Nash Equilibrium in a Duopoly

Vibhanshu Singh

Introduction

Optimizing prices is not a new problem. Businesses need to change their prices regularly to maximize profits. With data businesses come to know certain empirical relations between their prices and sales. The market is very responsive to the prices set by a certain supplier or seller. The competitors keep a close eye on each other's prices to be quick in response to the changes in prices. This results in a rather complex interplay of sales and prices of competitors in a market situations. In this paper I will attempt to explore how, under some reasonable assumptions of market conditions and business behavior, two competitors would behave optimally to maximize their profits and gradually attain a Nash equilibrium.

Problem Statement

For easier contextualization, let's assume two competitors with any arbitrary product, for instance, tires.

There are two shops in a small town. Both of them sell tires of different brands. Let's pick one tire which sells the most in the town. Obviously both of them would want to sell this tire and keep an enough inventory of it. Now they want to optimize the prices of the tire such that they are able to make maximum profit out of it. Now the total profit will depend on how much is the profit they are making per tire times the total number they sell. Intuitively, they can't set the price too high as it would lead to very few sales. Also keeping the price too low would lead to very small profit per tire. Moreover, their sales will also depend on the competitor prices. If the competitor raises its price too high, people will tend to buy cheaper tire at the competitor's shop and their sales will increase and thus the profit. Introducing a competitor makes the problem more complex as the profit of one shop depends on the behavior of its competitor. So in this case both the competitors will closely observe the behavior of their respective competitors to modify their pricing strategy.

Assumptions

We will try to solve this problem in a very special case. Thus, it comes with assumptions which our model needs to know:

1. Two identical competitors. Both use the same model for optimizing their prices. This is not an arbitrary assumption. Usually when competitors are competing closely, they are aware of each other's strategies. They keep modifying their strategies until they are very similar to each other.

2. Both shops observe the sales on a price for a fixed period which is same for both. Without loss of generality, we will assume it to be a day. This period is required, as we would see, to learn how the sales vary with the prices.
3. There is a linear relationship between the sales of a shop and the prices of both the competitors:

   $$\mathbf{S_1 = a_1x_1 + b_1x_2 + c_1} \tag{1}$$

   $$\mathbf{S_2 = a_2x_2 + b_2x_1 + c_2} \tag{2}$$

   Where, $\mathbf{S_1}$ and $\mathbf{S_2}$ are the sales volume of shop 1 and shop 2 respectively in a day. $\mathbf{x_1}$ and $\mathbf{x_2}$ are the selling price of shop 1 and shop 2 respectively. $\mathbf{c}$ is a constant. Intuitively $\mathbf{a_1}$ and $\mathbf{a_2}$ should be negative coefficients as sales should drop with increasing selling prices of a shop. Moreover, $\mathbf{b_1}$ and $\mathbf{b_2}$ should be positive coefficients as the sales of a shop should increase with the increase in selling price of the competitor.
4. The shops have no prior knowledge of the coefficients of equation (1) and (2). They just know the equations in the above form.
5. With time and enough data, both competitors have come to know that the general market relation between average price and demand of the tire is given as below:

   $$\mathbf{D = Ax_{av} + B} \tag{3}$$

   Where $\mathbf{D}$ is the total demand. $\mathbf{A}$ is a non-positive coefficient due to the similar reasoning as $\mathbf{a}_1$ and $\mathbf{a_2}$. $\mathbf{B}$ is the demand when the average price is zero.
6. Cost price of each tire is the steady for each shop. Let us assume $\mathbf{k_1}$ and $\mathbf{k_2}$ to be the cost price of each tire at shop 1 and 2 respectively.
7. Shops have the access to only the daily prices set by their competitors. They don't know the cost price, sales or the coefficients of the equations (1) or (2) of the competitor.

Using above assumptions, we will proceed to solve this problem

Approach

If a shop knows its sales equation (1 or 2), then they can find out the expression for total profit in a day. Maximizing this expression of profit we get the optimal price of a shop as a function of its competitor's selling price, shop's own cost price and the coefficients of equations (1) or (2). As we will see soon, this comes out to be a linear relationship with respect to the competitor's selling price. This is obtained for both the shops, by both shops. The intersection of these two relations of each shops gives them their respective Nash equilibriums.

But in our case, the shops have no prior knowledge of coefficients of equations (1) and (2). So they will assume the coefficients of the equation randomly and then daily observe customer and competitor behavior to modify their coefficients. Once they know their own coefficients, finding the coefficients of the competitor is easy as we know the total market behavior. Once all the coefficients of equations (1) and (2) are known in this way, a shop can find the Nash Equilibrium if the cost price of competitor is known. The cost price is learnt iteratively along with other coefficients as we would see. Eventually, all the coefficients and cost price converge to their actual values and the actual Nash Equilibrium is obtained.

We will see how this iterative process will lead to setting of actual optimal prices if they both knew their sales equations.

Solution

We have to think about the problem from the perspective of each shop. None of them knows how their sales would vary with their own prices and their competitor prices.

We can break the problem in two parts. First part is to find out the equation of sales. And after finding the equation, setting the optimal price. First part can be solved by fitting a regression plane on daily observed data.

For example, shop 1 can use its own selling price $\mathbf{x_1}$ and competitor price $\mathbf{x_2}$ as inputs to linear regression model and the number of tires sold in that day as the output. After some days the data would be enough to predict the coefficients of equation 1 with a fair enough accuracy.

Before finding the equations, let's find out how the second step of finding the optimal prices would look like once we know the equations 1 and 2. Jumping to this step will make sense when we go back to first step from here.

Assuming both the shops know their sales equations (1) and (2) with fair enough accuracy, they can write the expression of their profit as below:

$$\mathbf{P_1 = (x_1-k_1)(S_1) = (x_1-k_1)(a_1x_1 + b_1x_2 + c_1)} \tag{4}$$

$$\mathbf{P_2 = (x_2-k_2)(S_2) = (x_2-k_2)(a_2x_2 + b_2x_1 + c_2)} \tag{5}$$

Differentiating each expression with respect to $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively, we get the optimal price on which the profit is maximum:

$$\mathbf{x_1 = [(k_1/2) – (c_1/2a_1)] – (b_1/2a_1)x_2} \tag{6}$$

$$\mathbf{x_2 = [(k_2/2) – (c_2/2a_2)] – (b_2/2a_2)x_1} \tag{7}$$

Now look at equation 6. It is a linear relation between $\mathbf{x_1}$ and $\mathbf{x_2}$. $\mathbf{(b_1/2a_1)}$ has a negative sign. We know that $\mathbf{b_1}$ is a positive coefficient and $\mathbf{a_1}$ is a negative coefficient from above discussion. Thus the actual sign of the coefficient of $\mathbf{x_2}$ is a positive term. Therefore, if $\mathbf{x_2}$ increases, $\mathbf{x_1}$ has to increase to be optimal. Figure 1 illustrates the two relations.

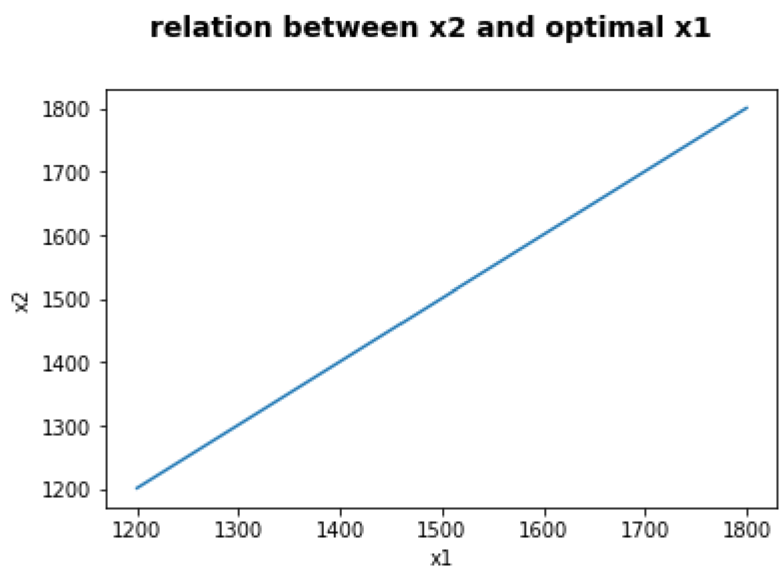

Figure 1

This is an arbitrary line for equation 6 with conditions of positive and negative coefficients as mentioned before. You can see that if $\mathbf{x_2}$ increases, $\mathbf{x_1}$ has to increase. On this line shop1 gets the value for $\mathbf{x_1}$ for each value of $\mathbf{x_2}$ such that it gains maximum profit given $\mathbf{x_2}$. This is how the behavior of shop1 changes depending on the behavior of shop 2. The same condition is with shop2. Shop 2 also has an optimal line on which it gets the maximum profit. We will see now that if these lines aren't parallel, then their intersection is the Nash Equilibrium.

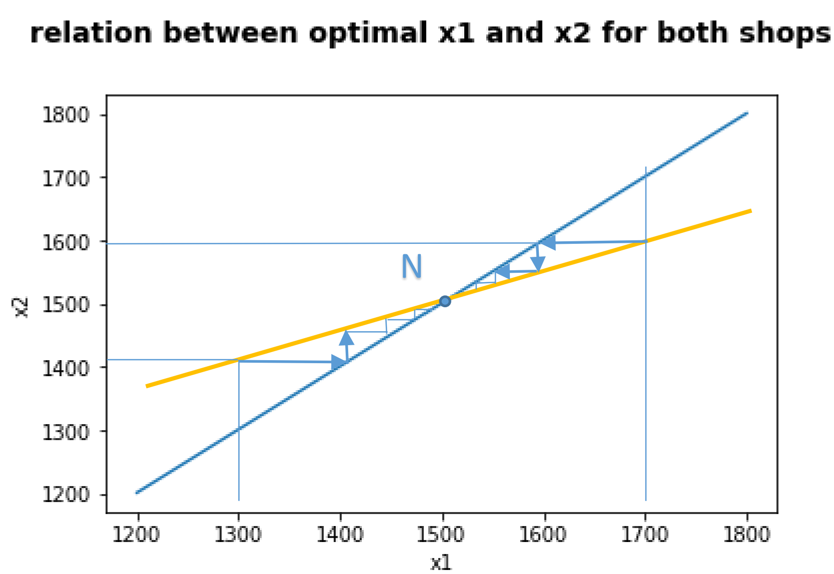

Figure 2

Here the blue line is for the shop 1 for its price optimization. The yellow line is for shop 2. Suppose on day 1, shop 1 sets its selling price $\mathbf{x_1}$ as 1300 as shown in the figure 2. Looking at the price of shop 1, the best price for shop 2 is 1400. So shop 2 will set its selling price at 1400. But then, for shop 1, the new best price changes to nearly 1400. This again changes the optimal price for shop 2. This will continue until both the shop prices converge to the point N. The same would happen if the initial point would start from the right of the point N, which is also illustrated in Figure 2.

This is the Nash Equilibrium in this scenario. Nash Equilibrium can be defined as following:

A stable state of a system involving the interaction of different participants, in which no participant can gain by a unilateral change of strategy if the strategies of the others remain unchanged.

We can see that no shop has any incentive on this point to change their prices as it would only reduce its profit. So, if a shop knows both the sales equations 1 and 2, then they can find out the Nash Equilibrium point N and set its price according to it.

But there are more problems here. First, the prices are converging here to the intersecting point. This is not always the case with two intersecting lines. The graph in figure 2 is drawn from the perspective of shop 1 as it has $\mathbf{x_1}$ on the x axis. From this perspective, it can be shown that any two blue and yellow lines will lead to a convergence if the slope of blue line is greater than the yellow line. Similarly, from the perspective of shop 2, the axes switch with each other, and the yellow line's slope becomes greater than the blue line and it will also lead to the convergence. Second, we don't know why the lines will always have a solution, i.e. they always intersect at one point. Third, we don't know how one shop will find the equation (6) or (7) of the other shop.

Knowing equations (6) or (7) of the other shop is equivalent to knowing the coefficients occurring in the equation. From the perspective of shop 1, it needs to know coefficients $\mathbf{a_2}$, $\mathbf{b_2}$, $\mathbf{c_2}$ and $\mathbf{k_2}$. We can use the overall market equation (3) to find out $\mathbf{a_2}$, $\mathbf{b_2}$ and $\mathbf{c_2}$ as soon as shop 1 knows its own coefficients $\mathbf{a_1}$, $\mathbf{b_1}$ and $\mathbf{c_1}$:

$$\mathbf{S_1 = a_1x_1 + b_1x_2 + c_1} \quad (1)$$
$$\mathbf{S_2 = a_2x_2 + b_2x_1 + c_2} \quad (2)$$

Using equation (3), we can replace **D** with $\mathbf{S_1 + S_2}$, as this would be the total tires sold in a day in that region. Moreover we can replace $\mathbf{x_{av}}$ with $\mathbf{(x_1 + x_2)/2}$

$$\mathbf{S_1 + S_2 = Ax_{av} + B}$$

- ⇨ $\mathbf{(a_1x_1 + b_1x_2 + c_1) + (a_2x_2 + b_2x_1 + c_2) = A(x_1 + x_2)/2 + B}$
- ⇨ $\mathbf{a_2x_2 + b_2x_1 + c_2 = ((A/2) - b_1)x_2 + ((A/2) - a_1)x_1 + (B - c_1)}$
- ⇨ $\mathbf{a_2 = (A/2) - b_1,\ b_2 = (A/2) - a_1,\ c_2 = B - c_1}$ (8)

Thus, shop 1 now knows equation (2). But it still doesn't know equation (7) as $\mathbf{k_2}$ is still unknown. We will see how we can estimate $\mathbf{k_2}$ soon.

For the first problem of converging prices, we want to make sure that both the slopes are positive and the slope of blue line in with respect to shop 1 is larger than the slope of yellow line. From equation (6) and (7) we can get the two slopes after converting the two equations to the form **y = mx + c**, where **m** would be the slope:

$$\mathbf{m_1 = -(2a_1/b_1)} \quad (9)$$

$$\mathbf{m_2 = -(b_2/2a_2)} \quad (10)$$

Where subscript $_1$ is for the blue line and $_2$ for the yellow line. Here we can see that both $\mathbf{m_1}$ and $\mathbf{m_2}$ are positive. After substituting $\mathbf{b_2}$ and $\mathbf{a_2}$ from equation (8), we can compare $\mathbf{m_1}$ and $\mathbf{m_2}$:

$$\mathbf{m_2 = -((A/2) - a_1)/2((A/2) - b_1)}$$

Putting $\mathbf{m_2 < m_1}$ and modifying the inequality, we get

$$\mathbf{A < (-6a_1b_1)/(b_1 - 4a_1)}$$

Right side is a positive value and left side is negative. Thus, $\mathbf{m_2 < m_1}$ is always true and hence the prices will converge at Nash Equilibrium. This also solves the problem of intersection. Since the slopes $\mathbf{m_1}$ and $\mathbf{m_2}$ are different, the lines will definitely intersect. This point can be obtained by solving equations (6) and (7) for $\mathbf{x_2}$ and $\mathbf{x_1}$.

$$\mathbf{x_1 = [(k_1/2) - (c_1/2a_1) - (k_2b_1/4a_1) + (c_2b_1/4a_2a_1)]/[1 - (b_2b_1/4a_2a_1)]} \quad (11)$$

$$\mathbf{x_2 = [(k_2/2) - (c_2/2a_2) - (k_1b_2/4a_2) + (c_1b_2/4a_1a_2)]/[1 - (b_1b_2/4a_1a_2)]} \quad (12)$$

From above discussion, we found out how shop 1 can find out the coefficients of equation 2 of shop 2 once it knows its own coefficients. With similar reasoning shop 2 can find out the coefficients of equation 1. Now they will both find the Nash equilibrium point using equations (11) and (12), which may or may not be identical as it depends on coefficient **k** of their competitor shop. Shop 1 will obtain a Nash Equilibrium point $\mathbf{(x_1^1, x_2^1)}$, where the superscript ^{1} is for the price predicted by shop 1. Similarly Nash Equilibrium point predicted by shop 2 is $\mathbf{(x_1^2, x_2^2)}$**.** The subscript ^{2} is not to be confused with the exponent. Shop 1 and Shop 2 set $\mathbf{x_1^1}$ and $\mathbf{x_2^2}$ as their prices for next day respectively. Prices are public, so they know each other's selling prices. But at the same time, $\mathbf{x_1^2}$ is hidden from shop 1 and $\mathbf{x_2^1}$ is hidden from shop 2. They can use the information available to find out **k**. Let $\mathbf{(k_1^2, k_2^1)}$ be the cost prices of shops 1 and 2 as assumed by shops 2 and 1 respectively**.** We can re-write equations (11) and (12) using this notation as follows from the perspectives of two shops:

Shop 1:

$$\mathbf{x_1^1 = [(k_1/2) - (c_1/2a_1) - (k_2^1b_1/4a_1) + (c_2b_1/4a_2a_1)]/[1 - (b_2b_1/4a_2a_1)]} \quad (11.1)$$

$$\mathbf{x_2^2 = [(k_2^1/2) - (c_2/2a_2) - (k_1\,b_2/4a_2) + (c_1b_2/4a_1a_2)]/[1 - (b_1b_2/4a_1a_2)]} \tag{12.1}$$

Shop 2:

$$\mathbf{x_1^1 = [(k_1^2/2) - (c_1/2a_1) - (k_2b_1/4a_1) + (c_2b_1/4a_2a_1)]/[1 - (b_2b_1/4a_2a_1)]} \tag{11.2}$$

$$\mathbf{x_2^2 = [(k_2/2) - (c_2/2a_2) - (k_1^2\,b_2/4a_2) + (c_1b_2/4a_1a_2)]/[1 - (b_1b_2/4a_1a_2)]} \tag{12.2}$$

Now, shop 1 knows $\mathbf{x_2^2}$ so it would want to apply equation (12.1) to find $\mathbf{k_2^1}$**.** But this might not be the actual $\mathbf{k_2}$ as Shop 2 has used equation 12.2 to find $\mathbf{x_2^2}$ which does not contain $\mathbf{k_2^1}$**.**

We see a problem here. Estimation of $\mathbf{k_2}$ by shop 1 depends recursively on the previous estimation of $\mathbf{k_2}$. Similarly the estimation of $\mathbf{k_1}$ by shop 2 depends on $\mathbf{k_2^1}$. All the other variables are obtained through regression fitting on daily data. These cost price variables can be obtained iteratively using the following algorithm if we can show that it converges:

Algorithm 1 (shop 1):

Assume $\boldsymbol{k_2^1}$*. Let's call it* $\boldsymbol{k_2^{1(old)}}$
For i in range (0, A):
    observe $\boldsymbol{x_2^2}$
    apply:
    $\mathbf{x_1^1 = [(\mathit{k_1}/2) - (c_1/2a_1) - (\mathit{k_2^{1(old)}}b_1/4a_1) + (c_2b_1/4a_2a_1)]/[1 - (b_2b_1/4a_2a_1)]}$
    $\mathbf{\mathit{k_2^{1(new)}} = 2x_2^2[1 - (b_1b_2/4a_1a_2)] + (k_1b_2/2a_2) - (c_1b_2/2a_1a_2) + (c_2/a_2)}$
    $\boldsymbol{k_2^{1(old)} = k_2^{1(new)}}$

Proof of Convergence of Algorithm 1

Now we just need to prove that the above algorithm converges in our case. We can generalize the above case with a general set of equations shown below and find the condition in which the solution obtained by using above algorithm converges:

$$\mathbf{x_1 = p_1y_1 + p_2y_2 + p_3} \tag{13}$$

$$\mathbf{x_2 = q_1y_1 + q_2y_2 + q_3} \tag{14}$$

Now imagine that there are two shops which are using above expressions to find their respective **x.** Shop 1 knows the value of $\mathbf{y_1}$ and shop 2 knows the value of $\mathbf{y_2}$**.** After calculating their values of **x** they share it with each other. Now they apply their version of Algorithm 1 as follows:

Shop 1

Assume $\mathbf{y_2}$*. Let's call the assumption on* r^{th} *day as* $\mathbf{y_2^r}$*.* $\mathbf{y_2^0}$ *be the assumption before starting the algorithm*
For $\mathbf{r}$ *in range (0, A):*
    observe $\mathbf{x_2}$
    apply:
$$\mathbf{x_1 = p_1y_1 + p_2\, y_2^r + p_3} \tag{A1.1}$$
$$\mathbf{y_2^{r+1} = (x_2 - q_1y_1 - q_3)/q_2} \tag{A1.2}$$

Shop 2

Assume $\mathbf{y_1}$*. Let's call the assumption on* r^{th} *day as* $\mathbf{y_1^r}$*.* $\mathbf{y_1^0}$ *be the assumption before starting the algorithm*
For $\mathbf{r}$ *in range (0, A):*
    observe $\mathbf{x_1}$
    apply:
$$\mathbf{x_2 = q_1\, y_1^r + q_2\, y_2 + q_3} \tag{A1.3}$$
$$\mathbf{y_1^{r+1} = (x_1 - p_2y_2 - p_3)/p_1} \tag{A1.4}$$

At any arbitrary day $\boldsymbol{r}$, shop 1 applies equation (A1.1) to find its selling price

$$\mathbf{x_1 = p_1y_1 + p_2\, y_2^r + p_3}$$

Then it applies (A1.2) to estimate $\boldsymbol{y_2^{r+1}}$:

$$\mathbf{y_2^{r+1} = (x_2 - q_1y_1 - q_3)/q_2}$$

This $\mathbf{x_2}$ is the one which shop 2 estimated after assuming $\boldsymbol{y_1^r}$ by the equation (A1.3). So, we can substitute $\mathbf{x_2}$ in equation (A1.2) to get:

$$\mathbf{y_2^{r+1} = ((q_1\, y_1^r + q_2\, y_2 + q_3) - q_1y_1 - q_3)/q_2} \tag{15}$$

Notice that $\mathbf{y_1^r}$ is not known by Shop 2. But the purpose here is to show that above relation holds between $\mathbf{y_2^{r+1}}$ and $\mathbf{y_1^r}$.

Similarly, Shop 1 finds the relation:

$$\mathbf{y_1^{r+1} = ((p_1\, y_1 + p_2\, y_2^r + p_3) - p_2y_2 - p_3)/p_1} \tag{16}$$

$\boldsymbol{y_1^r}$ in equation (15) can be obtained from equation (16) after replacing $\boldsymbol{r+1}$ with $\boldsymbol{r}$.

After substituting, we get the following relation between $\boldsymbol{y_2^{r+1}}$ and $\boldsymbol{y_2^{r-1}}$:

$\mathbf{y_2^{r+1} = ((q_1p_2)/(p_1q_2))(y_2^{r-1} - y_2) + y_2}$ **(17)**

It can be shown now that a series of the above recursive form converges if

$\mathbf{-1<((q_1p_2)/(p_1q_2))<1}$

Now we can substitute the values of $\mathbf{p_1, p_2, q_1, q_2}$ from equations (11) and (12) to obtain:

$\mathbf{((q_1p_2)/(p_1q_2)) = ((b_1b_2)/(4a_1a_2))}$ **(18)**

Substituting $\mathbf{a_2}$ and $\mathbf{b_2}$ from equation (8), it can be easily seen that $\mathbf{((b_1b_2)/(4a_1a_2)) < 1}$**.**

Thus, using Algorithm 1, the cost prices converge to their actual values and both shops find the real cost price of their competitor.

All of the above analysis is done assuming that both the shops know their coefficients **a, b** and **c.** But they would actually find them by fitting a linear regression plane with input variables $\mathbf{x_1}$ and $\mathbf{x_2}$ and output $\mathbf{S_1}$. The actual $\mathbf{S_1}$ is found out daily at the end of the day. Thus everyday variables $\mathbf{a_1}$**,** $\mathbf{b_1}$ and $\mathbf{c_1}$ are modified. Thus, after some observations, these three variables will be known with some certainty. Similarly, shop 2 will find out variables $\mathbf{a_2}$**,** $\mathbf{b_2}$ and $\mathbf{c_2}$.

Algorithm 2 (shop 1):

Assume a_1, b_1, c_1, and k_2 consistent with signs
For i in range (0, A):

1. *Find variables a_2, b_2 and c_2 using equation 8*
2. *Find Nash Equilibrium price x_1 using equation (11.1). Use current value of k_2 in the equation*
3. *Observe selling price x_2 of Shop 2.*
4. *Use equation (12.1) to re-estimate k_2*
5. *Observe S_1 for the current day*
6. *Run regression on historical data till date with input variables x_1, x_2 and output S_1*
7. *Modify a_1, b_1 and c_1*
8. *Stop if converged*

Find N with known coefficients (a_1, b_1, c_1, k_1, a_2, b_2, c_2 and k_2)

"A" in the above algorithm in the For loop is the model parameter. It should be chosen depending on the noise in data and when user is satisfied that regression is not leading to much change in variables.

Implementation

A simulation of above algorithm was performed in python with the actual value of coefficients as following:

$\mathbf{a_1}$ **=** -0.055
$\mathbf{b_1}$ **=** 0.058
$\mathbf{c_1}$ **=** 100
A = 0
B = 200
$\mathbf{a_2 = -b_1}$
$\mathbf{b_2 = -a_1}$
$\mathbf{c_2 = B - c_1}$

Actual cost prices were set as:

$\mathbf{k_1}$ **=** 1200
$\mathbf{k_2}$ **=** 1100

In the simulation of the algorithm, coefficients $\mathbf{a_1}$ and $\mathbf{b_1}$ are randomly assumed between 0.01 and 0.1. $\mathbf{c_1}$ is randomly assumed between 90 and 110. $\mathbf{k_2}$ was assumed to be equal to $\mathbf{k_1}$ by shop 1 and vice versa. Then steps 1 to 4 were ran for both the shops alternatively. Observed value of sales in step 5 was found using equations (1) and (2) using the actual values of coefficients with added normal noise with mean equal to S_1 or S_2 and standard deviation 1. Then regression plane was fitted for both the shops and coefficients were re-estimated. The value of the hyper parameter "A" was taken to be 100 but the algorithm converged within 10 steps.

The actual Nash Equilibrium that had to be reached according to the actual coefficients was:
$\mathbf{x_1}$**(actual)** = 3004.85
$\mathbf{x_2}$**(actual)** = 2836.78

The values of $\mathbf{x_1}$ and $\mathbf{x_2}$ obtained after convergence:
$\mathbf{x_1}$**(predicted)** = 2911.37
$\mathbf{x_2}$**(predicted)** = 2795.10

The values of other variables predicted using the algorithm:
$\mathbf{a_1}$ **(predicted)=** -0.055
$\mathbf{b_1}$ **(predicted)=** 0.057
$\mathbf{c_1}$ **(predicted)=** 99.34
$\mathbf{k_1}$ **(predicted)=** 1067.08
$\mathbf{k_2}$ **(predicted)=** 1061.87

The actual graph of convergence obtained after the algorithm converged is shown below:

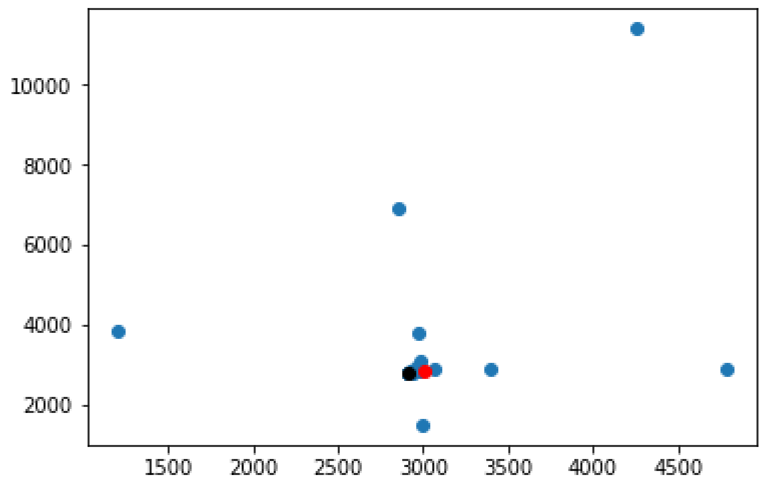

Figure 3

Where, the x-axis shows the selling price of Shop 1 and y-axis the selling price of Shop 2 at each iteration. The red dot is the actual Nash equilibrium. The black dot shows the last point at the end of 50^{th} iteration. So we can see that the converged point is quite close to the actual Nash equilibrium.