Mapping

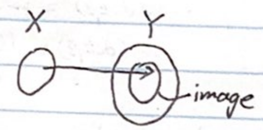
X (domain) $\longrightarrow$ Y (codomain)

Injection / 1-to-1: $f(v)=f(u) \rightarrow v=u$

OR $v\neq u \rightarrow f(v)\neq f(u)$

Surjection / onto: $\forall\, v' \in Y\ \exists\, v \in X$ s.t. $f(v)=v'$

Bijection: 1-to-1 + onto



Periodic Function

$\forall x \quad f(x+T)=f(x) \qquad (T\neq 0)$

Odd Function

$-f(x)=f(-x)$

Even Function

$f(-x)=f(x)$

$\langle f(x)=\frac{1}{2}[f(-x)+f(x)]+\frac{1}{2}[f(x)-f(-x)]\rangle$

Monotone Function

increasing: $\forall x_1, x_2$ if $x_1<x_2$, then $f(x_1)\leq f(x_2)$ OR $f'(x)\geq 0$

decreasing: $\forall x_1, x_2$ if $x_1>x_2$, then $f(x_1)\geq f(x_2)$ OR $f'(x)\leq 0$

also: strictly increasing / decreasing

Boundness

bdd above: $\forall x \quad f(x)\leq K$

Elementary Functions:

Power function $y=x^a \quad (a\in\mathbb{R})$

Exponential function $y=a^x \quad (a>0,\ a\neq 1)$

Logarithmic function $y=\log_a x \quad (a>0,\ a\neq 1)$

Trigonometric function sin cos tan

csc sec cot

arc__

Ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad c=\sqrt{|a^2-b^2|}$

Necessary Formulas for Trig Func

1. $\underset{\cos}{\sin}\left(\frac{\pi}{2} / \frac{3}{2}\pi \pm x\right)=\pm\cos x$

$\uparrow$ sin

$\cos\left(\frac{\pi}{2} / \frac{3}{2}\pi \pm x\right)$?

2. $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$

$\cos(\alpha \pm \beta) = \cancel{\sin} \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$

3. $\sin 2\alpha = 2\sin\alpha\cos\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha}$ $\quad \cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

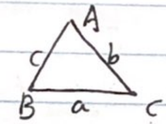
4. $\sin^2 x + \cos^2 x = 1$ $\quad \sec^2 x = \tan^2 x + 1$ $\quad \csc^2 x = \cot^2 x + 1$

5. $a \sin\alpha + b \frac{\cancel{\sin}}{\cos}\alpha = \sqrt{a^2+b^2} \sin(\alpha + \varphi)$ $\quad (\tan\varphi = \frac{b}{a})$

6. $\sin x \le x \le \tan x \quad x \in [0, \frac{\pi}{2})$

The Law of Sin

$a:b:c = \sin A : \sin B : \sin C$



The Law of Cos

$\cos A = (b^2 + c^2 - a^2)/2bc$

Def of Limits: $\lim_{x\to c} f(x) = L$: $\forall x \in D_f$, $\forall \underset{M>0}{\varepsilon > 0}$, $\exists \underset{N>0}{\delta > 0} \,|\, 0 < |x-c| < \delta \to |f(x) - L| < \varepsilon$

$\quad x > N, \; < -N \qquad f(x) > M, \; < -M$

Foundamental Limits $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$ (def of e) $\qquad \lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \dfrac{1-\cos x}{x} = 0$

Squeeze Thm. IF $l(x) \le f(x) \le u(x)$, $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$

THEN $\lim_{x\to c} f(x) = L$

Def of Continuity: $\lim_{x\to c} f(x) = f(c)$

removable
jump
infinite
} discontinuity

Indeterminate Form: $\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0\cdot\infty \quad \infty - \infty \quad 0^0 \quad 1^\infty \quad \infty^0$

$\to 0$: $\frac{1}{\infty} \quad \frac{0}{\infty} \quad \frac{0}{1} \quad 0^\infty \quad 0^1 \quad 0-0$

$\to \infty$: $\frac{1}{0^+} \quad \frac{\infty}{0^+} \quad \frac{\infty}{1} \quad \infty + \infty \quad \infty\cdot\infty \quad \infty^\infty \quad \infty^1$

Def of Derivative: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

tangent line: $y = f'(c)(x-c) + f(c)$

Important Rules for Basic Derivative

$(fg)'(x) = g'(x)f(x) + f'(x)g(x)$

$(\frac{f}{g})'(x) = [f'(x)g(x) - g'(x)f(x)] / g^2(x)$

$(f\circ g)'(x) = (f'(g)\, g')(x)$

2. $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cancel{\sin}\cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

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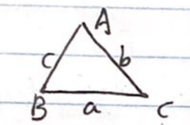
The Law of Sin

$a:b:c = \sin A : \sin B : \sin C$



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$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{x<-N}{x>N} \qquad \underset{f(x)<-M}{f(x)>M}$

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Indeterminate Form: $\frac{0}{0}$ $\quad \frac{\infty}{\infty}$ $\quad 0\cdot\infty$ $\quad \infty-\infty$ $\quad 0^0$ $\quad 1^\infty$ $\quad \infty^0$

$\to 0$: $\frac{1}{\infty}$ $\quad \frac{0}{\infty}$ $\quad \frac{0}{1}$ $\quad 0^\infty$ $\quad 0^1$ $\quad 0\cdot 0$

$\to\infty$: $\frac{1}{0^+}$ $\quad \frac{\infty}{0^+}$ $\quad \frac{\infty}{1}$ $\quad \infty+\infty$ $\quad \infty\cdot\infty$ $\quad \infty^\infty$ $\quad \infty^1$

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$(f\circ g)'(x) = (f'(g)g')(x)$

Derivative for Elementary Func

$(x^k)' = kx^{k-1}$

$\begin{cases} (a^x)' = a^x \cdot \ln a \\ (\log_a x)' = \frac{1}{x \ln a} \end{cases}$

$\sin' = \cos$	$\csc' = -\csc\cot$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\operatorname{arccsc} x)' = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\cos' = -\sin$	$\sec' = \sec \cdot \tan$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(\operatorname{arcsec} x)' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\tan' = \sec^2$	$\cot' = -\csc^2$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$

> for hyperbolic func: $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

$(\cosh^2 x - \sinh^2 x = 1)$

$(\sinh x)' = \cosh x$	$(\tanh x)' = \cancel{\operatorname{sech}^2} \operatorname{sech}^2 x$	$(\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$
$(\cosh x)' = \sinh x$	$(\coth x)' = -\operatorname{csch}^2 x$	$(\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$
$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$	$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$	$(\tanh^{-1} x)' = \frac{1}{1-x^2}$

$(\ln f(x))' = \frac{f'(x)}{f(x)}$

1. $f(x) = \frac{(x-1)^3(x-6)^6}{(x-2)^2(x^2+2x+6)^5}$ $f'(x) = f(x) \cdot (\ln f(x))' = \frac{(x-1)^3(x-6)^6}{(x-2)^2(x^2+2x+6)} \cdot \left(\frac{3}{x-1} + \frac{6}{x-6} - \frac{2}{x-2} - \frac{5}{x^2+2x+6}\right)$

2. $f(x) = x^x$ $f'(x) = x^x (x \ln x)' = x^x (\ln x + 1)$

Fermat's thm for local extrema

IF $f(x)$ has a local extremum at an interior point c, and $f'(x)$ exists

THEN $f'(c) = 0$

Rolle's Thm

IF $f(x)$ diff on (a,b), cont on $[a,b]$ with $f(a) = f(b)$

THEN $\exists\, c \in (a,b) \mid f'(c) = 0$

Lagrange's Thm (Mean Value Thm)

IF $f(x)$ diff on (a,b), cont on $[a,b]$

THEN $\exists\, c \in (a,b) \mid \frac{f(b)-f(a)}{b-a} = f'(c)$

L'Hôpital's Rule:

IF $\lim_{x \to c} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

THEN $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

> $\lim_{x \to c} f(x)^{g(x)} = e^{\lim_{x \to c} [g(x) \cdot \ln(f(x))]}$

The Meaning of a Derivative's Sign

$f'(x) > 0$	increasing ↑	$f''(x) > 0$	concave up ∪
< 0	decreasing ↓	< 0	concave down ∩

Antiderivative

thm: $f'(x) = g'(x) \rightarrow f(x) = g(x) + c$

Necessary Info for Σ (sigma)

$\sum_{k=1}^{n} 1 = n$ $\quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ $\quad \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ $\quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{k=1}^{n} k\right)^2$

$\sum_{k=0}^{n} (a_1 + kd) = na_1 + n(n-1)d$ $\quad \sum_{k=0}^{n} a_1 \cdot q^k = a_1 \frac{1-q^n}{1-q}$

$(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k}$ $\quad$ Note: $\binom{n}{k} = C_n^k = C(n,k) = \frac{n!}{k!(n-k)!}$

$A_n^k = P(n,k) = \frac{n!}{(n-k)!} = P_n^k$

Riemann's Sum: $\sum_{k=1}^{n} f(x_k^*) \Delta x$ where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$, $x_k^* \in [x_{k-1}, x_k]$

left ~: $\sum_{k=1}^{n} f(x_{k-1}) \Delta x$

right ~: $\sum_{k=1}^{n} f(x_k) \Delta x$

mid ~: $\sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right) dx$

trapozoid sum: $\sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2} \Delta x$

Upper/Lower Sum: $U(f,p) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ $\quad L(f,p) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$

where: $M_i = \sup\{f(x), x \in [x_i, x_{i+1}]\}$ $\quad m_i = \inf\{f(x), x \in [x_i, x_{i+1}]\}$

Definite Integral: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)^* \Delta x$

Integrability

Darboux Definition: $\sup\{L(f,p) \mid \forall p \text{ of } [a,b]\} = \inf\{U(f,p) \mid \forall p \text{ of } [a,b]\}$

Integrability Reformulation: $\forall \varepsilon > 0\ \exists p \text{ of } [a,b] \mid U(f,p) - L(f,p) < \varepsilon$

<f is bdd on [a,b]>

> Inte = cont OR f has a finite number of jump discontinuities

Foundamental Thm of Calculus

FTOC I: IF f cont on $[a,b]$, F is any antiderivative of f on $[a,b]$

THEN $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

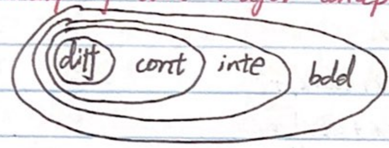
FTOC II: IF f cont on $[a,b]$, $F(x) = \int_a^x f(t)\,dt \ \forall x \in [a,b]$

THEN ①. F is diff on (a,b), cont on $[a,b]$

②. $F'(x) = f(x) \quad \forall x \in [a,b]$

Prove $a=b$: ①. $a \geq b$ $\quad a \leq b$ $\qquad$ ②. $\forall \varepsilon > 0$, $|a-b| < \varepsilon$

Relationship of some major concepts:



5 Integration Techniques

1. Inspection
2. Substitution: IF f, g' cont on $[a,b]$

   THEN $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ $(u=g(x))$

   $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ $(u=g(x))$

3. Integration by Parts: IF $u=f(x)$, $v=g(x)$ are diff

   THEN $\int u\,dv = uv - \int v\,du$

   $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

4. Partial Fraction Decomposition

   Linear Factor Partial Fraction:

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

   Quadratic Factor Partial Fraction:

   $$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

5. Trignometric Substitution

   a^2+u^2: $u=a\tan\theta$ $\quad \theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

   a^2-u^2: $u=a\sin\theta$ $\quad \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$ $\quad |u|\le a$

   u^2-a^2: $u=a\sec\theta$ $\quad \theta\in[0,\frac{\pi}{2})$ $\quad |u|\ge a$

Comparison Thm

f, g, h cont on $[a,b]$ where $a\in\mathbb{R}$

conv case: IF $0\le g(x)\le f(x)$ on $[a,b]$, $\int_a^b f(x)\,dx$ conv

THEN $\int_a^b g(x)\,dx$ conv

div case: IF $0\le h(x)\le g(x)$ on $[a,b]$, $\int_a^b h(x)\,dx$ div

THEN $\int_a^b g(x)\,dx$ div

Know-hows

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\int \tan x\,dx = -\ln|\cos x| + C$ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \cot x\,dx = \ln|\sin x| + C$ $\int \csc x\,dx = \ln|\csc x + \cot x| + C$

$\int \frac{-x}{\sqrt{1-x^2}}dx = \sqrt{1-x^2} + C$ $\int \frac{x}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + C$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$\begin{cases} \int_1^\infty \frac{dx}{x^p} \text{ conv iff } p>1 \ (\frac{1}{p-1}), \text{ div iff } 0<p\le 1 \\ \int_0^1 \frac{dx}{x^p} \text{ conv iff } 0\le p<1 \ (\frac{1}{1-p}), \text{ div iff } p\ge 1 \end{cases}$

Sequence

$a_n = f(n)$ or $\{a_n\}_{n=1}^\infty$ or $\{a_n\}$

$\{a_n\}$ conv to L: $\forall \varepsilon>0\ \exists N>0 \mid \forall n\in\mathbb{N}$, if $n>N$, then $|a_n - L|<\varepsilon$

$a_k \to \infty$: $\forall M>0\ \exists N>0 \mid$ if $k>N$, then $a_k > M$

$a_k \to -\infty$: . . . . . . $a_k < -M$

Bounded Monotone Convergence Thm BMCT

IF ① $\{a_n\}\uparrow$ & bdd above
   ② $\{a_n\}\downarrow$ & bdd below
THEN $\{a_n\}$ conv

> conv → bdd

Series

$\sum_{n=1}^\infty a_n$ or $\sum a_n$

geometric series: $\sum_{n=0}^\infty ar^n$ $(a, r \in \mathbb{R}-\{0\})$

harmonic series: $\sum_{n=1}^\infty \frac{1}{n}$

p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ $(p\in\mathbb{R}^+)$

power series: $\sum_{n=0}^\infty c_n (x-a)^n$

telescoping series

alternating series / Lebniz series: $\sum_{n=1}^\infty (-1)^{n+1} b_n$ $(b_n>0)$

10 Tests for Series

$\sum a_n$ $\sum b_n$ $\sum c_n$

1. GS Test: $\sum a_n = \sum_{n=0}^\infty ar^n$ $(a, r\in\mathbb{R}-\{0\})$
   - $|r|<1$: $\sum a_n$ conv w/ sum $= \frac{a}{1-r}$
   - $|r|\ge 1$: $\sum a_n$ div
2. p-series Test: $\sum a_n = \sum_{n=1}^\infty \frac{1}{n^p}$
   - $p>1$: $\sum a_n$ conv
   - $p\in(0,1]$: $\sum a_n$ div

3. Comparison Thm (CT)

IF $0 \le a_n \le b_n \ \forall n$ and Σb_n conv, THEN Σa_n conv

IF $0 \le c_n \le a_n \ \forall n$ and Σc_n div, THEN Σa_n div

4. Limit Comparison Thm (LCT)

$a_n, b_n > 0 \ \forall n \in \mathbb{N}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = p$

$p \in (0,\infty) \Rightarrow \Sigma a_n, \Sigma b_n$ are both conv or both div

$p = 0$, Σb_n conv $\Rightarrow \Sigma a_n$ conv

$p = \infty$, Σb_n div $\Rightarrow \Sigma a_n$ div

5. Integral Test (IT)

IF $a_n = f(n) \ \forall n \in \mathbb{N}$, $f(x)$ cont, +, ↓ on $[1,\infty)$

THEN Σa_n conv $\Leftrightarrow \int_1^\infty f(x)\,dx$ conv

6. Ratio Test

Σa_n w/ $a_n \in \mathbb{R} - \{0\}$, $p = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$p < 1 \Rightarrow \Sigma a_n$ AC and so also conv

$p > 1 \Rightarrow \Sigma a_n$ div

$p = 1 \Rightarrow$ inconclusive

7. Root Test

$p = \lim_{n\to\infty} (|a_n|)^{\frac{1}{n}}$

$p < 1 \Rightarrow \Sigma a_n$ AC and so also conv

$p > 1 \Rightarrow \Sigma a_n$ div

$p = 1 \Rightarrow$ inconclusive

8. Absolute Convergence Test

$\Sigma |a_n|$ conv $\Rightarrow \Sigma a_n$ conv

> AC absolutely conv: $\Sigma |a_n|$ conv

CC conditionally conv: $\Sigma |a_n|$ ~~conv~~ div, Σa_n conv

9. Divergence Test

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \Sigma a_n$ div

10. Alternating Series Test (AST)

$\Sigma (-1)^{n+1} b_n$ $b_n > 0$

IF b_n +, ↓, $\lim_{n\to\infty} b_n = 0$

THEN $\Sigma (-1)^n b_n$ conv