MODULE-1
SYSTEM OF LINEAR EQUATIONS

MAT3004

APPLIED LINEAR ALGEBRA

VIT

Topics to be discussed

System of Linear Equations:

- Gaussian elimination
- Gauss Jordan methods
- Elementary matrices
- Permutation matrix
- Inverse matrices
- System of linear equations
- LU factorizations
- LDU factorizations

List of Definition and terms

- Linear equation (LE)
- System of linear equation (SLE)
- Matrix representation of a SLE ($AX = B$)
- Homogeneous and nonhomogeneous system
- Solution, Trivial solution, Unique solution, Infinite solution, No solution
- Consistency and Inconsistency of a system
- Augmented matrix, variable matrix, solution matrix
- Pivot element or key element
- Rows and Columns of a matrix, Elimination and reduction of a rows and columns
- 10. Forward and backward elimination
- 11. Row equivalent, Row reduction of 1^{st}, 2^{nd}, 3^{rd} kinds
- ECHELON FORM of a matrix
- Triangular form of a matrix (Upper and Lower triangular form)
- Elementary matrix, Permutation matrix
- Block matrix, submatrix
- Inverse of a matrix, left and right inverse, invertible, non-singular, singular

Some problem types

1. Solve the following (given) system of equations using
   - Gauss Elimination method
   - Gauss Jordan Elimination method
   - Matrix inversion method
   - LU factorization method
   - LDU factorization method
2. Find the conditions for the consistency of a (given) system $A\boldsymbol{X} = B$.
3. Find the values of a, b for which a (given) system $A\boldsymbol{X} = B$ has infinite solutions, no solution, unique solution.
4. Reduce the matrix (given) to Echelon form.
5. Find the inverse of a (given) matrix by Gauss Jordan method.
6. Find the LU and LDU decomposition of a given matrix.
7. Find the elementary matrices in I_n.
8. Write the matrix A as a product of elementary matrices.

Some Theorems

1. If a $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal i.e. $B = C$. **Proof required**.
2. The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reversed order i.e. $(AB)^{-1} = B^{-1}A^{-1}$. **Proof required**.
3. Let A be a $n \times n$ matrix. The following are equivalent :
   a) A has a left inverse;
   b) $A\boldsymbol{X} = \mathbf{0}$ has only the trivial solution $\boldsymbol{X} = \mathbf{0}$;
   c) A is row equivalent to I_n;
   d) A is a product of elementary matrices;
   e) A is invertible;
   f) A has a right inverse. **Proof required.**

4. A triangular matrix is invertible if and only if it has no zero diagonal entry.

Statement only without proof.

Linear equation:-

Consider the equation $a_1x_1+a_2x_2+\cdots a_nx_n=b$ — (1)

where $a_1, a_2, \ldots a_n$ and b are constants is called linear equation.

System of linear equation:- (Group of linear equations)

A general system of m linear equations with n unknowns can be written as

$$\left.\begin{array}{l}
a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots a_{1n}x_n=b_1\\
a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots a_{2n}x_n=b_2\\
a_{3}\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\cdots a_{mn}x_n=b_m
\end{array}\right\} \longrightarrow (2)$$

where $x_1, x_2, \ldots x_n$ are the unknowns,

$a_{11}, a_{12}, \ldots a_{mn}$ are the coefficients of the system

and $b_1, b_2, \ldots b_m$ are the constant terms.

The system ② is said to be <u>homogeneous</u> if all the constant terms are zero, i.e. $b_1 = b_2 = \ldots b_m = 0$

otherwise ② is known as <u>non-homogeneous</u>.

Every homogeneous eq/sol has trivial solution.

$\Downarrow$ all sol = 0

$3x - 5y = 0 \Rightarrow x = 0, y = 0 \Rightarrow 0 = 0$

In general, a linear system may have in any one of three possible ways:

i) The system has a single (unique) solution.

ii) The system has infinitely many solution.

iii) The system has no solution.

The system is ② is said to be <u>consistent</u> if it has atleast one solution and <u>inconsistent</u> if it has no solution.

The system ② can be expressed as

$AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

$$[A : B] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & b_m \end{array}\right]$$

is called argumented matrix.

Elementary operations :-

The following operations on a argumented matrix (system of linear equations) are called elementary operations

i) Interchange two rows (equations)

i.e. $R_i \longleftrightarrow R_j$ ($l_i \longleftrightarrow l_j$)

ii) Multiply a non-zero constant throughout a row (an equation)

i.e. $R_i \longrightarrow aR_i$ ($l_i \rightarrow a l_i$) $a \neq 0$

iii) Add a constant multiple of an equation to another equation

i.e. $R_i \rightarrow aR_i + R_j$ (or) $R_j \rightarrow aR_j + R_i$

Row- echelon form of a matrix :- (Gauss elimination metho

Reduced row-echelon form (Gauss Jordan elimination)

A matrix is said to be in row-echelon form if it satisfies the following:

(i) The zero rows, if they exist, come last in the ordered rows.

(ii) The first non-zero entries, in the non-zero rows are 1, called leading ones.

(iii) In each column containing a leading non-zero element, the entries below that leading non-zero element are 0.

The reduced row-echelon form of an argumented matrix is of the form:

(iv) Above each leading is a column of zero's in addition to the row-echelon form

Gauss - elimination method :-

The gauss - elimination algorithm is as follows : (i) Write the argumented matrix of the system of linear equations.

(ii) Find an echelon form of the argumented matrix using elementary row operations.

(iii) Write the system of equations corresponding to the echelon form.

(iv) Use back substitution to get the solution.

Pivot element :-

The left most non-zero entries in the non-zero rows are called pivots.

Example: Solve the system of linear equations $2y+4z=2$, $x+2y+2z=3$, $3x+4y+6z=-1$ and write the pivots.

Sol:- The system of linear equations

$$0.x+2y+4z=2$$
$$x+2y+2z=3$$
$$3x+4y+6z=-1$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/4 \end{array}$$

pivots of $x=1$, $y=2$, $z=4$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

[Back substitution method]

$\Rightarrow \quad 1.x + 2y + 2z = 3$ —— ①

$\Rightarrow \quad 0x + y + 2z = 1$ —— ②

$0x + 0y + z = -2$ —— ③

From equation ③, we have $\boxed{z = -2}$

∴ sub $z = -2$ in equation ②, we get

$$y + 2(-2) = 1$$

$$\boxed{y = 5}$$

∴ Substitute $y = 5$, $z = -2$ in equation ①, we get

$$x + 10 - 4 = 3$$

$$\boxed{x = -3}$$

∴ $x = -3$

$y = 5$

$z = -2$

1. Solve the following system of equations by gaussian elimination. What are the pivots

(i) $-x+y+2z=0$, $3x+4y+z=0$, $2x+5y+3z=0$

(ii) $2y-z=1$, $4x-10y+3z=5$, $3x-3y=6$

(iii) $w+x+y=3$, $-3w-17x+y+2z=1$, $4w-17x+8y-5z=1$, $-5x-2y+z=1$

(iv) $x_1+2x_2+3x_3+2x_4=-1$, $-x_1-2x_2-2x_3+x_4=2$, $2x_1+4x_2+8x_3+12x_4=4$

(i) Sol :- The system of equations

$$-x+y+2z=0$$
$$3x+4y+z=0$$
$$2x+5y+3z=0$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix}$ $R_1 \to R_1 \times (-1)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2/7$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x - y - 2z = 0$ —— ①

$y + z = 0$ —— ②

Let $\boxed{z = t}$ sub in equ ②

$y + t = 0$

$\boxed{y = -t}$

Substitute $y = -t$, $z = t$ in equation ①

$x + t - 2t = 0$

$x = 2t - t$

$\boxed{x = t}$

$\therefore$ $x = t$

$y = -t$

$z = t$

2. Determine the condition on b_i. so that the following system has a solution.

(i) $x+2y+6z=b_1$, $2x-3y-2z=b_2$, $3x-y+4z=b_3$

(ii) $x+3y-2z=b_1$; $2x-y+3z=b_2$, $4x+2y+z=b_3$

(i) Sol: The system of equations

$$x+2y+6z=b_1$$

$$2x-3y-2z=b_2$$

$$3x-y+4z=b_3$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 2 & -3 & -2 & b_2 \\ 3 & -1 & 4 & b_3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & -7 & -14 & b_3-3b_1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix} R_2 \to R_2/-7$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & 1 & 1 & \frac{1}{7}(2b_1-b_2) \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

Since $\rho(A)=2$, $\rho(A \vdots B)=2$ if $b_3-b_2-b_1=0$

$\therefore \rho(A)=2=\rho(A \vdots B) <$ no of variables (3)

$\therefore$ The given system has an infinite no of solution when $b_3-b_1-b_2=0$

(ii) <u>Sol</u> : The system of equations

$$x+3y-2z=b_1$$
$$2x-y+3z=b_2$$
$$4x+2y+z=b_3$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 2 & -1 & 3 & b_2 \\ 4 & 2 & 1 & b_3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & -7 & 7 & b_2-2b_1 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix}$$

$R_2 \to R_2/-7$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix} R_3 \to R_3 + 10R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & 0 & -1 & b_3-10b_2+16b_1 \end{bmatrix}$$

Since $\rho(A)=3$, $\rho(A:B)=3$

$\therefore \rho(A) = 3 = \rho(A:B) =$ no of variables(3)

$\therefore$ The given system has an unique solution.

1. (ii) Sol: The system of equations

$$2y - z = 1$$
$$4x - 10y + 3z = 5$$
$$3x - 3y = 6$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 3 & -3 & 0 & 6 \end{bmatrix}$ $R_3 \to R_3/3$

$$\sim \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 1 & -1 & 0 & 2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_2 \to R_2 - 4R_1.$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -6 & 3 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_2 \to R_2/3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x - y = 2$ —— ①

$y - \frac{1}{2}z = \frac{1}{2}$ —— ②

let $\boxed{z = t}$ substitute in equation ②

$$y - \frac{1}{2}t = \frac{1}{2}$$

$$\frac{2y - t}{2} = \frac{1}{2}$$

$$2y - t = 1$$

$$2y = 1 + t$$

$$\boxed{y = \frac{1+t}{2}}$$

Put $y = \frac{1+t}{2}$, $z = t$ substitute in equation ①

$$x - \frac{1+t}{2} = 2$$

$$x = 2 + \frac{1+t}{2}$$

$$\boxed{x = \frac{5+t}{2}}$$

$$\therefore \quad x = \frac{5+t}{2}$$

$$y = \frac{1+t}{2}$$

$$z = t$$

(iii) Sol: The system of equations

$$w + x + y = 3$$

$$-3w - 17x + y + 2z = 1$$

$$4w - 17x + 8y - 5z = 1$$

$$-5x - 2y + z = 1$$

Argumented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ -3 & -17 & 1 & 2 & 1 \\ 4 & -17 & 8 & -5 & 1 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & -14 & 4 & 2 & 10 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$R_4 \to 7R_4 + 5R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & -2 & -8 & -26 \\ 0 & 0 & -24 & 2 & -18 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3/-2 \\ R_4 \to R_4/-2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & +1 & 4 & 13 \\ 0 & 0 & +12 & -1 & 9 \end{bmatrix} \quad \begin{array}{l} \cancel{R_4 \to R_4} \\ R_4 \to R_4 - 12R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & -49 & -147 \end{bmatrix}$$

-12×4

-48

$\Rightarrow \quad w + x + y + 0z = 3 \quad \text{——} ①$

$0w + 7x - 2y - z = -5 \quad \text{——} ②$

$y + 4z = 13 \quad \text{——} ③$

$-49z = -147 \quad \text{——} ④$

$\boxed{z = 3}$

Put $z = 3$ in equation ③

$y + 12 = 13$

$\boxed{y = 1}$

Put $z = 3, y = 1$ in equation ②

$7x - 2 - 3 = -5$

$7x - 5 = -5$

$\boxed{x = 0}$

Substitute $x = 0, y = 1, z = 3$ in equation ①

$$w+0+1=3$$

$$\boxed{w=2}$$

$\therefore$ $w=2$

$x=0$

$y=1$

$z=3$

(iv) <u>Sol</u>: The system of equations

$$x_1+2x_2+3x_3+2x_4=-1$$

$$-x_1-2x_2-2x_3+x_4=2$$

$$2x_1+4x_2+8x_3+12x_4=4$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} R_3 \to R_3/2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 1 & 2 & 4 & 6 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 + 2x_4 = -1 \quad \text{—— (1)}$$

$$x_3 + 3x_4 = 1 \quad \text{—— (2)}$$

$$\boxed{x_4 = 2}$$

Substitute $x_4 = 2$ in equation (2)

$$x_3 + 6 = 1$$

$$\boxed{x_3 = -5}$$

Substitute $x_3 = -5$, $x_4 = 2$ in equation (1)

$$x_1 + 2x_2 - 15 + 4 = -1$$

$$x_1 + 2x_2 = +10$$

Let $\boxed{x_2 = t}$

$$x_1 + 2t = 10$$

$$\boxed{x_1 = 10 - 2t}$$

Note:-

(i) If $\rho(A) = \rho(A \mid B)$ = number of variables, then the system has a unique solution

(ii) If $\rho(A) = \rho(A \mid B)$ < number of variables, then the system has an infinite number of solutions

(iii) If $\rho(A) \neq \rho(A \mid B)$, then the system has no solution

1. Determine all values of b_i that make the following system $x+y-z=b_1$, $2y+z=b_2$, $y-z=b_3$ consistent

2. Determine the condition b_i so that the following system has no solution $2x+y+7z=b_1$, $6x-2y+11z=b_2$,

$2x - y + 3z = b_3$

3. Which of the following system has a non-trivial solution.

(i) $x + 2y + 3z = 0$

$2y + 2z = 0$

$x + 2y + 3z = 0$

(ii) $2x + y - z = 0$

$x - 2y - 3z = 0$

$3x + y - 2z = 0$

4. For which values of "a" does each of the following system have no solution, exactly one solution or infinitely many solution

(i) $x + 2y - 3z = 4$, $3x - y + 5z = 2$, $4x + y + (a^2 - 14)z = a + 2$

(ii) $x - y + z = 1$, $x + 3y + az = 2$, $2x + ay + 3z = 3$

4.(i)

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

$R_2 \to R_2/-7$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

i) If $a^2-16=0$ and $a-4=0$ then $\rho(A)=2$, $\rho(A|B)=2$

i.e $a=\pm 4$ then the given system has an infinite number of solutions $[\because \rho(A)=2=\rho(A|B)<3]$

ii) If $a^2-16\neq 0$ and $a-4\neq 0$ then $\rho(A)=3$, $\rho(A|B)=3$

i.e $a\neq\pm 4$ then the given system has an unique solution $(\because \rho(A)=3=\rho(A|B)=$ no of unknown$)$

iii) If $a=-4$, then the given system has no solution

*** Gauss - Jordan elimination :-

(i) Write the argumented matrix for the given system of linear equations.

(ii) Derive the reduced row-echelon form for the argumented matrix by using elementary row operations

(iii) Write the system of equations corresponding to the reduced row-echelon form. This system gives the solution.

Solve the following system of linear equations by gauss jordan elimination

$x_1+3x_2-2x_3=3$, $2x_1+6x_2-2x_3+4x_4=18$, $x_2+x_3+3x_4=10$

Argumented matrix $[A \mid B] = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix} R_3 \rightarrow R_3/2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} R_1 \rightarrow R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & -4 & -27 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 0x_2 + 0x_3 + x_4 = 3$ ——— (1)

$0x_1 + x_2 + 0x_3 + x_4 = 4$ ——— (2)

$0x_1 + 0x_2 + x_3 + 2x_4 = 6$ ——— (3)

$x_1 + x_4 = 3$ ——— (4)

$x_2 + x_4 = 4$ ——— (5)

$x_3 + 2x_4 = 6$ ——— (6)

choose $x_4 = t$

From equ (4), we have $x_1 = 3 - t$

From equ (5), we have $x_2 = 4 - t$

From equ (6), we have $x_3 = 6 - 2t$

$$\therefore x_1 = 3 - t$$
$$x_2 = 4 - t$$
$$x_3 = 6 - 2t$$
$$x_4 = t \quad t \in R$$

2. Solve the following system of linear equation by Gauss-Jordan elimination.

(i) $2x - 3y = 8$, $4x - 5y + z = 15$, $2x + 4z = 1$

(ii) $x_1 + x_2 + x_3 - x_4 = -2$, $2x_1 - x_2 + x_3 + x_4 = 0$,

$3x_1 + 2x_2 - x_3 - x_4 = 1$, $x_1 + x_2 + 3x_3 - 3x_4 = -8$

1. Argumented matrix $[A:B] = \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{bmatrix}$ $R_3 \Rightarrow 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 0 & -3 & 2b_3 - b_2 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2/2 \\ R_3 \Rightarrow R_3/-3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 1 & 1/2 & b_2/2 \\ 0 & 0 & 1 & \frac{b_2 - 2b_3}{3} \end{bmatrix}$$

$\rightarrow x + y - z = b_1$ ——— (1)

$y + \frac{1}{2}z = b_2/2$ ——— (2)

$$\boxed{z = \frac{b_2 - 2b_3}{3}}$$

Sub $z = \frac{b_2 - 2b_3}{3}$ in equ (2)

$$y + \frac{b_2 - 2b_3}{6} = \frac{b_2}{2}$$

$$y + b_2 - 2b_3 = 3b_2$$

$$y = \frac{2b_2 + 2b_3}{6}$$

$$\boxed{y = \frac{b_2 + b_3}{3}}$$

Sub $y = \frac{b_2 + b_3}{3}$, $x = \frac{b_2 - 2b_3}{3}$ in equ (1)

$$x + \frac{b_2 + b_3}{3} - \frac{b_2 + 2b_3}{3} = b_1$$

$$x + \frac{\cancel{b_2} + b_3 - \cancel{b_2} + 2b_3}{3} = b_1$$

$$x + \frac{\cancel{3}b_3}{\cancel{3}} = b_1$$

$$\boxed{x = b_1 - b_3}$$

2. Argumented matrix $[A|B] = \begin{bmatrix} 2 & 1 & 7 & b_1 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix}$ $R_1 \Rightarrow R_1/2$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix} \begin{matrix} \\ R_2 \Rightarrow R_2 - 6R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & -5 & -10 & b_2 - 3b_1 \\ 0 & -2 & -4 & b_3 - b_1 \end{bmatrix} \begin{matrix} \\ R_2 \Rightarrow R_2/-5 \\ R_3 \Rightarrow R_3/-2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{-b_2+3b_1}{5} \\ 0 & 1 & 2 & \frac{-b_3+b_1}{2} \end{bmatrix} \begin{matrix} \\ \\ R_3 \Rightarrow R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{3b_1-b_2}{5} \\ 0 & 0 & 0 & \frac{-b_1+2b_2+5b_3}{10} \end{bmatrix}$$

Since $\rho(A) = 2$, $\rho(A:B) = 3$

$\therefore \rho(A) \neq \rho(A:B)$

$\therefore$ The given system has no solution.

3.(i) Argumented matrix $[A|B] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ $R_2 \Rightarrow R_2/2$

$R_3 \Rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x+2y+3z=0$ —— (1)

$y+z=0$ —— (2)

Sub $\boxed{z=t}$ in equ (2)

$y+t=0$

$\boxed{y=-t}$

Sub $y=-t$, $z=t$ in equ (1)

$x-2t+3t=0$

$x+t=0$

$\boxed{x=-t}$

$\therefore x=-t$

$y=-t$

$z=t$

(ii) Argumented matrix $[A \,\vdots\, B] = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & -2 & -3 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2/5 \\ R_3 \Rightarrow R_3/7 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x - 2y - 3z = 0$ —— (1)

$y + z = 0$ —— (2)

sub $\boxed{z = t}$ in equ (2)

$\boxed{y = -t}$

sub $y = -t$, $z = t$ in equ (1)

$x + 2t - 3t = 0$

$x - t = 0$

$\boxed{x = t}$

$\therefore x = t$

$y = -t$

$z = t$

2.(ii) Argumented matrix $[A \mid B] = \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 1 & 0 \\ 3 & 2 & -1 & -1 & 1 \\ 1 & 1 & 3 & -3 & -8 \end{bmatrix}$

$R_2 \Rightarrow R_2 - 2R_1$
$R_3 \Rightarrow R_3 - 3R_1$
$R_4 \Rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 2 & -2 & -6 \end{bmatrix} \quad R_2 \Leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & -3 & -1 & 3 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \Rightarrow R_2/2 \\ R_4 \rightarrow R_4 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 \\ R_4 \to R_4 - 11R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 8 & 16 \end{bmatrix} \quad R_4 \to R_4/8$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 \Rightarrow R_1 - R_2$$

$$\cancel{1 \quad 1 \quad 1 \quad -1 \quad -2}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 1 & 5 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - 4R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -4 \\ 0 & 1 & 0 & 2 & +5 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 2R_4 \\ R_2 \to R_2 - 2R_4 \\ R_3 \to R_3 + R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore \quad x_1 = 0$

$x_2 = 1$

$x_3 = -1$

$x_4 = 2$

2(i) Argumented matrix $[A \,;\, B] = \begin{bmatrix} 2 & -3 & 0 & 8 \\ 4 & -5 & 1 & 15 \\ 2 & 0 & 4 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & 0 & 4 & 1 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} R_1 \rightarrow R_1/2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & -5 & -7 & 13 \\ 0 & -3 & -4 & 7 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & -3 & -4 & 7 \end{bmatrix} \quad R_3 \to 5R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad R_2 \to R_2/5$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{7}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad R_1 \to R_1 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{2} \\ 0 & 1 & \frac{7}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad R_2 \to R_2 - \frac{7}{5} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$\therefore \; x = \frac{17}{2}$

$y = 3$

$z = -4$

Inverse matrix :-

An $n \times n$ square matrix A is said to be invertible or non-singular if there exists a square matrix B of the same size such that

$$AB = I_n = BA$$

Such that a matrix B is called the inverse of A, and is denoted by A^{-1}

Note:- i) A matrix 'A' is said to be singular if it is not invertible.

ii) Let A be an invertible matrix and K be any non-zero scalar, then (a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(b) the matrix KA is invertible and $(KA)^{-1} = \frac{1}{K} A^{-1}$

(c) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

(d) $AA^{-1} = A^{-1}A = I$

iii) Any matrix with a zero row or zero column cannot be invertible.

iv) The product of invertible matrices is also invertible whose inverse is the product of the individual inverses in reversed order. i.e $(AB)^{-1} = B^{-1}A^{-1}$

*** <u>Finding</u> the <u>inverse</u> of a <u>matrix</u> by <u>using</u> <u>elementary</u> <u>row</u> <u>operations</u> (<u>Gauss-Jordan</u> <u>elimination</u>):-

Let A be an $n \times n$ matrix, then

i) Write the matrix $[A \mid I_n]$

ii) Compute the reduced echelon form of $[A \,\vdots\, I_n]$

iii) If the reduced echelon form is of the type $[I_n \,\vdots\, B]$, then B is the inverse of A.

Ex: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ by using Gauss - Jordan elimination.

Consider $[A \,\vdots\, I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_2 \to R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] R_1 \to R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\sim \left[I_n \,\vdots\, B \right]$$

$$\therefore \quad B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\text{i.e } A^{-1} = B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

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1. Find the inverse of the following matrices

i) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

Consider $[A \,\vdots\, I_{3\times 3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 + 3R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{array}\right]$$

Since one row is zero so inverse of A does not exist.

i) consider $[A \mid I_{3\times 3}] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 0 & -11 & -3 & 2 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-11 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & 3 & -1 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 11R_3 \\ R_2 \to R_2 - 9R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -5/11 & 7/11 & 9/11 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right]$$

$$\sim [I_n : B]$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$$

i.e $A^{-1} = B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$

iii) Consider $[A \vdots I_{3\times 3}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\sim [I_n \vdots B]$$

$$B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

iv) Consider $[A \mid I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/-4 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 - \frac{3}{2}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

$$\sim [I_n \mid B]$$

$$B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

Block Matrix :-

A sub matrix "A" is a matrix obtained from A by deleting certain rows and/or columns of A.

Consider a matrix $A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$

divided up into four blocks (sub matrices) by the dotted lines shown.

Now, if we write $A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $A_{12} \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$

$A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$, $A_{22} = \begin{bmatrix} a_{34} \end{bmatrix}$ then

A can be written as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ called a block matrix.

Product of block matrices:-

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ are block matrices and the number of columns in A_{ik} is equal to the number of rows in B_{kj}, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Example: Compute AB using block multiplication where

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hline 0 & 0 & 2 & -1 \end{array}\right] \quad B = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

Consider $A_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ $A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 2 & -1 \end{bmatrix}$

$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $B_{21} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$ $B_{22} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$A_{12}B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix} 1 & 8 \end{bmatrix}$$

$$A_{21}B_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}$$

$$A_{22}B_{22} = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 5 & 12 \\ 0 & 2 & 7 \\ 1 & 8 & 7 \end{bmatrix}$$

Elementary matrix :-

An elementary matrix is a matrix, which is obtained from the identity matrix I_n by executing only one elementary row operation.

Example :-

$$\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_2$ $\quad R_2 \leftrightarrow R_4$ $\quad R_1 \to R_1 + 3R_3$

Properties :-

i) If E denotes an elementary matrix and $E'(E^{-1})$ denotes the elementary matrix corresponding to the inverse elementary row operation on E, then

$$EE' = I$$

ii) If E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$

iii) If E interchanges two rows, then E'

interchanges them again

iv) If E adds a multiple of one row to another, then E' subtracts it from the same row.

v) Every elementary matrix is invertible and inverse matrix $E' = E^{-1}$ is also an elementary matrix.

Example:-

1. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

3. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Express the following matrices as a product of elementary matrices

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$ $R_2 \to R_2/-2$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 3R_2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 2R_1$

$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Since $R_2 \to R_2/-2$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

Since $R_2 \to R_2 - 2R_1$.

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Since $R_2 \to -2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Since $R_1 \to R_1 + 3R_2$

$$E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 - 3R_2$

$$E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A = E_1 . E_2 . E_3$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 11/2 \end{bmatrix}$$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$ ✓

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 5R_3$

$R_2 \to R_2 - 4R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $R_1 \to R_1 + 2R_2$ (inverse)

$$E_1^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 - 5R_3$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_2 \to R_2 + 4R_3$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix A cannot be expressed as the product of elementary matrices since the third row is zero

Permutations :-

A permutation matrix is a square matrix obtained from the identity matrix if permuting <changing the order> the rows

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a permutation matrix but

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$ is not a permutation matrix.

Properties:-

i) Every permutation matrix is a elementary matrix but every elementary matrix need not be a permutation matrix.

ii) The product of any two permutation matrices is again a permutation matrix.

iii) The transpose of a permutation matrix is also a permutation matrix.

iv) Every permutation matrix P is invertible and $P^{-1} = P^{T}$

v) A permutation matrix is the product of a finite number of elementary matrices each of which corresponds to the row interchanging elementary row operation.

ii) $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$ $R_2 \to R_2 - 5R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $R_2 \to R_2/2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 5R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

Since $R_2 \to 2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = E_1^{-1} \cdot E_2^{-1}$

$= \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

LU Factorization :-

Let A be a square matrix that can be factorized into the form $A = LU$, where L is a lower triangular matrix & U is an upper triangular matrix. This factorization is called an LU factorization or LU decomposition of A

Note: i) Every matrix has an LU factorization and when it exists, it is not unique.

ii) If the matrix A is invertible & if the permutation matrix P is fixed then the matrix PA has a unique LDU factorization.

system of linear equatio

PA has a unique ...

Solving method for a given system of linear equations by LU factorization :-

Let $AX=B$ be a system of "n" linear equations in "n" unknowns then

i) Find the LU decomposition of A

ii) Solve $LY=B$ by forward substitution

iii) Solve $UX=Y$ by back substitution

Example: 1. Solve the following system of equations using LU decomposition

$$2x_1+x_2+3x_3=-1$$
$$4x_1+x_2+7x_3=5$$
$$-6x_1-2x_2-12x_3=-2$$

The given system of linear equations can be expressed as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \cancel{\begin{matrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 4 & 1 & 7 \end{matrix}}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

The inverse elementary matrices that corresponds to the row operations.

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$L = E_1^{-1} \cdot E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$$A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Consider $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad \boxed{y_1 = -1}$ —— (1)

$2y_1 + y_2 = 5$ —— (2)

$-3y_1 - y_2 + y_3 = -2$ —— (3)

sub $y_1 = -1$ in equ (2)

$-2 + y_2 = 5$

$\boxed{y_2 = 7}$

Sub $y_1=-1$, $y_2=7$ in equ (3)

$$3-7+y_3=-2$$

$$-4+y_3=-2$$

$$\boxed{y_3=2}$$

$y_1=-1$, $y_2=7$, $y_3=2$

Since $UX=Y$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$2x_1+x_2+3x_3=-1 \quad \text{—— (1)}$$

$$-x_2+x_3=7 \quad \text{—— (2)}$$

$$-2x_3=2 \quad \text{—— (3)}$$

$$(3) \Rightarrow \boxed{x_3=-1}$$

Sub $x_3=-1$ in equ (2)

$$-x_2-1=7$$

$$\boxed{x_2=-8}$$

Sub $x_2=-8$, $x_3=-1$ in equ (1)

$$2x_1-8-3=-1$$

$$2x_1-11=-1$$

$$2x_1=10$$

$$\boxed{x_1=5}$$

$x_1=5$, $x_2=-8$, $x_3=-1$

Find the LDU factorization for each of the following matrices

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$

$\sim \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$

Since $R_2 \to R_2 - 4R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$ $R_2 \to R_2 - 8R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U$

since $R_2 \to R_2 - 8R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \to R_2 + 8R_1$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

PA

If $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is any permutation matrix then express $\boxed{PA}$ as LDU factorization

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\boxed{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

Since $R_3 \to R_3 - R_2$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- $PA = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(A) LDU

For all possible permutation matrices P, find the LDU factorization of PA for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$P_2\ (R_1 - R_3)$

$R_2\ (R_1\ R_2 \ldots)$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2 \qquad R_1 \leftrightarrow R_3 \qquad R_2 \leftrightarrow R_3$

$$P_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_1A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_1 \Leftrightarrow R_3$

$$E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_1$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$P_1A = LU$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_1A = LDU$

$$P_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} R_3 \Leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \Rightarrow R_2 - 2R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \Leftrightarrow R_2$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P_2 A = L U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 A = L D U$$

$$P_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \Rightarrow R_2 - R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$P_3A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & +1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3A = LDU$$

Gauss elimination

4. ii) $x-y+z=1,\ x+3y+az=2,\ 2x+ay+3z=3$

$$[A:B]=\begin{bmatrix}1&-1&1&1\\1&3&a&2\\2&a&3&3\end{bmatrix} R_3\to R_3-2R_1$$

$$\sim\begin{bmatrix}1&-1&1&1\\0&4&a-1&1\\0&a+2&1&1\end{bmatrix} R_3\to-4R_3+(a+2)R_2$$

$$\sim\begin{bmatrix}1&-1&1&1\\0&4&a-1&1\\0&0&-4+(a+2)(a-1)&a-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-1&1&1\\0&4&a-1&1\\0&0&(a+3)(a-2)&a-2\end{bmatrix}$$

i) If $a\neq-3,\ a\neq2$

$\rho(A)=\rho(A,B)=$ no of unknowns

$\therefore$ It has unique solution

ii) If $a\neq-3,\ a=2$

$\rho(A)=\rho(A,B)\neq$ no of unknowns

$\therefore$ It has infinite no of solutions

iii) If $a=-3,\ a\neq2$

$\rho(A)\neq\rho(A,B)$

$\therefore$ It has no solution