Introduction to Image Processing

Lecture 3
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Fundamentals

Fourier Series

- Time: Continuous + *Periodic* $\rightarrow$ Frequency: *Digital (coefficients)*

Fourier Transform

- Time: Continuous + *Non-periodic* $\rightarrow$ Frequency: *Continuous*

Discrete Time Fourier Transform

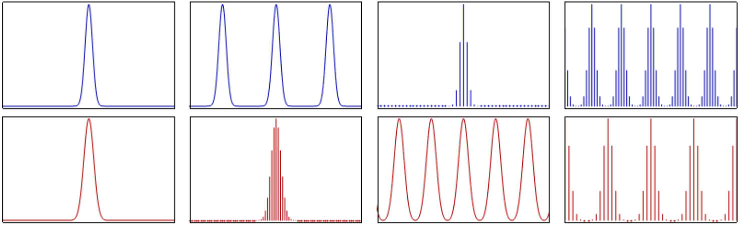
- Time: **Discrete** + *Non-periodic* $\rightarrow$ Frequency: *Continuous* + **Periodic**

Discrete Fourier Transform

- Time: **Discrete** + *Periodic* $\rightarrow$ Frequency: *Digital* + **Periodic**

Fundamentals

The different transforms



Introduction to Fourier Transform

Fourier transform of continuous function $f(x)$

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp(-2j\pi ux)dx$$

- Integral shows $F(u)$ composed of infinite sum of sine and cosine
- Each u value determines the frequency of its corresponding sin and cos pair

Introduction to Fourier Transform

Inverse Fourier transform of $F(u)$

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u) \exp(2j\pi ux) du$$

- The above equations ($\Im\{f(x)\}$ and $\Im^{-1}\{F(u)\}$) represent the Fourier transform pair

Introduction to Fourier Transform

Fourier transform pair for $f(x, y)$ of two variables

$$\Im\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int f(x,y)\exp(-2j\pi(ux+vy))dxdy$$

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int F(u,v)\exp(2j\pi(ux+vy))dudv$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Fourier transform pair - 1D

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \exp(-2\pi ux/M)$$

for $u = 0, 1, 2, ..., M-1$

$$f(x) = \sum_{u=0}^{M-1} f(u) \exp(2\pi ux/M)$$

for $x = 0, 1, 2, ..., M-1$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Fourier transform pair - 2D

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp(-2\pi(ux/M + uy/N))$$

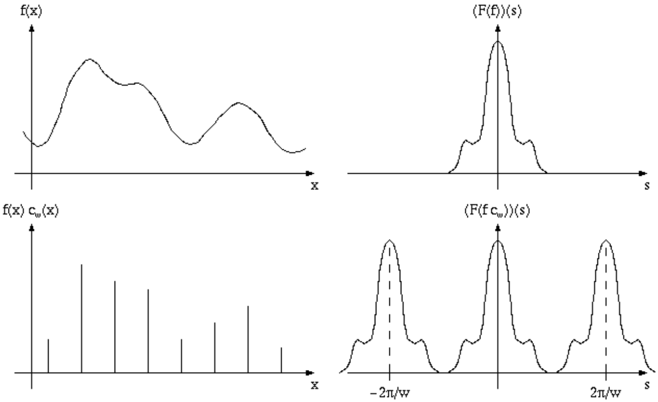
for $u = 0, 1, 2, ..., M-1$, and $v = 0, 1, 2, ..., N-1$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(u,v)\exp(2\pi(ux/M + uv/N))$$

for $x = 0, 1, 2, ..., M-1$ and $y = 0, 1, 2, ..., N-1$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Aliasing



Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Sampling theorem

- A continuous function can be *recovered uniquely* from a set of its samples
- Under the sampling condition that:

$$\frac{1}{\Delta T} > 2\mu_{\mathsf{max}}$$

Fourier transform of sampled function

$$\widetilde{F}(\mu) = F(\mu) * S(\mu) \ ,$$

with

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

$$\widetilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Aliasing

- A band-limited function $f(t,z)$ can be recovered if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{\text{max}}}$$

and

$$\Delta Z < \frac{1}{2\nu_{\text{max}}}$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Aliasing

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

- The Fourier transform of a real function is generally complex and we use polar coordinates:

1D - DFT

$$F(u) = R(u) + jI(u)$$

$$F(u) = |F(u)|e^{j\phi(u)}$$

Magnitude spectrum:

$$|F(u)| = [R^2(u) + I^2(u)]^{0.5}$$

Phase spectrum:

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

2D - DFT

$$F(u,v) = R(u,v) + jI(u,v)$$

$$F(u,v) = |F(u,v)|e^{j\phi(u,v}$$

Magnitude spectrum:

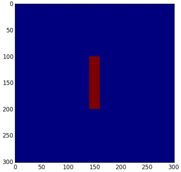
$$|F(u)| = [R^2(u,v) + I^2(u,v)]^{0.5}$$

Phase spectrum:

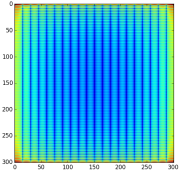
$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

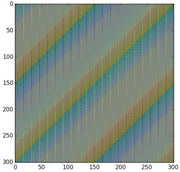
Phase and Magnitude Spectrum



(a) Original



(b) Magnitude



(c) Phase

Introduction to Fourier Transform
DFT properties

Effect of Translation in Spatial Domain

$$f(x-x0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(\frac{x_0 u}{M}+\frac{y_0 v}{N})}$$

$$f(u-u0, v-v_0) \Leftrightarrow f(x,y)e^{j2\pi(\frac{u_0 x}{M}+\frac{v_0 y}{N})}$$

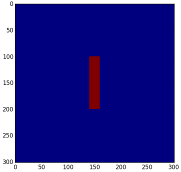
Periodicity

- The DFT and its inverse are periodic with period N

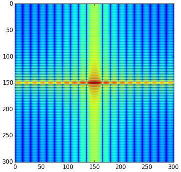
$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

- Due to the periodicity, we can shift the spectrum such that $f(x)e^{jpix+y} = f(x,y)(-1)^{x+y}$. It is corresponding to the function *fftshift*.

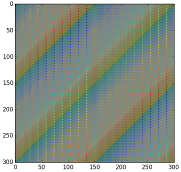
Phase and Magnitude Spectrum



(d) Original



(e) Magnitude



(f) Phase

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

2D- DFT - Basic properties

- Power spectrum:

$$P(u,v) = |F(u,v)^2|$$

- Average gray level of the image:

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

- Symmetric spectrum:

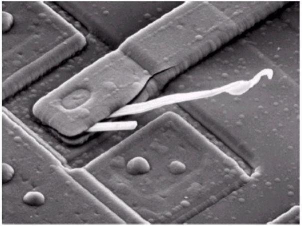
$$F(u,v) = F*(-u,-v)$$

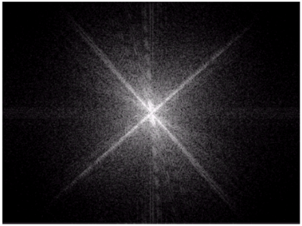
$$|F(u,v)| = |F(-u,-v)|$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Magnitude and phase interpretation

- Magnitude spectrum tells the amplitude of the sinusoids that forms the image
- For any given frequency, large amplitude indicates high influence of that frequency, while the low amplitude indicate the opposite
- Phase indicate the displacement of the sinusoids with respect to their origin





Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Information in magnitude spectrum

- Cancelling the magnitude remove information about the pixel intensities in the spatial domain
- See notebook

Information in phase spectrum

- Cancelling the phase remove information about the spatial information in the spatial domain
- See notebook

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Relationship between spatial and frequency intervals

Spatial and frequency are inversely proportional such that:

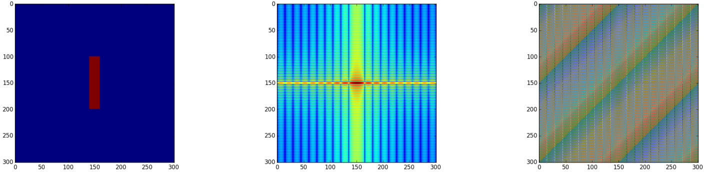
$$\Delta u = \frac{1}{M \Delta T}$$

and

$$\Delta v = \frac{1}{N \Delta Z}$$

Introduction to Fourier Transform
Discrete Fourier Transform (DFT)

Phase and Magnitude Spectrum



(g) Original (h) Magnitude (i) Phase

Introduction to Fourier Transform
DFT Properties

Effect of rotation in Spatial Domain

$$f(x,y) \Leftrightarrow f(r,\theta)$$

$$F(u,v) \Leftrightarrow F(w,\varphi)$$

$$f(r,\theta+\theta_0) \Leftrightarrow F(w,\varphi+\theta)$$

- Rotating $f(x,y)$ by θ rotates $F(u,v)$ by the same angle and vice versa.

Effect of Translation in Spatial Domain

$$f(x-x0,y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(\frac{x_0 u}{M}+\frac{y_0 v}{N})}$$

$$f(u-u0,v-v_0) \Leftrightarrow f(x,y)e^{j2\pi(\frac{u_0 x}{M}+\frac{v_0 y}{N})}$$

- No changes in the magnitude.

Introduction to Fourier Transform
DFT Properties

Distributing and Scaling

- Distributive over addition but not over multiplication

$$\Im\{f_1(x,y) + f_2(x,y)\} = \Im\{f_1(x,y)\} + \Im\{f_2(x,y)\}$$

$$\Im\{f_1(x,y).f_2(x,y)\} \neq \Im\{f_1(x,y)\}.\Im\{f_2(x,y)\}$$

- For two scalars a and b

$$af(x,y) \Leftrightarrow aF(u,v)$$

$$f(ax,by) \Leftrightarrow \frac{1}{|ab|}F(u/a,v/b)$$

Introduction to Fourier Transform
DFT Properties

Conjugate Symmetry

- Conjugate symmetry

$$F(u,v) = F*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

Separability

- DFT pair can be expressed in separable forms:

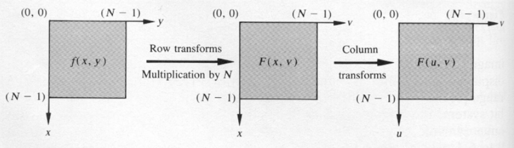
$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)exp[-j2\pi ux/M]$$

$$F(x,v) = \left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)exp[-2\pi vy/N]\right]$$

Introduction to Fourier Transform
DFT Properties

Separability

- For each value of x, the expression inside the brackets is a 1-D transform
- 2-D $F(x, v)$ is obtained by taking a transform along each row of $f(x, y)$ and multiplying the result by N
- $F(u, v)$ is obtained by making a transform along each column of $F(x, v)$



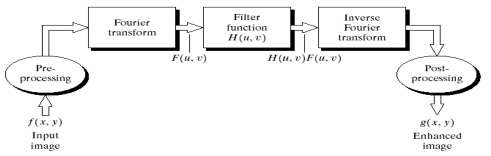
Introduction to Fourier Transform
DFT Properties

Convolution

$$f(x,y) * h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$

$$f(x,y) * h(x,y) \Longleftrightarrow F(u,v)H(u,v)$$

Image Enhancement
Basic Filtering in Frequency Domain



- Compute Fourier transform of image $F(u,v)$
- Multiply the result by a filter transfer function $H(u,v)$
- Take the inverse transform to produce the enhanced image

$$G(u,v) = H(u,v)F(u,v)$$

$$g(x,y) = \Im^{-1}[G(u,v)]$$

Attention!!

Because of periodicity when taking DFT we have to avoid wraparound error or aliasing

Image Enhancement
Zero-padding

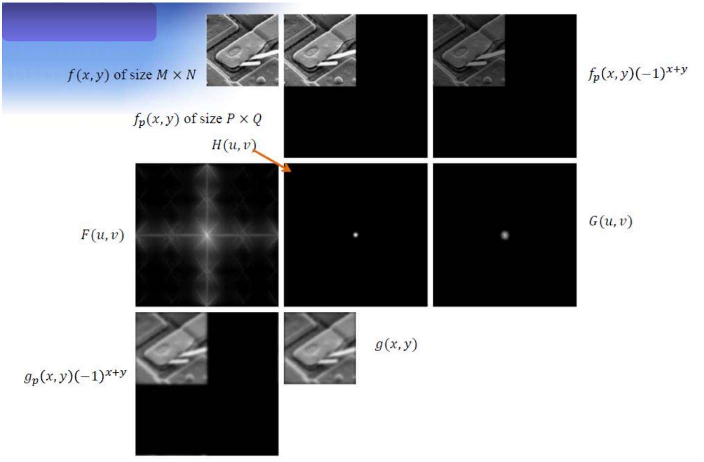


Image Enhancement
Filtering

Notch filter

- It forces the average of the image to be 0
- F(0,0) = 0 and then take the inverse

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = M/2, N/2 \\ 1 & \text{otherwise} \end{cases}$$

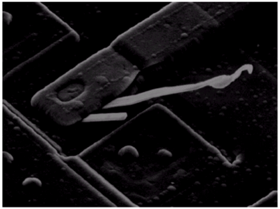
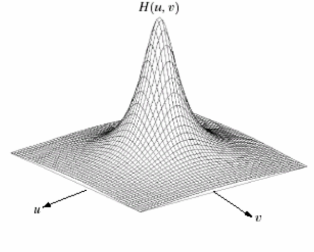


Image Enhancement
Filtering

Low-pass filter

- Reduces the high frequency contents (blurring or smoothing)



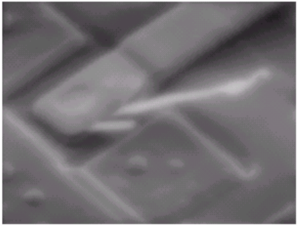
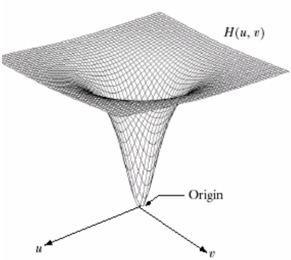


Image Enhancement
Filtering

High-pass filter

- Increase the magnitude of the high frequency components relative to low frequency components (sharpening)



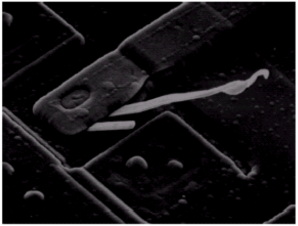


Image Enhancement
Low-pass filter (Smoothing)

- Edges, noise contribute significantly to the high-frequency content of the FT of an image
- Blurring/smoothing is achieved by reducing a specified range of high-frequency components

$$G(u,v) = H(u,v)F(u,v)$$

Different Types

- Ideal
- Butterworth
- Gaussian

Image Enhancement
Low-pass filter

Ideal low-pass filter

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

D_0 is a specified non-negative quantity (Cutoff frequency)
D(u,v) is the distance from point (u, v) to the center of frequency rectangle
$F_c(u,v) = (M/2, N/2)$
$D(u,v) = (u^2 + v^2)^{1/2}$

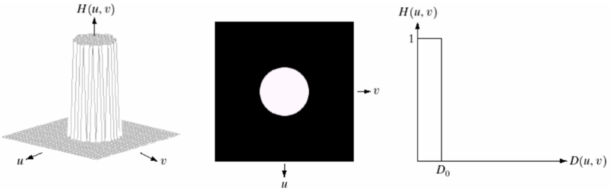


Image Enhancement
Low-pass filter

Ideal low-pass filter

Original image and results of ideal low-pass with cutoff frequencies {5,15,30,80,230}

Image Enhancement
Low-pass filter

Butterworth low-pass filter

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

- n is the order of the filter
- D_0 cutoff frequency locus (distance from the origin)
- $D(u,v) = (u^2 + v^2)^{1/2}$ filter characteristics

- Does not have a sharp discontinuity
- Does not establish a clear cutoff between passed and filtered frequencies
- When $D(u,v) = D_0$, $H(u,v) = 0.5$

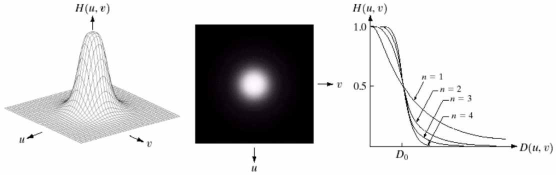


Image Enhancement
Low-pass filter

Butterworth low-pass filter

Original image and results of Butterworth low-pass with order 2 and cutoff frequencies {5,15,30,80,230}

Image Enhancement
Low-pass filter

Butterworth filter

Spatial representation of Butterworth low-pass with orders of {1,2,5,20}

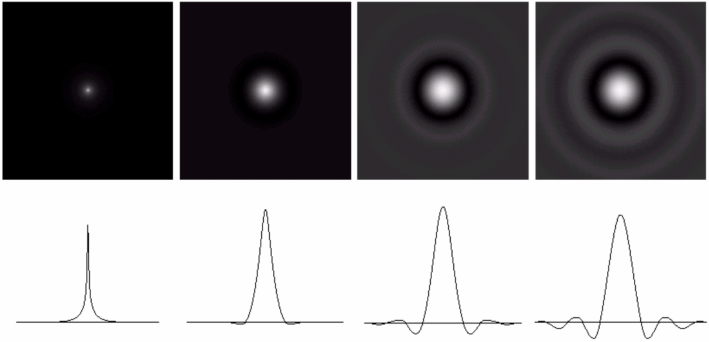


Image Enhancement
Low-pass filter

Gaussian low-pass filter

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

- $\sigma = D_0$ cutoff frequency
- $D(u,v) = (u^2 + v^2)^{1/2}$ Distance from the FT center

- The inverse FT of Gaussian is also a Gaussian

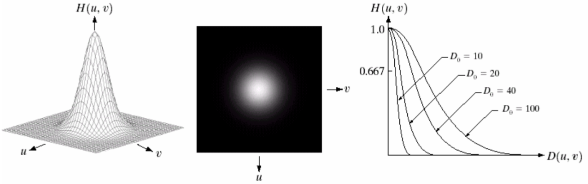


Image Enhancement
Low-pass filter

Gaussian low-pass filter

Original image and results of Gaussian low-pass with cutoff frequencies {5,15,30,80,230}

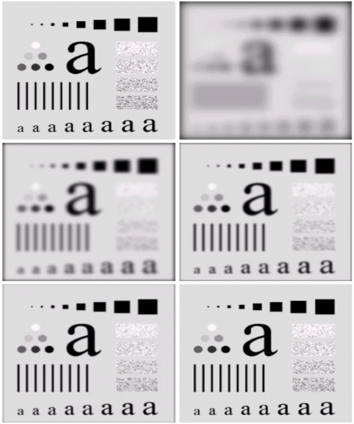


Image Enhancement
High-pass filter (Sharpening)

- Attenuating low-frequency components without disturbing high-frequency information.

$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

Different Types

- Ideal
- Butterworth
- Gaussian

Image Enhancement
High-pass filter

Ideal high-pass filter

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- Opposite of the ideal low-pass

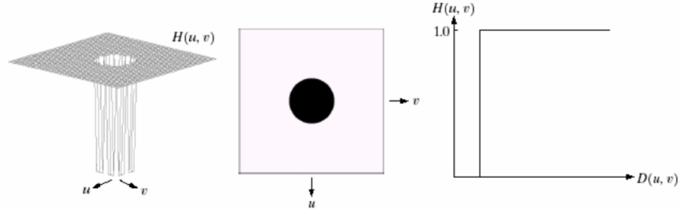


Image Enhancement
High-pass filter

Butterworth high-pass filter

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$

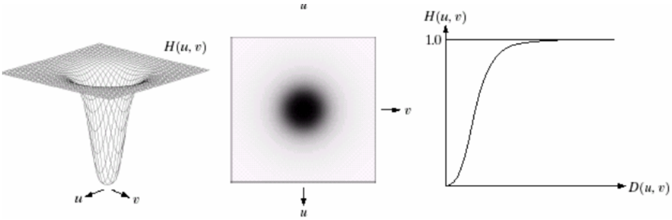


Image Enhancement
High-pass filter

Gaussian high-pass filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2\sigma^2}$$

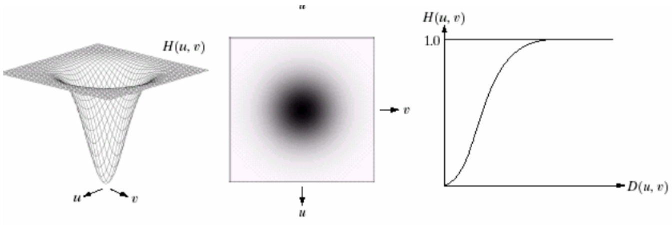


Image Enhancement
High-pass filter

Example of Ideal, Butterworth and Gaussian high-pass

$D_0 = \{30,\ 60,\ 160\ \}$

Ideal High-pass

Butterworth High-pass, $n = 2$

Gaussian High-pass

Image Enhancement
High-pass filter

Recall

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\nabla^2 f = [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - 4f(x,y)$$

Laplacian in FD

$$\Im[\nabla^2 f(x,y)] = -(u^2+v^2)F(u,v)$$

$\rightarrow$ The Laplacian can be implemented in FD by using a filter

$$H(u,v) = -4\pi(u^2+v^2)$$

- FT pair:

$$\nabla^2 f(x,y) \Leftrightarrow [(u-M/2)^2(v-N/2)^2]F(u,v)$$