0,\infty) \times [0,\infty)$ by

$$H(s,t) \coloneqq |s+t|^p - |s-t|^p. \tag{14.83}$$

For every $(s,t) \in (0,\infty)$, the functions $H(\cdot,t)$ and $H(s,\cdot)$ are continuous and strictly increasing: indeed if $s,t \in (0,\infty)$,

$$\frac{d}{ds}H(s,t) = p\big(|s+t|^{p-1} - \operatorname{sgn}(s-t)|s-t|^{p-1}\big) > 0 \tag{14.84}$$

Let $\varepsilon \in (0,2)$. Since $H(2,0) = 2^{p+1}$, we have $H(2,\varepsilon) > 2^{p+1}$. Since moreover $H(0,\varepsilon) = \varepsilon^{p+1} < 2^{p+1}$, there exists $\delta \in (0,2)$ such that $H(2-\delta,\varepsilon) = 2^{p+1}$. Hence if $\|f+g\|_{L^p(\Omega,\mu)} \ge 2-\delta$, we have by (14.82)

$$\begin{aligned} H\big(2-\delta, \|f-g\|_{L^p(\Omega,\mu)}\big) &\le H\big(\|f+g\|_{L^p(\Omega,\mu)}, \|f-g\|_{L^p(\Omega,\mu)}\big) \\ &\le 2^{p+1} = H(2-\delta,\varepsilon). \end{aligned} \tag{14.85}$$

Since H is strictly increasing with respect to its second argument, this implies that $\|f - g\|_{L^p(\Omega,\mu)} \le \delta$. □

Proposition 14.19. *Let $\mu : \Sigma \to [0,\infty]$ be a measure on Ω. The space $L^p(\Omega,\mu)$ is smooth and for every $f, g \in L^p(\Omega,\mu)$*

$$\langle F'(f), g\rangle = \frac{1}{\|f\|^{p-2}_{L^p(\Omega,\mu)}} \int_\Omega \operatorname{sgn}(f)|f|^{p-1} g \,\mathrm{d}\mu. \tag{14.86}$$

Proof. Let $f, g \in L^p(\Omega, \mu)$. For every $t \in \mathbb{R} \setminus \{0\}$, we have

$$\int_\Omega \frac{|f + tg|^p - |f|^p}{pt} \,\mathrm{d}\mu = \int_\Omega \int_0^1 \operatorname{sgn}(f + \tau t g)|f + \tau t g|^{p-1} g(x) \,\mathrm{d}\tau \,\mathrm{d}\mu. \tag{14.87}$$

Since if $|t| \le 1$,

$$\bigl|\operatorname{sgn}(f + \tau t g)\,|f + \tau t g|^{p-1} g\bigr| \le \bigl(|f| + |g|\bigr)^{p-1}|g|, \tag{14.88}$$

and since by Hölder's and Minkowski inequalities

$$\int_\Omega \bigl(|f| + |g|\bigr)^{p-1}|g| \,\mathrm{d}\mu \le \bigl(\|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)}\bigr)^{p-1}\|g\|_{L^p(\Omega,\mu)}, \tag{14.89}$$

we have by Lebesgue's dominated convergence theorem

$$\lim_{t \to 0} \int_\Omega \frac{|f + tg|^p - |f|^p}{pt} \,\mathrm{d}\mu = \int_\Omega \operatorname{sgn}(f)|f|^{p-1} g \,\mathrm{d}\mu. \tag{14.90}$$

In order to conclude, we note that

$$F(f) = \frac{\|f\|^2_{L^p(\Omega,\mu)}}{2} = \frac{p^{2/p}}{2}\Bigl(\int_\Omega \frac{|f|^p}{p} \,\mathrm{d}\mu\Bigr)^{2/p}, \tag{14.91}$$

and therefore

$$\begin{aligned} \langle F'(f), g\rangle &= p^{-\frac{p-2}{2}}\Bigl(\int_\Omega \frac{|f|^p}{p} \,\mathrm{d}\mu\Bigr)^{-\frac{p-2}{p}} \int_\Omega \operatorname{sgn}(f)|f|^{p-1} g \,\mathrm{d}\mu \\ &= \frac{1}{\|f\|^{p-2}_{L^p(\Omega,\mu)}} \int_\Omega \operatorname{sgn}(f)|f|^{p-1} g \,\mathrm{d}\mu. \end{aligned} \tag{14.92}$$

□

Theorem 14.20. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. For every $\ell \in L^p(\Omega, \mu)^*$, there exists a unique $g \in L^{\frac{p}{p-1}}(\Omega, \mu)$ such that for every $f \in L^p(\Omega, \mu)$,*

$$\langle \ell, f\rangle = \int_\Omega gf \,\mathrm{d}\mu. \tag{14.93}$$

Moreover,

$$\|g\|_{L^{\frac{p}{p-1}}(\Omega,\mu)} = \|\ell\|_{L^p(\Omega,\mu)^*}. \tag{14.94}$$

Proof. This follows from proposition 14.16, proposition 14.19 and theorem 14.15. □

14.3 Extension of linear functionals

Theorem 14.21 (Hahn–Banach). *Let X endowed with $\|\cdot\|_X$ be a normed space and let $Y \subseteq X$ be a linear subspace. For every $k \in \mathscr{L}(Y, \mathbb{R})$, there exists $\ell \in \mathscr{L}(X, \mathbb{R})$ such that for every $y \in Y$, $\langle \ell, y \rangle = \langle k, y \rangle$ and $\|\ell\|_{\mathscr{L}(X,\mathbb{R})} = \|\ell\|_{\mathscr{L}(Y,\mathbb{R})}$.*

Proof of theorem 14.21 when X is an inner product space. By replacing the space X by its completion if necessary, we can assume that X is complete. Moreover, by proposition 7.4, we can assume that the space Y is closed. The space Y endowed with the same inner product as X is thus a complete inner product space. By proposition 14.7, there exists $z \in Y$ such that for every $y \in Y$, $\langle k, y \rangle = (z \,|\, y)$ and such that $\|z\|_X = \|k\|_{\mathscr{L}(Y,\mathbb{R})}$. Defining $\ell : X \to \mathbb{R}$ for $x \in X$ by $\langle \ell, x \rangle := (z \,|\, x)$, we conclude. □

Proof of theorem 14.21 when X is complete and uniformly convex. By replacing the space X by its completion if necessary, we can assume that X is complete. Moreover, by proposition 7.4, we can assume that the space Y is closed. Moreover Y is uniformly convex and smooth, with $F'_Y : y \in Y \mapsto F'(y)|_Y$ the corresponding mapping. By theorem 14.15, there exists $y \in Y$ such that $k = F'(y)$. Setting $k = F'(y)$, we reach the conclusion. □

Definition 14.22. Let X be a linear space. The function $\phi : X \to \mathbb{R}$ is *convex* whenever for every $x, y \in X$ and $t \in [0, 1]$, $\phi((1-t)x + ty) \le (1-t)\phi(x) + t\phi(y)$.

Definition 14.23. Let X be a linear space. The function $\phi : X \to \mathbb{R}$ is *positively homogeneous* whenever for every $x \in X$ and $t \in [0, \infty)$, $\phi(tx) = t\phi(x)$.

Theorem 14.24. *Let X be a linear space and let $\phi : X \to \mathbb{R}$ be convex and positively homogeneous. If $Y \subseteq X$ is a linear subspace, if $k : Y \to \mathbb{R}$ is linear and if for every $y \in Y$, $k(y) \le \phi(y)$, then there exists a linear mapping $\ell : X \to \mathbb{R}$ such that for every $y \in Y$, $\ell(y) = k(y)$ and such that for every $x \in X$, $\ell(x) \le \phi(x)$.*

Lemma 14.25. *Theorem 14.24 holds when $X = Y \oplus \mathbb{R}z$.*

Proof. We set $\ell(y + tz) := k(y) + \alpha t$, where $\alpha \in \mathbb{R}$ is chosen in such a way that for each $t \in \mathbb{R}$ and $y \in Y$, we have

$$k(y) + \alpha t \le \phi(y + tz). \tag{14.95}$$

This inequality (14.95) holds for $t = 0$. By positive homogeneity of ϕ and linearity of k, we need to check that for every $t \in (0, \infty)$

$$k(y/t) + \alpha \le \phi(y/t + z). \tag{14.96}$$

and that for every $t \in (-\infty, 0)$

$$k(-y/t) - \alpha \le \phi(-y/t - z). \tag{14.97}$$

Since Y is a linear space, this is equivalent to check that for every $u, v \in Y$,

$$k(v) - \phi(v - z) \le \alpha \le \phi(u + z) - k(u). \tag{14.98}$$

We have for every $u, v \in Y$, we have by our assumptions on k and on ϕ,

$$
\begin{aligned}
k(u)+k(v) = k(u+v) \le \phi(u+v) &= 2\phi\Big(\frac{u+z+v-z}{2}\Big)\\
&\le 2\Big(\phi\Big(\frac{u+z}{2}\Big)+\phi\Big(\frac{v-z}{2}\Big)\Big) = \phi(u+z)+\phi(v-z).
\end{aligned}
\tag{14.99}
$$

and thus

$$
k(v)-\phi(v-z) \le \phi(u+z)-k(u). \tag{14.100}
$$

Hence

$$
\sup\{k(v)-\phi(v-z) \mid v \in Y\} \le \inf\{\phi(u+z)-k(u) \mid u \in Y\}, \tag{14.101}
$$

and there exists $\alpha \in \mathbb{R}$ such that (14.98) holds. This concludes the proof. □

Proof of theorem 14.21 when X is separable. Let $\{x_n \mid n \in \mathbb{N}\} \subseteq X$ be a dense subset of the space X. We define $X_0 := Y$ and $\ell_0 := k$. For $n \in \mathbb{N}$, we set $X_{n+1} := X_n + \mathbb{R}x_n$. We define inductively $\ell_n \in \mathscr{L}(X_n, \mathbb{R})$ so that $\ell_n|_Y = k$ and $\|\ell_n\|_{\mathscr{L}(X_n,\mathbb{R})} = \|k\|_{\mathscr{L}(Y,\mathbb{R})}$. If $X_{n+1} = X_n$, we set $\ell_{n+1} := \ell_n$. Otherwise, we have $X_{n+1} = X_n \oplus \mathbb{R}x_n$. Letting for $x \in X_{n+1}$, $\phi_{n+1}(x) := \|k\|_{\mathscr{L}(Y,\mathbb{R})}\|x\|_X$, by lemma 14.25, there exists $\ell_{n+1} : X_{n+1} \to \mathbb{R}$ such that $\ell_{n+1}|_{X_n} = \ell_n$ and for every $x \in X_{n+1}$, $\ell_{n+1}(x) \le \phi_{n+1}(x) = \|k\|_{\mathscr{L}(Y,\mathbb{R})}\|x\|_X$. Hence $\ell_{n+1} \in \mathscr{L}(X_{n+1}, \mathbb{R})$ and $\|\ell_{n+1}\|_{\mathscr{L}(X_{n+1},\mathbb{R})} = \|k\|_{\mathscr{L}(Y,\mathbb{R})}$.

We define $X_* := \bigcup_{n\in\mathbb{N}} X_n$. For every $x \in X_*$, we set $\ell_*(x) := \ell_n(x)$, for any $n \in \mathbb{N}$ such that $X_n \ni x$. We have $\ell_* \in \mathscr{L}(X_*, \mathbb{R})$, $\|\ell_*\|_{\mathscr{L}(X_*,\mathbb{R})} = \|k\|_{\mathscr{L}(Y,\mathbb{R})}$ and for every $x \in Y$, $\langle \ell_*, x\rangle = \langle k, x\rangle$.

Since $\{x_n \mid n \in \mathbb{N}\} \subseteq X_*$, the set X_* is dense in X. In view of proposition 7.4, we can take $\ell \in \mathscr{L}(X, \mathbb{R})$ such that $\|\ell\|_{\mathscr{L}(X,\mathbb{R})} = \|\ell_*\|_{\mathscr{L}(X_*,\mathbb{R})} = \|k\|_{\mathscr{L}(Y,\mathbb{R})}$ and $\ell|_{X_*} = \ell_*$, so that $\ell|_Y = \ell_*|_Y = k$. □

Proof of theorem 14.24. We define the set

$$
\begin{aligned}
\mathscr{S}\{(Z,h) \mid Y \subseteq Z \subseteq X,\ &Z \text{ is a linear subspace,}\\
&h : Z \to \mathbb{R} \text{ is linear and for every } z \in Z,\ h(z) \le \phi(z)\}
\end{aligned}
\tag{14.102}
$$

We define $(Z_1, h_1) \preccurlyeq (Z_2, h_2)$ whenever $Z_1 \subseteq Z_2$ and $h_2|_{Z_1} = h_1$. One can readily check that $\preccurlyeq$ is an order on $\mathscr{S}$ and that $(k, Y) \in \mathscr{S}$.

If $\mathscr{R} \subseteq \mathscr{S}$, if the order $\preccurlyeq$ is total on $\mathscr{R}$ and if $\mathscr{R} \ne \emptyset$ then we set

$$
Z_* := \bigcup_{(Z,h)\in\mathscr{R}} Z, \tag{14.103}
$$

and for each $z \in Z_*$, $h_*(z) := h(z)$, where $h \in Z$ for some $(Z,h) \in \mathscr{R}$. Since the order $\preccurlyeq$ is total on $\mathscr{R}$, h_* is well-defined.

By Zorn's lemma (theorem E.5), there exists $(Z, \ell) \in \mathscr{S}$ which is maximal for $\preccurlyeq$. We claim that $Z = X$. Otherwise, there would exist $z \in X \setminus Z$ and by lemma 14.25, there would exist $h : Z + \mathbb{R}z \to \mathbb{R}$ such that $h|_Z = \ell$ and $h \le \phi$, in contradiction with the maximality of (Z, ℓ). Hence $Z = X$ and the theorem is proved. □

Proof of theorem 14.21 in the general case. Defining $\phi : X \to \mathbb{R}$ by $\phi(x) := \|\ell\|_{\mathscr{L}(X,\mathbb{R})}$, we apply theorem 14.24 and we conclude. □

Proposition 14.26. *Let X endowed with $\|\cdot\|_X$ be a normed space. For every $x \in X$, there exists $\ell \in X^*$ such that*

$$\|\ell\|_{X^*}^2 = \langle \ell, x\rangle = \|x\|_X^2. \tag{14.104}$$

Proof. We set $Y := \mathbb{R}x$ and we define $k : X \to \mathbb{R}$, by setting for every $t \in \mathbb{R}$, $\langle k, tx\rangle := t\|x\|_X^2$. By theorem 14.21, there exists $\ell \in \mathscr{L}(X,\mathbb{R})$ such that $\|\ell\|_{X^*} = \|k\|_{\mathbb{R}x} = \|x\|_X$ and $\ell|_Y = k$, so that $\langle \ell, x\rangle = \langle k, x\rangle = \|x\|_X^2$. □

Remark 14.27. In many cases, one does not need theorem 14.21 to get the existence of ℓ in proposition 14.26:

a. if $x \in X$ with X is an inner product space, one can define for $y \in X$, $\langle \ell, y\rangle = (x \,|\, y)$,

b. if $x \in X$ with X a smooth space, one can take $\ell = F'(x)$ in view of proposition 14.13,

c. if $f \in L^p(\Omega,\mu)$, where $\mu : \Omega \to [0,\infty]$ is a measure and $1 \le p < \infty$, one can take

$$\langle \ell, g\rangle := \frac{1}{\|f\|_{L^p(\Omega,\mu)}^{p-2}} \int_\Omega |f|^{p-1} \operatorname{sgn}(f) g \,\mathrm{d}\mu. \tag{14.105}$$

Proposition 14.28. *Let X be a normed space. If* $\dim X = \infty$*, then* $\dim X^* = \infty$.

Proof. Let $\ell_1, \dots, \ell_d \in X^*$. We need to show that $\ell_1, \dots, \ell_d$ do not form a basis of the space X^*. We define

$$V := \bigcap_{i=1}^d \ker \ell_i. \tag{14.106}$$

Since $\dim X = \infty$, we have $V \ne \{0\}$. We fix $x \in V \setminus \{0\}$. By proposition 14.26, there exists $k \in V^*$ such that

$$\|k\|_{V^*}^2 = \langle k, x\rangle = \|x\|_X^2 > 0. \tag{14.107}$$

In particular $k \ne 0$.

By theorem 14.21, there exists $\ell \in X^*$ such that $\|\ell\|_{X^*} = \|k\|_{V^*}$ and $\ell|_V = k$. Since $\ell|_V \ne 0$, ℓ is not a linear combination of $\ell_1, \dots, \ell_d$ and thus $\ell_1, \dots, \ell_d$ is not a basis of X^*. □

Proposition 14.29. *Let X be a normed space and let $Y \subseteq X$ be a linear space. Then*

$$\bar{Y} = \left\{ x \in X \;\middle|\; \text{if } \ell \in X^* \text{ and } \ell|_Y = 0, \text{ then } \langle \ell, x\rangle = 0 \right\} \tag{14.108}$$

Proof. If $x \in \bar{Y}$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in Y converging to x. If $\ell \in X^*$ and $\ell|_Y = 0$, then for every $n \in \mathbb{N}$, $\langle \ell, x_n\rangle = 0$ and thus $\langle \ell, x\rangle = 0$.

Conversely, let $x \in X \setminus \bar{Y}$. Since $\bar{Y}$ is closed, by lemma 3.18 and theorem 14.21, there exists $\ell \in X^*$ such that $\langle \ell, x\rangle = 1$. This concludes the proof. □

Proposition 14.30. *Let X be a normed space and let $Y \subseteq X$ be a linear space. For every $x \in X$,*

$$\inf\{\|x-y\|_X \mid y \in Y\} = \sup\{\langle \ell, x\rangle \mid \ell \in X^*, \|\ell\|_{X^*} \le 1 \text{ and } \ell|_Y = 0\}. \tag{14.109}$$

Proof. If $\ell \in X^*$, $\|\ell\|_{X^*} \le 1$ and $\ell|_Y = 0$, then for every $y \in Y$, we have

$$\langle \ell, x\rangle = \langle \ell, x-y\rangle \le \|\ell\|_{X^*}\|x-y\|_X \le \|x-y\|_X. \tag{14.110}$$

Conversely, we define for each $z \in X$,

$$\phi(z) := \inf\{\|z-y\|_X \mid y \in Y\}. \tag{14.111}$$

The function ϕ is a seminorm. Given $x \in X$, we define the function $k : Y + \mathbb{R}x \to \mathbb{R}$ by requiring for $y \in Y$ and $t \in \mathbb{R}$ that $k(y+tx) = \phi(x)t$. We have $|k(y+tx)| = \phi(tx) \le \|y+tx\|_X$, and thus $\|k\|_{(Y+\mathbb{R}x)^*} \le 1$. By the Hahn–Banach theorem (theorem 14.21), there exists $\ell : X \to \mathbb{R}$ such that $\ell|_{Y+\mathbb{R}x} = k$ and for every $\|\ell\|_{X^*} \le 1$. In particular $\ell|_Y = 0$ and $\ell(x) = \phi(x)$. □

Proposition 14.31. *Let X be normed space. If X^* is separable, then X is separable.*

Proof. Let $\{\ell_n \mid n \in \mathbb{N}\} \subseteq X^*$ be dense in X^*. By definition of norm in X^*, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\|x_n\|_X = 1$ and $\langle \ell_n, x_n\rangle \ge \|\ell_n\|_{X^*}/2$. Let Y be the linear space spanned by $\{x_n \mid n \in \mathbb{N}\}$.

Assume that $\ell \in X^*$ and that $\ell|_Y = 0$. By construction, we have for every $n \in \mathbb{N}$,

$$\begin{aligned} \|\ell\|_{X^*} &\le \|\ell-\ell_n\|_{X^*} + \|\ell_n\|_{X^*} \le \|\ell-\ell_n\|_{X^*} + 2\langle \ell_n, x_n\rangle \\ &= \|\ell-\ell_n\|_{X^*} + 2\langle \ell_n - \ell, x_n\rangle \le 3\|\ell_n-\ell\|_{X^*}. \end{aligned} \tag{14.112}$$

Since the right-hand side can be made arbitrarily small, we have $\ell = 0$. By proposition 14.29, we have thus that $\bar{Y} = X$, and hence the space X is separable. □

14.4 Bidual space and reflexivity

14.4.1 Definition and elementary properties

Definition 14.32. Given a normed space X, the *bidual* of X is the space

$$X^{**} := (X^*)^* = \mathscr{L}(\mathscr{L}(X,\mathbb{R}),\mathbb{R}), \tag{14.113}$$

endowed with the norm defined for $\xi \in X^{**}$ by

$$\|\xi\|_{X^{**}} := \|\xi\|_{(X^*)^*} = \|\xi\|_{\mathscr{L}(\mathscr{L}(X,\mathbb{R}),\mathbb{R})}. \tag{14.114}$$

Proposition 14.33. *Let X be a normed space. The mapping $J : X \to X^{**}$ defined for $x \in X$ and $\ell \in X^*$ by*

$$\langle J(x), \ell \rangle = \langle \ell, x \rangle, \tag{14.115}$$

is linear and for every $x \in X$,

$$\|J(x)\|_{X^{**}} = \|x\|_X. \tag{14.116}$$

In particular, J is injective.

Proof. The linearity follows from the definition. The identity (14.116) follows from (14.114) and proposition 14.26:

$$\begin{aligned} \|J(x)\|_{X^{**}} &= \sup\{\langle J(x), \ell \rangle \mid \ell \in X^* \text{ and } \|\ell\|_{X^*} \le 1\} \\ &= \sup\{\langle \ell, x \rangle \mid \ell \in X^* \text{ and } \|\ell\|_{X^*} \le 1\} = \|x\|_X. \end{aligned}$$ □

Definition 14.34. Let X be a normed space. The space X is *reflexive* whenever for every $\xi \in X^{**}$, there exists $x \in X$ such that for every $\ell \in X^*$,

$$\langle \xi, \ell \rangle = \langle \ell, x \rangle. \tag{14.117}$$

In other words, X is reflexive if and only if $J : X \to X^{**}$ is surjective.

Proposition 14.35. *Let X be a normed space. If* $\dim X < \infty$*, then X is reflexive.*

Proof. Since the space X is finite-dimensional, it has a basis $x_1, \dots, x_d \in X$. Moreover, there exist $\ell_1, \dots, \ell_d \in X^*$ such that $\langle \ell_i, x_j \rangle = \delta_{ij}$. In particular, for every $\ell \in X^*$, we have $\ell = \sum_{i=1}^d \langle \ell, x_i \rangle \ell$. Given $\xi \in X^{**}$, we have for every $\ell \in X^*$,

$$\langle \xi, \ell \rangle = \sum_{i=1}^d \langle \xi, \ell_i \rangle \langle \ell, x_i \rangle = \langle \ell, x \rangle, \tag{14.118}$$

with

$$x := \sum_{i=1}^d \langle \xi, \ell_i \rangle x_i. \tag{14.119}$$

Hence the space X is reflexive. □

Proposition 14.36. *Let X endowed with $\|\cdot\|_X$ be a normed space. If X is reflexive, then X is complete.*

Proof. Let $J : X \to X^{**}$ be the bidual mapping given by proposition 14.33.

Assume now that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X. We then have for $m, n \in \mathbb{N}$, since J is a linear isometry,

$$\|J(x_n) - J(x_m)\|_{X^{**}} = \|J(x_n - x_m)\|_{X^{**}} = \|x_n - x_m\|_X, \tag{14.120}$$

so that $(J(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in X^{**}. Since $X^{**} = \mathscr{L}(X^*, \mathbb{R})$, by proposition 5.19, X^{**} endowed with $\|\cdot\|_{X^{**}} = \|\cdot\|_{\mathscr{L}(X^*,\mathbb{R})}$ is complete. Therefore, there exists $\xi_* \in X^{**}$ such that $(J(x_n))_{n\in\mathbb{N}}$ converges to ξ_* in X^{**}. In view of definition 14.34, the reflexivity assumption means that $J(X) = X^{**}$. Hence there exists $x_* \in X$ such that $J(x_*) = \xi_*$. We have then for each $n \in \mathbb{N}$

$$\|x_n - x_*\|_X = \|J(x_n - x_*)\|_{X^{**}} = \|J(x_n) - J(x_*)\|_{X^{**}} = \|J(x_n) - \xi_*\|_{X^{**}}, \tag{14.121}$$

so that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_*. Hence X is complete. □

Proposition 14.37. *Let X be a normed space. If X is reflexive, then for every $\ell \in X^*$ there exists $x \in X$ such that*

$$\|x\|_X^2 = \langle \ell, x\rangle = \|\ell\|_{X^*}^2. \tag{14.122}$$

Proof. By proposition 14.26, there exists $\xi \in X^{**}$ such that

$$\|\xi\|_{X^{**}}^2 = \langle \xi, \ell\rangle = \|\ell\|_{X^*}^2. \tag{14.123}$$

Since X is reflexive, there exists $x \in X$ such that $J(x) = \xi$. We have then

$$\|x\|_X^2 = \|J(x)\|_{X^{**}}^2 = \|\xi\|_{X^{**}}^2 \tag{14.124}$$

and

$$\langle \ell, x\rangle = \langle J(x), \ell\rangle = \langle \xi, \ell\rangle, \tag{14.125}$$

and the conclusion follows. □

Proposition 14.38. *Let X endowed with $\|\cdot\|_X$ be a normed space and let $Y \subseteq X$ be a linear subspace. If X is reflexive and Y is closed, then Y is reflexive.*

Proof. For $\xi \in Y^{**}$, we define $\tilde{\xi} \in X^{**}$ by setting for every $\tilde{\ell} \in X^*$,

$$\langle \tilde{\xi}, \tilde{\ell}\rangle := \langle \xi, \tilde{\ell}|_Y\rangle. \tag{14.126}$$

Since X is reflexive, by definition 14.34 there exists $x \in X$ such that for every $\tilde{\ell} \in X^*$,

$$\langle \tilde{\xi}, \tilde{\ell}\rangle = \langle \tilde{\ell}, x\rangle. \tag{14.127}$$

If $\tilde{\ell} \in X^*$ and $\tilde{\ell}|_Y = 0$, then by (14.126) and (14.127) $\langle \tilde{\ell}, x\rangle = 0$. Since Y is closed, by proposition 14.29, we have $x \in Y$. Assuming now that $\ell \in Y^*$, by theorem 14.21, there exists $\tilde{\ell} \in Y$, such that $\tilde{\ell}|_Y = \ell$, and thus by (14.126) and (14.127)

$$\langle \xi, \ell\rangle = \langle \xi, \tilde{\ell}|_Y\rangle = \langle \tilde{\xi}, \tilde{\ell}\rangle = \langle \tilde{\ell}, x\rangle = \langle \ell, x\rangle. \tag{14.128}$$ □

Proposition 14.39. *Let X be a normed space. The space X is reflexive if and only if X is complete and X^* is reflexive.*

Proof. Assume that X is reflexive. Then X is complete by proposition 14.36. For every $\lambda \in X^{***} = (X^*)^{**}$, we define $\ell \in X^*$ by

$$\langle \ell, x\rangle := \langle \lambda, J(x)\rangle. \tag{14.129}$$

By definition of J, we have

$$\langle \lambda, J(x)\rangle = \langle J(x), \ell\rangle. \tag{14.130}$$

Since J is surjective, we have for every $\xi \in X^{**}$,

$$\langle \lambda, \xi\rangle = \langle \xi, \ell\rangle, \tag{14.131}$$

and thus X^* is reflexive by definition 14.34.

Conversely, if X^* is reflexive and if X is complete, then X^{**} is reflexive and $J(X) \subseteq X^{**}$ is a closed subspace; hence by proposition 14.38 $J(X)$ is reflexive and thus X is reflexive. □

14.4.2 Reflexivity of concrete spaces

Proposition 14.40 (Reflexivity of Hilbert spaces)**.** *Let X be an inner product space. The space X is reflexive if and only if X is complete.*

Proof. We consider the linear operator $T : X \to X^*$ defined for $x \in X$ and $y \in X$, by

$$\langle T(y), x\rangle = (y \,|\, x). \tag{14.132}$$

Given $\xi \in X^{**} = \mathscr{L}(X^*, \mathbb{R})$, we define $k : X \to \mathbb{R}$ for $y \in X$ by

$$\langle k, y\rangle = \langle \xi, T(y)\rangle. \tag{14.133}$$

By the representation of linear functionals on Hilbert spaces (proposition 14.7), there exists $x \in X$, such that for every $y \in X$, we have

$$\langle k, y\rangle = (x \,|\, y). \tag{14.134}$$

Combining (14.132), (14.133) and (14.134), we get

$$\langle \xi, T(y)\rangle = \langle T(y), x\rangle. \tag{14.135}$$

Since X is complete, by proposition 14.7, T is surjective and for every $\ell \in X^*$,

$$\langle \xi, \ell\rangle = \langle \ell, x\rangle. \tag{14.136}$$

□

Proposition 14.41. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If $1 < p < \infty$, then $L^p(\Omega, \mu)$ is reflexive.*

Proof. We consider the linear operator $L : L^{p'}(\Omega,\mu) \mapsto L^p(\Omega,\mu)^*$, with $\frac{1}{p} + \frac{1}{p'}$, defined for $g \in L^{p'}(\Omega,\mu)$ and $f \in L^p(\Omega,\mu)$, by

$$\langle T(g), f\rangle = \int_\Omega g f \,\mathrm{d}\mu. \tag{14.137}$$

By Hölder's inequality, T is a well-defined bounded linear operator.
Given ξ in $L^p(\Omega,\mu)^{**}$, we define $\ell \in (L^{p'}(\Omega,\mu))^*$ for $g \in L^{p'}(\Omega,\mu)$

$$\langle \ell, g\rangle = \langle \xi, T(g)\rangle. \tag{14.138}$$

By the representation theorem of $L^{p'}(\Omega,\mu)^*$ (theorem 14.20), there exists $f \in L^p(\Omega,\mu)$, such that for every $g \in L^{p'}(\Omega,\mu)$,

$$\langle \ell, g\rangle = \int_\Omega f\, g \,\mathrm{d}\mu. \tag{14.139}$$

Combining (14.137), (14.138) and (14.139), we get

$$\langle \xi, T(g)\rangle = \langle T(g), f\rangle. \tag{14.140}$$

Since by theorem 14.20, T is surjective, we get that for every $\ell \in L^p(\Omega,\mu)^*$,

$$\langle \xi, \ell\rangle = \langle \ell, f\rangle. \tag{14.141}$$

In view of definition 14.34, the space $L^p(\Omega,\mu)$ is reflexive. □

14.5 Comments

By the Hahn-Banach theorem theorem 14.24 and by proposition 14.13, the smoothness of X is equivalent to the uniqueness for every given $x \in X$ of $\ell \in \mathscr{L}(X,\mathbb{R})$ such that $\langle \ell, x\rangle = \|x\|_X^2 = \|\ell\|_{\mathscr{L}(X,\mathbb{R})}^2$.

When $p \neq 2$, equality in proposition 14.17 holds if and only if there exist $s, t \in \mathbb{R}$ such that for every $x, y \in \Omega$, either $su(x) + tv(x) = 0$ or $su(x) - tv(x) = 0$. This follows from the fact that equality holds in lemma 14.18 if and only if $ta = sb$.

Proposition 14.11 can be generalised to

Proposition 14.42. *Let X, Y endowed with $\|\cdot\|_X$ be a normed space. If X is uniformly convex and complete and if $L \in \mathscr{L}_c(X,Y)$, then there exists $x \in X$ such that*

$$\|x\|_X^2 = \|L(x)\|_Y = \|L\|_{\mathscr{L}(X,Y)}^2. \tag{14.142}$$

Proof. By definition of the norm of a bounded linear operator supremum there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for every $n \in \mathbb{N}$, $\|x_n\|_X \le \|L\|_{\mathscr{L}(X,Y)}$ and the sequence $(\|L(x_n)\|_Y)_{n\in\mathbb{N}}$ converges to $\|L\|_{\mathscr{L}(X,Y)}$ in $\mathbb{R}$. Since L is a compact operator, in view of definition 13.10 we can assume without loss of generality that $(L(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

For every $n, m \in \mathbb{N}$, we have

$$\begin{aligned}(14.143)\qquad \|L(x_n)\|_Y - \|L\|_{\mathscr{L}(X,Y)}\|x_n - x_m\|_X &\le \|L(x_n + x_m)\|_Y \\ &\le \|L\|_{\mathscr{L}(X,Y)}\|x_n + x_m\|_X \le \|L\|^2_{\mathscr{L}(X,Y)},\end{aligned}$$

so that

$$(14.144)\qquad \lim_{m,n\to\infty} \|x_n + x_m\|_X = \|L\|_{\mathscr{L}(X,Y)},$$

and thus by proposition 14.9, we have the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. By completeness of X, there exists $x \in X$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x in X. We then have $\|x\|_X \le \|L\|_{\mathscr{L}(X,Y)}$ and

$$(14.145)\qquad \|L(x)\|_Y = \lim_{n\to\infty} \|L(x_n)\|_Y = \|L\|^2_{\mathscr{L}(X,Y)}. \qquad \square$$

The proof of proposition 14.17 follows the strategy of Lieb and Loss [LL01, §2.5] (see also [DiB16, Prop. 12.1]). The original proof proposition 14.17 by Olof Hanner's (which he presents as a reconstruction of a proof of Beurling) [Han56], which consists in showing that the function $\Theta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for $(t_1, t_2) \in [0,\infty) \times [0,\infty)$ by

$$(14.146)\qquad \Theta(t_1, t_2) \coloneqq \left|t_1^{1/p} + t_2^{1/p}\right|^p + \left|t_1^{1/p} - t_2^{1/p}\right|,$$

is convex and positively homogeneous and deducing that

$$(14.147)\qquad \int_\Omega \Theta(|f|^p, |g|^p)\,\mathrm{d}\mu \le \Theta\Big(\int_\Omega |f|^p\,\mathrm{d}\mu, \int_\Omega |g|^p\,\mathrm{d}\mu\Big);$$

this proof can be presented in the general framework of convexity inequalities by Paolo Roselli and Michel Willem [RW02] (see also [Wil13]).

Proposition 14.17 also holds for $f, g : \Omega \to V$, where V is an inner product space, with associated norm $|\cdot|$; this follows from the fact that if u, v are orthonormal vectors, then

$$\begin{aligned}(14.148)\qquad |ru + s(\cos\theta u + \sin\theta v)|^p &+ |ru + s(\cos\theta u + \sin\theta v)|^p \\ &= |r^2 + s^2 - 2rs\cos\theta|^{p/2} + |r^2 + s^2 - 2rs\cos\theta|^{p/2},\end{aligned}$$

which achieves its maximum when $p \le 2$ and its maximum when $\theta = k\pi$ with $k \in \mathbb{Z}$ (when $p = 2$, it is constant as a consequence of the parallelogram identity).

The choice of δ in the proof of proposition 14.16 is optimal ([Han56]).

The original proof of the uniform convexity of $L^p(\Omega, \mu)$ due to Clarkson [Cla36] (see also [Bre11; DiB16, Prop. 12.2]) was based on Clarkson's inequalities. In a general form due to Ralph Philip Boas they hold [Boa40, th. 1]

$$(14.149)\qquad \Big(\|f+g\|^r_{L^p(\Omega,\mu)} + \|f+g\|^r_{L^p(\Omega,\mu)}\Big)^{1/r} \le 2^{1-1/s}\Big(\|f\|^s_{L^p(\Omega,\mu)} + \|g\|^s_{L^p(\Omega,\mu)}\Big)^{1/s},$$

under the assumptions that $r \ge p \ge s > 1$ and $\frac{1}{s} + \frac{1}{p} \le 1$.

For the smoothness of the L^p norm, see [Wil13, th. 5.4.1; LL01, §2.6].

15 Weak* and weak topologies and convergences

15.1 Weak* topology and convergence

Definition 15.1. Let X be a normed space. A set $U \subseteq X^*$ is *weakly* open* whenever for every $\ell \in U$, there exist $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that

$$\Big\{k \in X^* \;\Big|\; \text{for each } i \in \{1, \dots, m\},\ |\langle k - \ell, x_i\rangle| < 1\Big\} \subseteq U. \tag{15.1}$$

The *weak* topology* $\sigma(X^*, X)$ is the set of weakly* open sets of X^*.

Proposition 15.2. *Let X be a normed space. For every $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$, the set*

$$\Big\{k \in X^* \;\Big|\; \textit{for each } i \in \{1, \dots, m\},\ |\langle k - \ell, x_i\rangle| < 1\Big\} \subseteq U. \tag{15.2}$$

is weakly open.*

Proof. Given $k, h, \ell \in X^*$, we observe that for every $i \in \{1, \dots, m\}$, we have

$$|\langle k - \ell, x_i\rangle| \le |\langle k - h, x_i\rangle| + |\langle h - \ell, x_i\rangle|. \tag{15.3}$$

We assume now that for for each $i \in \{1, \dots, m\}$, $|\langle h - \ell, x_i\rangle| < 1$. Setting

$$y_i := \frac{x_i}{1 - |\langle \ell - h, x_i\rangle|}, \tag{15.4}$$

we have

$$|\langle k - \ell, x_i\rangle| \le |\langle k - h, x_i\rangle| + (1 - |\langle \ell - h, x_i\rangle|)|\langle h - \ell, y_i\rangle|, \tag{15.5}$$

and

$$\begin{aligned}\Big\{k \in X^* \;\Big|\; \text{for each } i \in \{1, \dots, m\},\ |\langle k - h, y_i\rangle| < 1\Big\}&\\ \subseteq \Big\{k \in X^* \;\Big|\; \text{for each } i \in \{1, \dots, m\},\ &|\langle k - \ell, x_i\rangle| < 1\Big\}.\end{aligned}$$

□

Proposition 15.3. *Let X be a normed space. If the sets $U \subseteq X^*$ and $V \subseteq X^*$ are weakly* open, then the set $U \cap V$ is weakly* open.*

Proposition 15.4. *Let X be a normed space. If for every $i \in I$, the set $U_i \subseteq X^*$ is weakly* open, then $\bigcup_{i\in I} U_i$ is weakly* open.*

Proposition 15.5. *Let X be a normed space. If $\xi : X^* \to \mathbb{R}$ is linear. Then ξ is weakly* continuous if and only if there exists $x \in X$ such that for every $\ell \in X$,*

$$\xi(\ell) = \langle \ell, x\rangle. \tag{15.6}$$

Proof. By continuity of ξ at 0, there exist $x_1, \dots, x_m \in X$ such that having for each $i \in \{1, \dots, m\}$, $|\langle \ell, x_i\rangle| < 1$ implies $|\xi(\ell)| \le 1$. In particular, for every $t \in (0, \infty)$, $|\langle \ell, x_i\rangle|/t = |\langle \ell/t, x_i\rangle| < 1$ implies $|\xi(\ell)|/t = |\xi(\ell/t)| \le 1$. Thus $\langle \ell, x_i\rangle = 0$ for each $i \in \{1, \dots, m\}$, implies $\xi(\ell) = 0$.

Letting

$$\Lambda := \big\{\ell \in X^* \bigm| \text{for every } i \in \{1, \dots, m\}\ \langle \ell, x_i\rangle = 0\big\}, \tag{15.7}$$

we have $\xi|_\Lambda = 0$. Choosing $\ell_i \in X^*$ such that $\langle \ell_i, x_j\rangle = \delta_{ij}$, we have $\ell - \sum_{i=1}^m \langle \ell, x_i\rangle \ell_i \in \Lambda$, and

$$\xi(\ell) = \sum_{i=1}^m \langle \ell, x_i\rangle \xi(\ell_i), \tag{15.8}$$

we have the conclusion with $x := \sum_{i=1}^m \xi(\ell_i) x_i$. □

Proposition 15.6. *Let X be a normed space. If the set $U \subseteq X^*$ is weakly* open, then U is open (with respect to the norm topology of X^*).*

Proof. By definition of weakly* open set (definition 15.14), for every $\ell \in U$, there exist $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that

$$\big\{k \in X^* \bigm| \text{for each } i \in \{1, \dots, m\}\ |\langle k - \ell, x_i\rangle| < 1\big\} \subseteq U. \tag{15.9}$$

Since for every $i \in \{1, \dots, m\}$ and $k \in X^*$,

$$|\langle k - \ell, x_i\rangle| \le \|k - \ell\|_{X_*} \|x_i\|_X, \tag{15.10}$$

choosing $\delta \in (0, \infty)$ such that for every $i \in \{1, \dots, m\}$ and $k \in X^*$,

$$\delta \|x_i\|_X \le 1, \tag{15.11}$$

we get that $\|k - h\|_{X^*} \le \delta$ implies that $k \in U$. This proves that the set U is open with in the norm topology of X^*. □

Proposition 15.7. *Let X be a normed space. If $U \subseteq X^*$ is weakly* open and bounded, and if $\dim X = \infty$, then $U = \emptyset$.*

Proof. Assume that $\ell \in U$. By definition, there exist $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that

$$\Bigl\{\ell \in X^* \Bigm| \text{for each } i \in \{1, \dots, m\},\ |\langle \ell - h, x_i\rangle| < 1\Bigr\} \subseteq U. \tag{15.12}$$

In particular,

$$\Bigl\{\ell \in X^* \Bigm| \text{for each } i \in \{1, \dots, m\},\ \langle \ell - h, x_i\rangle = 0\Bigr\} \subseteq U. \tag{15.13}$$

If $\dim X = \infty$, there exists x_* which does not belong to the linear subspace generated by $x_1, \dots, x_m$. By lemma 3.18 and theorem 14.21, there exists $h \in X^*$ such that $\langle h, x_i\rangle = 0$ and $\langle h, x_*\rangle$, which contradicts the fact that the set U is bounded. □

Definition 15.8. Let X be a normed space. The sequence $(\ell_n)_{n \in \mathbb{N}}$ in X^* *converges weakly* to* $\ell \in X^*$, whenever for every $U \in \sigma(X^*, X)$ with $\ell \in U$, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \ge N$ implies $\ell_n \in U$.

One also writes $\ell_n \overset{*}{\rightharpoonup} \ell$ as $n \to \infty$ to denote that the sequence $(\ell_n)_{n \in \mathbb{N}}$ in X^* converges weakly* to $\ell \in X^*$.

Proposition 15.9. *Let X be a normed space. If the sequence $(\ell_n)_{n \in \mathbb{N}}$ in X^* converges in norm to $\ell \in X$, then the sequence $(\ell_n)_{n \in \mathbb{N}}$ in X^* converges weakly* to $\ell \in X^*$.*

Proof. This follows from the definitions of convergence in norm and weak* convergence and from the fact that weakly* open sets are open (proposition 15.6). □

Proposition 15.10. *Let X be a normed space and $(\ell_n)_{n \in \mathbb{N}}$ be a sequence in X^*. The sequence $(\ell_n)_{n \in \mathbb{N}}$ converges weakly* to ℓ in X^* if and only if for every $x \in X$, $(\langle \ell_n, x\rangle)_{n \in \mathbb{N}}$ converges to $\langle \ell, x\rangle$.*
Moreover,

$$\|\ell\|_{X^*} \le \liminf_{n \to \infty} \|\ell_n\|_{X^*}. \tag{15.14}$$

Proof. If the sequence $(\ell_n)_{n \in \mathbb{N}}$ converges weakly* to ℓ in X^*, if $x \in X$ and $\varepsilon > 0$, we define the set

$$U := \Bigl\{k \in X^* \Bigm| |\langle k - \ell, x/\varepsilon\rangle| < 1\Bigr\}. \tag{15.15}$$

Since the set U is weakly* open, there exists $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \ge N$, $|\langle \ell_n - \ell, x\rangle| < \varepsilon$, which implies the convergence.

Conversely, if $U \in \sigma(X^*, X)$, then by definition of weakly* open sets (definition 15.1) there exist $x_1, \dots, x_m \in X$ such that

$$\{k \in X^* \mid \text{for each } i \in \{1, \dots, m\}\ |\langle k - \ell, x_i\rangle| < 1\} \subseteq U. \tag{15.16}$$

By assumption, the sequence $(\langle \ell_n - \ell, x_i\rangle)_{n \in \mathbb{N}}$ converges in $\mathbb{R}$ to 0 and we conclude.

The last conclusion follows from proposition 9.1. □

Proposition 15.11. *Let X be a normed space. If the sequence $(\ell_n)_{n\in\mathbb{N}}$ in X^* converges weakly* to $\ell \in X^*$ and converges weakly* to $k \in X^*$, then $k = \ell$.*

Proof. By proposition 15.10, for every $x \in X$, we have

$$\langle k, x\rangle = \lim_{n\to\infty} \langle \ell_n, x\rangle = \langle \ell, x\rangle, \tag{15.17}$$

from which it follows that $k = \ell$ by definition of the duality product in (14.3). □

Proposition 15.12. *Let X be a normed space. Let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence in X^* and let $Z \subseteq X$. If* $\operatorname{span}(Z)$ *is dense in X, if for every $x \in Z$, $(\langle \ell_n, x\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence and if $(\|\ell_n\|_{X^*})$ is bounded, then the sequence $(\ell_n)_{n\in\mathbb{N}}$ converges weakly* to some $\ell \in X^*$.*

Proof. This follows from proposition 9.2 and proposition 9.1, since $\mathbb{R}$ is complete. □

Proposition 15.13. *Let X be a normed space. If X is complete and if the sequence $(\ell_n)_{n\in\mathbb{N}}$ in X^* converges weakly* to $\ell \in X^*$, then the sequence $(\|\ell_n\|_{X^*})_{n\in\mathbb{N}}$ is bounded.*

Proof. This follows from proposition 9.5. □

15.2 Weak topology and convergence

Definition 15.14. Let X be a normed space. A set $U \subseteq X$ is weakly open whenever for every $x \in U$, there exist $m \in \mathbb{N}$ and $\ell_1, \dots, \ell_m \in X^*$ such that

$$\{y \in X \mid \text{for each } i \in \{1, \dots, m\}, |\langle \ell_i, x - y\rangle| < 1\} \subseteq U. \tag{15.18}$$

The weak topology $\sigma(X, X^*)$ is the set of weakly open subsets of X.

Proposition 15.15. *Let X be a normed space and let $\ell : X \to \mathbb{R}$ be linear. Then ℓ is weakly continuous if and only if $\ell \in X^*$.*

Proof. If ℓ is weakly continuous on X, then there exist $\ell_1, \dots, \ell_m \in X^*$ such that $|\langle \ell_i, x\rangle| < 1$ implies $|\ell(x)| \le 1$. In particular, $\langle \ell_i, x\rangle = 0$ implies that $\ell(x) = 0$. Equivalently we have $\bigcap_{i=1}^m \ker \ell_i \subseteq \ker \ell$. Assuming that $\ell_1, \dots, \ell_m$ are linearly independent, there exist $x_1, \dots, x_m \in X$ such that $\langle \ell_i, x_j\rangle = \delta_{ij}$. We then have $\ell = \sum_{i=1}^m \langle \ell, x_i\rangle \ell_i \in X^*$.
The converse implication is direct. □

Proposition 15.16. *Let X be a normed space. A set $U \subseteq X$ is weakly open if and only if $U = J^{-1}(V)$, for $V \subseteq X^{**}$ weakly* open, where $J : X \to X^{**}$ is the canonical embedding into the bidual of proposition 14.33.*

Proof. Assume that the set U is weakly open. For every $x \in U$, by definition of weakly open set (definition 15.14),there exist $m \in \mathbb{N}$ and $\ell_1, \dots, \ell_m \in X^*$ such that

$$\{y \in X \mid \text{for each } i \in \{1, \dots, m\}, |\langle \ell_i, x - y\rangle| < 1\} \subseteq U. \tag{15.19}$$

Defining the set

$$V_x := \left\{\xi \in X^{**} \,\middle|\, \text{for each } i \in \{1,\dots,m\}, |\langle J(x)-\xi,\ell_i\rangle| < 1\right\} \subseteq X^{**}, \tag{15.20}$$

we have that V_x is weakly* open and that $x \in J^{-1}(V_x) \subseteq U$. Hence, the set $V := \bigcup_{x\in U} V_x$ is weakly* open and $U = J^{-1}(V)$.

Conversely, assume that $V \subseteq X^{**}$ is weakly* open and that $U = J^{-1}(V)$. If $x \in U$, then $J(x) \in V$. Since the set V is weakly* open, in view of definition 15.1 there exist $m \in \mathbb{N}$ and $\ell_1,\dots,\ell_m \in X^*$ such that

$$\left\{\xi \in X^{**} \,\middle|\, \text{for each } i \in \{1,\dots,m\}, |\langle J(x)-\xi,\ell_i\rangle| < 1\right\} \subseteq V. \tag{15.21}$$

In particular,

$$\left\{y \in X \,\middle|\, \text{for each } i \in \{1,\dots,m\}, |\langle J(y)-J(x),\ell_i\rangle| < 1\right\} \subseteq J^{-1}(V) = U, \tag{15.22}$$

which proves that the set U is weakly open. □

Proposition 15.17. *Let X be a normed space. If the set $U \subseteq X$ is weakly open, then U is open (with respect to the norm topology of X).*

Proof. Assume that the set $U \subseteq X$ is weakly open. By proposition 15.16, there exists a weakly* open set $V \subseteq X^{**}$ such that $U = J^{-1}(V)$. By proposition 15.6, the set V is open in the norm topology of X^{**}. By proposition 14.33, the set $U = J^{-1}(V)$ is open in the norm topology of X. □

Proposition 15.18. *Let X be a normed space. The set $B(0,1) \subseteq X$ is weakly open if and only if $\dim X < \infty$.*

Proof. This follows from proposition 15.7 and proposition 15.16. □

Definition 15.19. Let X be a normed space. The sequence $(x_n)_{n\in\mathbb{N}}$ in X *converges weakly to $x \in X$*, whenever for every $U \in \sigma(X,X^*)$ with $\ell \in U$, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \ge N$ implies $\ell_n \in U$.

One also writes $x_n \rightharpoonup x$ as $n \to \infty$ to denote that the sequence $(x_n)_{n\in\mathbb{N}}$ in X converges weakly to $x \in X$.

Proposition 15.20. *Let X be a normed space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x if and only if the sequence $(J(x_n))_{n\in\mathbb{N}}$ converges weakly* to $J(x)$ in X^{**}.*

Proof. This follows from the definitions and from proposition 15.16. □

Proposition 15.21. *Let X be a normed space. If the sequence $(x_n)_{n\in\mathbb{N}}$ in X converges weakly in X to $x \in X$ and converges weakly in X to $y \in X$, then $x = y$.*

Proof. By proposition 15.20, the sequence $(J(x_n))_{n\in\mathbb{N}}$ in X^{**} converges weakly* to $J(x) \in X^{**}$ and to $J(y) \in X^{**}$, and thus by proposition 15.11, we have $J(x) = J(y)$. By the injectivity of the canonical injection J into the bidual X^{**} (proposition 14.33), we conclude that $x = y$. □

Proposition 15.22. *Let X be a normed space. If the sequence $(x_n)_{n\in\mathbb{N}}$ in X converges in norm in X to $x \in X$, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly in X to $x \in X$.*

Proof. This follows from proposition 15.9 and proposition 15.20. □

Proposition 15.23. *Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x in X if and only if for every $\ell \in X^*$, the sequence $(\langle \ell, x_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle \ell, x\rangle$.*
Moreover, we have then

$$\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X. \tag{15.23}$$

Proof of proposition 15.23. By proposition 15.16, the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x in X if and only if the sequence $(J(x_n))_{n\in\mathbb{N}}$ converges weakly* to $J(x)$ in X^*. By proposition 15.10, this is equivalent to the convergence of $(\langle J(x_n), \ell\rangle)_{n\in\mathbb{N}}$ to $\langle J(x), \ell\rangle)$. Since $\langle J(x_n), \ell\rangle = \langle \ell, x_n\rangle$, the equivalence follows.
Finally, by proposition 14.33 and proposition 15.10, we have

$$\|x\|_X = \|J(x)\|_{X^{**}} \le \liminf_{n\to\infty} \|J(x_n)\|_{X^{**}} = \liminf_{n\to\infty} \|x_n\|_X. \tag{15.24}$$
□

Proposition 15.24. *Let X be a normed space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X^* and let $Z \subseteq X^*$ and let $x \in X$. If $\operatorname{span}(Z)$ is dense in X^*, if for every $\ell \in Z$, the sequence $(\langle \ell, x_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle \ell, x\rangle$ and if the sequence $(\|x_n\|_X)_{n\in\mathbb{N}}$ is bounded, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly* to some x.*

Proof. This follows from proposition 15.20 and proposition 9.2. □

Proposition 15.25. *Let X be a normed space. If the sequence $(x_n)_{n\in\mathbb{N}}$ in X converges weakly in X to $x \in X$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. By proposition 15.16, we have that $(J(x_n))_{n\in\mathbb{N}}$ converges weakly* in X^{**} to $J(x)$. By proposition 5.19, the space X^* is complete, and thus by proposition 15.13, the sequence $(J(x_n))_{n\in\mathbb{N}}$ is bounded in X^{**}, and thus by proposition 14.33, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X. □

Proposition 15.26. *Let X be a normed space. If $U \subseteq X^*$ is weakly* open, then U is weakly open. If X is reflexive, then the converse holds.*

Proof. By definition 15.1, for each $\ell \in X^*$, there exist $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that

$$\Big\{k \in X^* \;\Big|\; \text{for each } i \in \{1, \dots, m\},\ |\langle k - \ell, x_i\rangle| < 1\Big\} \subseteq U, \tag{15.25}$$

setting for each $i \in \{1, \dots, m\}$, $\xi_i := J(x_i)$, we get

$$\Big\{k \in X^* \;\Big|\; \text{for each } i \in \{1, \dots, m\}, |\langle \xi_i, k - \ell\rangle| < 1\Big\} \subseteq U, \tag{15.26}$$

so that the set U is open in view of definition 15.14. Conversely, if the space X is reflexive, given $\xi_1, \dots, \xi_m$ satisfying (15.26), setting for each $i \in \{1, \dots, m\}$, $x_i := J^{-1}(\xi_i)$ we get (15.25). □

Proposition 15.27. *Let X be a normed space and let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence in X^* and let $\ell \in X^*$. If $(\ell_n)_{n\in\mathbb{N}}$ converges weakly to ℓ, then $(\ell_n)_{n\in\mathbb{N}}$ converges weakly* to ℓ.*

15.3 Weak and weak* topological properties of some sets

Proposition 15.28. *Let X be a normed space and let $V \subseteq X$ be a linear subspace. If V is closed in the norm topology, then V is weakly closed.*

Proof. By proposition 14.29, if $x \in X \setminus V$, there exists $\ell \in X^*$ such that $\langle \ell, x\rangle \neq 0$ and $\ell|_V = 0$. Hence $X \setminus V$ is weakly open and thus V is weakly closed. □

Proposition 15.29 (Goldstine theorem)**.** *Let X be a normed space. The set $J(X) \cap B[0,1]$ is weakly* dense in $X^{**} \cap B[0,1]$.*

Lemma 15.30. *Let X be a normed space and let $V \subseteq X^*$ be a finite-dimensional space. For every $\xi \in X^{**}$,*

$$\|\xi|_V\|_{\mathscr{L}(V,\mathbb{R})} = \inf\Big\{x \in X \;\Big|\; \textit{for every } \ell \in V, \langle \ell, x\rangle = \langle \xi, \ell\rangle\Big\}. \tag{15.27}$$

Proof. Since V is finite-dimensional, let $\ell_1, \dots, \ell_d \in V$ with $d = \dim V$ and let $x_1, \dots, x_d \in X$ such that $\langle \ell_i, x_j\rangle = \delta_{ij}$. We have then for every $\ell \in V$,

$$\ell = \sum_{i=1}^{d} \langle \ell, x_i\rangle \ell_i, \tag{15.28}$$

and thus for every $\ell \in V$,

$$\langle \xi, \ell\rangle = \sum_{i=1}^{d} \langle \ell, x_i\rangle \langle \xi, \ell_i\rangle = \langle \ell, x_*\rangle, \tag{15.29}$$

where

$$x_* := \sum_{i=1}^{d} \langle \xi, \ell_i\rangle x_i. \tag{15.30}$$

Setting

$$W := \bigcap_{i=1}^{d} \ker \ell_i, \tag{15.31}$$

we have for every $w \in W$,

$$\langle \xi, \ell \rangle = \langle \ell, x_* - w \rangle. \tag{15.32}$$

Moreover, we have

$$V = \{\ell \in X^* \mid \ell|_W = 0\}; \tag{15.33}$$

the inclusion of V in the right-hand side follows from (15.28); for the converse inequality if $\ell|_W = 0$, then (15.28) holds.

Moreover, by proposition 14.30,

$$\begin{aligned} \inf\big\{\|x_* - w\|_X \bigm| w \in W\big\} &= \sup\big\{\langle \ell, x_* \rangle \bigm| \ell \in X^*, \|\ell\|_{X^*} \le 1 \text{ and } \ell|_W = 0\big\} \\ &= \sup\big\{\langle \xi, \ell \rangle \bigm| \ell \in V \text{ and } \|\ell\|_{X^*} \le 1\big\}. \end{aligned} \tag{15.34}$$

The conclusion follows. □

Proof of proposition 15.29. Let $\xi \in X^{**} \setminus \{0\}$. Given $\ell_1, \dots, \ell_m \in X^*$, for every $\varepsilon > 0$, by lemma 15.30 there exists $x \in X$ such that for every $i \in \{1, \dots, m\}$, $\langle \ell_i, x \rangle = \langle \xi, \ell_i \rangle$ and $\|x\|_X \le \|\xi\|_{X^{**}}(1+\varepsilon)$. We then have

$$J(x/(1+\varepsilon)) \in \big\{\zeta \in X^{**} \bigm| \text{for every } i \in \{1, \dots, m\}, |\langle \zeta - \xi, \ell_i \rangle| \le \varepsilon \langle \xi, \ell_i \rangle/(1+\varepsilon)\big\}. \tag{15.35}$$

Taking $\varepsilon > 0$ small enough, we have

$$J(x/(1+\varepsilon)) \in \big\{\zeta \in X^{**} \bigm| \text{for every } i \in \{1, \dots, m\}, |\langle \zeta - \xi, \ell_i \rangle| \le 1\big\}. \tag{15.36}$$ □

Proposition 15.31. *Let X be a normed space. The following are equivalent*

(i) X is reflexive,

(ii) $J(X) \subseteq X^{**}$ is weakly closed,*

(iii) $J(X) \cap B[0,1] \subseteq X^{**}$ is weakly closed.*

Proof. This follows from propositions 14.33, 15.28 and 15.29. □

15.4 Weak topology and convergence in concrete spaces

15.4.1 Inner product spaces

Proposition 15.32. *Let X be a complete inner product space. A set $U \subseteq X$ is weakly open if and only if for every $x \in X$, there exist $m \in \mathbb{N}$ and $y_1, \dots, y_m \in X$ such that*

$$\big\{z \in X \bigm| \text{for each } i \in \{1, \dots, m\},\ |(x - z \,|\, y_i)| < 1\big\} \subseteq U. \tag{15.37}$$

Proof. This follows by the representation of linear functionals on complete inner product spaces (proposition 14.7). □

Proposition 15.33. *Let X be a complete inner product space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x in X if and only if for every $y \in X$, $((x_n \,|\, y))_{n\in\mathbb{N}}$ converges to $(x \,|\, y)$. Moreover, we have then*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|. \tag{15.38}$$

Proof. This follows from proposition 15.23 and proposition 14.7. □

Proposition 15.34. *Let X be a complete inner product space and let $(e_n)_{n\in\mathbb{N}}$ be sequence in X. If for every $m, n \in \mathbb{N}$, $(e_n \,|\, e_m) = 0$, then $(e_n)_{n\in\mathbb{N}}$ converges weakly to 0 in X.*

Proof. By Bessel's inequality (proposition 10.3), for every $x \in X$, we have

$$\sum_{n\in\mathbb{N}} |(e_n \,|\, x)|^2 \le \|x\|^2 < \infty, \tag{15.39}$$

and thus $\lim_{n\to\infty} (e_n \mid x) = 0 = (0 \mid x)$. By proposition 15.33, the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to 0. □

Proposition 15.35. *Let X be an complete inner product space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $E \subseteq X$ be an orthonormal basis of X. The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x in X if and only if $(x_n)_{n\in\mathbb{N}}$ is bounded in X and for every $e \in E$, $((e \,|\, x_n))_{n\in\mathbb{N}}$ converges to $(e \,|\, x)$.*

Proof. This follows from proposition 10.5, proposition 14.7 and proposition 15.24. □

15.4.2 Weak convergence in $\ell^p(\Gamma)$ and $c_0(\Gamma)$

Proposition 15.36. *Let $p \in (1, \infty)$ and let Γ be a set. A sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $\ell^p(\Gamma)$ if and only if for every $g \in \ell^{p'}(\Gamma)$,*

$$\lim_{n\to\infty} \sum_{x\in\Gamma} g(x) f_n(x) = \sum_{x\in\Gamma} g(x) f(x). \tag{15.40}$$

Proof. This follows from the characterisation of weak convergence (proposition 15.23) and from the characterisation of the dual space of $\ell^p(\Gamma)$ (proposition 14.5). □

Proposition 15.37. *Let $p \in (1, \infty)$ and let Γ be a set. A sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $\ell^p(\Gamma)$ if and only if $(f_n)_{n\in\mathbb{N}}$ is bounded and for every $x \in \Gamma$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.*

Proof. Assuming that $(f_n)_{n\in\mathbb{N}}$ is bounded and for every $x \in \Gamma$, the sequence $(f_n(x))_{n\in\mathbb{N}}$, we are going to prove that $(f_n)_{n\in\mathbb{N}}$ converges weakly.

We define $\ell_n \in \ell^{p'}(\Gamma)^*$ by

$$\langle \ell_n, g \rangle = \sum_{x\in\Gamma} f_n(x) g(x) \tag{15.41}$$

and

$$\langle \ell, g\rangle = \sum_{x \in \Gamma} f(x) g(x). \tag{15.42}$$

By assumption and by linearity, for every $g \in c_c(\Gamma)$, $(\langle \ell_n, g\rangle)_{n\in\mathbb{N}}$ converges to $(\langle \ell, g\rangle)_{n\in\mathbb{N}}$. Since $c_c(\Gamma)$ is dense in $\ell^p(\Gamma)$ (proposition 7.7) and since $(\ell_n)_{n\in\mathbb{N}}$ is bounded, we have that for every $g \in \ell^p(\Gamma)$, the sequence $(\langle \ell_n, g\rangle)_{n\in\mathbb{N}}$ converges to $\langle \ell, g\rangle$. In view of (15.41) and (15.42) and of proposition 15.36, sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $\ell^p(\Gamma)$. □

Proposition 15.38. *Let Γ be a set. A sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $c_0(\Gamma)$ if and only if for every $g \in \ell^1(\Gamma)$,*

$$\lim_{n\to\infty} \sum_{x\in\Gamma} g(x) f_n(x) = \sum_{x \in \Gamma} g(x) f(x). \tag{15.43}$$

Proof. This follows from the characterisation of weak convergence (proposition 15.23) and from the characterisation of the dual space of $\ell^p(\Gamma)$ (proposition 14.6). □

Proposition 15.39. *Let Γ be a set. A sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $c_0(\Gamma)$ if and only if $(f_n)_{n\in\mathbb{N}}$ is bounded in $c_0(\Gamma)$ and for every $x \in \Gamma$, $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.*

Proof. Assuming that $(f_n)_{n\in\mathbb{N}}$ is bounded and for every $x \in \Gamma$, the sequence $(f_n(x))_{n\in\mathbb{N}}$, we are going to prove that $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $c_0(\Gamma)$.

We define $\ell_n \in \ell^1(\Gamma)^*$ by

$$\langle \ell_n, g\rangle = \sum_{x\in\Gamma} f_n(x) g(x) \tag{15.44}$$

and $\ell \in \ell^1(\Gamma)^*$ by

$$\langle \ell, g\rangle = \sum_{x\in\Gamma} f(x) g(x). \tag{15.45}$$

By assumption and by linearity, for every $g \in c_c(\Gamma)$, $(\langle \ell_n, g\rangle)_{n\in\mathbb{N}}$ converges to $(\langle \ell, g\rangle)_{n\in\mathbb{N}}$. Since $c_c(\Gamma)$ is dense in $c_0(\Gamma)$ (proposition 7.7) and since $(\ell_n)_{n\in\mathbb{N}}$ is bounded, we have that for every $g \in c_0(\Gamma)$, the sequence $(\langle \ell_n, g\rangle)_{n\in\mathbb{N}}$ converges to $\langle \ell, g\rangle$. In view of (15.44) and (15.45) and of proposition 15.38, the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $c_0(\Gamma)$. □

Proposition 15.40. *Let Γ be a set. A sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $\ell^1(\Gamma)$ if and only it $(f_n)_{n\in\mathbb{N}}$ converges in norm to f in $\ell^1(\Gamma)$.*

Proof. The convergence in norm implies the weak convergence by proposition 15.22.

Conversely, we assume that the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $\ell^1(\Gamma)$. Without loss of generality, we can assume that $f = 0$. If $(f_n)_{n\in\mathbb{N}}$ does not converge strongly to 0, we have

$$\nu := \limsup_{n\to\infty} \|f_n\|_{\ell^1(\Gamma)} > 0. \tag{15.46}$$

There exists a finite set $F_0 \subseteq \Gamma$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{x \in F_0} |f_{n_0}(x)| > \frac{\nu}{3}. \tag{15.47}$$

We construct then inductively, disjoint sets $F_n \subseteq \Gamma$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that

$$\sum_{x \in F_k} |f_{n_k}(x)| \geq \frac{\nu}{2} \tag{15.48}$$

and for every $i \in \mathbb{N} \setminus \{k\}$,

$$\sum_{x \in F_i} |f_{n_k}(x)| \leq \frac{\nu}{2^{i+3}} \tag{15.49}$$

Indeed assuming that the construction has been performed for some $k \in \mathbb{N}$, there exists a finite set H_{k+1} such that for every $i \in \{0, \dots, k\}$

$$\sum_{E \setminus H_{k+1}} |f_{n_i}(x)| \leq \frac{\nu}{2^{k+3}}. \tag{15.50}$$

There exists $n_{k+1} > n_k$ such that

$$\sum_{H_{k+1}} |f(x)| \leq \frac{\nu}{8}. \tag{15.51}$$

for every $i \in \{0, \dots, k\}$,

$$\sum_{x \in F_i} |f_{n_k}(x)| \leq \frac{\nu}{2^{i+3}} \tag{15.52}$$

and

$$\sum_{x \in E} |f_{n_{k+1}}(x)| \geq \frac{3\nu}{4}. \tag{15.53}$$

There exists thus a finite set $F_{k+1} \subseteq \Gamma \setminus H_{k+1}$ such that

$$\sum_{F_k} |f_{n_{k+1}}(x)| \geq \sum_{x \in E \setminus H_{k+1}} |f_{n_{k+1}}(x)| - \frac{\nu}{8} = \frac{\nu}{2}. \tag{15.54}$$

In order to conclude, we define $\ell : \ell^1(\Gamma) \to \mathbb{R}$ for $f \in \ell^1(\Gamma)$ by

$$\langle \ell, f \rangle = \sum_{k \in \mathbb{N}} \sum_{x \in F_k} \operatorname{sgn}(f_{n_k}(x)) f(x). \tag{15.55}$$

Since the sets F_k are disjoint, we have $\|\ell\|_{\ell^1(\Gamma)'} \leq 1$. By construction we have for every $k \in \mathbb{N}$

$$\langle \ell, f_{n_k} \rangle \geq \sum_{x \in F_k} |f(x)| - \sum_{i \in \mathbb{N} \setminus \{k\}} |f_{n_k}(x)| \geq \frac{\nu}{4}, \tag{15.56}$$

in contradiction with the fact that the sequence $(f_n)_{n \in \mathbb{N}}$ converges weakly to 0 in $\ell^1(\Gamma)$. □

15.4.3 Weak convergence in L^p spaces

Proposition 15.41. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If $1 < p < \infty$, a set $U \subseteq L^p(\Omega, \mu)$ is weakly open if and only if for every $f \in U$, there exist $m \in \mathbb{N}$, $g_1, \dots, g_m \in L^{p'}(\Omega, \mu)$ such that*

$$\left\{h \in L^p(\Omega, \mu) \;\middle|\; \text{for each } i \in \{1, \dots, m\}, \left|\int_\Omega g_i(f - h)\,\mathrm{d}\mu\right| < 1\right\} \subseteq U. \tag{15.57}$$

Proof. This follows from theorem 14.20. □

Proposition 15.42. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, let $p \in (1, \infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega, \mu)$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^p(\Omega, \mu)$ if and only if for every $g \in L^{p'}(\Omega, \mu)$,*

$$\lim_{n\to\infty} \int_\Omega g f_n \,\mathrm{d}\mu = \int_\Omega g f \,\mathrm{d}\mu. \tag{15.58}$$

Moreover, we have then

$$\int_\Omega |f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_\Omega |f_n|^p \,\mathrm{d}\mu. \tag{15.59}$$

Proof. This follows from proposition 15.23 and theorem 14.20. □

Proposition 15.43. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, let $p \in (1, \infty)$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega, \mu)$, let $f \in L^p(\Omega, \mu)$ and let $Z \subseteq L^{p'}(\Omega, \mu)$. If for every for every $g \in Z$,*

$$\lim_{n\to\infty} \int_\Omega g f_n \,\mathrm{d}\mu = \int_\Omega g f \,\mathrm{d}\mu, \tag{15.60}$$

if $\operatorname{span}(Z)$ *is dense in $L^{p'}(\Omega, \mu)$ and if*

$$\sup_{n\in\mathbb{N}} \int_\Omega |f_n|^p \,\mathrm{d}\mu < \infty, \tag{15.61}$$

then $(f_n)_{n\in\mathbb{N}}$ converges weakly to f.

Proof. This follows from theorem 14.20 and proposition 15.24. □

Proposition 15.44. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, let $p \in (1, \infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega, \mu)$. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega, \mu)$ and if $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to f in Ω, then $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^p(\Omega, \mu)$.*

Proof. By Fatou's lemma, we have

$$\int_\Omega |f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_\Omega |f_n|^p \,\mathrm{d}\mu. \tag{15.62}$$

Let $g \in L^{p'}(\Omega, \mu)$. We define for each $n \in \mathbb{N}$ the set

$$A_n := \bigl\{ x \in \Omega \bigm| |f_n(x) - f(x)| \le |g(x)|^{\frac{1}{p-1}} \bigr\}. \tag{15.63}$$

We have for each $n \in \mathbb{N}$,

$$\begin{aligned}
\left| \int_\Omega f_n g \,\mathrm{d}\mu - \int_\Omega f\, g \,\mathrm{d}\mu \right| &\le \int_\Omega |f_n - f|\,|g| \,\mathrm{d}\mu \\
&\le \int_{A_n} |f_n - f|\,|g| \,\mathrm{d}\mu + \int_{\Omega \setminus A_n} |f_n - f|\,|g| \,\mathrm{d}\mu \\
&\le \int_{A_n} |f_n - f|\,|g| \,\mathrm{d}\mu + \left(\int_\Omega |f_n - f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_{\Omega \setminus A_n} |g|^{\frac{p}{p-1}} \,\mathrm{d}\mu \right)^{1-\frac{1}{p}}.
\end{aligned} \tag{15.64}$$

Since $|f_n - f|\,|g| \le |g|^{\frac{p}{p-1}}$, we have

$$\lim_{n \to \infty} \int_{A_n} |f_n - f|\,|g| \,\mathrm{d}\mu = 0, \tag{15.65}$$

and

$$\lim_{n \to \infty} \int_{\Omega \setminus A_n} |g|^{\frac{p}{p-1}} \,\mathrm{d}\mu \tag{15.66}$$

by Lebesgue's dominated convergence theorem. We conclude by (15.64) and by the characterisation of weak convergence in $L^p(\Omega, \mu)$ of proposition 15.42. □

Proposition 15.45. *Define for $n \in \mathbb{N}$,*

$$f_n(x) = \sin(2\pi n x). \tag{15.67}$$

For every $p \in (1, \infty)$, the sequence $(f_n)_{n \in \mathbb{N}}$ converges weakly to 0 in $L^p((0,1))$.

Proof of proposition 15.45. For $p = 2$, $L^2((0,1))$ is a complete inner-product space, $(f_n)_{n \in \mathbb{N}}$ converges weakly to 0 by proposition 15.34. By proposition 15.42, we have for every $g \in L^2((0,1))$,

$$\lim_{n \to \infty} \int_0^1 g f_n = 0. \tag{15.68}$$

Since $L^1((0,1)) \cap L^\infty((0,1)) \subseteq L^2((0,1))$, (15.68) holds in particular for every function $g \in L^1((0,1)) \cap L^\infty((0,1))$. The conclusion follows from proposition 7.9 and proposition 15.43. □

15.5 Comments

The weak* convergence is described by Michel Willem [Wil13, §5.1] under the name of weak convergence.

16 Weak compactness

16.1 Sequential weak compactness

Proposition 16.1 (Sequential weak* compactness). *Let X be a normed space and let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence in X^*. If the space X is separable and if the sequence $(\ell_n)_{n\in\mathbb{N}}$ is bounded, then $(\ell_n)_{n\in\mathbb{N}}$ has a weakly* converging subsequence.*

Proof. Since the space X is separable, by definition of separable space definition 7.14, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that the set $\{x_n \mid n \in \mathbb{N}\}$ is dense in X. Since $(\ell_n)_{n\in\mathbb{N}}$ is bounded and since

$$|\langle \ell_n, x_0\rangle| \le \|\ell_n\|_{X^*}\|x_0\|_X, \tag{16.1}$$

the sequence $(|\langle \ell_n, x_0\rangle|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$; there exists a subsequence $(\ell_n^0)_{n\in\mathbb{N}}$ such that $(\langle \ell_n^0, x_0\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence. We continue as previously, constructing sequences $(\ell_n^m)_{n\in\mathbb{N}}$ such that $(\langle \ell_n^m, x_n\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence and $(\ell_n^{m+1})_{n\in\mathbb{N}}$ is a subsequence of $(\ell_n^m)_{n\in\mathbb{N}}$.

Defining, $\ell_n^* := \ell_n^n$, we get that $(\ell_n^*)_{n\in\mathbb{N}}$ is a subsequence of $(\ell_n)_{n\in\mathbb{N}}$ and for every $m \in \mathbb{N}$, $(\langle \ell_n^m, x_m\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(\ell_n)_{n\in\mathbb{N}}$ is bounded, by proposition 15.12, the sequence $(\ell_n^*)_{n\in\mathbb{N}}$ converges weakly* to some $\ell_* \in X^*$. □

Proposition 16.2. *Let X be a normed space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If X is reflexive and $(x_n)_{n\in\mathbb{N}}$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ has a weakly converging subsequence.*

Proof. Let $Y \subseteq X$ be the closure of the linear space spanned by $\{x_n \mid n \in \mathbb{N}\}$. By construction the space Y is separable. By proposition 14.38, Y is reflexive and hence $Y^{**} = J(Y)$ is separable; thus by proposition 14.31, Y^* is separable. By proposition 16.1, there exists a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $\xi \in Y^{**}$ such that $(J(x_n))_{n\in\mathbb{N}}$ converges weakly* to ξ. By reflexivity of Y, $\xi = J(x)$ for some $x \in Y$, and for every $\ell \in X$, we have

$$\begin{aligned}\lim_{n\to\infty} \langle \ell, x_n\rangle = \lim_{n\to\infty} \langle \ell|_Y, x_n\rangle = \lim_{n\to\infty} \langle J(x_n), \ell\rangle = \langle \xi, \ell\rangle\\ = \langle J(x), \ell\rangle = \langle \ell|_Y, x\rangle = \langle \ell, x\rangle.\end{aligned}$$ □

16.2 Sequential weak compactness in concrete spaces

Proposition 16.3. *Let X be a an inner product space space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If X is complete and $(x_n)_{n\in\mathbb{N}}$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ has a weakly converging subsequence.*

Proof. Since X is complete, by proposition 14.40, X is reflexive. The result then follows from proposition 16.2. □

Proposition 16.4. *Let $\mu : \Sigma \to [0, \infty]$ and let $p \in (1, \infty)$. If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega, \mu)$, then $(f_n)_{n\in\mathbb{N}}$ has a weakly converging subsequence.*

Proof. By proposition 14.41, $L^p(\Omega, \mu)$ is reflexive. The result then follows from proposition 16.2. □

16.3 Applications of weak compactness

Proposition 16.5. *Let X be a normed space. If X is reflexive and if $V \subseteq X$ is closed linear subspace, then for every $x \in X$, there exists $v \in V$ such that*

$$\|x - v\| = \inf\{\|x - w\| \mid w \in V\}. \tag{16.2}$$

Proof. Let

$$\nu := \inf\{\|x - w\| \mid w \in V\}. \tag{16.3}$$

By definition of infimum, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $v_n \in V$ and $(\|v_n - x\|)_{n\in\mathbb{N}}$ converges to ν. In particular, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in X. Since X is reflexive, by proposition 16.2, we can assume, up to a subsequence, that the sequence $(v_n)_{n\in\mathbb{N}}$ converges weakly to some v. By proposition 15.28, since the set V is a closed linear subspace, we have $v \in V$ and thus $\|x - v\| \geq \nu$. By proposition 15.23, we have

$$\|x - v\| \leq \liminf_{n\to\infty} \|x - v_n\| = \nu. \tag{16.4}$$

□

A Inequalities

A.1 Sums of powers

Lemma A.1. *If $p \geq 1$ and $t_1, \dots, t_n \in [0, \infty)$, then*

$$\sum_{i=1}^{n} t_i^p \leq \Bigl(\sum_{i=1}^{n} t_i\Bigr)^p. \tag{A.1}$$

Proof. For every $i \in \{1, \dots, n\}$,

$$t_i \leq \sum_{j=1}^{n} t_j. \tag{A.2}$$

Therefore

$$t_i^p \leq t_i \Bigl(\sum_{j=1}^{n} t_j\Bigr)^{p-1}, \tag{A.3}$$

and the inequality follows from summing (A.3). □

A.2 Convexity of superlinear power functions

Lemma A.2. *If $p \geq 1$, then for every $a, b \in [0, \infty)$ and $t \in [0, 1]$,*

$$((1-t)a + tb)^p \leq (1-t)|a|^p + t|b|^p.$$

Proof. We let $c \coloneqq (1-t)a + tb$. We have

$$\begin{aligned} a^p &= c^p + p \int_c^a x^{p-1} \,\mathrm{d}x \\ &\geq c^p + p \int_c^a c^{p-1} \,\mathrm{d}x = c^p + pc^{p-1}(a-c) = c^p + ptc^{p-1}(a-b). \end{aligned} \tag{A.4}$$

Similarly, we have

$$b^p \geq c^p + pc^{p-1}(b-c) = c^p + p(1-t)c^{p-1}(b-a). \tag{A.5}$$

Summing (A.4) and (A.5)

$$(1-t)a^p + tb^p \geq c^p = ((1-t)a + tb)^p, \tag{A.6}$$

which is the conclusion. □

A.3 Young's inequality

Young's inequality for a single product

Proposition A.3. *If $p \in (1, \infty)$ and $p' := \frac{p}{p-1}$, for every $a, b \in [0, \infty)$,*

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}. \tag{A.7}$$

Proof. Defining the sets

$$\begin{aligned} A :=&\{(x, y) \in [0, a] \times [0, \infty) \mid y^{p'} \le x^p\} \\ =&\{(x, y) \in [0, a] \times [0, \infty) \mid y \le x^{p-1}\} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} B :=&\{(x, y) \in [0, \infty] \times [0, b] \mid x^p \le y^{p'}\} \\ =&\{(x, y) \in [0, \infty] \times [0, b] \mid x \le y^{p'-1}\} \end{aligned} \tag{A.9}$$

We have

$$\begin{aligned} ab = \mathscr{L}^2([0, a] \times [0, b]) \le \mathscr{L}^2(A) + \mathscr{L}^2(B) \\ = \int_0^a x^{p-1} \,\mathrm{d}x + \int_0^b y^{p'-1} \,\mathrm{d}y = \frac{a^p}{p} + \frac{b^{p'}}{p'}. \end{aligned} \tag{A.10}$$

□

A.4 Comments

Lemma A.1 appears in [HLP52, §2.10, Th. 19].

B Inferior and superior limits

Definition B.1 (Inferior limit)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. The *inferior limit* of $(u_n)_{n\in\mathbb{N}}$ is

$$\liminf_{n\to\infty} u_n := \sup\{\inf\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\} \mid k \in \mathbb{N}\} \in [-\infty, \infty] \tag{B.1}$$

Definition B.2 (Superior limit)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. The *superior limit* of $(u_n)_{n\in\mathbb{N}}$ is

$$\limsup_{n\to\infty} u_n := \inf\{\sup\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\} \mid k \in \mathbb{N}\} \in [-\infty, \infty]. \tag{B.2}$$

Proposition B.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. One has*

$$\liminf_{n\to\infty}(-u_n) = -\limsup_{n\to\infty} u_n.$$

Proof. For every $k \in \mathbb{N}$, we have

$$\inf\{-u_n \mid n \in \mathbb{N} \text{ and } n \geq k\} = -\sup\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\}; \tag{B.3}$$

the conclusion then follows from definition B.1 and definition B.2. □

Proposition B.4. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. One has*

$$\liminf_{n\to\infty} u_n \leq \limsup_{n\to\infty} u_n.$$

Proof. For every $k \in \mathbb{N}$, we have

$$\inf\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\} \leq \sup\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\}; \tag{B.4}$$

the conclusion then follows from definition B.1 and definition B.2. □

Proposition B.5. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $(u_n)_{n\in\mathbb{N}}$ converges if and only if $-\infty < \liminf_{n\to\infty} u_n = \limsup_{n\to\infty} u_n < \infty$. One has then*

$$\lim_{n\to\infty} u_n = \liminf_{n\to\infty} u_n = \limsup_{n\to\infty} u_n. \tag{B.5}$$

Proof. Let $\varepsilon \in (0, \infty)$. By definition B.1, there exist $k_- \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n \geq k_-$, then

$$u_n \geq \liminf_{m\to\infty} u_m - \varepsilon. \tag{B.6}$$

Similarly, by definition B.2, there exists $k_+ \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n \ge k_+$, then

(B.7) $$u_n \le \limsup_{m\to\infty} u_m + \varepsilon.$$

If $n \ge \max\{k_-, k_+\}$, we have

(B.8) $$\liminf_{m\to\infty} u_m - \varepsilon \le u_n \le \limsup_{m\to\infty} u_m + \varepsilon,$$

which proves the convergence. □

Proposition B.6. *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$.*

(i) If $\liminf_{n\to\infty} u_n > -\infty$ and $\liminf_{n\to\infty} v_n > -\infty$, then

(B.9) $$\liminf_{n\to\infty} u_n + \liminf_{n\to\infty} v_n \le \liminf_{n\to\infty}(u_n + v_n)$$

(ii) If $\limsup_{n\to\infty} u_n < \infty$ and $\limsup_{n\to\infty} v_n > \infty$, then

(B.10) $$\limsup_{n\to\infty}(u_n + v_n) \le \limsup_{n\to\infty} u_n + \limsup_{n\to\infty} v_n$$

In general, the inequalities in (i) and (ii) are strict, as can be seen from

(B.11) $$\liminf_{n\to\infty}(-1)^n + \liminf_{n\to\infty}(-1)^{n+1} = -2 < 0 = \liminf_{n\to\infty}(-1)^n + (-1)^{n+1}$$

and

(B.12) $$\limsup_{n\to\infty}(-1)^n + (-1)^{n+1} = 0 < 2 = \limsup_{n\to\infty}(-1)^n + \limsup_{n\to\infty}(-1)^{n+1}.$$

Proof of proposition B.6. We prove (i). For every $k \in \mathbb{N}$, we have

(B.13) $$\begin{aligned}\inf\{u_n \mid n \in \mathbb{N} \text{ and } n \ge k\} &+ \inf\{v_n \mid n \in \mathbb{N} \text{ and } n \ge k\}\\ &= \inf\{u_n + v_m \mid n, m \in \mathbb{N} \text{ and } n, m \ge k\}\\ &\le \inf\{u_n + v_n \mid n \in \mathbb{N} \text{ and } n \ge k\}\end{aligned}$$

Since both terms on the left-hand side and the right-hand side are nondecreasing, the conclusion follows from the properties of sums of limits.

The proof of (ii) is similar. □

Proposition B.7. *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$.*

(i) If $\liminf_{n\to\infty} u_n < \infty$ and $\limsup_{n\to\infty} v_n < \infty$ then

(B.14) $$\liminf_{n\to\infty}(u_n + v_n) \le \liminf_{n\to\infty} u_n + \limsup_{n\to\infty} v_n.$$

(ii) If $\liminf_{n\to\infty} u_n > -\infty$ and $\limsup_{n\to\infty} v_n > -\infty$ then

(B.15) $$\liminf_{n\to\infty} u_n + \limsup_{n\to\infty} v_n \le \limsup_{n\to\infty}(u_n + v_n).$$

In general the inequalities in (i) and (ii) are strict:

$$\liminf_{n\to\infty}(0 + (-1)^n) = -1 < 1 = \liminf_{n\to\infty} 0 + \limsup_{n\to\infty}(-1)^n \tag{B.16}$$

and

$$\liminf_{n\to\infty}(-1)^n + \limsup_{n\to\infty} 0 = -1 < 1 = \limsup_{n\to\infty}((-1)^n + 0). \tag{B.17}$$

Proof of proposition B.7. We prove (i). By proposition B.6 (i) and by proposition B.3, we have

$$\liminf_{n\to\infty} u_n \geq \liminf_{n\to\infty}(u_n + v_n) + \liminf_{n\to\infty}(-v_n) = \liminf_{n\to\infty}(u_n + v_n) - \limsup_{n\to\infty} v_n. \tag{B.18}$$

The proof of (ii) is similar. □

Proposition B.8. *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. If for each $n \in \mathbb{N}$, $u_n \leq v_n$, then*

$$\liminf_{n\to\infty} u_n \leq \liminf_{n\to\infty} v_n \tag{B.19}$$

and

$$\limsup_{n\to\infty} u_n \leq \limsup_{n\to\infty} v_n. \tag{B.20}$$

Proof. For every $k \in \mathbb{N}$, we have

$$\inf\{u_n \mid n \in \mathbb{N} \text{ and } n \geq k\} \leq \inf\{v_n \mid n \in \mathbb{N} \text{ and } n \geq k\}; \tag{B.21}$$

(B.19) then follows from definition B.1. The inequality (B.20) follows similarly from definition B.2. □

C Infinite sums

C.1 Sums of nonnegative numbers

Definition C.1. Let J be a set and let $f : J \to [0, \infty]$.

$$\sum_{j \in J} f(j) = \sup \Big\{ \sum_{j \in F} f(j) \;\Big|\; F \subseteq J \text{ is finite} \Big\}, \tag{C.1}$$

with the convention that $\sum_J f(j) = 0$ if $J = \emptyset$.

Proposition C.2. *Let J be a set and let $f, g : J \to [0, \infty]$. If for every $j \in J$, $f(j) \le g(j)$, then*

$$\sum_{j \in J} f(j) \le \sum_{j \in J} g(j). \tag{C.2}$$

Proof. If $F \subseteq J$ is finite, then

$$\sum_{j \in F} f(j) \le \sum_{j \in F} g(j), \tag{C.3}$$

the conclusion then follows from definition C.1. □

Proposition C.3. *Let J be a set and let $f : J \to [0, \infty]$. If $I \subseteq J$, then*

$$\sum_{j \in I} f(j) \le \sum_{j \in J} f(j). \tag{C.4}$$

Proof. Since $I \subseteq J$, if $F \subseteq I$ is finite, then $F \subseteq J$. □

Proposition C.4. *Let J be a set and let $f : J \to [0, \infty]$,*

$$\sum_{j \in J} t f(j) = t \sum_{j \in J} f(j). \tag{C.5}$$

Proof. Given a finite set $F \subseteq J$, we have

$$\sum_{j \in F} t f(j) = t \sum_{j \in F} f(j), \tag{C.6}$$

and the conclusion follows from definition C.1. □

Proposition C.5. *Let J be a set and let $f, g : J \to [0, \infty]$,*

$$\sum_{j \in J} (f(j) + g(j)) = \sum_{j \in J} f(j) + \sum_{j \in J} g(j). \tag{C.7}$$

Proof. Given finite sets $F, G \subseteq J$, we have

$$\sum_{j \in F} f(j) + \sum_{j \in G} g(j) \le \sum_{j \in F \cup G} (f(j) + g(j)) \le \sum_{j \in J} (f(j) + g(j)), \tag{C.8}$$

so that by definition C.1,

$$\sum_{j \in J} f(j) + \sum_{j \in J} g(j) \le \sum_{j \in J} (f(j) + g(j)). \tag{C.9}$$

Conversely, given a finite set $H \subseteq J$, we have

$$\sum_{j \in H} (f(j) + g(j)) \le \sum_{j \in H} f(j) + \sum_{j \in H} g(j) \le \sum_{j \in J} f(j) + \sum_{j \in J} g(j), \tag{C.10}$$

so that

$$\sum_{j \in J} (f(j) + g(j)) \le \sum_{j \in J} f(j) + \sum_{j \in J} g(j). \tag{C.11}$$

□

Proposition C.6. *Let J be a set, assume that $\mathscr{I} \subseteq \mathscr{P}(J)$ partitions J, let $f : J \to [0, \infty]$. One has*

$$\sum_{j \in J} f(j) = \sum_{I \in \mathscr{I}} \biggl(\sum_{i \in I} f(i) \biggr). \tag{C.12}$$

Proposition C.7. *Let J be a set and let $f_n : J \to [0, \infty]$.*

$$\sum_{j \in J} \liminf_{n \to \infty} f_n(j) \le \liminf_{j \in J} \sum_{j \in J} f_n(j). \tag{C.13}$$

Proof. If $F \subseteq J$ is finite, we have by the sum property for inferior limits (proposition B.6) and their comparison property (proposition B.8),

$$\sum_{j \in F} \liminf_{n \to \infty} f_n(j) \le \liminf_{n \to \infty} \sum_{j \in F} f_n(j) \le \liminf_{n \to \infty} \sum_{j \in J} f_n(j), \tag{C.14}$$

and therefore by definition C.1

$$\sum_{j \in J} \liminf_{n \to \infty} f_n(j) \le \liminf_{n \to \infty} \sum_{j \in J} f_n(j). \tag{C.15}$$

□

m

Proposition C.8. *Let J be a set and let $f : J \to [0,\infty]$. If $\sum_{j\in J} f(j) < \infty$, then the set*

$$\{j \in J \mid f(j) \neq 0\} \tag{C.16}$$

is countable.

Proof. For every $n \in \mathbb{N}$, we have

$$\#\big\{j \in J \;\big|\; f(j) \geq 2^{-n}\big\} \leq 2^n \sum_{j\in J} f(j) < \infty. \tag{C.17}$$

Thus the set

$$\{j \in J \mid f(j) > 0\} = \bigcup_{n\in\mathbb{N}} \big\{j \in J \;\big|\; |f(j)| \geq 2^{-n}\big\} \tag{C.18}$$

is countable as a countable union of finite sets. □

C.2 Unconditional summation of vectors

Definition C.9. Let J be a set and $|\cdot|_V$ be a norm on V. The function $f : J \to V$ is *unconditionally summable* on J whenever there exists $S \in V$ such that for every $\varepsilon > 0$ there exists a finite set $F_0 \subseteq J$ such that for every finite set $F \subseteq J$ satisfying $F_0 \subseteq F$, we have

$$\Big|S - \sum_{j\in F} f(j)\Big|_V \leq \varepsilon. \tag{C.19}$$

Proposition C.10. *If $S_0, S_1 \in V$, satisfy definition C.9, then $S_0 = S_1$.*

Definition C.11. If $f : J \to V$ is unconditionally summable,

$$\sum_{j\in J} f(j), \tag{C.20}$$

denotes the unique vector $S \in V$ satisfying definition C.9.

Proposition C.12. *Let J be a set. If the functionf $: J \to V$ is unconditionally summable on $\mathbb{R}$, then*

$$\sup\Big\{\Big|\sum_{j\in F} f(j)\Big|_V \;\Big|\; F \subseteq J \text{ is finite}\Big\} < \infty \tag{C.21}$$

and for every $\varepsilon > 0$ there exists a finite set $F_0 \subseteq J$ such that

$$\sup\Big\{\Big|\sum_{j\in F} f(j)\Big|_V \;\Big|\; F \subseteq J \setminus F_0 \text{ is finite}\Big\} \leq \varepsilon. \tag{C.22}$$

Proposition C.13. *Let J be a set. If the function $f : J \to V$ is unconditionally summable on $\mathbb{R}$, then the set*

$$\{j \in J \mid f(j) \neq 0\} \tag{C.23}$$

is countable.

Proposition C.14. *Let J be a set. If $f, g : J \to V$ are unconditionally summable on J, then $f + g$ is unconditionally summable on J and*

$$\sum_{j \in J} (f(j) + g(j)) = \sum_{j \in J} f(j) + \sum_{j \in J} g(j). \tag{C.24}$$

Proposition C.15. *Let J be a set. If $f : J \to \mathbb{R}$ is unconditionally summable on J and $t \in \mathbb{R}$, then tf is unconditionally summable on J and*

$$\sum_{j \in J} t f(j) = t \sum_{j \in J} f(j). \tag{C.25}$$

Proposition C.16. *Let I, J be disjoint set and let $f : I \cup J \to V$. If $f|_I$ and $f|_J$ are unconditionally summable, then f is unconditionally summable and*

$$\sum_{j \in I \cup J} f(i) = \sum_{j \in I} f(j) + \sum_{j \in J} f(j). \tag{C.26}$$

Proof. If $H \subseteq I \cup J$ is finite, then

$$\Bigl| \sum_{j \in H} f(j) - \sum_{j \in I} f(j) - \sum_{j \in J} f(j) \Bigr| \le \Bigl| \sum_{j \in H \cap I} f(j) - \sum_{j \in I} f(j) \Bigr| + \Bigl| \sum_{j \in H \cap J} f(j) - \sum_{j \in J} f(j) \Bigr|. \tag{C.27}$$

By definition, there exist a finite set $F_0 \subseteq I$ such that if $H \cap I \supseteq F_0$, then the first term on the right-hand side is small and a finite set $G_0 \subseteq J$ such that if $H \cap J \supseteq G_0$, then the second term on the right-hand side is small. Defining $H_0 \coloneqq F_0 \cup G_0$, which is finite, if $F \subseteq I \cup J$ is finite and if $F \supseteq H_0$, then the left-hand side is small. Hence f is unconditionally summable. □

Proposition C.17. *Let J be a set. The function $f : J \to [0, \infty) \subseteq \mathbb{R}$ is unconditionally summable if and only if*

$$\sum_{j \in J} f(j) < \infty. \tag{C.28}$$

in the sense of definition C.1. Moreover the sums in the sense of definition C.1 and definition C.11 are identical.

Proposition C.18. *Let J be a set and let $f : J \to V$. If V is complete and if for every $\varepsilon > 0$ there exists a finite set $F_0 \subseteq J$ such that*

$$\sup \Bigl\{ \Bigl| \sum_{j \in F} f(j) \Bigr|_V \Bigm| F \subseteq J \setminus F_0 \text{ is finite} \Bigr\} \le \varepsilon, \tag{C.29}$$

then f is unconditionally summable.

Proof. Given a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, we construct finite sets $F_n \subseteq J$ such that $F_n \subseteq F_{n+1}$ and for every finite set $F \subseteq J \setminus F_n$,

$$\Big|\sum_{j\in F} f(j)\Big|_V \le \varepsilon_n. \tag{C.30}$$

In particular, we have if $m \in \mathbb{N}$ and $m > n$,

$$\Big|\sum_{j\in F_n} f(j) - \sum_{j\in F_m} f(j)\Big|_V = \Big|\sum_{j\in F_m\setminus F_n} f(j)\Big|_V \le \varepsilon_n, \tag{C.31}$$

so that $(\sum_{j\in F_n} f(j))_{j\in\mathbb{N}}$ is a Cauchy sequence. By completeness of V it converges to some $S \in V$, which is the sum of the family. □

Proposition C.19. *Let J be a set and let $f : J \to V$. If $W \subseteq V$ is a closed linear subspace and if $f(J) \subseteq W$, then*

$$\sum_{j\in J} f(j) \in W. \tag{C.32}$$

Proof. Given $\varepsilon \in (0,\infty)$, there exists a finite set $F \subseteq J$ such that

$$\Big\|\sum_{j\in F} f(j) - \sum_{j\in J} f(j)\Big\| \le \varepsilon. \tag{C.33}$$

Since F is finite, we have

$$\sum_{j\in F} f(j) \in W. \tag{C.34}$$

Since W is closed, we conclude that $\sum_{j\in J} f(j) \in W$. □

Proposition C.20. *Let J be a set and let $f : J \to V$. If f is unconditionally summable, if V is complete and if $I \subseteq J$, then $f|_I$ is unconditionally summable.*

Proof. This follows from proposition C.12 and proposition C.18. □

Proposition C.21. *Let J be a set. The function $f : J \to \mathbb{R}$ is unconditionally summable if and only if*

$$\sum_{j\in J} |f(j)| < \infty. \tag{C.35}$$

Proof. Defining

$$J_+ \coloneqq \{j \in J \mid f(j) > 0\} \tag{C.36}$$

and

$$J_- \coloneqq \{j \in J \mid f(j) < 0\}, \tag{C.37}$$

by proposition C.20 and proposition C.17, the function f is unconditionnaly summable if and only the functions $f|_{J_+}$ and $f|_{J_-}$ are absolutely summable (f being trivially unconditionally summable on $J \setminus (J_+ \cup J_-)$. □

Proposition C.22. *Let J be a set, assume that $\mathscr{I} \subseteq \mathscr{P}(J)$ partitions J, let $f : J \to V$. If f is summable on J, then the function $I \in \mathscr{I} \mapsto \sum_{i\in I} f(i)$ is summable on $\mathscr{I}$ and one has*

$$\sum_{j\in J} f(j) = \sum_{I\in\mathscr{I}}\Bigl(\sum_{i\in I} f(i)\Bigr). \tag{C.38}$$

Proof. Since f is unconditionally summable, by definition C.9, there exists a finite set $F_0 \subseteq J$ such that if $F \subseteq J$ is finite and $F \subseteq F_0$, then

$$\Bigl|\sum_{j\in F} f(j) - \sum_{j\in J} f(j)\Bigr| \le \varepsilon. \tag{C.39}$$

We set

$$\mathscr{F}_0 := \{I \in \mathscr{I} \mid I \cap F_0 \ne \emptyset\} \tag{C.40}$$

Assume now that $\mathscr{F} \subseteq \mathscr{I}$ is finite and that $\mathscr{F} \supseteq \mathscr{F}_0$. If $F \subseteq J$ is finite and $F \subseteq \bigcup_{I\in\mathscr{F}_0} I$, then

$$\Bigl|\sum_{i\in F} f(i) - \sum_{I\in\mathscr{F}}\sum_{i\in I} f(i)\Bigr| \le \sum_{I\in\mathscr{F}} |\sum_{i\in I\cap F} f(i) - \sum_{i\in I} f(i)|. \tag{C.41}$$

Taking F large enough so that the right-hand side of (C.41) is small enough and $F \supseteq F_0$, we deduce from (C.39) that

$$\Bigl|\sum_{j\in J} f(j) - \sum_{I\in\mathscr{F}}\sum_{i\in I} f(i)\Bigr| \le \varepsilon, \tag{C.42}$$

which is the conclusion. □

Proposition C.23. *Let J be a set, V be normed by $|\cdot|_V$. Assume that for every $j \in J$, $H_j \subseteq V$ and that every $f : J \to V$ such that for each $j \in J$ one has $f(j) \in H_j$ is summable and that the set*

$$\Bigl\{\sum_{j\in J} f(j) \Bigm| f : J \to V \text{ and for every } j \in J,\ f(j) \in H_j\Bigr\} \tag{C.43}$$

is bounded in V, then if for each $j \in J$, we have a set I_j, a function $\lambda_j : I_j \to [0,1]$ such that $\sum_{i\in I_j} \lambda_j(i) = 1$ and a function $g_j : I \to H_i$, defining $K = \{(j,i) \mid j \in J \text{ and } i \in I_j\}$, then $(j,i) \mapsto \lambda_j(i)g_j(i)$ is summable and

$$\sum_{\substack{j\in J\\ i\in I_j}} \lambda_j(i)g_j(i) \in \overline{\operatorname{conv}\Bigl\{\sum_{j\in J} f(j) \Bigm| f : J \to V \text{ and for every } j \in J,\ f(j) \in H_j\Bigr\}}. \tag{C.44}$$

Sketch of the proof. The boundedness assumption implies that H_j is bounded for every $j \in J$.

We claim that for every $\varepsilon > 0$ there exists F_0 such that for every $f : J \to V$ such that $f(j) \in H_j$, we have if $F \subseteq J \setminus F_0$ is finite

$$\left|\sum_{j\in J} f(j)\right| \le \varepsilon. \tag{C.45}$$

Otherwise, there exist a sequence of disjoint sets $(F_n)_{n\in\mathbb{N}}$ and functions $f_n : J \to V$ such that

$$\left|\sum_{j\in F_n} f_n(j)\right| > \varepsilon. \tag{C.46}$$

Taking f such that $f|_{F_n} = f_n|_{F_n}$, we reach a contradiction.

By convexity, it follows thus that $(j,i) \in K \mapsto \lambda_j(i) g_j(i)$ is summable.

In view of proposition C.14, without loss of generality, we can assume that for every $j \in J$ one has $0 \in H_j$. Given a finite set $F \subseteq K$, we have

(C.47)
$$\sum_{(i,j)\in F} \lambda_j(i) f_j(i) \in \overline{\operatorname{conv}\left\{\sum_{j\in G} f(j) \;\middle|\; G \subseteq K \text{ is finite}, f : J \to V \text{ and for every } j \in J,\, f(j) \in H_j\right\}}.$$

The conclusion follows. □

C.3 Comments

For unconditional convergence, see [Dix84, §9.1–9.2].

If V is not finite-dimensional, then $f : J \to V$ can be unconditionally summable while $\sum_{j\in J} |f(j)|_V = \infty$. For instance of V is a Hilbert space and if $(e_n)_{n\in\mathbb{N}}$ is an orthonormal set, then $n \mapsto e_n/n$ is unconditionally summable, but $\sum_{n\in\mathbb{N}} \|e_n/n\| = \infty$. In general, A. Dvoretzky and C. A. Rogers have showed that if X is a complete normed space, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $n \in \mathbb{N} \mapsto x_n$ is unconditionally summable but $\sum_{n\in\mathbb{N}} \|x_n\|_X = \infty$ [DR50] (see also [Hei11, §3.6]).

Proposition C.23 is due to Garrett Birkhoff [Bir35, th. 8].

Unconditionally summable functions form a complete normed space under the norm [Bir35, th. 6]

$$\|f\| := \sup\left\{\left|\sum_{j\in F} f(j)\right| \;\middle|\; F \subseteq J \text{ is finite}\right\} \tag{C.48}$$

D Measures and integrals

D.1 Measurable sets and functions

Definition D.1. A set $\Sigma \subseteq \mathscr{P}(\Omega)$ is a σ–algebra whenever

(i) if $A \in \Sigma$, then $\Omega \setminus A \in \Sigma$,

(ii) if for each $n \in \mathbb{N}$, $A_n \in \Sigma$, then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

Example D.2. If Ω is a set, then $\{\emptyset, \Omega\}$ is a σ–algebra.

Example D.3. If Ω is a set, then $\mathscr{P}(\Omega)$ is a σ–algebra.

Example D.4. If Ω is a set, then

$$\{A \in \mathscr{P}(\Omega) \mid \text{either } A \text{ is countable or } \Omega \setminus A \text{ is countable}\} \tag{D.1}$$

is a σ–algebra.

Proposition D.5. *If $\Sigma \subseteq \mathscr{P}(\Omega)$ is a σ–algebra and if for each $n \in \mathbb{N}$, $A_n \in \Sigma$, then $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.*

Proof. We have

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \Omega \setminus (\Omega \setminus A_n) = \Omega \setminus \bigcup_{n\in\mathbb{N}} (\Omega \setminus A_n). \tag{D.2}$$

□

Definition D.6. Let Σ be a σ–algebra on Ω and let $A \in \Sigma$. A function $f : A \to [0, \infty]$ is measurable with respect to Σ whenever for every $t \in [0, \infty]$, $\mu(f^{-1}((t, \infty])) \in \Sigma$.

Proposition D.7. *Let Σ be a σ–algebra on Ω and let $A \in \Sigma$. If $f, g : \Omega \to [0, \infty]$ is measurable with respect to a σ–algebra Σ, then $f + g$ is measurable.*

Proof. We have for every $t \in \mathbb{R}$

$$(f + g)^{-1}((t, \infty]) = \bigcup_{q\in\mathbb{Q}} f^{-1}((q, \infty]) \cap g^{-1}((t - q, \infty]). \tag{D.3}$$ □

Definition D.8. Let V be a normed linear space. A function $f : \Omega \to V$ is measurable with respect to a σ–algebra Σ whenever for every open set $U \subseteq V$, the $f^{-1}(U) \in \Sigma$.

Definition D.9. Let $(X, \mathscr{U})$ be a topological space. The Borel σ–algebra $\mathscr{B}(X)$ is defined as

$$\mathscr{B}(X) = \bigcap \{\Sigma \subseteq \mathscr{P}(X) \mid \mathscr{U} \subseteq \Sigma\}. \tag{D.4}$$

A Borel set is an element of $\mathscr{B}(X)$.

Proposition D.10. *If $(X, \mathscr{U})$ is a topological space, then $\mathscr{B}(X)$ is a σ–algebra.*

D.2 Measures

Definition D.11. Given $\Sigma \subseteq \mathscr{P}(\Omega)$, a function $\mu : \Sigma \to [0, \infty]$ is a measure whenever Σ is a σ–algebra and if for every $n \in \mathbb{N}$ $A_n \in \Sigma$ and if for $m > n$, $A_m \cap A_n = \emptyset$, then

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n). \tag{D.5}$$

Proposition D.12. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $A, B \in \Sigma$. If $A \subseteq B$, then $\mu(A) \le \mu(B)$.*

Proof. We have

$$\mu(A) \le \mu(A) + \mu(B \setminus A) = \mu(B). \tag{D.6}$$

□

Proposition D.13. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If for every $n \in \mathbb{N}$, $A_n \in \Sigma$, then*

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \mu(A_n). \tag{D.7}$$

Proof. We define for each $n \in \mathbb{N}$,

$$\tilde{A}_n \coloneqq A_n \setminus \bigcup_{k=0}^{n-1} A_k, \tag{D.8}$$

so that $\tilde{A}_n \subseteq A_n$, $\tilde{A}_n \in \Sigma$ and

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \tilde{A}_n. \tag{D.9}$$

We then have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} \tilde{A}_n\Big) = \sum_{n\in\mathbb{N}} \mu(\tilde{A}_n) \le \sum_{n\in\mathbb{N}} \mu(A_n). \tag{D.10}$$

□

Proposition D.14. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If for every $n \in \mathbb{N}$, $A_n \in \Sigma$ and $A_n \subseteq A_{n+1}$,, then*

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty} \mu(A_n). \tag{D.11}$$

Proof. Defining $B_0 = A_0$ and $B_n = A_n \setminus A_{n-1}$, we have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n\in\mathbb{N}} \mu(B_n) = \lim_{n\to\infty} \sum_{k=0}^{n} \mu(B_n) = \lim_{n\to\infty} \mu(A_n). \tag{D.12}$$

□

Proposition D.15. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If for every $n \in \mathbb{N}$, $A_n \in \Sigma$ and $A_n \supseteq A_{n+1}$, and if $\mu(A_0) < \infty$, then*

$$\mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty} \mu(A_n). \tag{D.13}$$

Proof. By proposition D.14, we have

$$\begin{aligned}\mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) &= \mu\Big(A_0 \setminus \bigcup_{n\in\mathbb{N}}(A_0 \setminus A_n)\Big) = \mu(A_0) - \mu\Big(\bigcup_{n\in\mathbb{N}}(A_0 \setminus A_n)\Big)\\ &= \mu(A_0) - \lim_{n\to\infty} \mu(A_0 \setminus A_n) = \lim_{n\to\infty} (\mu(A_0) - \mu(A_0 \setminus A_n)) = \lim_{n\to\infty} \mu(A_n).\end{aligned} \tag{D.14}$$

□

Definition D.16. Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. A set $A \subseteq \Omega$ is *negligible* with respect to μ whenever there exists a set $B \in \Sigma$ such that $A \subseteq B$ and $\mu(B) = 0$.

D.2.1 Semifiniteness

Definition D.17. A measure $\mu : \Sigma \to [0, \infty]$ on Ω is semifinite whenever for every $A \in \Sigma$ with $\mu(A) = \infty$ there exists $B \in \Sigma$ such that $0 < \mu(B) < \infty$ and $B \subseteq A$.

Proposition D.18. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If μ is semifinite, then for every $A \in \Sigma$,*

$$\mu(A) = \sup\{\mu(B) \mid B \in \Sigma \text{ and } \mu(B) < \infty\}. \tag{D.15}$$

Proof. If $\mu(A) < \infty$, the proposition is trivial. Otherwise, we assume that

$$\lambda := \sup\{\mu(B) \mid B \in \Sigma \text{ and } \mu(B) < \infty\} < \infty. \tag{D.16}$$

There exists then a sequence $(B_n)_{n\in\mathbb{N}}$ of sets in Σ such that for each $n \in \mathbb{N}$, one has $B_n \subseteq A$ and such that $\lim_{n\to\infty} \mu(B_n) = \lambda$. For every $m \in \mathbb{N}$, we have $\mu(\bigcup_{n=0}^m B_n) \le \sum_{n=0}^m \mu(B_n) < \infty$, and thus $\mu(\bigcup_{n=0}^m B_n) \le \lambda$. Setting $H := \bigcup_{n\in\mathbb{N}} B_n$, we have for every $n \in \mathbb{N}$

$$\mu(B_n) \le \mu(H) = \lim_{m\to\infty} \mu\Big(\bigcup_{n=0}^m B_n\Big) \le \lambda, \tag{D.17}$$

and thus

$$\mu(H) = \lambda. \tag{D.18}$$

we have $\mu(H) = \lambda$, and thus $\mu(A \setminus H) = \infty$. Since μ is semi-finite, there exists $E \in \Sigma$ such that $E \subseteq A \setminus H$ and $0 < \mu(E) < \infty$, and therefore,

$$\lambda < \mu(H \cup E) < \infty, \tag{D.19}$$

in contradiction with the definition of λ in (D.16). □

Proposition D.19. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. Define $\mu_0 : \Sigma \to [0, \infty]$ for $A \in \Sigma$ by*

$$\mu_0(A) := \sup\{\mu(E) \mid E \in \Sigma, E \subseteq A \text{ and } \mu(A) < \infty\}. \tag{D.20}$$

Then μ_0 is a semi-finite measure and for every $A \in \Sigma$ with $\mu(A) < \infty$, we have $\mu_0(A) = \mu(A)$.

D.2.2 Non-atomicity

Definition D.20. A measure $\mu : \Sigma \to [0, \infty]$ on Ω is *non-atomic* whenever for every $A \in \Sigma$ such that $\mu(A) > 0$, there exists $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \mu(A)$.

Proposition D.21. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If μ is non-atomic, then μ is semi-finite.*

Proof. If $\mu(A) = \infty$, then by definition of non-atomic measure there exists $B \in \Sigma$ such that $0 < \mu(B) < \mu(A) = \infty$, and thus μ is semi-finite. □

Proposition D.22. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If μ non-atomic, then for every $t \in [0, \mu(\Omega)]$ there exists $A \in \Sigma$ such that $\mu(A) = t$.*

Lemma D.23. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If μ is non-atomic, then for every $A \in \Sigma$ such that $0 < \mu(A) < \infty$,*

$$\inf\{\mu(B) \mid B \in \Sigma, B \subseteq A \text{ and } \mu(B) > 0\} = 0. \tag{D.21}$$

Proof. We set $B_0 := A$. Assuming that $B_n \in \mathbb{N}$ has been constructed for some $n \in \mathbb{N}$, by non-atomicity, there exists a set $B_n \in \Sigma$ such that $B_{n+1} \subseteq B_n$ and $0 < \mu(B_{n+1}) < \mu(B_n)/2$. Indeed, by definition there exists E_{n+1} such that $0 < \mu(E_{n+1}) < \mu(B_n)$. Since $\mu(E_{n+1}) + \mu(B_n \setminus E_{n+1} = \mu(B_n)$ we conclude taking either $B_{n+1} = E_{n+1}$ or $B_{n+1} = B_n \setminus E_{n+1}$. □

Lemma D.24. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If μ is non-atomic, then for every $A \in \Sigma$ and $t \in (0, \mu(\Omega))$ there exists $B \in \Sigma$ such that $B \subseteq \Sigma$ and $t/2 \le \mu(B) \le t$.*

Proof. Let

$$s := \sup\{\mu(B) \mid B \in \Sigma, B \subseteq A \text{ and } \mu(A) \le t/2\}. \tag{D.22}$$

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence in Σ such that for every $n \in \mathbb{N}$, $B_n \subseteq A$ and $\mu(B_n) \le t/2$, and such that $\lim_{n\to\infty} \mu(B_n) = s$. We define $H_m = \bigcup_{n=0}^{m} B_m$.

If $\mu(H_m) \le t/2$ for every $m \in \mathbb{N}$, then we set $B := \bigcup_{n\in\mathbb{N}} B_n$.

$$s \le \mu(B) = \lim_{m\to\infty} \mu(H_m) \le t/2. \tag{D.23}$$

By definition of s, this implies that $\mu(B) = s$. If $s < t/2$, then by lemma D.23 there exists $E \in \Sigma$ such that $E \subseteq A \setminus B$ and $0 < \mu(E) < t/2 - s$ and thus $s < \mu(B \cup E) < t/2$, in contradiction with the definition of s.

Otherwise, there exists $m \in \mathbb{N}$ such that $\mu(H_m) \le t/2 < \mu(H_{m+1})$. By construction, we have $\mu(H_{m+1}) \le t$. We set $B := H_{m+1}$. □

Proof of proposition D.22. By lemma D.24, there exists a set $B_0 \in \Sigma$ such that $B_0 \subseteq A$ and $t/2 \le \mu(B_0) \le t$.

Next, we construct inductively $B_{n+1} \in \Sigma$ such that $B_{n+1} \supseteq B_n$ and $\frac{\mu(B_n)+t}{2} \le \mu(B_{n+1}) \le t$. Indeed, by lemma D.24, there exists a set $E_{n+1} \in \Sigma$ such that $E_{n+1} \subseteq A \setminus B_n$ and $\frac{t-\mu(B_n)}{2} \le \mu(E_{n+1}) \le t - \mu(B_n)$. We set $B_{n+1} = B_n \cup E_{n+1}$.

By induction, we have $\mu(B_n) \ge (1 - 2^{-n-1})t$. Letting $B := \bigcup_{n\in\mathbb{N}} B_n$, we reach the conclusion. □

D.3 Integrating nonnegative functions

Definition D.25. Let $\mu : \Sigma \to [0, \infty]$ be a measure on Σ, let $A \in \Sigma$. If $f : A \to [0, \infty]$ is measurable, then

$$\int_\Omega f \,\mathrm{d}\mu := \inf\left\{ \sum_{j=1}^n t_j \mu(E_j) \;\middle|\; E_1, \dots, E_n \in \Sigma, \text{ are disjoint and } f \ge t_j \text{ on } E_j \subseteq A \right\} \in [0, \infty]. \tag{D.24}$$

We use here the convention that $0 \cdot \infty = \infty \cdot 0 = 0$.

Proposition D.26. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Σ and let $A \in \Sigma$. If $f, g : A \to [0, \infty]$ are measurable and if for every $x \in A$, $f(x) \le g(x)$, then*

$$\int_A f \,\mathrm{d}\mu \le \int_A g \,\mathrm{d}\mu. \tag{D.25}$$

Proof. If $E_1, \dots, E_n \in \Sigma$ are disjoint and $g \ge t_j$ on $E_j \subseteq A$, then $f \ge t_j$ on E_j. □

Proposition D.27. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Σ and let $A \in \Sigma$. If $f : A \to [0, \infty]$ is measurable, then for every $t \in [0, \infty)$,*

$$\int_A t f \,\mathrm{d}\mu = t \int_A f \,\mathrm{d}\mu. \tag{D.26}$$

Proof. The proposition is trivial when $t = 0$.

If $t > 0$, if $E_1, \dots, E_n \in \Sigma$ are disjoint, we have $tf \ge tt_j$ on $E_j \subseteq A$ if and only if $f \ge t_j$ on E_j. □

Proposition D.28. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Σ and let $A \in \Sigma$. If $f, g : A \to [0, \infty]$ are measurable, then*

$$\int_A (f + g) \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu. \tag{D.27}$$

Proof. Let $E_1, \dots, E_n \in \Sigma$ and $D_1, \dots, D_m \in \Sigma$ be disjoint subsets of A. If $f \ge t_j$ on E_j and $g \ge s_i$ on D_i, then $(E_j \cap D_i)_{1 \le j \le m, 1 \le i \le n} \in \Sigma$ are disjoint subsets of A, and $f + g \ge t_j + s_i$ on $C_{j,i}$. Hence,

$$\sum_{j=1}^{m} t_j \mu(E_j) + \sum_{i=1}^{n} s_i \mu(D_i) = \sum_{j=1}^{n} \sum_{j=1}^{n} (t_j + s_i) \mu(D_j \cap E_i) \le \int_A (f + g) \,\mathrm{d}\mu. \tag{D.28}$$

Hence,

$$\int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu \le \int_A (f + g) \,\mathrm{d}\mu. \tag{D.29}$$

Conversely, let $E_1, \dots, E_n \in \Sigma$ be disjoint subsets of A such that $f + g \ge t_j$ on E_j and let $\ell \in \mathbb{N} \setminus \{0\}$. We define for $k \in \{1, \dots, \ell - 1\}$, the sets

$$E_j^k = \left\{ x \in E_j \;\middle|\; \frac{k-1}{\ell} t_j \le f(x) < \frac{k}{\ell} t_j \right\}. \tag{D.30}$$

and

$$E_j^\ell = \left\{ x \in E_j \;\middle|\; \frac{\ell-1}{\ell} t_j \le f(x) \right\}. \tag{D.31}$$

Since f is measurable, the sets E_k are disjoint and $\bigcup_{k=0}^{\ell} E_j^k = E_j$. We have $f \ge \frac{k-1}{\ell} t_j$ and $g \ge \frac{\ell-k}{\ell} t_j$ in E_j. Hence,

$$\begin{aligned} \int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu &\ge \sum_{j=1}^{m} \sum_{k=1}^{\ell} \frac{k-1}{\ell} t_j \mu(E_j^k) + \sum_{j=1}^{m} \sum_{k=1}^{\ell} \frac{\ell-k}{\ell} t_j \mu(E_j^k) \\ &= \sum_{j=1}^{m} \sum_{k=1}^{\ell} \frac{\ell-1}{\ell} t_j \mu(E_j^k) = \frac{\ell-1}{\ell} \sum_{j=1}^{m} t_j \mu(E_j), \end{aligned} \tag{D.32}$$

and thus

$$\int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu \ge \frac{\ell-1}{\ell} \int_A (f + g) \,\mathrm{d}\mu. \tag{D.33}$$

Since $\ell \in \mathbb{N} \setminus \{0\}$ is arbitrary, this implies that

$$\int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu \ge \int_A (f + g) \,\mathrm{d}\mu. \tag{D.34}$$

□

Proposition D.29. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $f : \Omega \to [0, \infty]$ be measurable. The function $\mu_f : \Sigma \to [0, \infty]$ defined for $A \in \Sigma$ by*

$$\mu_f(A) = \int_A f \,\mathrm{d}\mu \tag{D.35}$$

is a measure on Ω.

Proposition D.30 (Chebyshev's inequality)**.** *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, let $A \in \Sigma$ and let $f : \Omega \to [0, \infty]$ be measurable. For every $t > 0$,*

$$\mu(\{x \in A \mid f(x) \geq t\}) \leq \frac{1}{t} \int_A f \,\mathrm{d}\mu. \tag{D.36}$$

Proof. We define the set

$$E := \{x \in \Omega \mid f(x) \geq t\}. \tag{D.37}$$

By definition D.25,

$$t\mu(E) \leq \int_E f \,\mathrm{d}\mu, \tag{D.38}$$

and the conclusion follows. □

Proposition D.31. *Let $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, let $A \in \Sigma$ and let $f : A \to [0, \infty]$ be measurable. If*

$$\int_A f \,\mathrm{d}\mu < \infty, \tag{D.39}$$

then

$$\mu(\{x \in A \mid f(x) = \infty\}) = 0. \tag{D.40}$$

Proof. We have

$$\{x \in A \mid f(x) = \infty\} = \bigcap_{n \in \mathbb{N}} \{x \in A \mid f(x) \geq 2^n\}, \tag{D.41}$$

and by proposition D.30,

$$\mu(\{x \in A \mid f(x) \geq 2^n\}) \leq 2^{-n} \int_A f \,\mathrm{d}\mu < \infty. \tag{D.42}$$

We conclude by proposition D.15. □

Proposition D.32. *Let $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, let $A \in \Sigma$ and let $f : A \to [0, \infty]$ be measurable. If*

$$\int_A f \,\mathrm{d}\mu = 0, \tag{D.43}$$

then

$$\mu(\{x \in A \mid f(x) > 0\}) = 0. \tag{D.44}$$

Proof. We have

$$\{x \in A \mid f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \{x \in A \mid f(x) \ge 2^{-n}\}. \tag{D.45}$$

By proposition D.30,

$$\mu\big(\{x \in A \mid f(x) \ge 2^{-n}\}\big) = 0, \tag{D.46}$$

and thus the conclusion follows from proposition D.14. □

Proposition D.33. *Let $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, let $A \in \Sigma$ and let $f : A \to [0, \infty]$ be measurable. If*

$$\int_\Omega f \,\mathrm{d}\mu < \infty, \tag{D.47}$$

then there exists a sequence of sets $(E_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $E_n \in \Sigma$ and $\mu(E_n) < \infty$

$$\mu(\{x \in A \mid f(x) > 0\}) = \bigcup_{n \in \mathbb{N}} E_n. \tag{D.48}$$

Proof. We define for $n \in \mathbb{N}$,

$$E_n := \{x \in \Omega \,\big|\, f(x) > 2^{-n}\}. \tag{D.49}$$

We clearly have (D.48) and $E_n \in \Sigma$. Moreover for every $n \in \mathbb{N}$, by proposition D.30

$$\mu(E_n) \le 2^n \int_A f \,\mathrm{d}\mu < \infty. \tag{D.50}$$

□

Proposition D.34 (Absolute continuity)**.** *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $A \in \Sigma$. If*

$$\int_A f \,\mathrm{d}\mu < \infty, \tag{D.51}$$

then

(i) for every $\varepsilon > 0$, there exists $F \in \Sigma$ such that $\mu(F) < \infty$, $F \subseteq A$ and $\int_{A \setminus F} f \,\mathrm{d}\mu \le \varepsilon$,

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \in \Sigma$, $E \subseteq A$ and $\mu(E) \le \delta$, then $\int_E f \,\mathrm{d}\mu \le \varepsilon$.

Proof. By definition D.25, there exists disjoint sets $E_1, \dots, E_n \in \Sigma$, such that $E_1, \dots, E_n \subseteq A$ and $t_1, \dots, t_j \in (0, \infty)$ such that $f \ge t_j$ on E_j and

$$\sum_{j=1}^n t_j \mu(E_j) \ge \int_A f \,\mathrm{d}\mu - \varepsilon. \tag{D.52}$$

In order to prove (i), we define $F := \bigcup_{j=1}^m E_i$. We then have

$$\int_F f \, \mathrm{d}\mu \geq \sum_{j=1}^n t_j \mu(E_j) \geq \int_A f \, \mathrm{d}\mu - \varepsilon, \tag{D.53}$$

and therefore

$$\int_{A \setminus F} f \, \mathrm{d}\mu \leq \varepsilon. \tag{D.54}$$

This proves (i).

Next if $E \in \Sigma$, then

$$\int_{A \setminus E} f \, \mathrm{d}\mu \geq \sum_{j=1}^n t_j \mu(E_j \setminus E) \geq \int_A f \, \mathrm{d}\mu - \varepsilon - \sum_{j=1}^n t_j \mu(E_j \cap E). \tag{D.55}$$

Taking $\delta > 0$ such that for every $j \in \mathbb{N}$, $t_j \delta \leq \varepsilon$, we get

$$\int_{A \setminus E} f \, \mathrm{d}\mu \geq \int_A f \, \mathrm{d}\mu - \varepsilon - \sum_{j=1}^m \frac{\varepsilon}{\delta} \mu(E_j \cap E) \geq \int_A f \, \mathrm{d}\mu - \varepsilon - \frac{\varepsilon \mu(E)}{\delta}. \tag{D.56}$$

It follows, thus that if $\mu(E) \leq \delta$, then

$$\int_E f \, \mathrm{d}\mu \leq 2\varepsilon. \tag{D.57}$$

This proves (ii). □

Proposition D.35. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $A \in \Sigma$. If μ is semifinite and if $f : A \to [0, \infty]$ is measurable, then*

$$\int_A f \, \mathrm{d}\mu = \sup \Big\{ \int_B f \, \mathrm{d}\mu \;\Big|\; B \in \Sigma,\ B \subseteq A \text{ and } \mu(B) < \infty \Big\}. \tag{D.58}$$

Proof. Since follows from definition D.25 and proposition D.18. □

Proposition D.36 (Fatou's lemma). *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω and let $A \in \Sigma$. If for every $n \in \mathbb{N}$, $f_n : A \to [0, \infty]$ is measurable, then*

$$\liminf_{n \to \infty} \int_A f_n \, \mathrm{d}\mu \geq \int_A \liminf_{n \to \infty} f_n \, \mathrm{d}\mu. \tag{D.59}$$

Proposition D.37 (Fatou's lemma for convergence in measure). *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω. If for every $n \in \mathbb{N}$, the function $f_n : \Omega \to [0, \infty]$ is measurable and if $f : \Omega \to [0, \infty]$ is measurable and for every $\varepsilon > 0$ and $E \in \Sigma$ with $\inf\{f(x) \mid x \in E\} > 0$,*

$$\lim_{n \to \infty} \mu(\{x \in E \mid f_n(x) \geq f(x) - \varepsilon\}) = \mu(E), \tag{D.60}$$

then

$$\liminf_{n \to \infty} \int_A f_n \, \mathrm{d}\mu \geq \int_A f \, \mathrm{d}\mu. \tag{D.61}$$

Proof. Let $E_1, \dots, E_\ell \in \Sigma$ be disjoints sets with $\mu(E_i) < \infty$ and $t_1, \dots, t_\ell \in (0, \infty)$ such that for every $j \in \{1, \dots, \ell\}$ and $x \in E_j$, $f(x) \ge t_j$. For $\eta \in (0, 1)$, we define

$$E^\eta_{j,n} := \{x \in E_j \mid f_n(x) \ge \eta t_j\}. \tag{D.62}$$

It follows that

$$\int_A f_n \ge \eta \sum_{j=1}^{\ell} t_j \mu\Big(E^\eta_{j,n}\Big). \tag{D.63}$$

Since

$$E^\eta_{j,n} \supseteq \{x \in E_j \mid f_n(x) \ge f(x) - (1-\eta) t_j\}, \tag{D.64}$$

we have by assumption (D.60)

$$\lim_{n \to \infty} \mu(E^\eta_{j,n}) = \mu(E_j). \tag{D.65}$$

Hence, by (D.63) and (D.65)

$$\liminf_{n \to \infty} \int_A f_n \,\mathrm{d}\mu \ge \eta \sum_{j=1}^{n} t_j \mu(E_j). \tag{D.66}$$

Therefore,

$$\liminf_{n \to \infty} \int_A f_n \,\mathrm{d}\mu \ge \eta \int_A f \,\mathrm{d}\mu, \tag{D.67}$$

and since $\eta \in (0, 1)$ is arbitrary,

$$\liminf_{n \to \infty} \int_A f_n \,\mathrm{d}\mu \ge \int_A f \,\mathrm{d}\mu, \tag{D.68}$$

which is the conclusion. □

Proof of proposition D.36. We define the function $f : A \to [0, \infty]$ for $x \in A$ by

$$f(x) := \liminf_{n \to \infty} f_n(x). \tag{D.69}$$

Let $E \in \Sigma$, and define for $n \in \mathbb{N}$,

$$E_n := \{x \in E \mid f_n(x) \ge f(x) - \varepsilon\}. \tag{D.70}$$

and

$$F_n := \bigcap_{k \ge n} E_k. \tag{D.71}$$

By assumption, we have $F_n \subseteq F_{n+1}$ and $\bigcup_{n \in \mathbb{N}} F_n = E$. We have proposition D.14,

$$\lim_{n \to \infty} \mu(F_n) = \mu(E). \tag{D.72}$$

Since $E_n \supseteq F_n$, this implies that $\lim_{n \to \infty} \mu(E_n) = \mu(E)$. We are thus in position to apply proposition D.37 and reach thus the conclusion. □

Proposition D.38 (Monotone convergence). *Let $\mu : \Sigma \to [0, \infty]$ is a measure on Ω. If for every $n \in \mathbb{N}$, $f_n : \Omega \to [0, \infty]$ is measurable and $f_n \le f_{n+1}$, then*

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega \lim_{n\to\infty} f_n \,\mathrm{d}\mu. \tag{D.73}$$

Proof. This follows from proposition D.36. □

D.4 Integrating vector functions

Definition D.39. A collection $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ *partitions* Ω whenever for every $x \in \Omega$, there exists one and only one $E \in \mathscr{E}$ such that $x \in A$.

Definition D.40. A function $\tau : \mathscr{E} \to \Omega$ *tags* $\mathscr{E}$ whenever for every $E \in \mathscr{E}$, $\tau(E) \in E$.

Definition D.41. Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, $|\cdot|_V$ be a seminorm on V, and let $A \in \Sigma$. The function $f : A \to V$ is *integrable* with respect to μ whenever there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a the countable set $\mathscr{E} \subseteq \Sigma \setminus \{\emptyset\}$ that partitions A such that for every function $\tau : \mathscr{E} \to A$ that tags $\mathscr{E}$, $(f \circ \tau)\mu$ is unconditionally summable on $\mathscr{E}$ and

$$\Bigl|\sum_{E\in\mathscr{E}} f(\tau(E))\mu(E) - I\Bigr|_V \le \varepsilon. \tag{D.74}$$

Definition D.42. A collection $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ refines $\mathscr{F} \subseteq \mathscr{P}(\Omega)$ whenever for every $E \in \mathscr{E}$, there exists $F \in \mathscr{F}$ such that $F \supseteq E$.

Proposition D.43. *Let $\mu : \Sigma \to [0, \infty]$ be a measure on Ω, $|\cdot|_V$ be a seminorm on V, let $A \in \Sigma$ and let $f : A \to V$. If $\mathscr{E} \subseteq \Sigma\setminus\{\emptyset\}$ partitions A, if for every $\tau : \mathscr{E} \to \Omega$, $(f \circ \tau)\mu$ is unconditionally summable and if $\mathscr{E}'$ is a refinement of $\mathscr{E}$, then for every tagging $\tau' : \mathscr{E}' \to \Omega$,$(f \circ \tau)\mu$ is unconditionally summable and*

$$\sum_{E'\in\mathscr{E}'} f(\tau(E'))\mu(E') \subseteq \overline{\operatorname{conv}\Bigl\{\sum_{E\in\mathscr{E}} f(\tau(E))\mu(E) \Bigm| \tau : \mathscr{E} \to \Omega \text{ tags } \mathscr{E}\Bigr\}}. \tag{D.75}$$

Proof. This follows from proposition C.23. □

Proposition D.44. *If $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, if $A \in \Sigma$, if $|\cdot|_V$ is a on the vector space V and if $f : A \to V$ is integrable with respect to μ on A and if $t \in \mathbb{R}$, then tf is integrable on A and*

$$\int_A tf \,\mathrm{d}\mu = t \int_A f \,\mathrm{d}\mu. \tag{D.76}$$

Proposition D.45. *If $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, if $A \in \Sigma$, if $|\cdot|_V$ is a on the vector space V and if $f, g : A \to V$ are integrable with respect to μ on A then $f + g$ is integrable on A and*

$$\int_A (f + g)\,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu + \int_A g \,\mathrm{d}\mu. \tag{D.77}$$

Proposition D.46. *If $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, if V, W are normed spaces, $f : A \to V$ is integrable with respect to μ on A and $L \in \mathscr{L}(V, W)$, then $L \circ f$ is integrable on A and*

$$\int_A L \circ f \, \mathrm{d}\mu = L\Big(\int_A f \, \mathrm{d}\mu\Big). \tag{D.78}$$

Ingredient of proof. If $\mathscr{E} \subseteq \Sigma$ is disjoint and countable and $\tau : \mathscr{E} \to A$ satisfies $\tau(E) \in A$, we have

$$\sum_{E \in \mathscr{E}} (L \circ f)(\tau(E))\mu(E) = L\Bigg(\sum_{E \in \mathscr{E}} f(\tau(E))\mu(E)\Bigg). \tag{D.79}$$

□

Proposition D.47. *If $\mu : \Sigma \to [0, \infty]$ is a measure on Ω, if V is a normed spaces, $f, g : \Omega \to V$ are integrable with respect to μ on Ω then $f + g$ is integrable with respect to μ on Ω and*

$$\int_\Omega (f + g) \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu + \int_\Omega g \, \mathrm{d}\mu. \tag{D.80}$$

Proposition D.48. *If V endowed with $|\cdot|_V$ is complete, then $f : \Omega \to X$ is integrable if and only if for every $\varepsilon > 0$ there exists $\mathscr{E} \subseteq \Sigma \setminus \{\emptyset\}$ that partitions Ω such that if $\tau_0, \tau_1 : \mathscr{E} \to \Omega$ both tag A, then*

$$\Bigg|\sum_{E \in \mathscr{E}} (f(\tau_0(E)) - f(\tau_1(E)))\mu(A)\Bigg| \le \varepsilon. \tag{D.81}$$

Sketch of the proof. Use proposition D.43 to have definition D.41 for refinements and then apply a classical Cauchy argument. □

Proposition D.49. *If V endowed with $|\cdot|_V$ is complete, if $f : A \to V$ is measurable, if there exists a separable set $S \subseteq V$ such that $\mu(\{x \in \Omega \mid f(x) \neq S\}) = 0$ and if $\int_A |f| \, \mathrm{d}\mu < \infty$, then f is integrable and*

$$\Big|\int_A f \, \mathrm{d}\mu\Big|_V \le \int_A |f| \, \mathrm{d}\mu. \tag{D.82}$$

Proof. Given $\varepsilon > 0$, we consider a sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$V = \bigcup_{n \in \mathbb{N}} B[a_n, \varepsilon |a_n|_V]. \tag{D.83}$$

There exists also $\mathscr{G} \subseteq \Sigma \setminus \{\emptyset\}$ that partitions A such that such that

$$\sum_{G \in \mathscr{G}} \mu(G) \sup_G |f|_V \le \int_A |f|_V + \varepsilon. \tag{D.84}$$

We define for every $n \in \mathbb{N}$

$$E_n := \left\{ x \in A \;\middle|\; f(x) \in B[a_n, \varepsilon |a_n|_V] \setminus \bigcup_{\ell=0}^{n-1} B[a_\ell, \varepsilon |a_\ell|_V] \right\}. \tag{D.85}$$

and we define

$$\mathscr{E} := \{G \cap E_n \mid G \in \mathscr{G} \text{ and } n \in E_n\} \setminus \{\emptyset\} \tag{D.86}$$

If $\tau_0, \tau_1 : \mathscr{E} \to A$ both tag A, then

$$\begin{aligned} \left| \sum_{E \in \mathscr{E}} (f(\tau_0(E)) - f(\tau_1(E)))\mu(E) \right|_V &\le \sum_{E \in \mathscr{E}} |f(\tau_0(E)) - f(\tau_1(E))|_V \mu(E) \\ &\le \frac{2\varepsilon}{1-\varepsilon} \sum_{E \in \mathscr{E}} \sup_A |f|_V \mu(E) \\ &\le \frac{2\varepsilon}{1-\varepsilon} \sum_{G \in \mathscr{G}} \sup_G |f|_V \mu(G) \\ &\le \frac{2\varepsilon}{1-\varepsilon} \left(\int_A |f|_V + \varepsilon \right). \end{aligned} \tag{D.87}$$

We conclude with proposition D.48. □

Proposition D.50. *If $\mu : \Sigma \to [0, \infty]$ is a measure and let $A \in \Sigma$. If for every $n \in \mathbb{N}$, $|f_n| \le g_n$ almost everywhere in Ω, and if $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to f in Ω, if*

$$|f_n|_V \le g_n \tag{D.88}$$

$(g_n)_{n \in \mathbb{N}}$ converges almost everywhere to g and if

$$\limsup_{n \to \infty} \int_A g_n \,\mathrm{d}\mu \le \int_A \liminf_{n \to \infty} g \,\mathrm{d}\mu < \infty, \tag{D.89}$$

then

$$\lim_{n \to \infty} \int_A |f_n - f|_V \,\mathrm{d}\mu = 0 \tag{D.90}$$

and

$$\lim_{n \to \infty} \int_A f_n \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu. \tag{D.91}$$

Proof. We have

$$2g_n - |f_n - f| \ge 0. \tag{D.92}$$

Hence by Fatou's lemma (proposition D.36), we have

(D.93)
$$\begin{aligned}\int_A 2\liminf_{n\to\infty} g \,\mathrm{d}\mu &\le \liminf_{n\to\infty} 2\int_A g_n - |f_n - f|\,\mathrm{d}\mu\\ &\le \limsup_{n\to\infty} 2\int_A g_n \,\mathrm{d}\mu - \limsup_{n\to\infty}\int_A |f_n - f|\,\mathrm{d}\mu,\end{aligned}$$

and thus

(D.94)
$$\limsup_{n\to\infty}\int_A |f_n - f|\,\mathrm{d}\mu \le 2\Bigl(\lim_{n\to\infty}\int_A g_n\,\mathrm{d}\mu - \int_A \liminf_{n\to\infty} g_n\,\mathrm{d}\mu\Bigr) \le 0,$$

and the conclusion follows. □

D.5 Comments

For the definition of semifinite measure see [Fol99, §1.3].

Definition D.41 is due to Garett Birkhoff as an integral on Banach spaces [Bir35].

E Zorn's lemma

E.1 Definitions and statement

Definition E.1. Let X be a set. The relation $\preccurlyeq$ is a partial order on X whenever

(i) for every $x \in X$, $x \preccurlyeq x$,

(ii) for every $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$,

(iii) for every $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$.

Definition E.2. Let X endowed with a partial order $\preccurlyeq$. The element $x \in X$ is an upper bound on Y whenever for every $y \in Y$, $y \preccurlyeq x$.

Definition E.3. Let X be endowed with a partial order $\preccurlyeq$. The set X is totally ordered whenever for every $x, y \in X$, either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition E.4. Let X endowed with a partial order $\preccurlyeq$. The element $x \in X$ is maximal whenever for every $y \in X$, $x \preccurlyeq y$ implies $x = y$.

Theorem E.5 (Zorn's lemma)**.** *Let X be endowed with a partial order $\preccurlyeq$. If X is not empty and if any $Y \subseteq X$ which is totally ordered has an upper bound, then X has a maximal element.*

E.2 Hamel bases

Definition E.6. Let X be a linear space. A set $E \subseteq X$ is a basis of X whenever every $x \in X$ can be written uniquely as finite linear combinations of elements of E.

Theorem E.7. *Every linear space has a basis.*

Proof. We consider

$$\mathscr{E} = \{E \subseteq X \mid E \text{ is linearly independent}\}, \tag{E.1}$$

and we define $E_1 \preccurlyeq E_2$ whenever $E_1 \subseteq E_2$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered, then $E := \bigcup_{F \in \mathscr{F}} F \in \mathscr{E}$ and E is an upper bound for $\mathscr{F}$.

The set $\mathscr{E}$ is not empty since $\emptyset \in \mathscr{E}$. By Zorn's lemma (theorem E.5), $\mathscr{E}$ has a maximal element E. We claim that E is a basis. Indeed, assume by contradiction that there exists $x \in X$ which is not a finite linear combination of elements of E. Then $E \cup \{x\}$ is linearly independent, and thus $E \cup \{x\} \in \mathscr{E}$, in contradiction with the maximality of E. □

E.3 Connection with the axiom of choice

Axiom of choice For every set X, there exists a function $f : \mathfrak{P}(X) \setminus \{\emptyset\} \to X$ such that for every $A \in \mathfrak{P}(A)$, $f(A) \in A$.

Theorem E.8. *Zorn's lemma is equivalent with the axiom of choice.*

Proof. We first assume that Zorn's lemma holds. Given a set X, we define the set

$$\mathscr{E} = \{(\mathscr{A}, f) \mid \mathscr{A} \subseteq \mathfrak{P}(X), f : \mathscr{A} \to X \text{ and for each } A \in \mathscr{A},\ f(A) \in A\}. \tag{E.2}$$

For $(\mathscr{A}_1, f_1), (\mathscr{A}_1, f_1) \in \mathscr{E}$ we define $(\mathscr{A}_1, f_1) \preccurlyeq (\mathscr{A}_1, f_1)$ whenever $\mathscr{A}_1 \subseteq \mathscr{A}_2$ and $f_2|_{\mathscr{A}_1} = f_1$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered by $\preccurlyeq$, then we set $\mathscr{A}_* := \bigcup_{(\mathscr{A},f)\in\mathscr{F}} A$ and $f_*(A) = f(A)$ for $(\mathscr{A}, f) \in \mathscr{F}$. The pair $(\mathscr{A}_*, f_*)$ is then an upper bound on $\mathscr{F}$. By Zorn's lemma, $\mathscr{E}$ has a maximal element $(\mathscr{A}, f)$ for $\preccurlyeq$. We claim that $\mathscr{A} = \mathfrak{P}(X) \setminus \{\emptyset\}$. Otherwise, there exists $A_* \in \mathfrak{P}(X) \setminus \{\emptyset\} \setminus \mathscr{A}$. Setting $\mathscr{A}_* = \mathscr{A} \cup \{A_*\}$ and defining $f_* : \mathscr{A}_* \to X$ by $f_*(A) = A$ if $A \in \mathscr{A}$ and $f_*(A_*)$ to be some element of $A_* \neq \emptyset$. We have $(\mathscr{A}, f) \preccurlyeq (\mathscr{A}_*, f_*)$, in contradiction with the maximality of $(\mathscr{A}, f)$.

Conversely, we assume that the axiom of choice holds and that $\preccurlyeq$ is an order on X. Let $f : \mathfrak{P}(X) \setminus \{\emptyset\} \to X$ be a function such that for every $A \in \mathfrak{P}(X) \setminus \{\emptyset\}$, $f(A) \in A$. We define a set $A \subseteq X$ to be conforming whenever,

(a) the order $\preccurlyeq$ is total on A,

(b) for every nonempty $B \subseteq A$, there exists $b \in B$ such that for every $x \in B$, $b \preccurlyeq x$,

(c) for every $x \in A$, $x = f(\{z \in X \mid \text{for each } y \in A_{\prec x},\ y \prec z\})$.

(Here and in the sequel $y \prec x$ whenever $y \preccurlyeq x$ and $y \neq x$.) If A, B are conforming and if $A \setminus B \neq \emptyset$, we define $x = \min(A \setminus B)$ in view of (b). By construction, we have $A_{\prec x} := \{y \in A \mid y \prec x\} \subseteq B$. We claim that $A_{\prec x} = B$. Otherwise, let $y := \min(B \setminus A_{\prec x})$ and $z := \min(A \setminus B_{\prec y})$. We have $B_{\prec y} = A_{\prec z}$. First, if $w \in A_{\prec z}$, then by definition of z, $w \in B_{\prec y}$. Next if $w \in B_{\prec y}$, then by definition of y, $w \in A_{\prec x}$; moreover if $z \preccurlyeq w$, then $z \prec x$ and thus $z \prec y$ in contradiction with the definition of z; hence $w \prec z$. By (c), we have $y = z \in A \cap B$. Since $x \notin B$, we have $z \prec x$. Since $z = y$, we have $y = z \in A_x$, in contradiction with the choice of y. Hence we have proved that $B = \{y \in A \mid y \prec x\}$.

In order to conclude, we take the union U of conforming set; this set is conforming. Since U is totally orderered, if U has no maximal element then $U \cup \{f(\{x \in X \mid \text{for each } u \in U,\ u \prec x\})\}$ is conforming, in contradiction with the construction. □

E.4 Comments

The proof in theorem E.8 Zorn's lemma from the axiom of choice is due to Jonathan Lewin [Lew91] (for other proofs see [Lan02, app. 2 §2][DS58, §I.2]).

F Cardinality of sets

F.1 Definitions

Definition F.1. The sets X and Y have the same cardinality whenever there exists a bijection $f : X \to Y$.

Definition F.2. A set X is *finite* whenever there exists $n \in \mathbb{N}$ such that X and $\{1, \dots, n\}$ have the same cardinality. The set X is *infinite* whenever X is not finite.

Taking $n = 0$, we have $\{1, \dots, n\} = \emptyset$; hence the empty set $\emptyset$ is finite.

Proposition F.3. *Let X be a set. If X is finite and if $A \subseteq X$, then A is finite.*

Definition F.4. A set X is *countably infinite* whenever X and $\mathbb{N}$ have the same cardinality.

Definition F.5. A set X is *countable* whenever either X is finite or countably infinite.

Proposition F.6. *If the set X is infinite, then there there exists $A \subseteq X$ such that A is countably infinite.*

Proof. We define a function $f : \mathbb{N} \to X$ as follows. Assuming that f is defined on $\{0, \dots, n-1\}$, for some $n \in \mathbb{N}$ (if $n = 0$, this means that f is not defined anywhere), since the set X is not finite, $f(\{0, \dots, n-1\}) \neq X$, and we take $f(n) \in X \setminus f(\{0, \dots, n-1\})$. $\square$

Proposition F.7 (Cantor–Schröder–Bernstein)**.** *Let X and Y be sets. If there exist injections $f : X \to Y$ and $g : Y \to X$, then there exists a bijection $h : X \to Y$.*

Proof. We define the set

$$A := \{(g \circ f)^n(x) \mid x \in X \setminus g(Y) \text{ and } n \in \mathbb{N}\}. \tag{F.1}$$

By definition, we have $g(Y) \cap A = \emptyset$. We define then $h : X \to Y$ for $x \in X$ by

$$h(x) := \begin{cases} f(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \notin A. \end{cases} \tag{F.2}$$

We first claim that h is injective. Since $f|_A$ and $g^{-1}|_{X \setminus A}$ are injective, we can assume that $f(x) = g^{-1}(y)$ for some $x \in A$, which implies that $y = g(f(x))$, and thus by definition of the set A in (F.1), we have $y \in A$, which is a contradiction.

We now claim that h is surjective. We have $h(X) = f(A) \cup g^{-1}(X \setminus A)$. We have

$$f(A) = \{f((f \circ g)^n(x)) \mid x \in X \setminus g(Y) \text{ and } n \in \mathbb{N}\}. \tag{F.3}$$

and

$$g^{-1}(X \setminus A) = Y \setminus g^{-1}\{(g \circ f)^n(x) \mid x \in X \setminus g(Y) \text{ and } n \in \mathbb{N}\}$$
$$= Y \setminus \big\{f(g \circ f)^{n-1}(x) \,\big|\, x \in X \setminus g(Y) \text{ and } n \in \mathbb{N}_*\big\}, \tag{F.4}$$

since $g^{-1}(X \setminus g(Y)) = \emptyset$. □

Proposition F.8. *If the set X is countable and if $A \subseteq X$, then A is countable.*

Proof. If A is finite, then A is countable by definition. Assume now that A is infinite. By proposition F.6, there exits an injective map $f : \mathbb{N} \to A$. Moreover, the canonical injection from A to X is injective. Since X has the same cardinality as $\mathbb{N}$, we reach the conclusion by proposition F.7. □

Proposition F.9 (Cantor). *If X is a set and $f : X \to \mathfrak{P}(X)$ is a function, then f is not surjective.*

Proof. If $X = \emptyset$, then $f(X) = \emptyset \subsetneq \{\emptyset\}$.
We define

$$A = \{x \in X \mid x \notin f(x)\}. \tag{F.5}$$

Assume that for some $a \in X$, we have $A = f(a)$. By definition of the set A, if $a \in A$, then $a \notin f(a) = A$; if $a \notin A$, then $a \in f(a) = A$. Which is a contradiction. □

Proposition F.10. *Let X a set. The set X is infinite, if and only if there exists a map $f : X \to X$ which is injective but not surjective.*

Proof. If such an f exists, then the set X cannot be finite.
Assume now that X is infinite. By proposition F.6, there exists an injective map $g : \mathbb{N} \to X$. We define

$$f(x) := \begin{cases} x & \text{if } x \in X \setminus g(\mathbb{N}), \\ g(n+1) & \text{if } x \in g(\mathbb{N}). \end{cases} \tag{F.6}$$

□

F.2 Enters Zorn's lemma

Proposition F.11. *If X and Y are sets, then either there exists an injection $f : X \to Y$ or there exists an injection $g : Y \to X$.*

Proof. We define the set

$$\mathscr{E} = \{(A, f) \mid A \subseteq X \text{ and } f \text{ is injective}\}. \tag{F.7}$$

We endow $\mathscr{E}$ with the order $(A_1, f_1) \preccurlyeq (A_2, f_2)$ whenever $A_1 \subseteq A_2$ and $f_2|_{A_1} = f_1$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered, then setting

$$A = \bigcup_{(B,g)\in\mathscr{F}} B, \tag{F.8}$$

and defining $f : A \to X$ by $f|_B = g$ for every $(B, g) \in \mathscr{F}$, we have that f is well-defined an injective, and (A, f) is an upper bound on $\mathscr{F}$. Moreover, taking $A = \emptyset$, we check that $\mathscr{E}$ is not empty. By Zorn's lemma, there exists a maximal element (A, f) in $\mathscr{E}$. We claim that either $A = X$ or $f(A) = Y$. Otherwise, there exists $x \in X \setminus A$ and $y = Y \setminus f(A)$. Defining $\bar{A} = A \cup \{x\}$ and $\bar{f}$ by $\bar{f}|_A = f$ and $f(x) = y$, we reach a contradiction. □

Proposition F.12. *If the set X is infinite, then there is a bijection between X and $X \times \{0, 1\}$.*

Proof. We define the set

$$\mathscr{E} = \{(A, f) \mid A \subseteq X \text{ and } f : A \times \{0, 1\} \to A \text{ is a bijection}\}. \tag{F.9}$$

We endow $\mathscr{E}$ with the order $(A_1, f_1) \preccurlyeq (A_2, f_2)$ whenever $A_1 \subseteq A_2$ and $f_2|_{A_1 \times \{0,1\}} = f_1$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered, then setting

$$A = \bigcup_{(B,g)\in\mathscr{F}} B, \tag{F.10}$$

and defining $f : A \times \{0, 1\} \to A$ by $f|_B = g$ for every $(B, g) \in \mathscr{F}$, we have that f is well-defined and bijection, and (A, f) is an upper bound on $\mathscr{F}$. Moreover, taking $A = \emptyset$, we check that $\mathscr{E}$ is not empty. By Zorn's lemma, there exists a maximal element (A, f) in $\mathscr{E}$.

We claim that $X \setminus A$ is finite. Otherwise, there exists an injective map $g : \mathbb{N} \to X \setminus A$. Defining $\bar{A} := A \times \cup g(\mathbb{N})$ and $\bar{f} : \bar{A} \times \{0, 1\} \to \bar{A}$ so that $\bar{f}|_{A \times \{0,1\}} = f$ and for every $n \in \mathbb{N}$ and $k \in (0, 1)$, $\bar{f}(g(n), k) = g(2n + k)$. We have that $(\bar{A}, \bar{f}) \in \mathscr{E}$ and $(A, f) \preccurlyeq (\bar{A}, \bar{f})$, in contradiction with the maximality of (A, f). We have thus proved that $X \setminus A$ is finite.

Since $X \setminus A$ is finite and since X is infinite, A is infinite, there exists an injection $g : \mathbb{N} \to X$ sucht that $X \setminus A \subseteq g(\mathbb{N})$. We define now $\tilde{f} : X \times \{0, 1\} \to X$ by $\tilde{f}|_{(X \setminus g(\mathbb{N})) \times \{0,1\}} = f|_{(A \setminus g(\mathbb{N})) \times \{0,1\}}$ and for every $n \in \mathbb{N}$, $\tilde{f}(g(n), k) = g(2n + k)$. Since $\tilde{f}$ is a bijection the proposition is proven. □

Proposition F.13. *If the set X is infinite, then there is a bijection between X and $X \times X$.*

Proof. We define the set

$$\mathscr{E} = \{(A, f) \mid A \subseteq X \text{ and } f : A \times X \to X \text{ is a bijection}\}. \tag{F.11}$$

We endow $\mathscr{E}$ with the order $(A_1, f_1) \preccurlyeq (A_2, f_2)$ whenever $A_1 \subseteq A_2$ and $f_2|_{A_1 \times X} = f_1$. If $\mathscr{F} \subseteq \mathscr{E}$ is totally ordered, then setting

$$A = \bigcup_{(B,g)\in\mathscr{F}} B, \tag{F.12}$$

and defining $f : A \times \{0,1\} \to A$ by $f|_B = g$ for every $(B,g) \in \mathscr{F}$, we have that f is well-defined and bijection, and (A,f) is an upper bound on $\mathscr{F}$. Moreover, taking $A = \emptyset$, we check that $\mathscr{E}$ is not empty. By Zorn's lemma, there exists a maximal element (A,f) in $\mathscr{E}$.

We first claim, that there is no injection $g : A \to X \setminus A$. Otherwise, defining $\bar{A} = A \cup g(A)$ and $\bar{f} : \bar{A} \times X$ to X for $(x,y) \in \bar{A} \times X$ by

$$\bar{f}(x,y) = \begin{cases} h(f(x,y),0) & \text{if } x \in A, \\ h(f(g^{-1}(x),y),1) & \text{if } x \in g(A), \end{cases} \tag{F.13}$$

where $h : A \times \{0,1\} \to A$ is a bijection that exists in view of proposition F.12, we contradict the maximality of (A,f).

By proposition F.11, there exists thus an injective map $g : X \setminus A \to A$. Without loss of generality, we can assume that $X \setminus A$ is infinite. Taking $h : (X \setminus A) \times \{0,1\} \to X \setminus A$ to be a bijection given by proposition F.12, we define $\tilde{f} : X \times X \to X$ for $(x,y) \in X \times X$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } x \in A \setminus g(X \setminus A), \\ f(h(x,0),y) & \text{if } x \in g(X \setminus A), \\ f(h(g(x),1),y) & \text{if } x \in X \setminus A; \end{cases} \tag{F.14}$$

the mapping $\tilde{f}$ is the required bijection. □

Proposition F.14. *If the set X is infinite, then there is a bijection between X and $X \times \mathbb{N}$.*

References

[Bir35] G. Birkhoff, *Integration of functions with values in a Banach space*, Trans. Amer. Math. Soc. **38** (1935), no. 2, 357–378, doi:10.2307/1989687. ↑195, 210

[Boa40] R. P. Boas Jr., *Some uniformly convex spaces*, Bull. Amer. Math. Soc. **46** (1940), 304–311, doi:10.1090/S0002-9904-1940-07207-6. ↑165

[Bre11] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. ↑33, 165

[BBT97] A. M. Bruckner, J. B. Bruckner, and B. S. Thomson, *Real analysis*, Prentice-Hall, 1997. ↑44

[Cla36] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414, doi:10.2307/1989630. ↑165

[DiB16] E. DiBenedetto, *Real analysis*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, New York, 2016. ↑165

[Dix84] J. Dixmier, *General topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1984. Translated from the French by Sterling K. Berberian. ↑195

[DS58] N. Dunford and J. T. Schwartz, *Linear Operators*. I: *General Theory*, with assistance by W. G. Bade and R. G. Bartle, Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York/London, 1958. ↑212

[DR50] A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197, doi:10.1073/pnas.36.3.192. ↑195

[Fol99] G. B. Folland, *Real analysis*: *Modern techniques and their applications*, 2nd ed., Pure and Applied Mathematics, John Wiley & Sons, New York, 1999. ↑67, 210

[For51] M. K. Fort Jr., *A note on pointwise convergence*, Proc. Amer. Math. Soc. **2** (1951), 34–35, doi:10.2307/2032617. ↑44

[MF08] M. Fréchet, *Essai de géométrie analytique à une infinité de coordonnées*, Nouv. Ann.(4) **8** (1908), 97–116, 289–317. ↑134

[Han56] O. Hanner, *On the uniform convexity of L^p and l^p*, Ark. Mat. **3** (1956), 239–244, doi:10.1007/BF02589410. ↑165

[HLP52] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, at the University Press, 1952. 2d ed. ↑184

[Hei11] C. Heil, *A basis theory primer*, Expanded edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2011. ↑195

[JVN35] P. Jordan and J. Von Neumann, *On inner products in linear, metric spaces*, Ann. of Math. (2) **36** (1935), no. 3, 719–723, doi:10.2307/1968653. ↑33

[KA82] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, 2nd ed., translated by H. L. Silcock, Pergamon Press, Oxford-Elmsford, N.Y., 1982. ↑

[Lan02] S. Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer, New York, 2002. ↑212

[Lew91] J. Lewin, *A simple proof of Zorn's lemma*, Amer. Math. Monthly **98** (1991), no. 4, 353–354, doi:10.2307/2323807. ↑212

[LL01] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. ↑165

[Meg98] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, vol. 183, Springer, New York, 1998. ↑142

[Phi40] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. **48** (1940), 516–541, doi:10.2307/1990096. ↑134

[18] *Über lineare Funktionalgleichungen*, Acta Math. **41** (1918), 71–98. ↑142

[RW02] P. Roselli and M. Willem, *A convexity inequality*, Amer. Math. Monthly **109** (2002), no. 1, 64–70, doi:10.2307/2695768. ↑33, 165

[Sut09] W. A. Sutherland, *Introduction to metric and topological spaces*, 2nd ed., Oxford University Press, Oxford, 2009. ↑50

[TE05] L. N. Trefethen and M. Embree, *Spectra and pseudospectra*: *The behavior of nonnormal matrices and operators*, Princeton University Press, Princeton, N.J., 2005. ↑142

[Wil13] M. Willem, *Functional analysis*: *Fundamentals and applications*, Cornerstones, Birkhäuser/Springer, New York, 2013. ↑33, 165, 179