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A NOTE ON DENAVIT-HARTENBERG NOTATION IN ROBOTICS

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ABSTRACT

The Denavit-Hartenberg conventions model chains of bodies connected by joints. Originally they were applied to single-loop chains but are now almost universally applied to open-loop serial chains such as robotic manipulators. Unfortunately there are several popular variations of the notation: the original, the distal variant, and the proximal variant. These three cases are compared for their application to serial robots. The proximal variate is advanced as the most notationally transparent for the mechanical analysis of serial manipulators.

1 INTRODUCTION

It has been some fifty years since Jacques Denavit and Richard S. Hartenberg introduced the ubiquitous and celebrated kinematic notation bearing their names, [Denavit and Hartenberg (1955)]. Denavit-Hartenberg notation is used to model kinematic chains of bodies connected by joints. Originally they were applied to single-loop chains but are now almost universally applied to open-loop serial chains such as robotic manipulators.

Unfortunately there are several popular variations of the notation: the original, the distal variant popularized by the textbook [Paul (1981)], and the proximal variant popularized by the textbook [Craig (1986)]. Often times researchers and students are not aware of this multiplicity, and those who are, may not fully perceive the details of how they are related. The goal of this paper is to provide a detailed comparison of these three cases for their application to serial robots. As such, the paper develops no new notation and is primarily pedagogical; yet the author is unaware of any such a comparison in the literature of this fundamental subject.

Other notational variations are not considered which are substantially similar to these but use double subscripted indices. Further, the three presented cases do not preclude other distinct variants. The proximal variate is advanced as the most transparent notation for the mechanical analysis of serial manipulators.

2 TOPOLOGY

A kinematic chain is a set of bodies connected by joints. The bodies are assumed rigid and are referred to as links. By convention, a joint imposes a kinematic constraint between a pair of links. Thus, though many joints may be connected to a single link, exactly two links are connected to each joint. This suggests a unique labeling scheme: links are numbered 0, 1, 2, ... and joints are double numbered by the links they connect, 01, 12, ... Figure 1a shows an undirected graph representing a general chain where the links are represented by nodes and the joints are represented by lines.

Single loop chains are simpler since every link is connected by exactly two joints, see Figure 1b. The links are numbered and connected consecutively giving a directed graph. The previous double numbering of the joints can be replaced with single numbers by dropping, for example, the first number. Removing a joint opens the single loop into an open-loop serial chain with similar labeling.

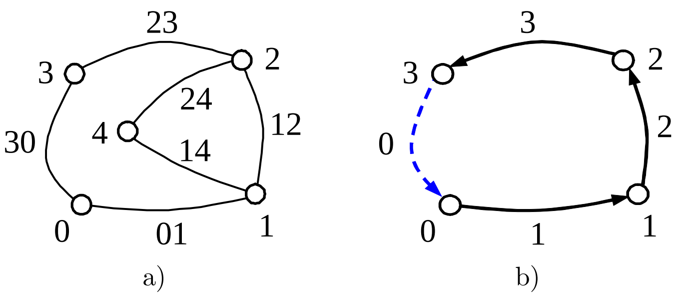

Figure 1. a) Undirected graph of a general kinematic chain. b) Directed graph of an open-loop or single loop mechanism.

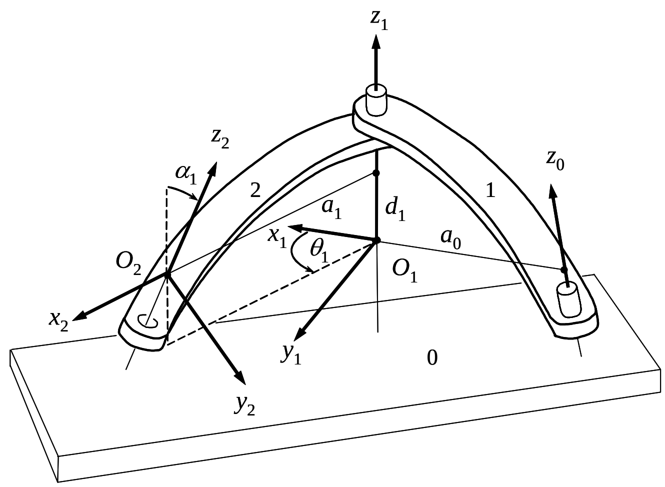

Figure 2. Original Denavit-Hartenberg convention.

3 ORIGINAL DENAVIT-HARTENBERG NOTATION

The Denavit-Hartenberg parameters describe the kinematic model dimensions of a closed-loop or open-loop serial chain with one degree-of-freedom joints. The joints are assumed to be revolute or prismatic but can also be helical. Multiple degree-of-freedom joints are frequently modelled by combinations of single degree-of-freedom joints.

To illustrate the original Denavit-Hartenberg conventions, Figure 2 reproduces a drawing from [Denavit and Hartenberg (1964)], with minor alterations, that shows a portion of a kinematic chain. That in turn is actually a line drawing of a physical model that was photographed and annotated in [Denavit and Hartenberg (1955)]. There are two links, labeled 1 and 2, and three joints, shown as revolutes.

For link 2 the joint axis directions are z_1 and z_2. The common perpendicular line from z_1 to z_2 supplies the direction x_2 to construct frame 2 fixed to the distal end of link 2, or its kinematic extension. (Distal and proximal refer to frame locations on the respective end and beginning of each link, in the sense of increasing link indices.) The common perpendicular distance is called the *link length* a_1.

Similarly, the common perpendicular line from z_0 to z_1 supplies the direction x_1 to construct frame 1 fixed to the distal end of link 1. *Joint offset* d_1 is the distance between the two common perpendicular lines measured along z_1. *Joint angle* θ_1 is measured from x_1 to x_2 about z_1. *Twist angle* α_1 is measured from z_1 to z_2 about x_2 and a negative angle is shown in the figure.

It is noted that d_i has been substituted for the original joint offset notation s_i since it is a frequent abbreviation for $\sin\theta_i$. Also the base has been designated as link 0. The names link length, joint offset, and twist angle are not generally used by Denavit and Hartenberg except for a_i as a link length in reference to a planar linkage.

The four Denavit-Hartenberg parameters $(a_1, \alpha_1, \theta_1, d_1)$ can be expressed using vector forms which take into account the signs of the displacements,

$$\begin{aligned} \cos\alpha_1 &= z_1 \cdot z_2 \\ \sin\alpha_1 &= z_1 \times z_2 \cdot x_2 \end{aligned}$$

$$\begin{aligned} \cos\theta_1 &= x_1 \cdot x_2 \\ \sin\theta_1 &= x_1 \times x_2 \cdot z_1 \end{aligned}$$

$$\begin{aligned} a_1 &= \overrightarrow{O_1O_2} \cdot x_2 \\ d_1 &= \overrightarrow{O_1O_2} \cdot z_1 \end{aligned}$$

where $\overrightarrow{O_1O_2}$ is the vector from O_1 to O_2.

Frame 1 can be transformed into frame 2 by a screwing motion about the axis of z_1 by distance d_1 and angle θ_1 followed by a screwing motion about the axis of x_2 by distance a_1 and angle α_1,

$$B_{12} = \mathit{screw}(\theta_1, d_1, z_1) \cdot \mathit{screw}(\alpha_1, a_1, x_2)$$

In terms of homogeneous transformation matrices this is

$$\begin{aligned} B_{12} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ d_1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1\cos\theta_1 & \cos\theta_1 & -\sin\theta_1\cos\alpha_1 & \sin\theta_1\sin\alpha_1 \\ a_1\sin\theta_1 & \sin\theta_1 & \cos\theta_1\cos\alpha_1 & -\cos\theta_1\sin\alpha_1 \\ d_1 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \end{aligned}$$

This form assumes that for a 4×1 array of position coordinates the first element is the homogeneous coordinate, $[1\ x\ y\ z]^T$, and for consistency this is used throughout. Another convention is to make the last element the homogeneous coordinate.

The four parameters, $(a_1, \alpha_1, \theta_1, d_1)$ specify the location of frame 2 relative to frame 1. Generally, six parameters are necessary to locate a frame with respect to another. However these frames are special since they have two independent conditions imposed on the axes of z_1 and x_2: *i*) they intersect, and *ii*) they are perpendicular. Thus only $6 - 2 = 4$ independent parameters are necessary to locate an adjacent frame.

It is interesting to note that [Denavit and Hartenberg (1955)] use a left-handed measurement for the twist angles α_i though it is never stated explicitly. Indeed, in Figure 2 if α_1 is measured left-handedly it is positive and if measured right-handedly it is negative. The homogeneous transformation matrix is actually given as

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 \cos\theta_1 & \cos\theta_1 & -\sin\theta_1 \cos\alpha_1 & -\sin\theta_1 \sin\alpha_1 \\ a_1 \sin\theta_1 & \sin\theta_1 & \cos\theta_1 \cos\alpha_1 & \cos\theta_1 \sin\alpha_1 \\ d_1 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix}$$

and replacing α_1 by $-\alpha_1$ yields back the right-handed B_{12}. (For consistency throughout the notation B_{12} is used rather than M_2.) In [Denavit and Hartenberg (1964)] this is changed to a purely right-handed system with the corresponding homogeneous transformation matrix given by B_{12} above.

A successful notation balances clarity against conciseness. Five subjective criteria are proposed to evaluate Denavit-Hartenberg notation and its variants for serial robot analyses in displacement, velocity, statics, and dynamics. In order of importance the criteria are deemed as:

1. Frame i is rigidly attached to link i.
2. Displacements θ_i and d_i are measured about z_i.
3. Displacements α_i and a_i are measured about x_i.
4. The first joint displacement is θ_1 or d_1.
5. The ground is link 0.

For the first criterion, a rigid frame is a very simple model for a rigid body. They are often used interchangeably, such as forming the vector derivative with respect to an observer modelled as a frame or body. Since a link is modeled as a rigid body, a common frame/link index enables a single expression, in context, to be valid for two closely related interpretations.

For the second criterion, the joint displacement and joint direction vector are often multiplied together such as in velocities, $\dot{\theta}_i z_i$ or $\dot{d}_i z_i$, and accelerations, $\ddot{\theta}_i z_i$ or $\ddot{d}_i z_i$. An association of θ_i and d_i with the axis of measurement z_i establishes a uniform index structure

For the third criterion, the link length and direction are often multiplied together such as in expressions for position vectors, $a_i x_i$. An association of α_i and a_i with the axis of measurement x_i also establishes a uniform index structure and diminishes introduction of index errors.

For the fourth criterion, the joints are usually counted and labelled from base to tip as 1, 2, ... so it is notationally consistent that the first joint displacement has index 1.

For the fifth criterion, the most the usual convention is that the robot base is designated as link 0. This is especially useful when the notation is adapted to several serial robots working together or in extending the modeling to the limbs of a parallel robot.

The original Denavit-Hartenberg notation satisfies criteria 1 and 2, violates criterion 3, and can exclusively satisfy either criteria 4 or 5 but not both simultaneously. If criterion 4 is satisfied then θ_1 and d_1 correspond to joint 1 making link 1 become the ground which violates criterion 5 (z_0 is not used). Otherwise, as shown in Figure 2, if criterion 5 is satisfied then link 0 is ground making θ_1 and d_1 correspond to joint 2 which violates criterion 4.

4 DISTAL VARIANT

The distal variant is currently the most popular form of the Denavit-Hartenberg notation found in the literature. The earliest references to the present form that the author is aware of occur in [Kahn (1969)], [Kahn (1970)], and [Kahn and Roth (1971)].

Referring to Figure 3, the original parameters $(a_1, \alpha_1, \theta_1, d_1)$ have been replaced by $(a_2, \alpha_2, \theta_2, d_2)$ while the coordinate frame indices and the link indices remain the same. This makes the displacement about the second joint θ_2 or d_2 and thus the displacements along the first joint axis are θ_1 and d_1 so criterion 4 is now satisfied. Since link 1 is the first moving link then link 0 can be selected as ground satisfying criterion 5. Criterion 1 is maintained since frame i is still attached to link i. Criterion 3 is now satisfied since α_i and a_i are now measured about x_i. However criterion 2 is now violated since θ_i and d_i are now measured about z_{i-1}.

The four Denavit-Hartenberg parameters $(a_2, \alpha_2, \theta_2, d_2)$ can be expressed using vector forms which take into account the signs of the displacements,

$$\cos\alpha_2 = z_1 \cdot z_2$$
$$\sin\alpha_2 = z_1 \times z_2 \cdot x_2$$

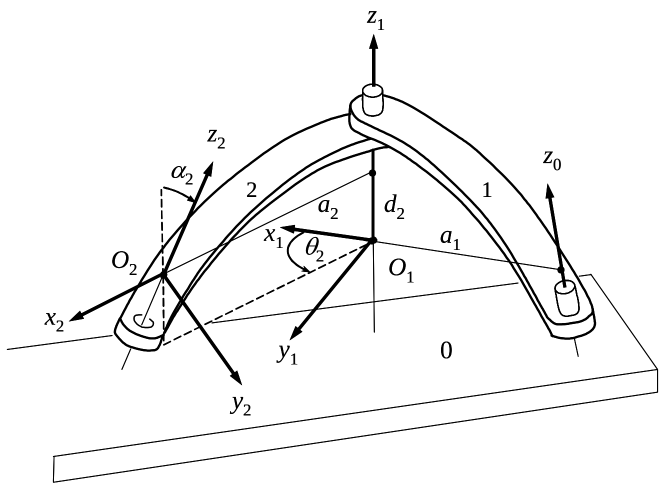

Figure 3. Distal variant of Denavit-Hartenberg notation.

$$\cos\theta_2 = x_1 \cdot x_2$$
$$\sin\theta_2 = x_1 \times x_2 \cdot z_1$$

$$a_2 = \overrightarrow{O_1O_2} \cdot x_2$$
$$d_2 = \overrightarrow{O_1O_2} \cdot z_1$$

Note that the right-hand sides of these equations are the same as for the original Denavit-Hartenberg notation but on the left-hand sides the index 2 has replaced the index 1.

Frame 1 can be transformed into frame 2 by a screwing motion about the axis of z_1 by distance d_2 and angle θ_2 followed by a screwing motion about the axis of x_2 by distance a_2 and angle α_2,

$$B_{12} = screw(\theta_2, d_2, z_1) \cdot screw(\alpha_2, a_2, x_2)$$

In terms of homogeneous transformation matrices this is

$$B_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & \sin\theta_2 & \cos\theta_2 & 0 \\ d_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 & 0 \\ 0 & \sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_2\cos\theta_2 & \cos\theta_2 & -\sin\theta_2\cos\alpha_2 & \sin\theta_2\sin\alpha_2 \\ a_2\sin\theta_2 & \sin\theta_2 & \cos\theta_2\cos\alpha_2 & -\cos\theta_2\sin\alpha_2 \\ d_2 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{bmatrix}$$

These are similar to the original forms except index 2 has replaced index 1 on the right-hand sides.

5 PROXIMAL VARIANT

The proximal variant is less widely found in present literature than the distal variant but occurs more frequently than the original notation. The earliest references to the present form that the author is aware of occur in [Featherstone (1982)], [Featherstone (1984)], [Featherstone (1987)].

Referring to Figure 4, the link indices have been retained but the axes and frame notation are, by consequence, distinct from the distal cases. The i^{th} joint in the chain is along z_i. Frame i is attached to the proximal end of link i to satisfy criterion 1. Since the displacements θ_i and d_i are measured about z_i criterion 2 is satisfied. Similarly displacements α_i and a_i are measured along x_i so criterion 3 is satisfied. Finally the first joint displacement is θ_1 or d_1 so criterion 4 is satisfied and the base is link 0 so criterion 5 is satisfied.

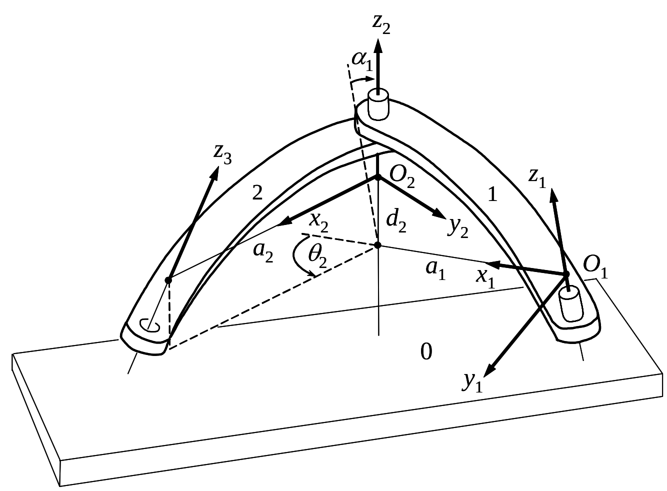

Figure 4. Proximal variant of Denavit-Hartenberg notation.

The four Denavit-Hartenberg parameters $(a_1, \alpha_1, \theta_2, d_2)$ can be expressed using vector forms which take into account the signs of the displacements,

$$\cos\alpha_1 = z_1 \cdot z_2$$
$$\sin\alpha_1 = z_1 \times z_2 \cdot x_1$$

$$\cos\theta_2 = x_1 \cdot x_2$$
$$\sin\theta_2 = x_1 \times x_2 \cdot z_2$$

$$a_2 = \overrightarrow{O_1O_2} \cdot x_2$$

$$d_2 = \overrightarrow{O_1O_2} \cdot z_2$$

Frame 1 can be transformed into frame 2 by a screwing motion about the axis of x_1 by distance a_1 and angle α_1 followed by a screwing motion about the axis of z_2 by distance d_2 and angle θ_2,

$$B_{12} = screw(\alpha_1, a_1, x_1) \cdot screw(\theta_2, d_2, z_2)$$

In terms of homogeneous transformation matrices this is

$$B_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & \sin\theta_2 & \cos\theta_2 & 0 \\ d_2 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & \cos\theta_2 & -\sin\theta_2 & 0 \\ -d_2\sin\alpha_1 & \cos\alpha_1\sin\theta_2 & \cos\alpha_1\cos\theta_2 & -\sin\alpha_1 \\ d_2\cos\alpha_1 & \sin\alpha_1\sin\theta_2 & \sin\alpha_1\cos\theta_2 & \cos\alpha_1 \end{bmatrix}$$

Unlike the original and the distal variant, each of the screw displacements are about quantities with the same index, (α_1, a_1, x_1) and (θ_2, d_2, z_2). Consequently, the transformation B_{12} between frames uses parameters with both 1 and 2 indices.

6 DISCUSSION

Table 1 compares the properties of the three notational conventions in general order of importance. All three have the desirable feature that frame i is attached to link i. The distal variant has the disadvantage that θ_i and d_i are measured along z_{i-1}. It is the personal experience of the author that this is a source of frequent student errors since this part of the distal variant convention is *notationally exceptional*. An analogous exception arises for the original convention where α_i and a_i are measured along x_{i+1}. For these reasons, the proximal conventions can be recommended as the most notationally transparent and the easiest to apply without undue attention.

For the original conventions is not possible to simultaneously have θ_1, d_1 measured along joint 1 and designate link 0 as the grounded link. It should however be noted that the original conventions were developed mainly for single loop mechanisms where it is common to number the links starting from 1.

For the proximal variant, frame 0 is not constrained lie on joint 1 like the distal variant or the original convention when criterion 4 is satisfied. However it is still necessary

	DH	Distal variant	Proximal variant
Criteria			
1. frame i on link i	✓	✓	✓
2. θ_i, d_i along z_i	×	×	✓
3. α_i, a_i along x_i	×	✓	✓
4. θ_1, d_1 along joint 1	⊕	✓	✓
5. link 0 is ground	⊕	✓	✓

Table 1. Comparison of properties for the three notational conventions. ⊕ denotes mutually exclusive properties.

that the x_0 axis intersect and be perpendicular to the z_1 axis. For an n link chain the distal cases (original and variant) only require $n+1$ frames to determine the all parameters $(a_i, \alpha_i, \theta_i, d_i)$ whereas the proximal variant typically uses an additional frame at the distal portion of link n to specify the parameters for this final link. These two features somewhat balance out each other.

In forming the homogeneous transformation matrices both distal cases use the same index throughout, whereas the proximal case uses two indices.

An interesting feature of the proximal and distal variants is that a given chain will have the exact same parameters $(a_i, \alpha_i, \theta_i, d_i)$, $i = 1 \ldots n$ if frame 0 of the proximal variant is chosen to be the same as the distal variant.

Finally it is noted that all of the Denavit-Hartenberg conventions have similar drawbacks for special geometries such as when consecutive joint axes are parallel. Then the parameters are no longer unique unless additional rules are added. This case can also cause robustness problems in calibration situations where it is necessary to determine robot parameters from measuring a series of poses. A small deviation in the measurements can cause a large change in the parameters so sometimes other parameterizations are used such as the zero-reference position method, [Gupta (1997)].

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