LINMA2345 - Game Theory
Decision Theory
The Widget Problem

Caroline Sautelet 19471200

Leila Van Keirsbilck 6682 1200

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Introduction

The widget problem is an interesting application of decision theory. Here is the problem : Mr. A is in charge of a widget factory producing three colors of widgets : red, yellow and green. His marketing department advertised all customers that any order will be delivered in 24 hours. Mr. A is therefore forced to respect this claim. If he doesn't, the enterprise will loose 1 per undelivered widget. The major constraint due to production line, is that only one color of widgets can be produced each day. Each morning, Mr. A has to choose which color to produce. During his decision process, he will go through 4 stages, receiving each time another piece of information. However, this information is not a "new" information as used in Bayes' theorem but an addition of prior information. To simplify the problem, we'll consider that Mr. A takes decisions on a day to day basis without thoughts of tomorrow. This report is based on `http://omega.albany.edu:8008/ETJ-PS/cc14g.ps`.

General method

In this problem, all the uncertainty resides in the number of widgets of each color that will be ordered today. The probability distributions associated to those 3 random variables are unknown. Those probability distribution will be estimated using the information we receive.
The general method to solve this problem is the following :
— Compute for each color, the probability distribution of the number of ordered widgets in a day ;
— Establish a loss function for each possible decision ;
— Compute the expected value of the loss function for each possible decision ;
— Select the decision which achieves the minimum expected value.

Stage 1

First, when Mr. A arrives in the morning at his enterprise, the first think he can do is to check the remaining stock of widgets. Today's remaining stock is the following :

	Red	Yellow	Green
Number of widgets in stock	100	150	50

If we assume Mr. A to be rational, he will decide to produce green widgets since it corresponds to the lower stock. There is no need for mathematical development here. When asking people what they would decide if they were Mr. A, they find directly the right color to produce as it is at this stage pretty intuitive.

Stage 2

Second, Mr. A receives a call form his statistical department. They computed the average number of widgets of each color sold each day :

	Red	Yellow	Green
Number of widgets in stock	100	150	50
Daily average	50	100	10

The mean of the probability distribution is now fixed. Of course, it does not define it completely, and we will choose the probability distribution fulfilling this constraint and maximizing the entropy. The entropy principle is similar to the maximum likelihood principle. It allows determining the prior probabilities. Let n_1, n_2 and n_3 be the number of red, yellow and green widgets that will be ordered today respectively. The maximum entropy theory leads to the following result :

$$p(n_1, n_2, n_3) = p_(n_1)p_2(n_2)p_3(n_3)$$

$$p_i(n_i) = (1 - e^{-\lambda_i})e^{-\lambda_i n_i}, n_i = 1, 2, 3, ...$$

$$p_i(n_i) = \frac{1}{\langle n_i \rangle + 1}[\frac{\langle n_i \rangle}{\langle n_i \rangle + 1}]^{n_i}$$

where $\langle n_i \rangle$ are the expectations of orders for each color.

In order to compute the loss functions, we enumerate the possible decisions. Let D_1, D_2 and D_3 be the decisions to produce red, yellow and green widgets. Let S_1, S_2 and S_3 be the stocks of red, yellow and green widgets respectively. The loss functions are then

$$L(D_1; n_1 n_2 n_3) = g(n_1 - S_1 - 200) + g(n_2 - S_2) + g(n_3 - S_3)$$

$$L(D_2; n_1 n_2 n_3) = g(n_1 - S_1) + g(n_2 - S_2 - 200) + g(n_3 - S_3)$$

$$L(D_3; n_1 n_2 n_3) = g(n_1 - S_1) + g(n_2 - S_2) + g(n_3 - S_3 - 200)$$

where g is a ramp function. The expected loss will be

$$\langle L \rangle_1 = \sum_{N_i} p(n_1 n_2 n_3) L(D_1; n_1 n_2 n_3) = \sum_{n_1=0}^{\infty} p_1(n_1) g(n_1 - S_1 - 200) + \sum_{n_2=0}^{\infty} p_2(n_2) g(n_2 - S_2) + \sum_{n_3=0}^{\infty} p_3(n_3) g(n_3 - S_3)$$

Of course, the expected loss for decisions 2 and 3 are similar.

Let's compare the three expected loss :

$$\langle L_1 \rangle = 22.70$$

$$\langle L_2 \rangle = 10.6$$

$$\langle L_3 \rangle = 39.38$$

The best decision to take is the one minimizing the expected loss. Mr. A then has to produce yellow widgets today.

Stage 3

Mr. A is now given an addition piece of information, the average individual orders for red, yellow, and green widgets.

	Red	Yellow	Green
Number of widgets in stock	100	150	50
Daily average	50	100	10
Individual average	75	10	20

At this stage, calculations in order to get the prior probabilities are quite more complicated. But the loss functions are the same. Let us directly compare the expected loss. They are given by :

$$\langle L_1 \rangle = 3.04$$

$$\langle L_2 \rangle = 15.1$$

$$\langle L_3 \rangle = 17.8$$

The color that Mr. A has to produce is thus the red.

Stage 4

Finally, Mr. A gets a fourth information. His front office calls with the information that someone just ordered 40 green widgets.

	Red	Yellow	Green
Number of widgets in stock	100	150	50
Daily average	50	100	10
Individual average	75	10	20
Specific order			40

Stage 4 is quite similar to the previous one as the only difference is the fact that the stock is lowered by 40. If we now compare the expected loss, we get :

$$\langle L_1 \rangle = 10.9$$

$$\langle L_2 \rangle = 23.0$$

$$\langle L_3 \rangle = 17.8$$

Now, and it is far as intuitive as the stage 1, Mr. A has to produce red widgets.

Conclusion

The decisions at each stage are :

	Red	Yellow	Green	Decision
Number of widgets in stock	100	150	50	Green
Daily average	50	100	10	Yellow
Individual average	75	10	20	Red
Specific order			40	Red

We can see that with just a few information, our intuition can give us the correct decision to make. However, when the number of information starts to grow, we need to use the results of decision theory to find the best decision to make.