Decision Theory

LINMA2345 – Game Theory

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Introduction

Definitions

In decision theory we consider one player, who makes one decision. We assume that the player is:

Rational Makes decisions in pursuit of objectives

Intelligent Knows everything about the game and can make any deductions that we can make

The results today will form the basis for game theory where we consider multiple players.

Basic concepts of Decision Theory

Running example: Illness problem

Consider that you are a sick person. You have to chose between two pills: A and B. There are 3 possible results: Cured,Uncured, and Dead.

For pil A the probabilities are: (Cured: .90, Uncured: .00, Dead: .10)
For pil B the probabilities are: (Cured: .50, Uncured: .50, Dead: .00)

Which one would you prefer and why ?

Notations

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- The decision made by the player
- The realisation of the objective uncertainty

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- The decision made by the player
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Notation 1.2: Δ: For any finite set Z, $\Delta(Z)$ is *the set of probability distributions* over the elements of Z:

$$\Delta(Z) = \left\{q : Z \to \mathbb{R} | \sum_{y \in Z} q(y) = 1, \text{and } q(z) \geq 0, \forall z \in Z\right\}$$

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Example:
X are the prizes
For pill A: $\Delta(X) =$(Cured: .90, Uncured: .00, Dead: .10)

Lottery

Definition 1.4: Lottery

A lottery f is a function:

$$f : \Omega \to \Delta(X)$$

where:

- Ω denotes the (finite) set of all possibles *states of the world*, that are all the possible realizations of the uncertainty
- X denotes the (finite) set of *prizes*.

Example for the illness problem: X = ?

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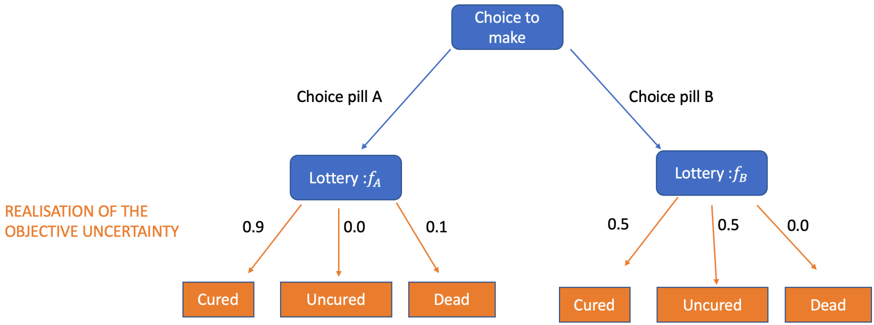
where:

- Ω denotes the (finite) set of all possibles *states of the world*, that are all the possible realizations of the uncertainty
- X denotes the (finite) set of *prizes*.

Example for the illness problem:

- X = {Cured, Uncured, Dead}
- Pill A and pill B are lotteries.
- Ω = there is only one state
- $f_A() = (Cured : .90, Uncured : .00, Dead : .10)$

Lottery



Choice to make
Choice pill A
Choice pill B
Lottery : f_A
Lottery : f_B
REALISATION OF THE OBJECTIVE UNCERTAINTY
0.9
0.0
0.1
0.5
0.5
0.0
Cured
Uncured
Dead
Cured
Uncured
Dead

Running example 2: Illness problem Extended

Consider that you are a sick person. You have to chose between two pills: A and B. There are 3 possible results: Cured,Uncured, and Dead.

Good quality:
Pill A: (Cured: .90, Uncured: .00, Dead: .10)
Pill B: (Cured: .50, Uncured: .50, Dead: .00)

Bad quality:
For pil A the probabilities are: (Cured: .70, Uncured: .00, Dead: .30)
For pil B the probabilities are: (Cured: .20, Uncured: .80, Dead: .00)

Where we have a chance of 0.9 for good quality and 0.1 for bad quality

Lottery

The probability of receiving a prize depends on three elements:

Lottery

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Given a lottery $f : \Omega \to \Delta(X)$

- $f(.|t) \in \Delta(X)$ = prob. distribution for the prizes if state = t
- $f(x|t) \in \Delta(X)$ = prob.of receiving the prize x if state = t

$[x]$ denote a lottery that will always give the prize x

What does Ω represent?

Lottery Conclusion

- Every decision process leads to a lottery
- Decisions are made, outcome = prob. distribution on prizes given state of the world t

Axioms of Decision Theory

Definition 1.6: $P(\Omega)$, **Event**: Let $P(\Omega)$ be the set containing all non-empty subsets of the set of states. An event is a set $S \in P(\Omega)$, i.e. a non-empty subset of Ω.

In our running example we have that
$P(\Omega) = \{\{\text{Good quality}, \text{Bad quality}\}, \{\text{Good quality}\}, \{\text{Bad quality}\}\}$.

An event could for example be $S = \{\text{Good quality}\}$.

Notation 1.7: $\succsim$, $\sim$, $\succ$: For any two lotteries f and g, and any event $S \in P(\Omega)$, we write $f \succsim_S g$ iff the player considers the lottery f at least as desirable as g, if the player learned that the true state of the world was in S. We then define:

$$f \sim_S g \text{ iff } f \succsim_S g \text{ and } g \succsim_S f,$$
$$f \succ_S g \text{ iff } f \succsim_S g \text{ and } f \not\precsim_S f.$$

Consider the extended illness problem with
Good quality:
pill A: (Cured: 0.50, Uncured: 0.50, Dead: .00)
Pill B: (Cured: .00, Uncured: 0.50, Dead: .50).

Then if $S = \{\text{Good quality}\}$ and we define f to be the lottery for pill A and g to be the lottery for pill B, it seems reasonable to prefer f over g in S, hence

$$f \succsim_S g.$$

Axioms

Axiom 1 (Completeness) $f \succsim_S g$ or $g \succsim_S f$.

Intuition We should have a preference about any combination of lotteries.

Axiom 2 (Transitivity) If $f \succsim_S g$ and $g \succsim_S h$ then $f \succsim_S h$.

Intuition If you prefer *pill A over pill B* and *pill B over pill C*, then you must prefer *pill A over pill C*.

Axiom 3 (Relevance) If $f(\cdot|t) = g(\cdot|t)\ \forall t \in S$, then $f \sim_S g$.

Intuition

State t	Lottery distribution
Good quality	$f(\cdot\|t) = g(\cdot\|t)$
Bad quality	$f(\cdot\|t) \neq g(\cdot\|t)$

Axioms

Axiom 4 (Monotonicity) If $f \succ_S h$ and $0 \leq \beta < \alpha \leq 1$, then

$$\alpha f + (1 - \alpha)h \succ_S \beta f + (1 - \beta)h.$$

Intuition A higher probability of getting a better lottery is always better.

Axiom 5 (Continuity) If $f \succsim_S g$ and $g \succsim_S h$, then $\exists \gamma$ such that $0 \leq \gamma \leq 1$ and $g \sim_S \gamma f + (1 - \gamma)h$.

Intuition There is some probability p where you are indifferent between

$$\text{[Uncured] and } p\text{[Cured]} + (1 - p)\text{[Dead]}.$$

Axioms

Axiom 6 (Objective substitution) If $e \succsim_S f$ and $g \succsim_S h$ and $0 \leq \alpha \leq 1$, then $\alpha e + (1-\alpha)g \geq_S \alpha f + (1-\alpha)h$.

Axiom 7 (Strict objective substitution) If $e \succ_S f$ and $g \succsim_S h$ and $0 < \alpha \leq 1$, then $\alpha e + (1-\alpha)g \succ_S \alpha f + (1-\alpha)h$.

Example: Breaking the objective substitution axioms

Consider a game with four prizes: w, x, y, z. The game is the following: A player can take w or toss a coin. If the coin comes up Heads the player gets z and for Tails the player can choose between x or y.

The player expresses the following preferences:

$$[x] \succ [y] \text{ and } 0.5[x] + 0.5[z] \prec [w] \prec 0.5[y] + 0.5[z].$$

What are the possible strategies?

Example: Breaking the objective substitution axioms

The player expresses the following preferences:

$$[x] \succ [y] \text{ and } 0.5[x] + 0.5[z] \prec [w] \prec 0.5[y] + 0.5[z].$$

The prizes are $X = \{w, x, y, z\}$. The players has three strategies:

Take w directly: $[w]$
Take x if Tails: $0.5[x] + 0.5[z]$
Take y if Tails: $0.5[y] + 0.5[z]$.

What is your advice to the player?

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The prizes are $X = \{w, x, y, z\}$. The players has three strategies:

Take *w* directly: $[w]$
Take *x* if Tails: $0.5[x] + 0.5[z]$
Take *y* if Tails: $0.5[y] + 0.5[z]$.

What is your advice to the player?

It is not the best choice to pick w since the player prefers lottery 3. However, if the coin comes up Tails, the first preference says that the player prefers x. So if the player refuses w, then the player will end up with lottery 2, which is worse than taking $[w]$.

This motivates the substitution axioms, since taking $\alpha = 0.5$, $e = [x]$, $f = [y]$, $g = h = [z]$ contradicts axiom 7 (Strict objective substitution).

Axioms

Axiom 8 (Subjective substitution) If $f \succsim_S g$ and $f \succsim_T g$ and $S \cap T = \emptyset$, then $f \succsim_{S \cup T} g$.

Axiom 9 (Strict subjective substitution) If $f \succ_S g$ and $f \succ_T g$ and $S \cap T = \emptyset$, then $f \succ_{S \cup T} g$.

Axioms

Axiom 10 (Interest) For every state $t \in \Omega$, there exist prizes y and z in X such that $[y] \succ_{\{t\}} [z]$.

Intuition There are some prizes that you prefer over others. Otherwise nothing interesting can happen.

Axiom 11 (State neutrality) For any two states r and t in Ω, if $f(\cdot|r) = f(\cdot|t)$ and $g(\cdot|r) = g(\cdot|t)$ and $f \succsim_{\{t\}} g$, then $f \succsim_{\{t\}} g$.

Intuition The preference order should be the same in all states of the world. Otherwise the prices has different amount of value to you in different states of the world.

The utility maximization theorem

Quick recap

We have seen:

- The concept of lottery
- Axioms: preference relationships between lotteries
- idea: relate prize with a utility
- This utility should agree with the axioms

Utility function

Definition 1.10: Utility function
A utility function is a function $u : X \to \mathbb{R}$, that assigns a value to each prize in each state of the world.

Example for the extended illness problem:

- Cured: 16 utils
- Uncured: 12 utils
- Dead: 0 utils

Conditional Probability Function

Definition 1.11: Conditional probability function

A conditional probability function on Ω is a function p : $P(\Omega) \mapsto \Delta(\Omega)$ that specifies for all $S \in P(\Omega)$ a probability distribution on Ω such that, for all $t \in \Omega$,

$$p(t|S) = 0 \text{ if } t \notin S, \text{and } \sum_{r \in S} p(r|S) = 1$$

Example for the extended illness problem:

- Ω = {good, bad }
- $P(\Omega)$ = { {good}, {bad}, {good,bad} }
- p(good|{good,bad}) = 0.9
- p(good|bad) = 0
- p(good|good) = 1

Expected Utility Value

Definition 1.12: Expected Utility Value

The expected utility value of a lottery f given an event S and a conditional probability function p is given by:

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in X} u(x,t) f(x,t)$$

What does p(t|S) represent ?

Expected Utility Value

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Subjective uncertainty

Utility

Objective uncertainty

Expected Utility Value

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Example for the illness problem (only good quality pills):

- $E_p(u(f_A)|good) = 1 * (.9 * (16) + 0 * 12 + .10(0)) = 14.4$
- $E_p(u(f_B)|good) = 1 * (.5 * (16) + 0.5 * 12 + .0(0)) = 14$

The utility maximization theorem

This leads us to the most important theorem:

- describes how an individual plays
- states that any rational player behaves exactly like if he was optimizing an expected payoff

The utility maximization theorem

Theorem 1.13: The utility maximization theorem

The Axioms 1.1 to 1.10 are satisfied if and only if

- $max_{x \in X} u(x, t) = 1$ and $min x_{x \in X} u(x, t) = 0, \forall t \in \Omega$
- for all sets $R \subseteq S \subseteq T \subseteq$ with $S \neq \emptyset$,

$$p(R|T) = p(R|S)p(S|T) \qquad \text{(Bayes' theorem)}$$

- . For any two lotteries f and g and for any event S, the preference relation $f \succsim_S g$ holds if and only if

$$E_p(u(f)|S) \geq E_p(u(g)|S)$$

$Ep(u(f_A)|good) > Ep(u(f_B)|good)$

Bayesian conditional probability system

Bayesian probability system

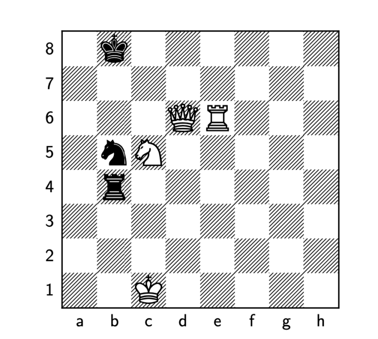
Conditional probability functions :

- satisfy Bayes formula
- right tools to encapsulate the behaviour of a rational agent

event S : not happen (fundamental theorem) , still needs to define $p(\cdot|S)$.

Chess game example

What is black going to play next?



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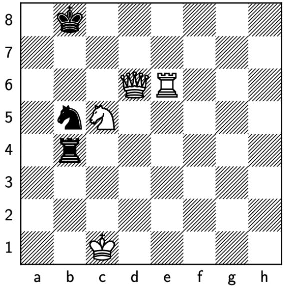

Mvmt m	$P(m)$	
`king to a8`	0	checkmate in 3 steps,
`king to a7`	0	checkmate in 2 steps,
`king to c8`	0	checkmate in 2 steps,
`knight to c7`	x	avoids checkmate,
`knight to d6`	$(1-x)$	avoids checkmate.

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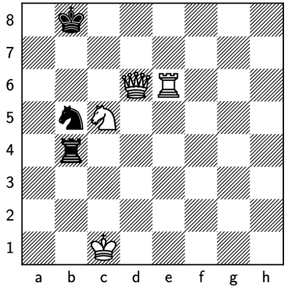

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- $S = \{ka8, ka7, kc8\} \subset \Omega_b$
- $P(ka8|\Omega_b) = P(ka8|S)P(S|\Omega_b) = 0$ (same for a7, c8)
- $p(ka7|S) = p(ka8|S) = p(kc8|S) = 1/3$
- An intelligent and rational decision maker always expects the unexpected!

Limits of the Bayesian framework

characterizing preferences between lotteries allows:

- predictive purpose. (help predict the behaviour of people)
- prescriptive purpose (help people take their decisions following their preferences)

This is a model, it has limits.

Application: The Widget Problem

The Widget Problem

A company markets that they can fulfil any order of widget where you can choose between red, yellow and green. However, this is not really true since the company only can produce 200 widgets per day in one single colour. **How do we decide which colour to produce to fulfil as many orders as possible?**

The director is given more and more information and we will see how this influences the decision made.

For simplicity we assume that the director disregards the future.

Modelling

We can use the maximum entropy principle to obtain a probability function for the demand. This can then be used to calculate the expected number of missed order depending on the choice of colour produced.

The choice of the distribution is a choice, and other choices could be considered.

Stage 1

The directors checks the stock of widgets and finds that

Information	R	Y	G	Decision
Stock	100	150	50	

Stage 1

The directors checks the stock of widgets and finds that

Information	R	Y	G	Decision
Stock	100	150	50	

The best decision is to produce green, assuming we have *no other information* about the demand.

Stage 2

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	

Stage 2

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	

Decision	R	Y	G
Expected loss	22.7	10.6	39.38

Hence producing yellow is the best decision.

Stage 3

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	Y
Average order size	75	10	20	

Stage 3

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	Y
Average order size	75	10	20	

Decision	R	Y	G
Expected loss	3.04	15.1	17.8

Hence producing red is the best decision.

Stage 4

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	Y
Average order size	75	10	20	R
Specific order	0	0	40	

Stage 4

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	Y
Average order size	75	10	20	R
Specific order	0	0	40	

Decision	R	Y	G
Expected loss	10.9	23.0	17.8

Hence the best choice is still to produce red.

Widget problem: Summery

As more and more information was available, the probabilities of different events changed.

Information	R	Y	G	Decision
Stock	100	150	50	G
Daily average	50	100	10	Y
Average order size	75	10	20	R
Actual order	0	0	40	R

Conclusion

Summary

- Definition of decision theory and an intelligent and rational player.
- Basic concepts, especially the notion of lottery.
- Preferences and axioms for these.
- The axioms are equivalent to maximising the expected utility.
- Bayesian probability to update probability distributions given new knowledge.

Questions?