Vectors

$\vec{v} = [a_1, a_2, \cdots, a_n]$

zero vector : $\vec{0} = [0, 0, \cdots, 0]$

parallel vector: $\vec{v} \parallel \vec{w}$ ($\vec{v}$ & $\vec{w}$ should be non-zero vectors), if $\vec{v} = r\vec{w}$ for some non-zero real number r)

transpose $\vec{v}^T$ $\quad [a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ column vector

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = [a, b, c]$ row vector

magnitude $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

span $\quad sp(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n})$ = the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$

dot product $\quad \vec{v} = [v_1, v_2, \cdots, v_n] \quad \vec{w} = [w_1, w_2, \cdots, w_n]$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_3 + \cdots + v_n w_n$

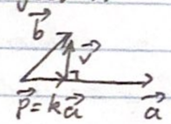
> 2 inequalities: 1° triangle inequality: $\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$

2° Cauchy-Schwarz inequality: $|\vec{v} \cdot \vec{w}| \le \|\vec{v}\| \|\vec{w}\|$

> $\langle \vec{v}, \vec{w} \rangle = \arccos \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \longrightarrow \frac{\vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{w}}{\|\vec{w}\|}$ duality!

perpendicular/orthogonal $\iff \vec{v} \cdot \vec{w} = 0$

projection $\quad \vec{p} = proj_{\vec{a}} \vec{b}$



$k = 0 \iff \vec{b} = \vec{0}$ or $\vec{b} \perp \vec{p}$

$k = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \quad \vec{p} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \cdot \vec{a}$

Matrices

An $m \times n$ matrix A: $\quad M_{m \times n}(R)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$m = n$: square matrix

all entries are 0: $O_{m \times n}$ zero matrix

For a square matrix; ① $\forall i \ne j$, $a_{ij} = 0$: diagonal matrix

② $a_{ij} = \begin{cases} 0, & \forall i \ne j \\ 1, & \forall i = j \end{cases}$: identity matrix

③ $a_{ij} = 0, \forall i > j$: upper triangular matrix

④ $a_{ij} = 0, \forall i < j$: lower triangular matrix

Transpose $\quad A$ & A^T

symmetric matrix: $A = A^T$ $\quad$ skew symmetric matrix: $A = -A^T$

$(A + A^T)^T = A + A^T$ $\quad \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$

$(AB)^T = B^T A^T$

Multiplication $(A: m\times n)(B: n\times k) = (C: m\times k)$

$(AB)_{ij} = (i^{th}\text{ row of } A)(j^{th}\text{ col of } B) = C_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$

$A(B+C) = AB+AC \qquad (A+B)C = AC+BC$

Trace $\quad A \in M_{m,n}(R)$

$tr(A) = \sum_{i=1}^{n} a_{ii}$

$tr(AB) = tr(BA)$

Linear System of Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

A: coefficient matrix

$[A|b]$: augmented/partitioned matrix

$A\vec{x} = \vec{b}$ has solution(s) iff b is in the span of the col of A

Elementary Row Operation

$\left.\begin{cases} \text{row interchange} & R_i \leftrightarrow R_j \\ \text{row scaling} & R_i \to aR_i \quad a\neq 0 \\ \text{row addition} & R_i \to R_i + sR_j \end{cases}\right\}$ $\quad A \sim B$ row equivalent

IF $[A|b] \sim [H|a]$, THEN $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{a}$ have the same soln set

> Row Echelon Form REF

pivot

Reduced Row Echelon Form RREF (unique)

> Solve linear equations: Gauss reduction with back substitution method

Gauss-Jordan method

$\begin{cases} \text{consistent solution : have soln} \\ \text{inconsistent solution : do not have soln} \end{cases}$

> Elementary Matrix

Inverse $\quad A \begin{cases} \text{invertible: if } \exists C \text{ s.t. } AC = CA = I \\ \text{singular} \end{cases}$

$(A^{-1})^T = (A^T)^{-1} \quad (AB)^{-1} = B^{-1}A^{-1} \qquad (A+B)^{-1}$ & $A^{-1}+B^{-1}$ are not necessarily the same

> A is invertible iff RREF of $[A|I]$ is $[I|C]$

A is invertible iff $|A| \neq 0$

A is invertible iff col vectors of A form a basis for R^n
.... iff $A\vec{x}=\vec{b}$ has a soln for all $\vec{b}\in R^n$
.... iff A can be expressed as a product of elementary matrices
.... iff span(row(A)) or span(col(A)) is R^n

Homogeneous System: $A\vec{x}=\vec{0}$

consistent and has a trivial soln $\vec{0}$

> null space: $null(A)=\{\vec{x}\in R^n \mid A\vec{x}=\vec{0}\}$

row/col space: row(A) col(A)

> linearly independent: $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]\vec{x}=\vec{0}$ only has the trivial soln

Dimension

Subspace $W\subseteq R^n$, W is a subspace of R^n if

1° W is nonempty

2° if $\vec{u},\vec{v}\in W$, then $\vec{u}+\vec{v}\in W$ (closure under vector addition)

3° if $\vec{u}\in W, r\in R$, then $r\vec{u}\in W$ (closure under scaler multiplication)

B is a Basis for W, if 1° $W=sp(B)$

2° $\forall \vec{w}\in W$, $B\vec{x}=\vec{w}$ has a unique soln

Dimension of W: $\dim(W)$ is the # of elements in a basis for W

$rank(A)=\dim(col(A))=\dim(row(A))$

$nullity(A)=\dim(null(A))$

☆ for $A: m\times n$ $A\sim H$

$\underbrace{rank(A)}_{\#\text{ of pivots in } H}+\underbrace{nullity(A)}_{\#\text{ of col in } H \text{ without pivots}}=n$

Linear Transformation

$T: R^n \to R^m$ ①. $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$

②. $T(r\vec{v})=rT(\vec{v})$

lines remain lines; origin does not move

R^n: domain R^m: codomain

$W\subseteq R^n$, the image of W under T: $T[W]=\{T(\vec{w}) \mid \vec{w}\in W\}$

$=\{\vec{y}\in R^m \mid \exists\, \vec{x}\in W \mid T(\vec{x})=\vec{y}\}$

$W'\subseteq R^m$, the inverse image of W' under T is $T^{-1}[W']=\{\vec{v}\in R^n \mid T(\vec{v})=\vec{w}\}$

range of T: $range(T)$: $\{T(\vec{v}) \mid \vec{v}\in R^n\}$

kernel of T: $\ker(T) = \{\vec{v} \in R^n \mid T(\vec{v}) = \vec{0}\}$

$\forall \vec{x} \in R^n$ $T(\vec{x}) = A\vec{x}$, A is the standard matrix representation of T

one-to-one: $T(\vec{v}) = T(\vec{u}) \rightarrow \vec{v} = \vec{u}$ $\qquad$ $null(A) = \{\vec{0}\}$

onto: $range(T) = R^m$ $\qquad$ $rank(A) = m$

isomorphism: one-to-one and onto

Determinant

determinant: 1×1 $\det(A) = |a| = a$

2×2 $\det(A) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$

3×3 $\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$> n\times n$: $A = [a_{ij}]$ $n\times n$

minor matrix A_{ij} is a $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} col of A

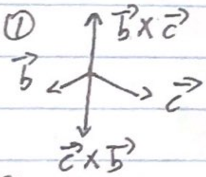
cofactor of a_{ij} of A is $a'_{ij} = (-1)^{i+j}|A_{ij}|$ $\qquad$ A'

$\det(A) = \sum_{i=1}^{n} a_{ij}' a_{ij}$ or $\sum_{j=1}^{n} a_{ij}' a_{ij} = |A|$

Cross Product $\quad \vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3] \in R^3$

$\vec{c} = \vec{a} \times \vec{b} = \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]$

$\vec{c} \perp \vec{a}$ $\quad$ $\vec{c} \perp \vec{b}$

① 

$\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$

②. $\vec{a} \times (\vec{b} \times \vec{c})$ not as $(\vec{a} \times \vec{b}) \times \vec{c}$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

③. area of a parallelgram: $\|\vec{a} \times \vec{b}\|$

volumn of a parallelepiped: $|\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

①. $A \ R_i \leftrightarrow R_j \ B$ $\quad$ $|B| = -|A|$

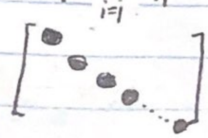
$A \ R_j \rightarrow rR_j \ B$ $\quad$ $|B| = r|A|$

$A \ R_i \rightarrow R_i + rR_j \ B$ $\quad$ $|B| = |A|$

②. $|A| = |A^T|$ $\quad$ $|A||A^{-1}| = 1$

③. $|rA| = r^n |A|$ $\quad$ $|adj(A)| = |A|^{n-1}$

④. if A is a triangular matrix, then $|A| = \prod_{i=1}^{n} a_{ii}$



adjacent matrix: $adj(A) = (A')^T$

> $adj(A)A = A\,adj(A) = |A|I$

$A^{-1} = \frac{adj(A)}{det(A)}$ $(|A| \neq 0)$

Cramer's Rule

$A\vec{x} = \vec{b}$ $|A| \neq 0$ n equations with n unknowns

then, $\vec{x} = [x_1, x_2, \cdots, x_n]$ is of the form $x_k = \frac{|B_k|}{|A|}$

(B_k is A with the k^{th} col replaced by $\vec{b}$)

Eigenvalues & Eigenvectors

if $\exists$ a non-zero vector $\vec{v}$, s.t. $A\vec{v} = \lambda\vec{v}$ λ: eigenvalue

$\vec{v}$: eigenvector

characteristic polynomial: $p(\lambda) = |\lambda I - A|$ algebraic multiplicity $\geq$ geometric multiplicity: $dim(E_{\lambda_i})$

characteristic equation: $|\lambda I - A| = 0$ (the soln are the eigenvalues of A)

eigenspace of λ: $E_\lambda = \{\vec{x} \in R^n \mid A\vec{x} = \lambda\vec{x}\}$ $(= null(A - \lambda I))$

> for A and λ, ① λ^k is an eigenvalue of A^k, and $\vec{v}$ is an eigenvector of A ($k \in N$)

② $\lambda = 0$ is not an eigenvalue of $A \Leftrightarrow$ A is invertible

③ $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, and $\vec{v}$ is an eigenvector of A (A is invertible)

Diagonalizable Matrix: if $\exists$ invertible matrix P s.t. $P^{-1}AP$ is diagonal then A is a diagonalizable matrix

A and B are similar matrices if $\exists$ invertible matrix s.t. $B = P^{-1}AP$

①. the matrix A has n eigenvalues (including each according to its algebraic multiplicity)

> A has n distinct eigenvalues $\Rightarrow \vec{v}_1, \cdots, \vec{v}_n$ are linearly independent

$\Updownarrow$

A is diagonalizable

②. the sum of the n eigenvalues of A = tr(A)

the product ······ of A = det(A)

③. For $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$

1°. $c_{n-1} = -tr(A)$ $c_0 = (-1)^n|A|$

2°. The Cayley-Hamilton Thm: $p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = 0$

> $A^{-1} = -\frac{1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I)$

④. symmetric matrices are diagonalizable

Diagonalization Process $\qquad A: n\times n$

1. $p(\lambda) = \det(A - \lambda I_n)$
2. $p(\lambda) = (\lambda-\lambda_1)^{n_1}\cdot(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}$

   [when $p(\lambda)$ cannot be written as the form above, A is not diagonalizable
3. from $(A-\lambda_i I)\vec{x} = \vec{0}$, find $E_{\lambda_i} = \text{null}(\lambda - \lambda_i I_n)$ $\qquad i = 1, \cdots, k$
4. $\vec{x} = s_1\vec{b_1} + s_2\vec{b_2} + \cdots + s_{m_i}\vec{b_{m_i}}$ $\qquad \underbrace{\{\vec{b_1}, \vec{b_2}, \cdots \vec{b_{m_i}}\}}$

   If the algebraic multiplicity of $\lambda_i = 1$ or the geometric multiplicity of λ_i, then A is diagonalizable [else not diagonalizable]
5. the col position of b_j in P is the col position of its associated eigenvalue on the diagonal of D $\qquad P^{-1}AP = D$

Linear Recurrence:

Fib seq: 1 1 2 3 5 8 13 ...

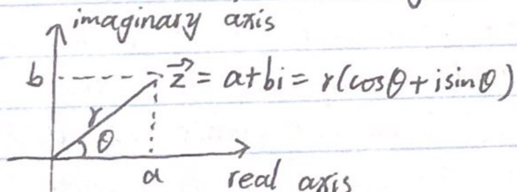
$$\begin{cases} x_{k+1} = x_{k+1} \\ x_k + x_{k+1} = x_{k+2} \end{cases} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$$

Complex Scaler

complex num: $z = \underset{\text{real part}}{x} + i\underset{\text{imaginary part}}{y}$ $\quad x, y \in R$

$z \in C \quad i^2 = -1 / \sqrt{-1} = i \quad Re\, z = x \quad Im\, z = y$

absolute value/modulus/magnitude of $z: |z| = \sqrt{x^2+y^2}$



θ: argument of z

if $-\pi < \theta < \pi$, θ is the principle argument of z or Arg(z)

$r(\cos\theta + i\sin\theta)$: the polar form of z

①. $z_1 z_2 = |z_1||z_2|[\cos(Arg(z_1) + Arg(z_2)) + i\sin(Arg(z_1) + Arg(z_2))]$

$z_1/z_2 = \frac{|z_1|}{|z_2|}[\cos(Arg(z_1) - Arg(z_2)) + i\sin(Arg(z_1) - Arg(z_2))]$

$Arg(z^n) = n\theta$

②. Conjugate: $\overline{a+ib} = a - ib$

$(a+ib)(\overline{a+ib}) = a^2 + b^2$ $\qquad$ real & non-negative

$> |z|^2 = z\bar{z} = a^2+b^2 \qquad \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \qquad z^{-1} = \frac{\bar{z}}{|z|^2}$

$|z_1 + z_2| \leq |z_1| + |z_2|$

If $z = r(\cos\theta + i\sin\theta) \in C$, then

① $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

②. the n^{th} roots of z are $r^{\frac{1}{n}}(\cos(\frac{\theta}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\theta}{n} + \frac{2k\pi}{n}))$ $\qquad k = 0, 1, 2, \cdots, n-1$