Automatica 65 (2016) 1–11

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Pursuit–evasion games in the presence of obstacles[☆]

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ARTICLE INFO

Article history:
Received 5 June 2014
Received in revised form
28 August 2015
Accepted 30 October 2015
Available online 5 December 2015

Keywords:
Pursuit–evasion games
Game theory
Dominance regions
Geometric approaches
Multi-agent systems
Autonomous mobile robots
Differential games

ABSTRACT

This paper studies planar pursuit–evasion games in the presence of obstacles that inhibit the motions of the players. The goal is to construct the dominance regions, where a point in the plane is said to be dominated by one of the players if that player is able to reach the point before the opposing players, regardless of the opposing players' actions. The key achievements of the paper are to provide the dominance regions and to show that an analysis of dominance provides a complete solution to the game. This paper also presents a study of the effects of obstacles by comparing the dominance regions in the presence and absence of obstacles. The obstacles considered include line segments and polygons as well as obstacles that have asymmetric effects on the players. As part of the discussion, a novel, multiplayer pursuit–evasion game is also presented. It features three players on two teams, and it can be used to model rescue scenarios and biological behaviors. The solution of this game cannot be determined from the previous literature, but the methods provided in this paper are used to determine dominance and solve the game.

1. Introduction

This paper studies planar pursuit–evasion (PE) games that feature mobile agents that move with fixed, but not necessarily equal, speeds, where some agents are pursuers and others are evaders. The goal of the pursuers is to capture the evaders, which requires the distance between a pursuer and an evader to become zero. The goal of the evaders is to avoid capture. Specifically, this paper studies PE games in the presence of obstacles that inhibit the motions of the players. The goal is to construct the dominance regions, where a point in the plane is said to be dominated by one of the players if that player is able to reach the point before the opposing players, regardless of the opposing players' actions. A dominance region is then the set of all points dominated by a particular player.

The key achievements of this paper are to provide the dominance regions for PE games with obstacles and to show that an analysis of dominance provides a complete solution to the game. Two methods are presented for constructing the dominance regions. The first involves finding the intersection of bundles of isochrones, where an isochrone is a level curve for the value function of a time-optimal control problem. The second method involves identifying singular surfaces that divide the plane into regions, and then determining closed form expressions for the portion of the dominance boundary that lies in each region. Optimal pursuit and evasion strategies are also provided.

As part of the development, a novel multiplayer PE game called the Prey, Protector and Predator (P3) game is also presented. This game has applications to search and rescue scenarios, recovery of military special forces teams, and biological behavior studies. This new game is included as an illustration because the current literature cannot determine its outcome, while the methods developed here can provide the solution.

1.1. Motivation

This paper is motivated by the prevalence of pursuit–evasion games with obstacles and the corresponding lack of a study of the dominance regions for these games. PE games can be used to model a variety of scenarios, including predator/prey relationships in biology, military confrontations, and worst-case scenarios for rescue missions. The addition of obstacles enables the study of more complex and realistic scenarios. For example, it provides a way to study the impacts of human infrastructure on predator/prey

☆ This research was supported in part by the United States Air Force grant FA 8650-07-2-3744. The material in this paper was partially presented at the 53rd IEEE Conference on Decision and Control, December 15–17, 2014, Los Angeles, CA, USA. This paper was recommended for publication in revised form by Associate Editor Hideaki Ishii under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2015.11.018
0005-1098/

relationships. There are also motion planning applications for teams of robots in obstacle-rich environments, such as urban search and rescue scenarios. Finally, PE with obstacles provides a framework to study military troop movements and surveillance missions in urban conflicts.

In some circumstances, obstacles have asymmetric effects on the players. For example, if an unmanned aerial vehicle (UAV) on a surveillance mission is attempting to take a picture of a mobile ground target, then the ground target is constrained by buildings and other ground obstacles that do not affect the UAV. Similarly, airspace structures such as no-fly zones affect the UAV, but not the ground target. Games in these types of environments cannot be analyzed with current methods.

Multiplayer PE games are also of current interest, and an understanding of dominance leads to valuable insights into these games. For example, the P3 game is applicable to combat search and recovery scenarios, and as Section 6 discusses, its solution is obtained through the use of dominance regions.

1.2. Problem statement

Two problems are of interest, and both are addressed in later sections.

P1: **Pursuit–evasion (PE) dominance regions**: Given two mobile agents, a pursuer, P, with constant speed v_P, and an evader, E, with constant speed v_E, moving in a plane in the presence of a known obstacle, where the players move with simple motion and can turn instantaneously, and where each player knows the location of its opponent at all times, find the locus of points, L, that separates the region of the plane dominated by P from the region dominated by E.

P2: **Prey, protector, and predator (P3) game**: Given two pursuers, a predator, P, and a protector, R, as well as an evader, E, known as the prey, moving with speeds v_P, v_R, and v_E, respectively, and separated into teams as follows:
- Red team: P,
  - Goal: capture E,
- Blue team: R and E,
  - Goal: rendezvous to rescue E before P achieves capture,

find the region of the plane where the blue team can rendezvous before capture, regardless of the actions of P.

Note that the solution to P1 is L, the boundary between the two players' dominance regions, and as Sections 3 and 4 show, the construction of L does not take the roles of pursuer and evader into account, but instead views both players simply as mobile agents. However, the construction of L provides all of the information required for P and E to implement their optimal pursuit and evasion strategies, respectively, for the PE game of degree with time to capture as the payoff. This is discussed in Sections 2.2 and 7.

Also note that the P3 game is similar to another multiplayer game known as the Lady, the Bandits and the Bodyguards (Rusnak, 2005), except that in that game the goal of the bodyguards is to intercept the bandit adversaries, and the bodyguards often (though not always) start from the location of the lady. Here the protector starts away from the prey, and the goal of the Blue team is to cooperate and rendezvous in order to rescue the prey.

1.3. Literature review

The literature contains a large body of work on PE games, including the problem formulation and solution of the single pursuer, single evader game in the absence of obstacles, where the solution is determined by construction of Apollonius circles which determine the dominance regions (Isaacs, 1965). The reader is referred to Basar and Olsder (1998), Isaacs (1965), and Kabamba and Girard (2014) for definitions of basic notions such as pursuit–evasion games, games of degree and kind, and value functions. Further analysis of PE games, including the determination of the role assignments of pursuer and evader, can be found in Getz and Pachter (1981) and Merz (1985). In many cases, analytical solutions for PE games are unknown, and numerical methods are used (Karaman & Frazzoli, 2011). For an in-depth survey of PE games, see Chung, Hollinger, and Isler (2011).

Pursuit–evasion games in the presence of obstacles are also covered in the literature (Bhadauria, Klein, Isler, & Suri, 2012; Bopardikar, Bullo, & Hespanha, 2008). For example, Isaacs (1965) introduces the Obstacle Tag game, but it is only used to illustrate the concept of dispersal surfaces, and to introduce the notion of instantaneous mixed strategies. In the literature, obstacles often create visibility constraints for the players (LaValle, Lin, Guibas, Latombe, & Motwani, 1997), and sometimes in PE games with obstacles, the pursuer's objective is not to capture the evader, but only to maintain an unbroken line of sight (Bhattacharya, Hutchinson, & Basar, 2009). However, even though the literature contains work on PE in the presence of obstacles, it does not contain a description of the dominance regions for these scenarios. Giovannangeli, Heymann, and Rivlin (2010) consider PE in an unknown environment with visibility constraints, and they use Apollonius circles to identify regions where capture occurs before loss of visibility. Since visibility constraints are included, the analysis considers only the portion of the standard Apollonius circle that can be reached by both players along straight-line paths, but it does not consider the overall structure of the dominance regions or how the dominance regions change due to the presence of the obstacle.

Similar to dominance regions, Voronoi diagrams divide a plane into regions of points that are closest to a predetermined set of seed points. They are often used to model crystal growth for crystals that grow at the same rate, but Schaudt (1991) extends the concept to crystals that grow at different rates. However, due to the focus on crystal structures, paths are not allowed to pass through the dominance regions of any other competitor, and instead crystals can only wrap around others as they grow. This differs from PE games, where an evader can flee directly away from the pursuer, and the pursuer can give chase and pass through the evader's dominance region. Bakolas and Tsiotras (2012) analyze a scenario with multiple pursuers and a maneuvering evader. A relay strategy is used which involves partitioning the plane with the use of a dynamic Voronoi diagram, where the dominance regions change as the players move. However, the approach requires pursuers with the same speeds. Neither Schaudt (1991) nor Bakolas and Tsiotras (2012) considers the effects of obstacles.

The concept of reach sets has also been applied to linear dynamic games in Hwang, Stipanovic, and Tomlin (2005). There, the two aircraft collision avoidance problem is considered, and backward reachable sets are used to determine a polytopic approximation to the set of points for which aircraft 2 can collide with aircraft 1, no matter how hard aircraft 1 tries to prevent the collision. Obstacles are not considered in this analysis.

The literature also contains PE applications with multiple players. For example, Isaacs (1965) discusses "The Two Cutters and the Fugitive Ship" which poses and solves a game with two cooperative pursuers against a single evader. Foley and Schmitendorf (1974) also address the problem of two pursuers and a single evader. The work by Jin and Qu (2010) is most similar to this work in its approach, and it considers multiple slow pursuers against a fast evader. Apollonius circles are used to determine the number of pursuers required to guarantee capture as well as a strategy that the pursuers can follow.

Another multiplayer pursuit–evasion game referred to as "The Lady, the Bandits and the Bodyguards" has also been studied

(Rusnak, 2005). This problem is often applied to the scenario of defense against homing missiles with a counter weapon (Boyell, 1976; Perelman, Shima, & Rusnak, 2011; Shima, 2011; Shinar & Silberman, 1995).

Biological occurrences similar to the P3 game are studied in White and Berger (2001), in which the tradeoff of maternal vigilance and foraging habits of Alaskan moose is studied. This tradeoff involves the moose leaving its young unattended to forage while remaining close enough to protect the calf from predators. One item of interest in the study is how the distance that the mother is willing to move away from the calf is related to protective cover, in other words, how the speed with which the mother can return to its young is related to the speed with which a predator can find and capture the calf. White and Berger (2001) provide an empirical study, but concepts from differential game theory are not addressed.

Although PE games, including games in the presence of obstacles, are of current interest and have recently been treated in the literature, a description of the dominance regions in the presence of obstacles is missing. Recent works have made use of classical techniques such as Apollonius circles and Voronoi diagrams for multiplayer games (Bakolas & Tsiotras, 2012; Jin & Qu, 2010) and visibility-constrained games (Giovannangeli et al., 2010), but a complete description of dominance regions for pursuit–evasion games in the presence of known obstacles is missing from the literature. Here, these dominance regions are provided, leading to a complete solution to the game.

In previous work (Oyler, Kabamba, & Girard, 2014), dominance regions are constructed for line segment obstacles. Polygonal obstacles are not considered, and obstacles always affect all players symmetrically. Additionally, the optimal pursuit and evasion strategies are not addressed.

Finally, the analysis that follows makes use of concepts from the Euclidean shortest path problem, which finds the shortest path between two points in the presence of polygonal obstacles. This path can be found with the continuous Dijkstra method (Mitchell, 1993), and computationally efficient algorithms exist to compute these paths (Hershberger & Suri, 1999).

1.4. Motivating example

Fig. 1 shows an initial configuration for a P3 game with a line segment obstacle. This scenario is used throughout the following sections to illustrate a number of concepts. In this scenario, the $\square$ represents the prey, the $\triangle$ represents the protector, and the $*$ represents the predator. The protector is twice as fast as the prey, and the predator is three times as fast as the prey. The literature is currently unable to answer the following questions:

Q1: What are the dominance regions if the obstacle is present?
Q2: Which players benefit from the presence of the obstacle and which players are affected negatively by it?
Q3: How can the game be analyzed if the obstacle affects the players asymmetrically?
Q4: Can the protector rescue the prey before the predator achieves capture?
Q5: What are the optimal pursuit and evasion strategies in the presence of obstacles?

Note that these questions are simply specific aspects of problems P1 and P2 that are not addressed in the current literature; i.e., Q1–Q3 are all answered if P1 is solved, and Q4 asks a simplified version of P2, namely, whether the solution of P2 is a non-empty set. Question Q5 identifies an application of the solution of P1.

The remainder of this paper starts from a reduced version of this example and slowly adds complexity, addressing each question in turn.

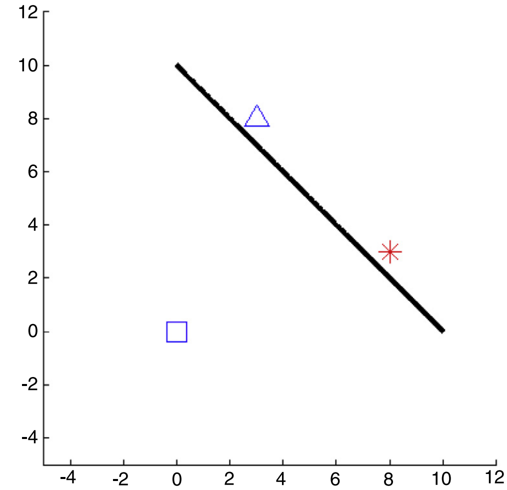


Fig. 1. Example scenario.

1.5. Original contributions

The original contributions of this paper are as follows:

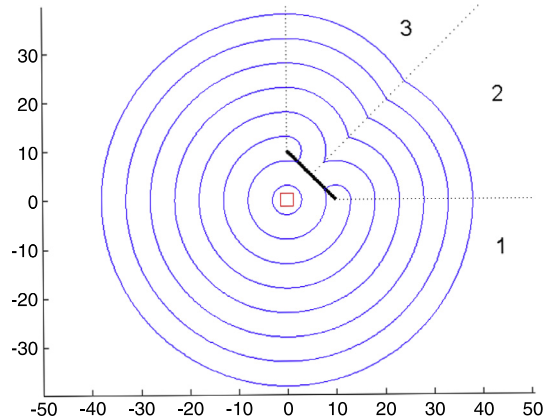
C1: Theorem 6, which gives the dominance regions in two player PE games with obstacles;
C2: Theorem 7 and Remark 1, which provide a convenient method to construct the dominance regions;
C3: A comparative study of the dominance regions in the presence and absence of obstacles;
C4: A study of the effects of obstacles that affect the players asymmetrically;
C5: The P3 Game, a novel PE game for rescue scenarios;
C6: Theorem 8, which gives the solution of the P3 game based on four dimensionless parameters in the absence of obstacles;
C7: Theorem 9, which gives the solution of the P3 game in the presence of obstacles;
C8: A study of the optimal pursuit and evasion strategies in the presence of obstacles for the two player game of degree with time to capture as the payoff.

Note that these contributions solve both problems stated previously and answer all of the questions posed above. Contributions C1 and C2 solve P1 and answer Question Q1; C3 answers Q2; C4 answers Q3; C6 and C7 solve P2 and answer Q4; and C8 answers Q5.

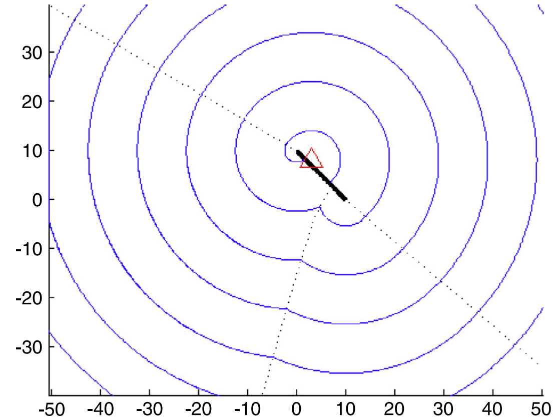
These contributions provide a complete solution to the PE game in the presence of obstacles when all players have full information. The solution to the P3 game provides a method to mathematically analyze rescue scenarios, combat search and recovery scenarios, and biological behaviors which have previously been studied empirically.

1.6. Organization

This paper is organized as follows: Section 2 provides background information from the literature which addresses two foundational versions of the motivating example. The first features a single player moving in the presence of an obstacle, and the second features two players and no obstacle. Next, Section 3 provides the dominance regions for a version of the motivating example that features two players in the presence of the simplest nontrivial obstacle, a line segment, and the analysis uses intersections of isochrones. Then Section 4 provides a convenient way to calculate the dominance regions in closed form. Section 5 addresses two player games in the presence of more complex obstacles, and solutions are provided for polygonal obstacles and obstacles with



(a) Prey.



(b) Protector.

Fig. 2. Bundles of isochrones.

asymmetric effects. In Section 6, the P3 game with no obstacle is analyzed, and then the motivating example is analyzed in full with all three players and the obstacle. Optimal pursuit and evasion strategies are considered in Section 7, and finally, Section 8 gives conclusions and future work.

2. Background

This section addresses two foundational versions of the motivating example. The first considers a single player moving in the presence of obstacles, and the second considers a two-player PE game with no obstacles.

2.1. Time-optimal paths and isochrones

The time-optimal control problem can be stated as follows: given an agent moving with simple motion and speed v, initial and final locations, (x_i, y_i) and (x_f, y_f), and a set of known obstacles, S, find a path that connects (x_i, y_i) to (x_f, y_f) without intersecting S such that the time required for the agent to reach (x_f, y_f) is minimized.

In this paper, the agents move with constant speeds, and therefore time-optimal paths correspond to paths with the smallest Euclidean distance. These paths can be determined using Hershberger and Suri (1999) and Mitchell (1993). For this work, we are interested in level curves for the value function of this time-optimal control problem, and these level curves are referred to as isochrones.

The following theorems are useful for future developments, and their proofs involve the propagation of a simulated wavefront in the plane (Mitchell, 1993).

Theorem 1. *In the absence of obstacles, the time-optimal paths are straight lines, and the isochrones are concentric circles centered at the agent's initial location.*

Theorem 2. *In the presence of a set of polygonal obstacles, the time-optimal paths are broken lines, breaking at obstacle vertices, and the isochrones form arcs of concentric circles centered at generating points, where a generating point is either an obstacle vertex or the agent's initial location.*

Theorem 3. *The curves that separate the plane into regions with unique generating points are either line segments or arcs of hyperbola.*

The preceding theorems are illustrated in Fig. 2(a), which shows a bundle of isochrones of different durations for the prey in the motivating example. The thick line is the obstacle, the thin lines are isochrones, and the dotted lines are the curves that separate the regions with different generating points. In region 1, the time-optimal paths are unaffected by the obstacle, and the isochrones form concentric circles centered at the agent's initial location. For destinations in regions 2 and 3, the time-optimal paths break at the obstacle's end points, and the isochrones form concentric circles centered at the end points. Regions 2 and 3 are separated by an arc of hyperbola where each point on the arc can be reached in equal time by traveling around the obstacle in either direction.

Fig. 2(b) shows the isochrones for the protector, and the isochrones are plotted for the same time durations as in Fig. 2(a). Note that the protector is faster than the prey, so the isochrones in Fig. 2(b) are spaced farther apart.

2.2. Pursuit–evasion games

For a single pursuer, single evader game with no obstacles where both players move with simple motion, the complete solution is obtained by constructing an Apollonius circle (Isaacs, 1965). The Apollonius circle is the set of all points for which the ratio of distances to two fixed points is constant, and for the game under consideration it is given by:

$$\left(\gamma^2 - 1\right) r^2 + \left(2d\cos(\theta)\right) r - d^2 = 0, \tag{1}$$

where (r, θ) are polar coordinates with origin at the evader's initial location and the direction of zero azimuth along the line of sight to the pursuer; $\gamma = v_P/v_E$ is the speed ratio, and d is the initial distance between the players.

Theorem 4. *In the absence of obstacles, the dominance regions of the PE game are divided by an Apollonius circle (Isaacs, 1965; Kabamba & Girard, 2014).*

Theorem 4 provides solutions to games of kind, for which there are a finite number of possible outcomes; e.g., the question of whether or not E can reach a safe haven before being captured. The dominance regions also provide the solution to the more general game of kind which asks simply whether or not P can capture E at all, given enough time. If E's dominance region is bounded, which is always the case if $v_P > v_E$, then P dominates this game of kind, and given the optimal strategy, P will always be able to capture E. This type of strategy that guarantees victory for one player is called a dominant strategy. In this case, since $v_P > v_E$, and since P is as maneuverable as E, the dominant strategy is that P simply travels to E's initial location and then follows the same path as E until capture occurs. However, the notion of dominance regions is not specific to the dynamics of simple motion, which this example considers, and knowledge of the dominance regions can be used to construct dominant strategies in other cases as well. Additionally, the usefulness of dominance regions is not limited to games of kind. The information they provide is sufficient to solve games of degree as well, which have a continuum of outcomes; e.g., the question of how long it takes for P to capture E.

Theorem 5. *In a two player PE game of degree with no obstacles where the payoff is the time to capture, which P seeks to minimize and E seeks to maximize, the optimal strategies are such that capture occurs at the point, C, on the Apollonius circle that is farthest from E's initial position, and optimal play dictates that P and E both travel to C in minimum time (Isaacs, 1965).*

Note that when no obstacles are present, P, E, and C are co-linear. Thus, the strategies that result from this construction are those typically referred to as classical pursuit and classical evasion, where P travels directly toward E along the line of sight and E flees directly away from P in the same direction. These strategies maximize the minimum time to capture, or, equivalently, minimize the maximum time to capture. An identical result is obtained by examining the rate of change of the range, ρ, between the players, which is as follows:

$$\dot{\rho} = v_E \cos(\beta - \psi_E) - v_P \cos(\beta - \psi_P), \tag{2}$$

where ψ_P and ψ_E are the headings of the pursuer and evader, respectively, and β is the line of sight angle from P to E. From (2), P minimizes $\dot{\rho}$ by choosing $\psi_P = \beta$, and E maximizes $\dot{\rho}$ by choosing $\psi_E = \beta$.

If P and E both play optimally, then they travel straight to the capture point, and the Apollonius circle at any intermediate time is tangent to the initial circle at C. If either player deviates from their optimal strategy and travels laterally, then the optimal capture point changes, and the resulting path for the other player curves.

3. Dominance boundary as an intersection of isochrone bundles

This section provides the dominance regions in two player PE games with obstacles. The method presented involves finding the intersection of bundles of isochrones, and this method is shown to agree with Theorem 4 when no obstacles are present. The method is then used to construct the dominance regions for a PE game with an obstacle, which the previous literature is unable to do.

3.1. Intersection of isochrone bundles

Section 2.1 describes how isochrones can be constructed for each player for a specified time duration. A bundle of isochrones, i.e., a set of curves parameterized by the time duration, t, is then formed for each player, as in Fig. 2(a) and (b), where the two bundles are parameterized by a common variable, t. Elimination of this common parameter leads to the region of points over all time where the players can meet if both follow time-optimal paths.

Theorem 6. *In the PE game with obstacles, the plane is exhaustively divided into three disjoint regions:*

(1) *A region where a player strictly dominates,*
(2) *A region where the other player strictly dominates,*
(3) *A region where neither player dominates.*

Moreover, the third region is obtained by intersecting bundles of isochrones.

Proof. Consider an arbitrary point in the plane. For that point, solve the two time-optimal control problems of moving from the initial locations of the two players to the arbitrary point. Only two outcomes are possible: either one transfer time is strictly smaller than the other, or the two transfer times are equal. In the first alternative, the arbitrary point is in the interior of one of the two dominance regions. In the second alternative, the arbitrary point is at the interface between the two dominance regions. However, that case is characterized by the equality of the transfer time; therefore, the point belongs to the intersection of the bundles of isochrones. □

As an example, consider the PE game with no obstacles in a Cartesian coordinate system with origin at the evader's initial location. If (x_E, y_E) and (x_P, y_P) are the positions of the evader and the pursuer, respectively, and if the pursuer's initial location is $(x_{P,0}, y_{P,0})$, then for a given time, t, the isochrones for each player are given by

$$x_E^2 + y_E^2 = v_E^2 t^2, \tag{3}$$

$$(x_P - x_{P,0})^2 + (y_P - y_{P,0})^2 = v_P^2 t^2. \tag{4}$$

After eliminating the common parameter, t, the intersections of the isochrones are the points where $x_E = x_P$ and $y_E = y_P$, so the subscripts are dropped, yielding

$$\frac{x^2 + y^2}{v_E^2} = \frac{(x - x_{P,0})^2 + (y - y_{P,0})^2}{v_P^2}. \tag{5}$$

Substituting $x_{P,0} = d$, $y_{P,0} = 0$, and $\gamma = v_P / v_E$ gives

$$(\gamma^2 - 1)(x^2 + y^2) + 2dx - d^2 = 0, \tag{6}$$

which agrees with (1).

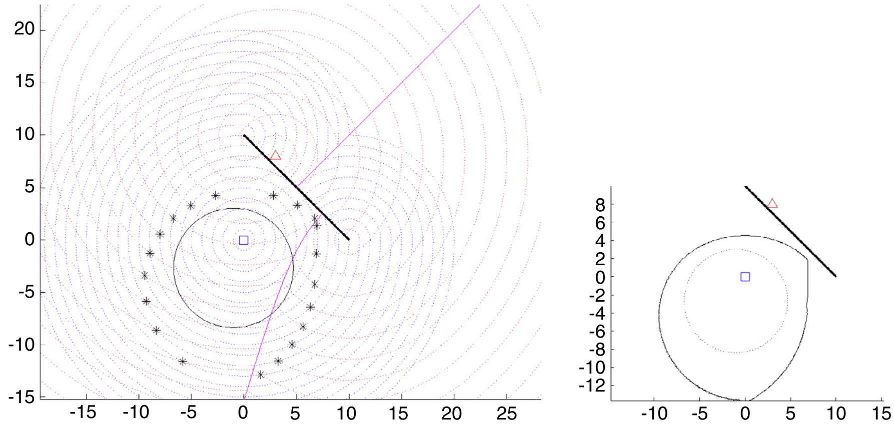
3.2. Two player PE with a line segment obstacle

Fig. 3 shows this method for a PE game in the presence of a line segment obstacle. In Fig. 3(a), the bundles of isochrones from Fig. 2(a) and (b) are shown, and each intersection of isochrones of the same duration is marked by a $*$. In Fig. 3(b), the isochrone bundles are removed for clarity, and a much larger number of intersections are plotted to form the boundary between the two dominance regions. In both figures, the circle is the Apollonius circle that would determine dominance if the obstacle was not present, and it is included to show how the dominance boundary changes due to the presence of the obstacle. The effects of obstacles are discussed in greater detail in Section 4.3.

4. Dominance boundary in closed form

The dominance boundary in Fig. 3(b) is piecewise smooth with two distinct cusps. As described in Isaacs (1965), this occurs in the solutions of many differential games. Following Isaacs (1965), in this section, the dominance boundaries are constructed analytically by determining singular surfaces and the solution in the small, where singular surfaces are curves that divide the plane into regions where the solution behaves differently in each region, and the solution "in the small" refers to the smooth part of the solution that occurs between singular surfaces. The solution in the small is presented in Section 4.1.

The solution need not be smooth when it crosses a singular surface, and in Section 4.2, the cusps in Fig. 3(b) are shown to occur at singular surfaces. The solution in its entirety is referred to as the solution "in the large", and it is obtained by identifying the singular surfaces and piecing together solutions in the small in regions delineated by singular surfaces. This is described in Section 4.2.



(a) Intersection of isochrone bundles.
(b) Dominance regions.

Fig. 3. Dominance regions formed by isochrone intersections.

4.1. Solution in the small

As stated in Theorem 2, in the presence of obstacles, the isochrones always form arcs of concentric circles centered at known points. Therefore, when considering the intersections of isochrone bundles, the solution in the small is as follows:

Theorem 7. *Each portion of the dominance boundary satisfies the following condition for a specific value of t_B and d:*

$$\left(\gamma^2-1\right)r^2+2\left(d\cos\theta-\gamma^2 v_A t_B\right)r+\left(\gamma^2 v_A^2 t_B^2-d^2\right)=0. \tag{7}$$

Proof. From Theorem 2, time-optimal paths are made up of a number of straight line segments. Consider two players moving along the final segments of their time-optimal paths to a candidate capture point. In general, the players begin their final segments at different times, so let A be the first player to begin its final segment, and for simplicity assume that A departs on this segment at $t=0$. Let B be the other player, and let t_B be the time that elapses before B departs on its final segment.

The locus of intersections of the isochrone bundles can be determined from Fig. 4, where d is the distance between the starting points of the final segments. From the law of cosines:

$$v_B^2(t-t_B)^2=v_A^2t^2+d^2-2v_A t d\cos(\theta). \tag{8}$$

Rearrange (8) and let $\gamma=v_B/v_A$ to obtain

$$(\gamma^2-1)t^2+2\left(\frac{d}{v_A}\cos(\theta)-\gamma^2 t_B\right)t+\left(\gamma^2 t_B^2-\frac{d^2}{v_A^2}\right)=0. \tag{9}$$

Define $r=v_A t$, substitute $t=r/v_A$ into (9), and multiply the resulting equation by v_A^2 to obtain

$$\left(\gamma^2-1\right)r^2+2\left(d\cos\theta-\gamma^2 v_A t_B\right)r+\left(\gamma^2 v_A^2 t_B^2-d^2\right)=0. \tag{10}$$

This quadratic equation is easily solved for r, and it defines, in polar form, the locus of points where A and B can meet at the end of the final segments of their paths, with the origin at the start of A's final segment and the direction of zero azimuth along the line of sight from the origin to B's position at $t=t_B$. □

Note that when $t_B=0$, the quadratic form of the standard Apollonius circle given by (1) is recovered.

Depending on the values of t_B and d, (7) can take the following three forms:

(1) A limaçon: when $t_B\neq 0$ and $d\neq 0$;
(2) An Apollonius circle: when $t_B=0$ and $d\neq 0$;
(3) A circle centered at an obstacle vertex: when $t_B\neq 0$ and $d=0$.

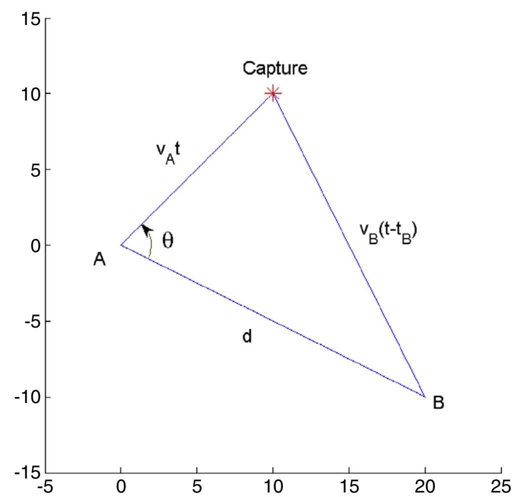


Fig. 4. Geometry of final segments of time-optimal paths.

Note that the fourth case, when $t_B=d=0$ at an obstacle vertex, is ignored because capture occurs when the players reach that location.

Consider the third form of the solution which applies to regions where both players travel past the same obstacle vertex at different times. Note that if A is faster than B, then A dominates the entire region. Therefore, isochrones only intersect in this type of region if $v_B > v_A$. Since both players' isochrones are concentric circles centered at the same location, the resulting locus of intersections is also a circle centered at the same vertex. The radius can be computed from (7) with $d=0$:

$$\left(\gamma^2-1\right)r^2-2\gamma^2 v_A t_B\, r+\gamma^2 v_A^2 t_B^2=0. \tag{11}$$

This equation has two solutions for r:

$$r_1=\frac{\gamma v_A t_B}{\gamma-1},\qquad r_2=\frac{\gamma v_A t_B}{\gamma+1}. \tag{12}$$

However, since $v_B > v_A$, $\gamma > 1$, and therefore $r_1, r_2 > 0$. Here, r_2 corresponds to a suboptimal path for A where A turns around and heads back toward the obstacle vertex at the moment that B reaches the vertex. Therefore, for time-optimal paths, the isochrones intersect along the circle with radius r_1.

4.2. Solution in the large

In this section, the solution in the large is constructed by identifying the singular surfaces in two player PE games and then assembling the portions of the solutions in the small that lie in regions delineated by these surfaces. The dominance boundary is continuous, but it need not be smooth, and this section explains why cusps often occur at the singular surfaces, as noted previously.

In the following remark, the singular surfaces are categorized using the taxonomy of Isaacs (1965). Here, a dispersal surface is a curve for which time-optimal paths move away from the surface on both sides, and in this context, it separates two regions of the plane and represents a choice for one of the players about which direction to travel around an obstacle. A surface of type (p, u, p) is one where time-optimal paths that are sufficiently close to the surface are parallel to it on both sides, and where time-optimal paths are allowed to coincide with the surface. These surfaces represent a difference of behavior in the time-optimal paths of nearby points, where on one side of the surface the time-optimal paths require a turn at an obstacle, while on the other side no turn is required. The term "generating point" is used in the context of Theorem 2.

Remark 1. The curves described in Theorem 3, which separate the plane into regions with unique generating points, are the singular surfaces for the two-player pursuit–evasion game in the plane in the presence of obstacles. These singular surfaces consist of:

(1) Surfaces of type (p, u, p) which are portions of straight lines emanating from a generating obstacle vertex and extending away from another generating point parallel to the line of sight;
(2) Dispersal surfaces which are arcs of hyperbola.

For example, consider again the reduced version of the motivating example from Section 3.2. Fig. 5(a) shows this scenario with the singular surfaces depicted with dotted lines and the dominance regions determined by assembling the solutions in the small for each region. For this scenario, the surfaces of type (p, u, p) separate regions that can be reached by both players with straight-line paths (those labeled with "1" in the figure) from regions where one of the players must travel around the obstacle (those labeled with "2" in the figure). If a player's destination is a point located near one of these surfaces, then paths that are sufficiently close to the surface on either side are parallel to the surface. The only difference is that on one side of the surface the time-optimal path is straight, while on the other side of the surface the time-optimal path requires a slight bend at the obstacle vertex.

The dispersal surfaces in Fig. 5(a) are the arcs of hyperbola that separate two regions that are both labeled with "2". If a player's destination is a point located near a dispersal surface, then that player is faced with a decision about which way to travel around the obstacle. Points that are very close together but on opposite sides of the dispersal surface have significantly different time-optimal paths.

The following points are noteworthy:

- The cusps in the dominance boundary occur where the dominance boundary intersects the dispersal surface, and they are the result of the difference in behavior for points that are close together, but on opposite sides of the dispersal surface.
- The singular surfaces are the same as the curves described in Theorem 3 for a single agent in the presence of obstacles, but they take on additional significance in the context of the PE game.
- The dominance boundary in Fig. 5(a), which is formed by assembling the solutions in the small between singular surfaces, agrees with the result obtained by intersecting isochrone bundles in Fig. 3.

4.3. Effect of obstacles on PE games

Fig. 5 shows two example scenarios. For both scenarios, the pursuer is twice as fast as the evader, the pursuer's starting location is given by $\triangle$, and the evader's starting location is given by $\square$. The thick line represents the obstacle, and the dotted lines are the singular surfaces. Regions labeled with "1" can be reached by both players with straight line paths, regions labeled with "2" require one player to travel around the obstacle, and regions labeled with "3" require both players to travel around the obstacle. The thin curve represents the boundary between dominance regions. For comparison, the dashed circle is the Apollonius circle that defines dominance in the absence of the obstacle.

As Fig. 5 shows, in some cases the dominance region in the presence of the obstacle is contained in the original Apollonius circle, while in other cases it encompasses the Apollonius circle. This shows that the obstacle can be either a benefit or a hindrance to both the faster and the slower player, depending on the initial player locations. In Fig. 5(b), the faster pursuer benefits from the obstacle, while in Fig. 5(a) the slower evader benefits.

5. More complex obstacles

The motivating example includes a line segment obstacle which affects all players symmetrically, but the theorems and methods provided in previous sections are not limited to obstacles with these properties. This section provides dominance regions for two types of more complicated obstacles. The first scenario involves polygonal obstacles, and the second involves an obstacle that has asymmetric effects on the players. The key result of this section is that isochrones still determine dominance.

5.1. Polygonal obstacles

When more complex obstacles are introduced, the solution method remains unchanged. The number of singular surfaces increases due to the increased number of generating points, but the singular surfaces are still determined by Remark 1, and they can be constructed using Hershberger and Suri (1999) and Mitchell (1993). Similarly, the isochrones are still arcs of concentric circles, and therefore the solution in the small from Section 4.1 holds. Fig. 6 shows how the version of the motivating example used in Figs. 3 and 5(a) changes when a third vertex is added to make the obstacle triangular.

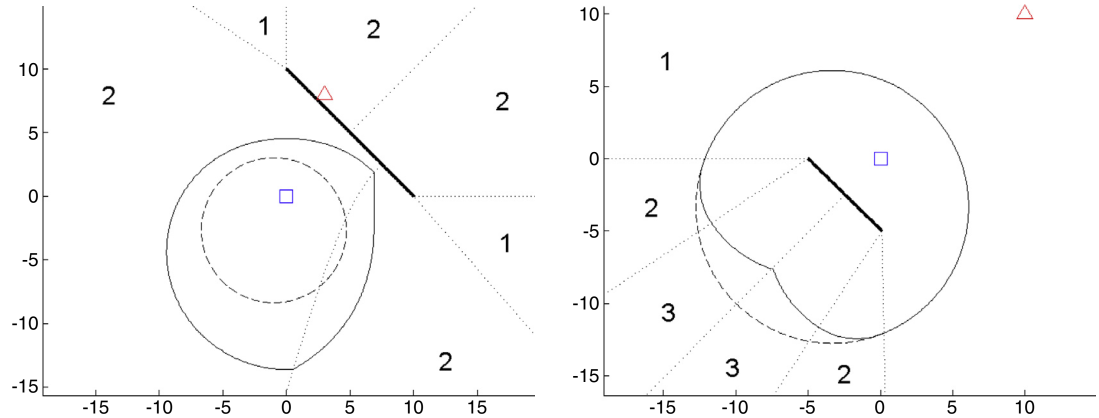
The dashed lines represent the singular surfaces, and the faint dotted lines show the previously determined dominance boundary and dispersal surface from the scenario where the obstacle consists of only the line segment between vertices 1 and 2.

In this case, the dispersal surface generated by the $\square$ consists of portions of two hyperbolas, labeled "a" and "b". Points on curve "b" can be reached in equal time by traveling past either vertex 1 or 3. Points on curve "a" can be reached in equal time by traveling past vertex 1 or past both vertices 3 and 2. As expected, these hyperbolas intersect at the singular surface extending upward from vertex 2.

5.2. Obstacles with asymmetric effects

Consider a scenario where the pursuer is an unmanned aerial vehicle (UAV) and the evader is an unmanned ground vehicle (UGV). Obstacles on the ground, such as streetside curbs or bushes, inhibit the motion of the UGV, but they do not affect the UAV.

Obstacles that have asymmetric effects on the players of a PE game can be analyzed using the techniques described previously. The isochrones are still arcs of concentric circles, and therefore the solution in the small from Section 4.1 holds. In fact, in this scenario



(a) Evader benefits from obstacle.

(b) Pursuer benefits from obstacle.

Fig. 5. Effect of obstacles on dominance regions.

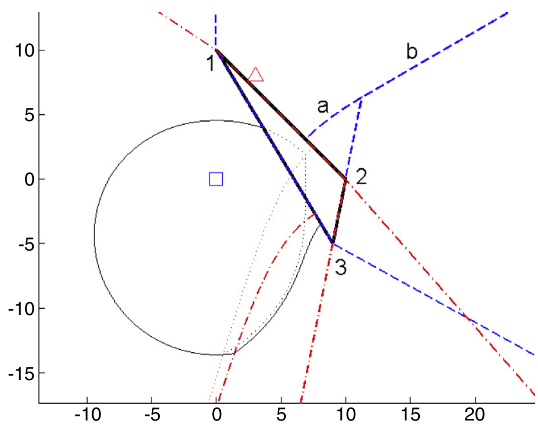


Fig. 6. Polygonal obstacle.

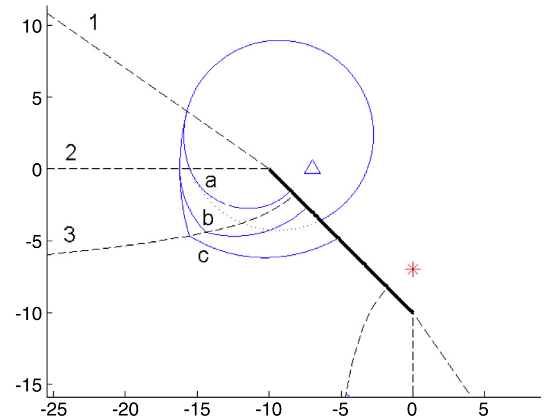


Fig. 7. Asymmetric PE game.

the analysis is simpler because only one player generates singular surfaces, and the number of singular surfaces therefore decreases.

Fig. 7 considers the motivating example with only two players where the $*$ is faster than the $\triangle$. The dashed lines show the 6 singular surfaces, and the dominance boundaries are shown for all four possible scenarios. These scenarios include the case when neither player is affected by the obstacle, when both are affected by the obstacle, and when only one player is affected by the obstacle.

In the region of points that both players can reach with straight-line paths, all four dominance boundaries coincide, and the dominance boundary forms a portion of an Apollonius circle. The remainder of the Apollonius circle is shown by the faint dotted line which gives the dominance boundary when neither player is affected by the obstacle. The other three dominance boundaries diverge from the Apollonius circle when the dominance boundary crosses a singular surface. The boundary labeled "a" represents the scenario where only the $\triangle$ is affected by the obstacle. Since the $*$ is unaffected, surface "1" is not a singular surface, and the dominance region agrees with the Apollonius circle until it reaches singular surface "2". As expected, since the $\triangle$ is the only player affected by the obstacle, this dominance region is the worst-case scenario for the $\triangle$, and it is the smallest of the four potential dominance regions.

When only the $*$ is affected by the obstacle, surface "1" is a singular surface, and the dominance boundary departs from the Apollonius circle when it crosses that surface. However, surface "2" is no longer a singular surface, so the dominance boundary does not deviate again until reaching the hyperbolic dispersal surface, "3", and this scenario leads to the dominance boundary labeled "c". Since the $*$ is the only player affected by the obstacle, this scenario is the best-case scenario for the $\triangle$, and it leads to the largest dominance region for the $\triangle$.

Finally, when both players are affected symmetrically by the obstacle, all six singular surfaces affect the solution, and the resulting dominance boundary is curve "b". As expected, this is an intermediate case in terms of the size of the dominance regions.

6. Games of kind: P3 game

In this section, the motivating example is solved in full using the methods developed in previous sections. First, a reduced version of the example is used which features all three players but no obstacle. A condition based on four dimensionless parameters is provided that determines the solution of the game. This result is also used to determine the solution to a larger game with n predators and m protectors. Finally, the motivating example is analyzed in full, and the effect of the obstacle is discussed. This section illustrates how dominance regions are used to solve games of kind, and games of degree are considered in Section 7.

6.1. Solution in the absence of obstacles

In the absence of obstacles, three Apollonius circles can be drawn using (1). Due to the transitivity of equality, any intersection between two of the circles must necessarily be an intersection of all three circles, and all of the dominance information can be obtained from any two of the Apollonius circles. Consider the protector/prey and predator/prey Apollonius circles. Place the origin at the prey and define the direction of zero azimuth as the line of sight from the prey to the protector. Then define the following four dimensionless parameters:

- γ_R: the ratio of speeds between protector and prey,
- γ_P: the ratio of speeds between predator and prey,
- α: the ratio of the initial distance between protector and prey to the initial distance between predator and prey,
- θ_P: the angle between the prey's lines of sight to the protector and the predator.

The solution to the game is as follows:

Theorem 8. *Given $\gamma_R > 1$, $\gamma_P > 1$, α, and θ_P, if $\exists\theta$ such that*

$$\alpha\,\frac{\gamma_P^2-1}{\gamma_R^2-1}\,\frac{-\cos(\theta)+\sqrt{\gamma_R^2-\sin^2(\theta)}}{-\cos(\theta-\theta_P)+\sqrt{\gamma_P^2-\sin^2(\theta-\theta_P)}}<1, \tag{13}$$

then the Blue team dominates the game of kind. If no such θ exists, then the Red team dominates the game of kind.

Proof. For a given angle θ, let the distance from the origin to the protector/prey Apollonius circle be r_R, and let the distance to the predator/prey Apollonius circle be r_P. These Apollonius circles can be expressed as:

$$r_R=\frac{-\cos(\theta)\pm\sqrt{\gamma_R^2-\sin^2(\theta)}}{\gamma_R^2-1}d_R, \tag{14}$$

$$r_P=\frac{-\cos(\theta-\theta_P)\pm\sqrt{\gamma_P^2-\sin^2(\theta-\theta_P)}}{\gamma_P^2-1}d_P, \tag{15}$$

where d_R is the initial distance between the prey and the protector, and d_P is the initial distance between the prey and the predator.

Eqs. (14) and (15) each provide two values of r for each θ. Since the prey is the slowest player, the dominance boundaries surround its initial location, so there are always one positive and one negative value of both r_R and r_P. Take the positive values, which correspond to the + sign in (14) and (15). Then the Blue team dominates if there exists a θ such that

$$r_R<r_P. \tag{16}$$

Since $r_P>0$, this is equivalent to

$$\frac{r_R}{r_P}<1. \tag{17}$$

Substitute (14) and (15) into (17), and substitute $\alpha = d_R/d_P$ to obtain (13). □

Consider the motivating example without an obstacle. The conditions are as follows:

- $\gamma_R = 2$,
- $\gamma_P = 3$,
- $\alpha = 1$,
- $\theta_P \approx 49°$.

Evaluating the left hand side of (13) yields a solution which is larger than 1 for all θ, and therefore the Red team dominates; that is, there is no location where the Blue team can rendezvous unless the predator acts suboptimally.

Theorem 8 follows from the Apollonius Circle Theorem, and indeed, Fig. 8 shows the same result. In Fig. 8, the predator/prey Apollonius circle is depicted with a dashed circle, and the protector/prey Apollonius circle is depicted with a solid circle. The protector/prey Apollonius circle lies entirely within the dominance region of the predator, and so there are no locations where the Blue team can rendezvous if the predator acts optimally.

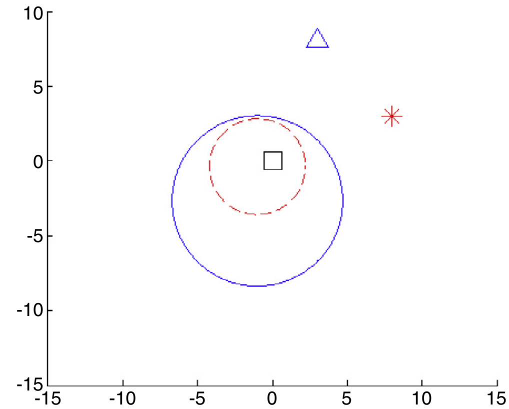


Fig. 8. P3 game with faster predator.

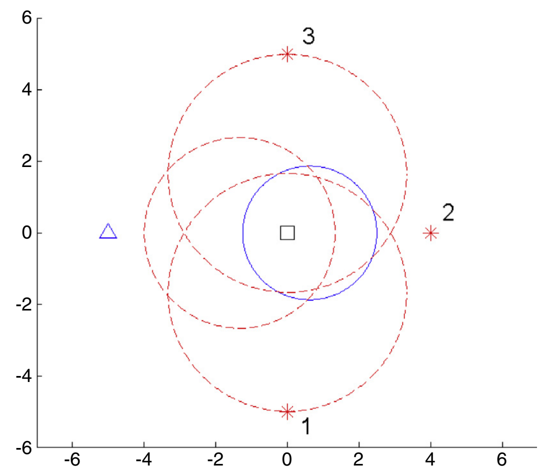


Fig. 9. P3 game with three predators.

This result easily generalizes to multiple players. Suppose there are n predators and m protectors. Condition (13) is evaluated at most mn times. If any protector is found to dominate all n predators, then the Blue team dominates the larger game, and the remaining protectors need not be evaluated. If all m protectors are dominated by at least one predator each, then the Red team dominates the larger game.

For example, consider the game in Fig. 9 where the prey is represented by □, the protector by △, and the three predators by $*$, with the predators numbered as shown. The parameters of the game are:

- $n = 3, m = 1$,
- $\gamma_R = 3$,
- $\gamma_{P1} = \gamma_{P2} = \gamma_{P3} = 2$,
- $d_R = d_{P1} = d_{P3} = 5, d_{P2} = 4$,
- $\theta_R = 0, \theta_{P1} = 90°, \theta_{P2} = 180°, \theta_{P3} = 270°$.

In this game, the Blue team dominates because for angles near $\theta = 0$, i.e., to the left in the figure, the protector and prey can rendezvous at a location that lies within the prey's dominance region for all three of the predator/prey Apollonius circles. Indeed, evaluating condition (13) for $\theta = 0$ and for each of the games involving a single predator vs. the protector yields:

$$\text{Predator 1}: \frac{\sqrt{3}}{4}<1, \tag{18}$$

$$\text{Predator 2}: \frac{5}{16}<1, \tag{19}$$

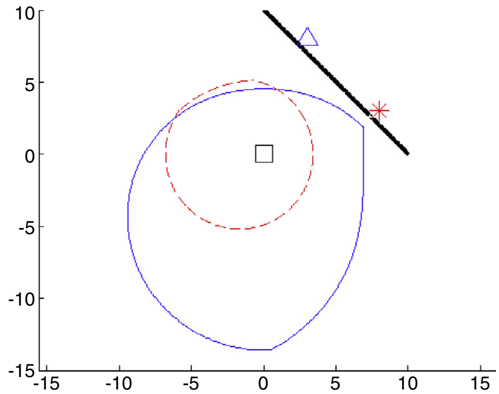


Fig. 10. P3 game with a line segment obstacle.

$$\text{Predator 3}: \frac{\sqrt{3}}{4} < 1. \tag{20}$$

Therefore, the protector dominates all individual predators, and the Blue team dominates the larger game.

6.2. Solution in the presence of obstacles

When the obstacle is introduced and the complete motivating example is considered, the solution is determined by constructing pairwise dominance regions as before, but here the dominance regions are constructed using Theorem 7 and Remark 1 instead of Apollonius circles. The solution of the game is as follows:

Theorem 9. *Given a* P3 *game with obstacles, if there exists a point where the predator is dominated by both the prey and the protector in their respective pairwise games, then the Blue team dominates the* P3 *game, and the Blue team can rendezvous at that point regardless of the actions of the Red team. If no such point exists, then the Red team dominates, and the Blue team can only rendezvous if the Red team acts suboptimally.*

Figs. 8 and 10 show the effect of an obstacle in the P3 game using the motivational example. The initial locations of the players are the same in both figures. The predator and the protector start at the same distance from the prey, but the predator is faster than the protector. In both figures, the dominance boundary for the predator/prey two player game is represented by the dashed line and the dominance boundary for the protector/prey game is represented by the solid line. In Fig. 10, the thick line represents the obstacle. The singular surfaces are not shown in these figures for clarity.

As stated previously, in Fig. 8, when there is no obstacle present, the predator dominates the game because for all directions that the prey could choose to travel, the predator can capture it before the protector rescues it. However, in the presence of the obstacle, a portion of the protector/prey dominance boundary lies within the prey's dominance region in the predator/prey game. The protector can therefore rescue the prey at any point along this section of the dominance boundary, and it is impossible for the predator to achieve capture first.

7. Games of degree and optimal strategies

As in Section 2.2, construction of the dominance boundary provides the complete solution to not only games of kind, but games of degree as well. If the payoff is the time to capture, then the solution is identical to Theorem 5; i.e., the optimal capture point, C, that minimizes the maximum time to capture (or respectively, maximizes the minimum), is the point on the dominance boundary with the longest minimum time path from E's initial location. The optimal strategies are such that both P and E travel to C in minimum time, though in this case P, E, and C are not necessarily co-linear, and the minimum time paths are not necessarily single line segments. If both players act optimally, then the dominance boundary at any intermediate point is tangent to the initial boundary at C. This is illustrated in Fig. 11(a), which shows a PE game with the same parameters as Figs. 3 and 5(a). The pursuer's initial location is given by the $\triangle$, and the evader's initial location is given by the $\square$. The minimum time paths are shown, and as expected, C remains in the same location as both players travel along these time-optimal paths to reach it. The dominance boundary is plotted for the initial time and three intermediate points in time, and as expected, all are tangent to the initial boundary at C.

Fig. 11(b) shows the same conditions as Fig. 11(a), except that in this case only the evader follows the optimal strategy. As before, the pursuer and evader begin at the points labeled P and E, respectively, leading to the initial dominance boundary marked DB with optimal capture point C. Here, the pursuer acts suboptimally and travels around the obstacle in the wrong direction. This causes E's path to curve as the minmax capture point moves, and E gains advantage due to P's suboptimal play. The locations of P and E at a later time are given by P' and E', and the dominance boundary at that time is marked DB'. Note that the final capture point, C', lies outside of the initial boundary, DB, and that capture at C' occurs at a later time than the minmax capture at C.

8. Conclusions and future work

8.1. Conclusions

This paper provides the dominance regions in pursuit–evasion games with obstacles, and two methods are presented for constructing these regions. The first method involves finding the intersection of bundles of isochrones, and the second method involves identifying singular surfaces that divide the plane into regions, and then determining closed form expressions for the solution in the small for each region. These methods are shown to agree with previous literature for scenarios without obstacles, but they are also able to analyze scenarios with obstacles, which the previous literature could not do.

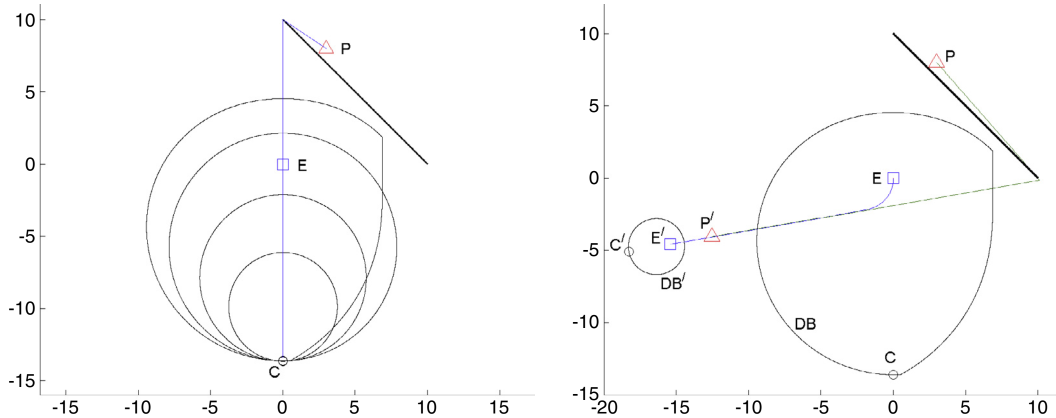
The effects of obstacles are studied by comparing the dominance regions in the presence of obstacles with the dominance regions in the absence of obstacles, and obstacles are shown to have potential benefits and drawbacks for both the faster and the slower players.

Finally, dominance regions are shown to provide the complete solution to PE games. A novel multiplayer pursuit–evasion game called the Prey, Protector and Predator Game is presented, and the methods provided in this paper are used to determine dominance criteria to solve the game of kind. Obstacles are shown to be able to change the outcome of the game. Similarly, in games of degree, dominance regions provide all of the information required to implement the optimal pursuit and evasion strategies.

8.2. Future work

For all of the games considered in this paper, the agents move with simple motion and are able to turn instantaneously. However, the method of intersecting isochrone bundles is not specific to these dynamics, and in future work, more complex models will be used for the agents. For example, turn radii will be bounded.

The P3 game can be extended to situations with partial information, such as the case where the prey's location is unknown to the predator at the beginning of the game. Another possibility is to restrict the communication abilities of the prey and protector.



(a) Optimal play.

(b) Suboptimal pursuit.

Fig. 11. Dominance regions during the game.

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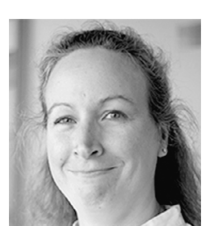

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