LINEAR EQUATIONS & MATRICES

Linear - Degree of the equation is 1.

Eg: $ax+by+c=0$

Quadratic - Degree of the equation is 2

Eg: $ax^2+by+c=0$

Linear equation:-

Consider the equation $a_1x_1+a_2x_2+\dots a_nx_n=b$ —①

where $a_1, a_2, \dots a_n$ and b are constants is called linear equation.

System of linear equation:- (Group of linear equations)

A general system of m linear equations with n unknowns can be written as

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\dots\dots a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\dots\dots a_{2n}x_n=b_2 \\ a_3\vdots \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\dots\ a_{mn}x_n=b_m \end{array}\right\} \text{—②}$$

where $x_1, x_2, \dots x_n$ are the unknowns, $a_{11}, a_{12}, \dots a_{mn}$ are the coefficients of the system and $b_1, b_2, \dots b_m$ are the constant terms.

The system ② is said to be homogeneous if all the constant terms are zero, i.e. $b_1=b_2=\dots b_m=0$ otherwise ② is known as non-homogeneous.

Every homogeneous sys/sol has trivial solution. i.e. all sol = 0

$3x-5y=0 \Rightarrow x=0, y=0 \Rightarrow 0=0$

In general, a linear system may have in any one of three possible ways:

i) The system has a single (unique) solution.

ii) The system has infinitely many solution.

iii) The system has no solution.

The system is ② is said to be consistent if it has atleast one solution and inconsistent if it has no solution.

The system ② can be expressed as

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$

$$[A : B] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & b_m \end{array}\right]$$

is called argumented matrix.

MAT202

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D14TD1

Open Hours:

Wed : 3-4

Tues : 9-10

SJT 511 A04

Linear Algebra is the branch of mathematics concerning vector spaces, often finite or countably infinite dimensional as well as linear mappings b/w such spaces. Such an investigation is initially motivated by a system of linear eq's containing several unknowns. Such eq's are naturally represented using the formalism of matrices and vectors.

y have

Elementary operations :-

The following operations on a augmented matrix (system of linear equations) are called elementary operations

i) Interchange two rows (equations)

i.e. $R_i \longleftrightarrow R_j$ ($l_i \longleftrightarrow l_j$)

ii) Multiply a non-zero constant throughout a row (an equation)

i.e. $R_i \longrightarrow aR_i$ ($l_i \rightarrow a l_i$) $a \neq 0$

iii) Add a constant multiple of an equation to another equation

i.e. $R_i \rightarrow aR_i + R_j$ (or) $R_j \rightarrow aR_j + R_i$

Row-echelon form of a matrix :- (Gauss elimination method)

Reduced row-echelon form (Gauss Jordan elimination)

A matrix is said to be in row-echelon form if it satisfies the following:

(i) The zero rows, if they exist, come last in the ordered rows.

(ii) The first non-zero entries, in the non-zero rows are 1, called leading ones

(iii) In each column containing a leading non-zero element, the entries below that leading non-zero element are 0.

The reduced row-echelon form of an augmented matrix is of the form:

(iv) Above each leading is a column of zeros in addition to the row-echelon form

Gauss - elimination method :-

The gauss - elimination algorithm is as follows : (i) Write the argumented matrix of the system of linear equations.

(ii) Find an echelon form of the argumented matrix using elementary row operations.

(iii) Write the system of equations corresponding to the echelon form.

(iv) Use back substitution to get the solution.

Pivot element :-

The left most non-zero entries in the non-zero rows are called pivots.

Example: Solve the system of linear equations $2y+4z=2$, $x+2y+2z=3$, $3x+4y+6z=-1$ and write the pivots.

Sol :- The system of linear equations

$$0x+2y+4z=2$$
$$x+2y+2z=3$$
$$3x+4y+6z=-1$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix}1&2&2&3\\0&2&4&2\\0&-2&0&-10\end{bmatrix} \quad R_3 \to R_3+R_2$$

$$\sim \begin{bmatrix}1&2&2&3\\0&2&4&2\\0&0&4&-8\end{bmatrix} \quad \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/4 \end{array}$$

pivots of $x=1$, $y=2$, $z=4$

$$\sim \begin{bmatrix}1&2&2&3\\0&1&2&1\\0&0&1&-2\end{bmatrix}$$

[Back substitution method]

$\Rightarrow \quad 1.x+2y+2z=3 \quad \text{—— ①}$

$\Rightarrow \quad 0x+y+2z=1 \quad \text{—— ②}$

$\quad 0x+0y+z=-2 \quad \text{—— ③}$

From equation ③, we have $\boxed{z=-2}$

Sub $z=-2$ in equation ②, we get

$$y+2(-2)=1$$

$$\boxed{y=5}$$

Substitute $y=5$, $z=-2$ in equation ①, we get

$$x+10-4=3$$

$$\boxed{x=-3}$$

$\therefore$ $x=-3$

$y=5$

$z=-2$

1. Solve the following system of equations by gaussian elimination. What are the pivots

(i) $-x+y+2z=0,\ 3x+4y+z=0,\ 2x+5y+3z=0$

(ii) $2y-z=1, 4x-10y+3z=5,\ 3x-3y=6$

(iii) $w+x+y=3,\ -3w-17x+y+2z=1, 4w-17x+8y-5z=1, -5x-2y+z=1$

(iv) $x_1+2x_2+3x_3+2x_4=-1,\ -x_1-2x_2-2x_3+x_4=2,$
$2x_1+4x_2+8x_3+12x_4=4$

(i) Sol:- The system of equations

$$-x+y+2z=0$$
$$3x+4y+z=0$$
$$2x+5y+3z=0$$

Argumented matrix $[A:B]=\begin{bmatrix}-1&1&2&0\\3&4&1&0\\2&5&3&0\end{bmatrix}$ $R_1\to R_1\times -1$

$$\sim\begin{bmatrix}1&-1&-2&0\\3&4&1&0\\2&5&3&0\end{bmatrix}\ R_2\to R_2-3R_1,\ R_3\to R_3-2R_1$$

$$\sim\begin{bmatrix}1&-1&-2&0\\0&7&7&0\\0&7&7&0\end{bmatrix}\ R_3\to R_3-R_2$$

$$\sim\begin{bmatrix}1&-1&-2&0\\0&7&7&0\\0&0&0&0\end{bmatrix}\ R_2\to R_2/7$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x - y - 2z = 0 \quad \text{——} \ (1)$$

$$y + z = 0 \quad \text{——} \ (2)$$

Let $\boxed{z = t}$ sub in equ (2)

$$y + t = 0$$

$$\boxed{y = -t}$$

Substitute $y = -t$, $z = t$ in equation (1)

$$x + t - 2t = 0$$

$$x = 2t - t$$

$$\boxed{x = t}$$

$$\therefore \quad x = t$$

$$y = -t$$

$$z = t$$

2. Determine the condition on b_i. So that the following system has a solution

(i) $x+2y+6z=b_1$, $2x-3y-2z=b_2$, $3x-y+4z=b_3$

(ii) $x+3y-2z=b_1$, $2x-y+3z=b_2$, $4x+2y+z=b_3$

(i) Sol: The system of equations

$$x+2y+6z=b_1$$
$$2x-3y-2z=b_2$$
$$3x-y+4z=b_3$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 2 & -3 & -2 & b_2 \\ 3 & -1 & 4 & b_3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & -7 & -14 & b_3-3b_1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix} R_2 \to R_2/-7$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & 1 & 1 & \frac{1}{7}(2b_1-b_2) \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

Since $\rho(A)=2$, $\rho(A \vdots B)=2$ if $b_3-b_2-b_1=0$

$\therefore$ $\rho(A)=2=\rho(A \vdots B) <$ no of variables (3)

$\therefore$ The given system has an infinite no of solution when $b_3-b_1-b_2=0$

(ii)

Arg

$\to R_2-2R$

$\to R_3-3R$

$\to R_3-$

$R_2 \to R_2$

(3)

solution

(ii) Sol : The system of equations

$$x+3y-2z=b_1$$
$$2x-y+3z=b_2$$
$$4x+2y+z=b_3$$

Argumented matrix $[A\,|\,B] = \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 2 & -1 & 3 & b_2 \\ 4 & 2 & 1 & b_3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & -7 & 7 & b_2-2b_1 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix}$$

$R_2 \to R_2/7$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix}$$

$R_3 \to R_3 + 10R_2$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & 0 & -1 & b_3-10b_2+16b_1 \end{bmatrix}$$

Since $\rho(A)=3$, $\rho(A\,|\,B)=3$

$\therefore$ $\rho(A)=3=\rho(A\,|\,B)=$ no of variables(3)

$\therefore$ The given system has an unique solution

1. (ii) Sol: The system of equations

$$2y - z = 1$$
$$4x - 10y + 3z = 5$$
$$3x - 3y = 6$$

Argumented matrix $[A|B] = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 3 & -3 & 0 & 6 \end{bmatrix}$ $R_3 \to R_3/3$

$$\sim \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 1 & -1 & 0 & 2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -6 & 3 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_2 \to R_2/3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x-y=2$ —— ①

$y-\frac{1}{2}z=\frac{1}{2}$ —— ②

Let $\boxed{z=t}$ substitute in equation ②

$y-\frac{1}{2}t=\frac{1}{2}$

$\frac{2y-t}{2}=\frac{1}{2}$

$2y-t=1$

$2y=1+t$

$\boxed{y=\frac{1+t}{2}}$

Put $y=\frac{1+t}{2}$, $z=t$ substitute in equation ①

$x-\frac{1+t}{2}=2$

$x=2+\frac{1+t}{2}$

$\boxed{x=\frac{5+t}{2}}$

$\therefore x=\frac{5+t}{2}$

$y=\frac{1+t}{2}$

$z=t$

(ii) Sol: The system of equations

$w+x+y=3$

$-3w-17x+y+2z=1$

$4w-17x+8y-5z=1$

$-5x-2y+z=1$

Argumented matrix $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ -3 & -17 & 1 & 2 & 1 \\ 4 & -17 & 8 & -5 & 1 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & -14 & 4 & 2 & 10 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + 3R_2, \; R_4 \to 7R_4 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & -2 & -8 & -26 \\ 0 & 0 & -24 & 2 & -18 \end{bmatrix} \quad R_3 \to R_3/-2, \; R_4 \to R_4/-2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & +1 & 4 & 13 \\ 0 & 0 & +12 & -1 & 9 \end{bmatrix} \quad \cancel{R_4 \to R_4}, \; R_4 \to R_4 - 12R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & -49 & -147 \end{bmatrix}$$

$\frac{-12\times4}{-48}$ $\frac{-12\times13}{-156}$ $\frac{9}{-147}$

$\Rightarrow \quad W+x+y+0z=3 \quad — ①$

$0W+7x-2y-z=-5 \quad — ②$

$y+4z=13 \quad — ③$

$-49z=-147 \quad — ④$

$\boxed{z=3}$

Put $z=3$ in equation ③

$y+12=13$

$\boxed{y=1}$

Put $z=3, y=1$ in equation ②

$7x-2-3=-5$

$7x-5=-5$

$\boxed{x=0}$

Substitute $x=0, y=1, z=3$ in equation ①

$W+0+1=3$

$\boxed{W=2}$

$\therefore W=2$
$x=0$
$y=1$
$z=3$

(iv) Sol: The system of equations

$x_1+2x_2+3x_3+2x_4=-1$

$-x_1-2x_2-2x_3+x_4=2$

$2x_1+4x_2+8x_3+12x_4=4$

Argumented matrix $[A:B] = \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} R_3 \to R_3/2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 1 & 2 & 4 & 6 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + 3x_3 + 2x_4 = -1$ —— (1)

$x_3 + 3x_4 = 1$ —— (2)

$\boxed{x_4 = 2}$

Substitute $x_4 = 2$ in equation (2)

$x_3 + 6 = 1$

$\boxed{x_3 = -5}$

Substitute $x_3 = -5,\ x_4 = 2$ in equation (1)

$x_1 + 2x_2 - 15 + 4 = -1$

$x_1 + 2x_2 = +10$

Let $\boxed{x_2 = t}$

$x_1 + 2t = 10$

$\boxed{x_1 = 10 - 2t}$

Note:-

(i) If $\rho(A) = \rho(A|B) =$ number of variables, then the system has a unique solution

(ii) If $\rho(A) = \rho(A|B) <$ number of variables, then the system has an infinite number of solutions

(iii) If $\rho(A) \neq \rho(A|B)$, then the system has no solution

1. Determine all values of b_i that make the following system $x+y-z=b_1$, $2y+z=b_2$, $y-z=b_3$ consistent

2. Determine the condition b_i so that the following system has no solution $2x+y+7z=b_1$, $6x-2y+11z=b_2$, $2x-y+3z=b_3$

3. Which of the following system has a non-trivial solution

(i)
$$x+2y+3z=0$$
$$2y+2z=0$$
$$x+2y+3z=0$$

(ii)
$$2x+y-z=0$$
$$x-2y-3z=0$$
$$3x+y-2z=0$$

4. For which values of "a" does each of the following system have no solution, exactly one solution or infinitely many solution

(i) $x+2y-3z=4$, $3x-y+5z=2$, $4x+y+(a^2-14)z=a+2$

(ii) $x-y+z=1$, $x+3y+az=2$, $2x+ay+3z=3$

4.(i)

Argumented matrix $[A \mid B] = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

$R_2 \to R_2/-7$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

i) If $a^2-16=0$ and $a-4=0$ then $\rho(A)=2$, $\rho(A \mid B)=2$
i.e $a=\pm 4$ then the given system has an infinite number of solutions $[\because \rho(A)=2=\rho(A \mid B)<3]$

ii) If $a^2-16 \neq 0$ and $a-4 \neq 0$ then $\rho(A)=3$, $\rho(A \mid B)=3$
i.e $a \neq \pm 4$ then the given system has an unique solution $(\because \rho(A)=3=\rho(A \mid B)=$ no of unknown$)$

iii) If $a=-4$, then the given system has no solution

*** Gauss - Jordan elimination :-

(i) Write the argumented matrix for the given system of linear equations.

(ii) Derive the reduced row-echelon form for the argumented matrix by using elementary row operation

(iii) Write the system of equations corresponding to the reduced row-echelon form. This system gives the solution.

Solve the following system of linear equations by gauss jordan elimination.

$x_1+3x_2-2x_3=3$, $2x_1+6x_2-2x_3+4x_4=18$, $x_2+x_3+3x_4=10$

$$\text{Argumented matrix } [A:B] = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}_{R_2\to R_2-2R_1}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}_{R_2\leftrightarrow R_3}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}_{R_3\to R_3/2}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}_{R_1\to R_1-3R_2}$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & -9 & -27 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 0x_2 + 0x_3 + x_4 = 3$ —— (1)

$0x_1 + x_2 + 0x_3 + x_4 = 4$ —— (2)

$0x_1 + 0x_2 + x_3 + 2x_4 = 6$ —— (3)

$x_1 + x_4 = 3$ —— (4)

$x_2 + x_4 = 4$ —— (5)

$x_3 + 2x_4 = 6$ —— (6)

choose $x_4 = t$

From equ (4), we have $x_1 = 3 - t$

From equ (5), we have $x_2 = 4 - t$

From equ (6), we have $x_3 = 6 - 2t$

$\therefore x_1 = 3 - t$

$x_2 = 4 - t$

$x_3 = 6 - 2t$

$x_4 = t \quad t \in R$

2. Solve the following system of linear equation by Gauss-Jordan elimination.

(i) $2x - 3y = 8,\ 4x - 5y + z = 15,\ 2x + 4z = 1$

(ii) $x_1 + x_2 + x_3 - x_4 = -2,\ 2x_1 - x_2 + x_3 + x_4 = 0,$
$3x_1 + 2x_2 - x_3 - x_4 = 1,\ x_1 + x_2 + 3x_3 - 3x_4 = -8$

R_1+5R_3
R_2-R_3

1. Argumented matrix $[A|B] = \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{bmatrix}$ $R_3 \Rightarrow 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 0 & -3 & 2b_3-b_2 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2/2 \\ R_3 \Rightarrow R_3/-3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 1 & 1/2 & b_2/2 \\ 0 & 0 & 1 & \frac{b_2-2b_3}{3} \end{bmatrix}$$

$\rightarrow x+y-z=b_1$ — ①

$y+\frac{1}{2}z = b_2/2$ — ②

$\boxed{z = \frac{b_2-2b_3}{3}}$

Sub $z = \frac{b_2-2b_3}{3}$ in equ ②

$$y + \frac{b_2-2b_3}{6} = \frac{b_2}{2}$$

$$y + b_2 - 2b_3 = 3b_2$$

$$y = \frac{2b_2+2b_3}{6}$$

$$\boxed{y = \frac{b_2+b_3}{3}}$$

quation by

Sub $y = \frac{b_2+b_3}{3}$, $z = \frac{b_2-2b_3}{3}$ in equ ①

$$x + \frac{b_2+b_3}{3} - \frac{b_2+2b_3}{3} = b_1$$

$$x + \frac{b_2+b_3-b_2+2b_3}{3} = b_1$$

$$x + \frac{3b_3}{3} = b_1$$

$$\boxed{x = b_1 - b_3}$$

2. Argumented matrix $[A|B] = \begin{bmatrix} 2 & 1 & 7 & b_1 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix}$ $R_1 \to R_1/2$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & -5 & -10 & b_2 - 3b_1 \\ 0 & -2 & -4 & b_3 - b_1 \end{bmatrix} \begin{matrix} R_2 \to R_2/-5 \\ R_3 \to R_3/-2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{-b_2 + 3b_1}{5} \\ 0 & 1 & 2 & \frac{-b_3 + b_1}{2} \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{3b_1 - b_2}{5} \\ 0 & 0 & 0 & \frac{-b_1 + 2b_2 + 5b_3}{10} \end{bmatrix}$$

Since $\rho(A) = 2$, $\rho(A|B) = 3$

$\therefore \rho(A) \neq \rho(A|B)$

$\therefore$ The given system has no solution.

3.(i) Argumented matrix $[A|B] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ $\begin{matrix} R_2 \to R_2/2 \\ R_3 \to R_3 - R_1 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x+2y+3z=0$ —— ①

$y+z=0$ —— ②

Sub $\boxed{z=t}$ in equ ②

$y+t=0$

$\boxed{y=-t}$

Sub $y=-t$, $z=t$ in equ ①

$x-2t+3t=0$

$x+t=0$

$\boxed{x=-t}$

$\therefore x=-t$

$y=-t$

$z=t$

(ii) Argumented matrix $[A \,\vdots\, B] = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & -2 & -3 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2/5 \\ R_3 \Rightarrow R_3/7 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x - 2y - 3z = 0$ —— ①

$y + z = 0$ —— ②

sub $\boxed{z = t}$ in equ ②

$\boxed{y = -t}$

sub $y = -t$, $z = t$ in equ ①

$x + 2t - 3t = 0$

$x - t = 0$

$\boxed{x = t}$

$\therefore x = t$

$y = -t$

$z = t$

2(ii) Argumented matrix $[A \,|\, B] = \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 1 & 0 \\ 3 & 2 & -1 & -1 & 1 \\ 1 & 1 & 3 & -3 & -8 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 2 & -2 & -6 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & -3 & -1 & 3 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/2 \\ R_4 \to R_4 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 \\ R_4 \to R_4 - 11R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 8 & 16 \end{bmatrix} \quad R_4 \to R_4/8$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 \Rightarrow R_1 - R_2$$

~~1 1 1 -1 -2~~

$$\sim \begin{bmatrix} 1 & 0 & -3 & 1 & 5 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - 4R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -4 \\ 0 & 1 & 0 & 2 & +5 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 2R_4 \\ R_2 \to R_2 - 2R_4 \\ R_3 \to R_3 + R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore x_1 = 0$

$x_2 = 1$

$x_3 = -1$

$x_4 = 2$

2 ii) Argumented matrix $[A \mid B] = \begin{bmatrix} 2 & -3 & 0 & 8 \\ 4 & -5 & 1 & 15 \\ 2 & 0 & 4 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & 0 & 4 & 1 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} R_1 \to R_1/2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & -5 & -7 & 13 \\ 0 & -3 & -4 & 7 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & -3 & -4 & 7 \end{bmatrix} R_3 \to 5R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & 0 & 1 & -4 \end{bmatrix} R_2 \to R_2/5$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 1 & 7/5 & -13/5 \\ 0 & 0 & 1 & -4 \end{bmatrix} R_1 \to R_1 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 17/2 \\ 0 & 1 & 7/5 & -13/5 \\ 0 & 0 & 1 & -4 \end{bmatrix} R_2 \to R_2 - \tfrac{7}{5} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 17/2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$\therefore\ x = 17/2$

$y = 3$

$z = -4$

<u>Inverse matrix</u> -

An $n \times n$ square matrix A is said to be <u>invertible</u> or <u>non-singular</u> if there exists a square matrix B of the same size such that

$$AB = I_n = BA$$

Such that a matrix B is called the <u>inverse of A</u>, and is denoted by A^{-1}

<u>Note</u>:- i) A matrix 'A' is said to be <u>singular</u> if it is not invertible.

ii) Let A be an invertible matrix and K be any non-zero scalar, then ⓐ A^{-1} is invertible and $(A^{-1})^{-1} = A$
ⓑ the matrix KA is invertible and $(KA)^{-1} = \frac{1}{K} A^{-1}$

ⓒ A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

ⓓ $AA^{-1} = A^{-1}A = I$

iii) Any matrix with a zero row or zero column cannot be invertible.

iv) The product of invertible matrices is also invertible whose inverse is the product of the individual inverses in reversed order. i.e $(AB)^{-1} = B^{-1}A^{-1}$

*** **Finding the inverse of a matrix by using elementary row operations (Gauss-Jordan elimination):-**

Let A be an $n\times n$ matrix, then

i) Write the matrix $[A \,\vdots\, I_n]$

ii) Compute the reduced echelon form of $[A \,\vdots\, I_n]$

iii) If the reduced echelon form is of the type $[I_n \,\vdots\, B]$, then B is the inverse of A.

*** **Ex:** Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ by using Gauss-Jordan elimination.

Consider $[A \,\vdots\, I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$ $R_2 \to R_2$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] R_1 \to R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\sim\left[\, I_n \,\vdots\, B \,\right]$$

$$\therefore\ B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\text{i.e. } A^{-1} = B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

1. Find the inverse of the following matrices

i) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

Consider $[A \,|\, I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{array}\right] R_3 \to R_3 + 3R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{array}\right]$

Since one row is zero so inverse of A does not exist.

i) Consider $[A \,|\, I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$

$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 0 & -11 & -3 & 2 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-11 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & 3 & -1 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 11R_3 \\ R_2 \to R_2 - 9R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -5/11 & 7/11 & 9/11 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right]$$

$\frac{-27}{11}+2$

$\frac{-27+22}{11}$

-5

$\frac{18}{11}-1 \quad \frac{18-11}{11} \quad \frac{7}{11}$

$$\sim [I_n \,|\, B]$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$$

i.e $A^{-1} = B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$

iii) Consider $[A \vdots I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\sim [I_n \vdots B]$$

$$B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

iv) consider $[A \mid I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/-4 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] R_1 \to R_1 - R_2$$

$$\sim \begin{array}{cccccc} 1 & \cancel{1} & 1 & 1 & 0 & \cancel{0} \\ 0 & \cancel{0} & -1/2 & 1 & \cancel{-1/2} & 0 \\ 0 \end{array}$$

$5/8 + 1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 - \frac{3}{2}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

$$\sim [I_n \mid B]$$

$$B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

Block Matrix:-

A sub matrix "A" is a matrix obtained from A by deleting certain rows and/or columns of A.

Consider a matrix $A = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$

divided up into four blocks (sub matrices) & by the dotted lines shown.

Now, if we write $A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $A_{12} \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$

$A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$, $A_{22} = \begin{bmatrix} a_{34} \end{bmatrix}$ then

A can be written as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ Called a block matrix.

Product of block matrices:-

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ are block matrices and the number of columns in A_{ik} is equal to the number of rows in B_{kj}, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Example: Compute AB using block multiplication where

$$A = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hdashline 0 & 0 & 2 & -1 \end{array}\right] \qquad B = \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hdashline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

Consider $A_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ $A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 2 & -1 \end{bmatrix}$

$B_{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $B_{21} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$ $B_{22} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$A_{12}B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix} 1 & 8 \end{bmatrix}$$

$$A_{21} B_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \end{bmatrix}$$

$$A_{22} B_{22} = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 5 & 12 \\ 0 & 2 & 7 \\ 1 & 8 & 7 \end{bmatrix}$$

Elementary matrix :-

An elementary matrix is a matrix, which is obtained from the identity matrix I_n by executing only one elementary row operation.

Example :-

$$\underset{R_2 \to R_2 - 5R_2}{\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}}, \quad \underset{R_2 \Leftrightarrow R_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}, \quad \underset{R_1 \to R_1 + 3R_3}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Properties :-

i) If E denotes an elementary matrix and $E'(E^{-1})$ denotes the elementary matrix corresponding to the inverse elementary row operation on E, then,

$$EE' = I$$

ii) If E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$

iii) If E interchanges two rows, then E'

interchanges them again

iv) If E adds a multiple of one row to another, then E' subtracts it from the same row.

v) Every elementary matrix is invertible and inverse matrix $E' = E^{-1}$ is also an elementary matrix.

<u>Example</u> :-

1. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

3. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

<u>Express</u> the following matrices as <u>a</u> <u>product</u> of <u>elementary</u> matrices

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$ $R_2 \to R_2/-2$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 3R_2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 2R_1$

$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Since $R_2 \to R_2 - 2R_1$.

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Since $R_2 \to R_2/-2$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

Since $R_2 \to -2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Since $R_1 \to R_1 + 3R_2$

$E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Since $R_1 \to R_1 - 3R_2$

$E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\therefore A = E_1 \cdot E_2 \cdot E_3$

$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 2 & 11/2 \end{bmatrix}$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ $R_2 \to R_2 - 5R_1$

$R_2/2$

$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 5 & 1/2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 5R_3$, $R_2 \to R_2 - 4R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $R_1 \to R_1 + 2R_2$ (inverse)

$$E_1^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 - 5R_3$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_2 \to R_2 + 4R_3$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A cannot be expressed as the product of elementary matrices since the third row is zero

Permutations:-

A permutation matrix is a square matrix obtained from the identity matrix by permuting <changing the order> the rows

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a permutation matrix but

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$ is not a permutation matrix.

Properties:-

i) Every permutation matrix is a elementary matrix but every elementary matrix need not be a permutation matrix.

ii) The product of any two permutation matrices is again a permutation matrix.

iii) The transpose of a permutation matrix is also a permutation matrix.

iv) Every permutation matrix P is invertible and $P^{-1} = P^T$

v) A permutation matrix is the product of a finite number of elementary matrices each of which corresponds to the row interchanging elementary row operation.

ii). $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$ $R_2 \to R_2 - 5R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $R_2 \to R_2/2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 5R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

Since $R_2 \to 2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = E_1^{-1} \cdot E_2^{-1}$

$= \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

LU Factorization :-

Let A be a square matrix that can be factorized into the form $A = LU$, where L is a lower triangular matrix & U is an upper triangular matrix. This factorization is called an LU factorization or LU decomposition of A

Note: i) Every matrix has an LU factorization and when it exists, it is not unique.

ii) If the matrix A is invertible & if the permutation matrix P is fixed then the matrix PA has a unique LDU factorization.

Solving method for a given system of linear equations by LU factorization :-

Let $AX=B$ be a system of "n" linear equations in "n" unknowns then

i) Find the LU decomposition of A

ii) Solve $LY=B$ by forward substitution

iii) Solve $UX=Y$ by back substitution

Example 1. Solve the following system of equations using LU decomposition

$$2x_1 + x_2 + 3x_3 = -1$$
$$4x_1 + x_2 + 7x_3 = 5$$
$$-6x_1 - 2x_2 + 12x_3 = -2$$

The given system of linear equations can be expressed as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\sim \cancel{\begin{matrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 4 & 1 & 7 \end{matrix}}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

The inverse elementary matrices that corresponds to the row operation

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$L = E_1^{-1} \cdot E_2^{-1} E_3^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{array}{r} -9 \\ -1 \\ \hline -2 \end{array}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Consider $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$\Rightarrow \boxed{y_1 = -1}$ — (1)

$2y_1 + y_2 = 5$ — (2)

$-3y_1 - y_2 + y_3 = -2$ — (3)

sub $y_1 = -1$ in equ (2)

$-2 + 1y_2 = 5$

$\boxed{y_2 = 7}$

Sub $y_1 = -1$, $y_2 = 7$ in equ ③

$$3 - 7 + y_3 = -2$$

$$-4 + y_3 = -2$$

$$\boxed{y_3 = 2}$$

$y_1 = -1$, $y_2 = 7$, $y_3 = 2$

Since $Ux = Y$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$2x_1 + x_2 + 3x_3 = -1 \quad \text{— ①}$$

$$-x_2 + x_3 = 7 \quad \text{— ②}$$

$$-2x_3 = 2 \quad \text{— ③}$$

③ $\Rightarrow$ $\boxed{x_3 = -1}$

Sub $x_3 = -1$ in equ ②

$$-x_2 - 1 = 7$$

$$\boxed{x_2 = -8}$$

Sub $x_2 = -8$, $x_3 = -1$ in equ ①

$$2x_1 - 8 - 3 = -1$$

$$2x_1 - 11 = -1$$

$$2x_1 = 10$$

$$\boxed{x_1 = 5}$$

$x_1 = 5$, $x_2 = -8$, $x_3 = -1$

2. Solve the system of linear equations

$$AX = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = b$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = U$$

The inverse elementary matrices that corresponds to the row operations.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \quad R_3 \to R_3 \mp 3R_2$$

$L = E_1 E_2 E_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

$A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}$$

-1
6
-6

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}$$

Consider $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$\boxed{y_1 = 1}$ — ①

$2y_1 + y_2 = -2$ — ②

$-y_1 - 3y_2 + y_3 = 7$ — ③

Sub $y_1 = 1$ in ②

$2 + y_2 = -2$

$\boxed{y_2 = -4}$

Sub $y_1 = 1$, $y_2 = -4$ in equ ③

$-1 + 12 + y_3 = 7$

$\boxed{y_3 = -4}$

$y_1 = 1$, $y_2 = -4$, $y_3 = -4$

Since $Ux = y$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}$$

$2x_1 + x_2 + x_3 + 0x_4 = 1$ —— ①

$-x_2 - 2x_3 + x_4 = -4$ —— ②

$-4x_3 + 4x_4 = -4$ —— ③

$\div 4 \quad -x_3 + x_4 = -1$

Sub $\boxed{x_3 = t}$ $\boxed{x_4 = -1-t}$

② $\Rightarrow$ $-x_2 - 2t - 1 - t = -4$

$-x_2 - 3t = -3$

$-x_2 = -3 + 3t$

$\boxed{x_2 = 3 - 3t}$

① $\Rightarrow$ $2x_1 + 3 - 3t + t = 1$

$2x_1 - 2t = -2$

$2x_1 = -2 + 2t$

$\boxed{x_1 = t - 1}$

$x_1 = t - 1$, $x_2 = 3(1-t)$, $x_3 = t$, $x_4 = -(1+t)$

Find the LDU factorization for each of the following matrices

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$ $R_2 \to R_2 - 4R_1$

$\sim \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$

Since $R_2 \to R_2 - 4R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_2 \to R_2 + 4R_1$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$ $R_2 \to R_2 - 8R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U$

since $R_2 \to R_2 - 8R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \to R_2 + 8R_1$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

$A' \; PA \; L$

If $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is any permutation matrix then express $\boxed{PA}$ as LDU factorization

$\boxed{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$A' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$

Since $R_3 \to R_3 - R_2$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\therefore PA = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = LDU$

For all possible permutation matrices P, find the LDU factorization of PA for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_2\ (P_1\ P_3)$ $P_4\ (P_2\ P_5)$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2 \qquad R_1 \leftrightarrow R_3 \qquad R_2 \leftrightarrow R_3$

$$P_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_1 \leftrightarrow R_3$

$$E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$P_1A = LU$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_1A = LDU$

$$P_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} R_3 \Leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \Rightarrow R_2 - 2R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \Leftrightarrow R_2$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$P_2 A = L U$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_2 A = L D U$

$$P_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \to R_2 - R_1$ $\qquad\qquad$ $R_3 \to R_3 - 2R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$P_3A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3A = LDU$$

Given equations

4. ii) $x - y + z = 1, \quad x + 3y + az = 2, \quad 2x + ay + 3z = 3$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & a & 2 \\ 2 & a & 3 & 3 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & a-1 & 1 \\ 0 & a+2 & 1 & 1 \end{bmatrix} \quad R_3 \to -4R_3 + (a+2)R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & a-1 & 1 \\ 0 & 0 & -4+(a+2)(a-1) & a-2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & a-1 & 1 \\ 0 & 0 & (a+3)(a-2) & a-2 \end{bmatrix}$$

i) If $a \neq 3, a \neq 2$

$\rho(A) = \rho(A,B) =$ no of unknowns

$\therefore$ It has unique solution

ii) If $a \neq -3, a = 2$

$\rho(A) = \rho(A,B) \neq$ no of unknowns

$\therefore$ It has infinite no of solutions

iii) If $a = -3, a \neq 2$

$\rho(A) \neq \rho(A,B)$

$\therefore$ It has no solution

Application:

- Cryptography

- Electrical network problem:

i) Ohm's law

ii) Kirchoff's current law (KCL)

iii) Kirchoff's voltage law (KVL)

Ohm's law:

$V = IR$

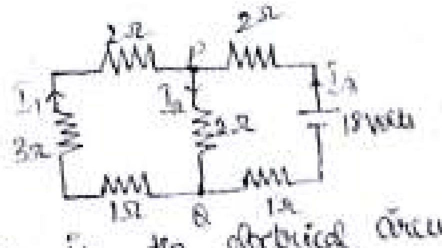
Kirchoff's current law:

The current flow into a node equals the current flow outside the node

Kirchoff's voltage law:

The algebraic sum of the voltage drop around a close loop equals the total voltage sources in the loop.

1.



Determine the currents in the network given in the above figure.

Determine in the electrical circuit I_1, I_2, I_3

Determine the current in the network

By KCL

at P, $I_1 + I_3 = I_2$

at Q, $I_2 = I_3 + I_1$

$I_1 - I_2 + I_3 = 0$ —— ①

By KVL,

$$2I_1 + 3I_1 + I_1 + 2I_2 = 0$$

$$6I_1 + 2I_2 = 0 \quad \text{---} (1)$$

$$2I_3 + I_3 + 2I_2 = 18$$

$$3I_3 + 2I_2 = 18 \quad \text{---} (2)$$

The system of linear equations

$$I_1 - I_2 + I_3 = 0$$

$$6I_1 + 2I_2 = 0$$

$$3I_3 + 2I_2 = 18$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 2 & 3 & 18 \end{bmatrix} \; R_2 \to R_2 - 6R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 8 & -6 & 0 \\ 0 & 2 & 3 & 18 \end{bmatrix} \; R_3 \to 4R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 8 & -6 & 0 \\ 0 & 0 & 18 & 72 \end{bmatrix} \; \begin{matrix} R_2 \to R_2/8 \\ R_3 \to R_3/18 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \; R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \; \begin{matrix} R_1 \to R_1 - \frac{1}{4} R_3 \\ R_2 \to R_2 + \frac{3}{4} R_3 \end{matrix}$$

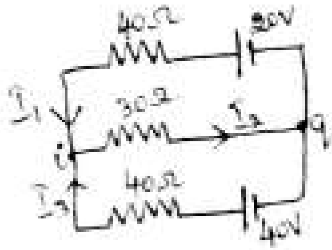
$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\therefore I_1 = -1$

$I_2 = 3$

$I_3 = 4$

2



By KCL,

at i, $I_1 + I_3 = I_2$

at q, $I_2 = I_1 + I_3$

$I_1 - I_2 + I_3 = 0$ — (1)

By KVL,

$40I_1 + 30I_2 = 20$ — (2)

$30I_2 + 40I_3 = 40$ — (3)

(2) & (3) ÷ by 10

$4I_1 + 3I_2 = 2$ — (4)

$3I_2 + 4I_3 = 4$ — (5)

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 4 \end{array}\right] \quad R_2 \to R_2 - 4R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 2 \\ 0 & 3 & 4 & 4 \end{array}\right] \quad R_3 \to 7R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 2 \\ 0 & 0 & 40 & 22 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2/7 \\ R_3 \to R_3/40 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -4/7 & 2/7 \\ 0 & 0 & 1 & 11/20 \end{array}\right] R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & 2/7 \\ 0 & 1 & -4/7 & 2/7 \\ 0 & 0 & 1 & 11/20 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 3/7 R_3 \\ R_2 \to R_2 + 4/7 R_3 \\ R_3 \to 1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/20 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 11/20 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.05 \\ 0 & 1 & 0 & 0.6 \\ 0 & 0 & 1 & 0.55 \end{array}\right]$$

$I_1 = 0.05$

$I_2 = 0.6$

$I_3 = 0.55$

Cryptography:-

In cryptography technique

$$\underset{\text{plain text}}{X} \xrightarrow[\text{coding}]{} \underset{\text{cypher text}}{C} \xrightarrow[\text{decoding}]{} \underset{\text{plain text}}{X}$$

By considering

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B \quad [A \text{ is Invertible}]$$

Also let A B C ... Z blankspace ? !
0 1 2 ... 26 27 28

Using cryptography technique sending the message "GOOD LUCK"

6 14 14 3 26 11 20 2 10

Q) Encode GOOD LUCK using $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Constructing 3×3 matrix for that take

$$C_1 = \begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix} \quad C_2 = \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix} \quad C_3 = \begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix}$$

Take any orbitrary non-singular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

By Gauss Jordan method

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B$$

$$AC_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix}$$

$$AC_1 = \begin{bmatrix} 6 \\ 26 \\ 34 \end{bmatrix}$$

$$AC_2 = \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix}$$

$$AC_3 = \begin{bmatrix} 20 \\ 42 \\ 32 \end{bmatrix}$$

The message sent is 6, 26, 34, 3, 26, 11, 20, 42, 32

To read the message $X = A^{-1}B$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 26 \\ 34 \end{bmatrix}$$

$$B = AC_1, AC_2, AC_3$$

$$X_1 = \begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 20 \\ 8 \\ 10 \end{bmatrix}$$

1. Decode the cipher text 19, 45, 26, 13, 36, 41 using

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

In the decoding first break the message into 2 vectors in R^3, we have

$$X_1 = \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix} \quad X_2 = \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}$$

Since $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$[A \vdots I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Now $A^{-1}X_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix}$

$$A^{-1}X_1 = \begin{bmatrix} 19 \\ 7 \\ 0 \end{bmatrix}$$

-38
45
7

-45
19
26

-45
45

$$A^{-1}X_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}$$

$$\begin{array}{r} 36 \\ -26 \\ \hline 10 \end{array} \qquad \begin{array}{r} 54 \\ -36 \\ \hline 18 \end{array}$$

$$A^{-1}X_2 = \begin{bmatrix} 13 \\ 10 \\ 18 \end{bmatrix}$$

The decoded numbers are 19, 7, 0, 13, 10, 18
T H A N K S

2. Encode "TAKE UFO" using the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Properties :-

i) If E_1 & E_2 are elementary matrices, then $E_1 \cdot E_2 = E_2 \cdot E_1$ which satisfies commutative property

ii) A consistent system has infinitely many solutions if it has atleast one free variable, and there has a unique solution if it has no free variable. In fact, if a consistent system has a free variable (which always happens when the number of equation is less than that of unknowns), then by assigning arbitrary value to the free variable, one always obtains infinitely many solutions.

Free Variables:

Among the variables in a system ones corresponding to the columns containing leading ones are called the basic variables, and ones corresponding to the columns without leading ones if there are any are called free variables.

Clearly the sum of the number of basic variables & that of free variables is equal to the total number of unknowns.

$$\begin{bmatrix} \underline{1} & 0 & 1 & 5 & 3 \\ 0 & \underline{1} & -2 & 1 & 0 \\ 0 & 0 & \underline{1} & 10 & 15 \end{bmatrix}$$

$\underbrace{\qquad\qquad}_{\text{Basic Variables}}\ \underbrace{\qquad}_{\text{Free Variables}}$

2. "TAKE UFO."

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 19 \\ 0 \\ 10 \end{bmatrix} \quad C_2 = \begin{bmatrix} 4 \\ 26 \\ 20 \end{bmatrix} \quad C_3 = \begin{bmatrix} 5 \\ 14 \\ 26 \end{bmatrix}$$

$$AC_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 19 \\ 38 \\ 29 \end{bmatrix}$$

$$AC_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 26 \\ 20 \end{bmatrix} = \begin{bmatrix} 4 \\ 34 \\ 50 \end{bmatrix}$$

$$AC_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \\ 26 \end{bmatrix} = \begin{bmatrix} 5 \\ 19 \\ 45 \end{bmatrix}$$

The message sent is 5, 14, 26, 19, 38, 29,

The message sent is 19, 38, 29, 4, 34, 50, 5, 19, 45

UNIT-2 VECTOR SPACE

Ring $(R, +, \cdot)$

i) $(R,+)$ abelian group

ii) $(R, \cdot)$ semi group

iii) Distributive properties

$a(b+c) = a \cdot b + a \cdot c$ (LDL)

$(b+c) \cdot a = b \cdot a + c \cdot a$ (RDL)

Field $(F, +, \cdot)$

i) $(F,+)$ abelian group

ii) $(F, \cdot)$ semi group

iii) Distributive properties

iv) Identity axiom w.r.t multiplication

v) Inverse axiom w.r.t multiplication for non zero elements

vi) Abelian group w.r.t multiplication

Vector space -

Let V be a non-empty set of vectors and F be a field, then $(V(F), +, \cdot)$ is said to be a vector space if it satisfies the following axioms

i) $(V, +)$ is an abelian group

i.e. i) $\bar{a}+\bar{b} \in V \quad \forall \bar{a}, \bar{b} \in V$ [closure axiom]

ii) $(\bar{a}+\bar{b})+\bar{c} = \bar{a}+(\bar{b}+\bar{c}), \ \forall \bar{a}, \bar{b}, \bar{c} \in V$ [Associative axiom]

iii) $\forall \bar{a} \in V$, $\exists \bar{e} = \bar{0} \in V$ such that [Identity axiom]

$\bar{a}+\bar{e} = \bar{e}+\bar{a} = \bar{a}$

where $\bar{e} = \bar{0}$ is called the identity element

iv) $\forall \bar{a} \in V$, $\exists \bar{b} \in V$ such that [Inverse axiom]

$\bar{a}+\bar{b} = \bar{b}+\bar{a} = \bar{e}$

where $\bar{b}$ is called the inverse element of $\bar{a}$

v) $\bar{a}+\bar{b} = \bar{b}+\bar{a} \quad , \forall \bar{a}, \bar{b} \in V$ [Abelian (commutative) axiom]

2) If $k_1, k_2 \in F$ and $\bar{a}, \bar{b} \in V$, then

$$k_1 \bar{a} \in V$$
$$k_1(\bar{a}+\bar{b}) = k_1\bar{a} + k_1\bar{b}$$
$$(k_1+k_2)\bar{a} = k_1\bar{a} + k_2\bar{a}$$
$$k_1(k_2 \bar{a}) = (k_1k_2)\bar{a}$$
$$1.\bar{a} \in V$$

NOTE:-

If the field F is a set of real numbers, then the vector space is called real vector space and simply it is denoted by V(R).

If the field F is a set of complex numbers, then the vector space is called complex vector space and simply it is denoted by V(C).

Examples:-

1. Let V be the set of all pairs (x, y) of real number defined $(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

$$k.(x,y) = (kx, ky)$$

Is the set V a vector space under these operations? Justify your answer

*** 2. Let V be the set of all pairs (x,y) of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x,y) + (u,v) = (x+2u, y+2v), \; k(x,y) = (kx, ky)$$

Is the set V a vector space under these operations? Justify your answer

3. Let $C(R)$ denote the set of real valued continuous functions defined on real line R. For two functions f and g and a real number K, the sum $f+g$ and the scalar multiplication Kf of them are defined by

$$(f+g)(x) = f(x)+g(x)$$

$$(Kf)(x) = Kf(x)$$

Then prove that $C(R)$ is a vector space under these operations

2. $V = \{(x,y) . x,y \in R\}$

and $(x,y)+(u,v) = (x+2u, y+2v)$, $K(x,y) = (Kx, Ky)$

i) closure axiom:-

Let $\bar{a} = (x,y), \bar{b} = (u,v)$

then $\bar{a}+\bar{b} = (x,y)+(u+v)$

$= (x+2v, y+2v) \in V$

i.e. $\bar{a}+\bar{b} \in V, \forall \bar{a}, \bar{b} \in V$

$\therefore V$ satisfies closure axiom

ii) Associative axiom:-

Let $\bar{a} = (x_1, y_1), \bar{b} = (x_2, y_2), \bar{c} = (x_3, y_3)$

$x_1, x_2, x_3, y_1, y_2, y_3 \in R$

Then $(\bar{a}+\bar{b})+\bar{c} = \{(x_1, y_1)+(x_2, y_2)\}+(x_3, y_3)$

$= (x_1+2x_2, y_1+2y_2)+(x_3, y_3)$

$= ((x_1+2x_2)+2x_3, (y_1+2y_2)+2y_3)$

$= (x_1+2x_2+2x_3, y_1+2y_2+2y_3)$

$\bar{a}+(\bar{b}+\bar{c}) = (x_1,y_1)+\{(x_2,y_2)+(x_3,y_3)\}$

$= (x_1,y_1)+(x_2+2x_3,\ y_2+2y_3)$

$= (x_1+2(x_2+2x_3),\ y_1+2(y_2+2y_3))$

$= (x_1+2x_2+4x_3,\ y_1+2y_2+4y_3)$

$\therefore\ (\bar{a}+\bar{b})+\bar{c} \neq \bar{a}+(\bar{b}+\bar{c})$

$\therefore$ V is not a vector space, since it is not satisfies associative axiom

1. iii) <u>Identity axiom</u> :-

Let $\bar{a} \in V$

Then $\bar{a} = (x_1,y_1)$, $x_1,y_1 \in R$

Since $0 \in R$

$\therefore\ (0,0) \in V$

i.e. $\bar{e} = (0,0) \in V$

Now $\bar{a}+\bar{e} = (x_1,y_1)+(0,0)$

$= (x_1+0,\ y_1+0)$

$= (x_1,y_1)$

$= \bar{a}$

Similarly $\bar{e}+\bar{a} = (0,0)+(x_1,y_1)$

$= (0+x_1,\ 0+y_1)$

$= (x_1,y_1)$

$= \bar{a}$

i.e., $\forall\ \bar{a} \in V$, $\exists\ \bar{e} \in V$ such that

$\bar{a}+\bar{e} = \bar{e}+\bar{a} = \bar{a}$

$\therefore$ V satisfies identity axiom

iv) Inverse axiom:-

Let $\bar{a} \in V$

Then $\bar{a} = (x_1, y_1)$; $x_1, y_1 \in R$

Since $x_1, y_1 \in R$

$\therefore -x_1, -y_1 \in R$

$\Rightarrow (-x_1, -y_1) \in V$

i.e., $\bar{b} = (-x_1, -y_1) \in V$

Now $\bar{a}+\bar{b} = (x_1, y_1) + (-x_1, -y_1)$

$= (x_1 - x_1, y_1 - y_1)$

$= (0,0) = \bar{e}$

Also $\bar{b}+\bar{a} = (-x_1, -y_1) + (x_1, y_1)$

$= (-x_1 + x_1, -y_1 + y_1)$

$= (0,0) = \bar{e}$

i.e., $\forall \bar{a} \in V, \exists \bar{b} \in V$ such that

$\bar{a}+\bar{b} = \bar{b}+\bar{a} = \bar{e}$

$\therefore$ V satisfies inverse axiom

v) Abelian (commutative) axiom:-

Let $\bar{a}, \bar{b} \in V$

Then $\bar{a} = (x_1, y_1)$, $\bar{b} = (x_2, y_2)$; $x_1, x_2, y_1, y_2 \in R$

Now $\bar{a}+\bar{b} = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

Now $\bar{b}+\bar{a} = (x_2, y_2) + (x_1, y_1)$

$= (x_2 + x_1, y_2 + y_1)$

$= (x_1 + x_2, y_1 + y_2)$

i.e., $\bar{a}+\bar{b}=\bar{b}+\bar{a}, \forall \bar{a}, \bar{b} \in V$

$\therefore$ V satisfies abelian axiom and hence $(V,+)$ is an abelian group

Let k_1, k_2 be any two scalars and $\bar{a}, \bar{b} \in V$

Then $\bar{a}=(x_1, y_1)$, $\bar{b}=(x_2, y_2)$; $x_1, x_2, y_1, y_2 \in R$

vi) $k_1\bar{a}=k_1(x_1, y_1)$
$=(k_1x_1, k_1y_1) \in V$

vii) $(k_1+k_2)\bar{a} = (k_1+k_2)(x_1, y_1)$
$=((k_1+k_2)x_1, (k_1+k_2)y_1)$
$=(k_1x_1+k_2x_1, k_1y_1+k_2y_1)$
$=(k_1x_1, k_1y_1)+(k_2x_1, k_2y_1)$
$=k_1(x_1, y_1)+k_2(x_1, y_1)$
$=k_1\bar{a}+k_2\bar{a}$

viii) $k_1.(k_2\bar{a}) = k_1.(k_2(x_1, y_1))$
$=k_1.(k_2x_1, k_2y_1)$
$=(k_1(k_2x_1), k_1(k_2y_1))$
$=((k_1k_2)x_1, (k_1k_2)y_1)$
$=(k_1k_2)(x_1, y_1)$
$=(k_1k_2)\bar{a}$

ix) $k_1(\bar{a}+\bar{b}) = k_1.((x_1, y_1)+(x_2, y_2))$
$=k_1.(x_1+x_2, y_1+y_2)$
$=(k_1(x_1+x_2), k_1(y_1+y_2))$

$= (k_1x_1 + k_1x_2 , k_1y_1 + k_1y_2)$

$= (k_1x_1 , k_1y_1) + (k_1x_2 , k_1y_2)$

$= k_1(x_1,y_1) + k_1(x_2,y_2)$

$= k_1\bar{a} + k_1\bar{b}$

x) $1\cdot\bar{a} = 1\cdot(x_1,y_1)$

$= (1\cdot x_1 , 1\cdot y_1)$

$= (x_1 , y_1) \in V$

$\therefore$ V satisfies all the properties of vector space w.r.t addition and and hence it is a vector space w.r.t addition and multiplication.

i) Closure axiom :-

Let $\bar{a} = (x,y)$, $\bar{b} = (x_1,y_1)$

then $\bar{a}+\bar{b} = (x,y)+(x_1,y_1)$

$= (x+x_1 , y+y_1) \in V$

i.e, $\bar{a}+\bar{b} \in V$, $\forall \bar{a},\bar{b} \in V$

$\therefore$ V satisfies closure axiom

ii) Associative axiom :-

Let $\bar{a} = (x_1,y_1)$, $\bar{b} = (x_2,y_2)$, $\bar{c} = (x_3,y_3)$: $x_1,x_2,x_3,$
$y_1,y_2,y_3 \in R$

Then $(\bar{a}+\bar{b})+\bar{c} = \{(x_1,y_1)+(x_2,y_2)\}+(x_3,y_3)$

$= (x_1+x_2 , y_1+y_2)+(x_3,y_3)$

$= ((x_1+x_2)+x_3 , (y_1+y_2)+y_3)$

$= (x_1+x_2+x_3, y_1+y_2+y_3)$

$\bar{a}+(\bar{b}+\bar{c}) = (x_1,y_1)+\{(x_2,y_2)+(x_3,y_3)\}$

$= (x_1,y_1)+(x_2+x_3, y_2+y_3)$

$= (x_1+(x_2+x_3), y_1+(y_2+y_3))$

$= (x_1+x_2+x_3, y_1+y_2+y_3)$

$\therefore (\bar{a}+\bar{b})+\bar{c} = \bar{a}+(\bar{b}+\bar{c})$

$\therefore$ V satisfies associative axiom

Subspace :-

Let (V,+,.) be a vector space and W be a non-empty set then W is called a subspace of V if W itself is a vector space and subset of V under the addition and the scalar multiplication defined in V.

Note :-

i) A vector space V itself and the zero vector $\{\bar{0}\}$ are trivially subspaces

ii) A non empty subset W of a vector space V is a subspace if and only if $\bar{x}+\bar{y}$ and $k\bar{x}$ are contained in W (or equivalently $\bar{x}+k\bar{y} \in W$) for any vectors $\bar{x}$ and $\bar{y}$ in W and any scalar $k \in R$

which of the following are the subspaces of

3-space (R^3)? Justify your answer.

i) $W = \{(x,y,z) \in R^3 : ax+by+cz=0\}$

ii) $W = \{(x,y,z) \in R^3 : xyz=0\}$

iii) $W = \{(2t, 3t, 4t) \in R^3 : t \in R\}$

iv) $W = \{(x,y,z) \in R^3 : x^2+y^2-z^2=0\}$

v) $W = \{x \in R^3 : x^T\bar{u} = \bar{0} = x^T\bar{v}\}$ where $\bar{u}$ and $\bar{v}$ are any two fixed non zero vectors in R^3

V = 3-space (R^3)

$= \{(x,y,z) : x,y,z \in R\}$

iv) clearly $W \neq \phi$ and $W \subseteq V$

let $\alpha, \beta \in W$ and K be any scalar

Then $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

where $x_1^2+y_1^2-z_1^2=0$ and $x_2^2+y_2^2-z_2^2=0$

$\alpha + K\beta = (x_1, y_1, z_1) + K(x_2, y_2, z_2)$

$= (x_1, y_1, z_1) + (Kx_2, Ky_2, Kz_2)$

$= (x_1+Kx_2, y_1+Ky_2, z_1+Kz_2) \in R^3$

now $(x_1+Kx_2)^2 + (y_1+Ky_2)^2 - (z_1+Kz_2)^2 =$

$= x_1^2 + K^2x_2^2 + 2Kx_1x_2 + y_1^2 + K^2y_2^2 + 2Ky_1y_2 - z_1^2 - K^2z_2^2 - 2Kz_1z_2$

$= (x_1^2+y_1^2-z^2) + K^2(x_2^2+y_2^2-z_2^2) + 2K(x_1x_2+y_1y_2 - z_1z_2)$

$= 0 + 2K(x_1x_2+y_1y_2-z_1z_2)$

$= 2K(x_1x_2 + y_1y_2 - z_1z_2)$

$\neq 0$

$\therefore \alpha + K\beta \notin W$ and hence W is not a subspace.

1. Let $V = C(R)$ be the vector space of all conditional function on R. which of the following sets W are subspaces? Justify your answer

i) W is the set of all differentiable functions on R

ii) W is the set of all bounded continuous functions on R

iii) W is the set of all continuous non negative - valued function on R i.e. $f(x) \geq 0$ for any $x \in R$

iv) W is the set of all continuous odd functions on R. i.e. $f(-x) = -f(x)$ for any $x \in R$.

v) W is the set of all polynomials with integer coefficients

iv) Let $V = C(R)$ be the vector space of all continuous function on R

continuous function

$= \{f(x) : f(x) \text{ is continuous on } R\}$

and W be the set of all continuous and odd function on R.

i.e. $W = \{f(x) : f(x) \text{ is continuous and } f(x) \text{ is odd function}\}$

Clearly $W \neq \phi$, $W \subseteq V$

Let $\alpha, \beta \in W$ and K be scalar

Then $\alpha = f(x)$, $f(x)$ is continuous, $f(-x) = -f(x)$

$\beta = g(x)$, $g(x)$ is continuous, $g(-x) = -g(x)$

Now $\alpha + K\beta = f(x) + Kg(x)$

Here $f(x) + Kg(x)$ is always continuous i.e, $\alpha + K\beta$ is continuous

[$\because$ Sum of two continuous functions is always continuous and f be continuous and c be any scalar, then cf is also continuous]

Now $(\alpha + K\beta)(-x) = (f + Kg)(-x)$

$$= f(-x) + (Kg)(-x)$$
$$= -f(x) + Kg(-x)$$
$$= -f(x) - Kg(x)$$
$$= -(f(x) + Kg(x))$$
$$= -(f + Kg)(x)$$
$$= -(\alpha + K\beta)(x)$$

$\therefore$ $\alpha + K\beta$ is an odd function

$\therefore$ $\alpha + K\beta \in W$ and hence W is a subspace of V

$\oplus \rightarrow$ Direct sum

Def:- Let U and W be the subspaces of a vector space V, then

i) the sum of U and W is defined by

$$U + W = \{\bar{u} + \bar{w} : \bar{u} \in U, \bar{w} \in W\}$$

ii) A vector space V is called the direct sum of two subspaces U and W, written as

$$V = U \oplus W \text{ if } V = U + W \text{ and } U \cap W = \{\bar{0}\}$$

Theorem :-

Let U and W be the two subspaces of a vector space V, then

i) $U \cap W$ is a subspace of V

ii) $U + W$ is a subspace of V

Proof :-

Let U and W are two subspaces of a vector space V

i) Then to prove that $U \cap W$ is a subspace of V

Since U and W are subspaces of V

$\therefore U \neq \phi$, $W \neq \phi$ and $U \subseteq V$, $W \subseteq V$

i.e. $\bar{0} \in V$, $\bar{0} \in W$ and $U \subseteq V$, $W \subseteq V$

$\Rightarrow \bar{0} \in U \cap W$ and $U \cap W \subseteq V$

i.e, $U \cap W \neq \phi$ and $U \cap W \subseteq V$

Let $\alpha, \beta \in U \cap W$ and K be any scalar.

Then $\alpha, \beta \in U$ and $\alpha, \beta \in W$

Now $\alpha, \beta \in U$, K be any scalar and U is a subspace of V.

$\Rightarrow \alpha + K\beta \in U$ — ①

Now $\alpha, \beta \in W$, K be any scalar and W is a subspace of V

$\Rightarrow \alpha + K\beta \in W$ — ②

For eqn ① & ② we get

$\alpha + K\beta \in U \cap W$

Hence $U \cap W$ is a subspace of V

ii) Then to prove that $U+W$ ...

Since U and W are subspaces of V

$\therefore U \neq \phi, \bar{W} \neq \phi$ and $U \subseteq V, W \subseteq V$

i.e $\bar{0} \in U, \bar{0} \in W$ and $U \subseteq V, W \subseteq V$

$\Rightarrow \bar{0} \in U+W$ and $U+W \subseteq V$

i.e $U+W \neq \phi$ and $U+W \subseteq V$

Let $\alpha, \beta \in U+W$ and K be any scalar

Since $\alpha \in U+W = \{\bar{u}_1 + \bar{w}_1 : \bar{u}_1 \in U, \bar{w}_1 \in W\}$

$\therefore \alpha = \bar{u}_1 + \bar{w}_1 ; \bar{u}_1 \in U, \bar{w}_1 \in W$

$\beta \in U+W = \{\bar{u}_2 + \bar{w}_2 : \bar{u}_2 \in U, \bar{w}_2 \in W\}$

$\beta = \bar{u}_2 + \bar{w}_2 ; \bar{u}_2 \in U, \bar{w}_2 \in W$

Now
$$\begin{aligned} \alpha + K\beta &= (\bar{u}_1 + \bar{w}_1) + K(\bar{u}_2 + \bar{w}_2) \\ &= \bar{u}_1 + \bar{w}_1 + K\bar{u}_2 + K\bar{w}_2 \\ &= (\bar{u}_1 + K\bar{u}_2) + (\bar{w}_1 + K\bar{w}_2) \\ &\in U+W \end{aligned}$$

$\therefore$ ~~U~~ $U+W$ is a subspace of V

Theorem:-

Let V be a vector space and let $\bar{x}, \bar{y}$ be vectors in V. Then

i) $\bar{x} + \bar{y} = \bar{y}$ implies $\bar{x} = \bar{0}$;

ii) $0\bar{x} = \bar{0}$

iii) $K\bar{0} = \bar{0}$ for any $K \in R$

iv) $-\bar{x}$ is unique and $-\bar{x} = (-1)\bar{x}$

v) If $K\bar{x} = \bar{0}$, then $k = 0$ or $\bar{x} = \bar{0}$

Let V be a vector space and $\bar{x}, \bar{y} \in V$

i) Suppose $\bar{x}+\bar{y}=\bar{y}$

Now $\bar{x} = \bar{x}+\bar{0}$

$= \bar{x}+(\bar{y}-\bar{y})$

$= \bar{x}+(\bar{y}+(-\bar{y}))$

$= (\bar{x}+\bar{y})+(-\bar{y})$, by associative axiom

$= \bar{y}+(-\bar{y}) \quad \langle \because \bar{x}+\bar{y}=\bar{y} \rangle$

$= \bar{y}-\bar{y}$

$= \bar{0}$

ii) $0\bar{x} = 0\bar{x}+\bar{0}$

$= 0\bar{x}+(\bar{x}-\bar{x})$

$= 0\bar{x}+(\bar{x}+(-\bar{x}))$

$= (0\bar{x}+\bar{x})+(-\bar{x})$, by associative axiom

$= (0\bar{x}+1.\bar{x})+(-\bar{x})$

$= (0+1)\bar{x}+(-\bar{x})$

$= 1.\bar{x}+(-\bar{x})$

$= \bar{x}-\bar{x}$

$= \bar{0}$

$0\bar{x} = (0+0)\bar{x}$

$0\bar{x}+0\bar{x}$

$0\bar{x}+\bar{0} = 0\bar{x}+0\bar{x}$

$\bar{0} = 0\bar{x}$

iii) $K\bar{0} = K(\bar{0}+\bar{0})$

$= K\bar{0}+K\bar{0}$

$K\bar{0}+\bar{0} = K\bar{0}+K\bar{0}$

$\bar{0} = K\bar{0}$

i.e., $K\bar{0}=\bar{0}$

iv) Suppose $\overline{x}_1$ is another negative of $\overline{x}$

Then $\overline{x}+\overline{x}_1=\overline{0}$ —— ①

Now $-\overline{x}=-\overline{x}+\overline{0}$

$=-\overline{x}+(\overline{x}+\overline{x}_1)$ by equation ①

$=(-\overline{x}+\overline{x})+\overline{x}_1$

$=\overline{0}+\overline{x}_1$

$=\overline{x}_1$

$\therefore -\overline{x}$ is unique

Now $\overline{x}+(-1)\overline{x}=1.\overline{x}+(-1)\overline{x}$

$=(1+(-1))\overline{x}$

$=(1-1)\overline{x}$

$=0.\overline{x}$

$=\overline{0}$

$\Rightarrow (-1)\overline{x}=-\overline{x}$

i.e., $-\overline{x}=(-1)\overline{x}$

v) Suppose $k\overline{x}=\overline{0}$

if $k=0$, then there is nothing to prove

Suppose $k\neq 0$

Now $\overline{x}=\frac{1}{k}(k\overline{x})$

$=\frac{1}{k}(\overline{0})$

$\overline{x}=\overline{0}$

if $k\overline{x}=\overline{0}$ then $k=0$ & $\overline{x}=\overline{0}$

Theorem: A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\bar{v} \in V$ there exist unique $\bar{u} \in U$ and $\bar{w} \in W$ such that $\bar{v} = \bar{u} + \bar{w}$

Proof:-

Let V be a vector space and U, W are subspaces of V.

Then to prove that $V = U \oplus W$

$V = U \oplus W \Leftrightarrow$ for any $\bar{v} \in V$ there exit unique $\bar{u} \in U$ and $\bar{w} \in W$ such that

$$\bar{v} = \bar{u} + \bar{w}$$

First suppose, $V = U \oplus W$

By the definition of direct sum, we have

$$V = U + W \text{ and } U \cap W = \{\bar{0}\}$$

$$U + W = \{\bar{u} + \bar{w} : \bar{u} \in U \text{ and } \bar{w} \in W\}$$

Since $V = U + W$

$\therefore$ for any $\bar{v} \in V$ such that

$\bar{v} \in V$ such that

$\bar{v} = \bar{u} + \bar{w}$ where $\bar{u} \in U, \bar{w} \in W$

To prove $\bar{v} = \bar{u} + \bar{w}$ is unique

For this assume $\bar{v} = \bar{u}_1 + \bar{w}_1$

Then $\bar{u} + \bar{w} = \bar{u}_1 + \bar{w}_1$

$\Rightarrow \bar{u} - \bar{u}_1 = \bar{w}_1 - \bar{w} \in U \cap W$

Since $U \cap W = \{\bar{0}\}$

$\therefore \bar{u}-\bar{u}_1=\bar{0}$ and $\bar{w}_1-\bar{w}=\bar{0}$

$\Rightarrow \bar{u}=\bar{u}_1$ and $\bar{w}=\bar{w}_1$

$\therefore \bar{v}=\bar{u}+\bar{w}$ is unique

Conversely, suppose for any $\bar{v}\in V$ there exists unique $\bar{u}\in U$ and $\bar{w}\in W$ such that $\bar{v}=\bar{u}+\bar{w}$

Then to prove that, $V=U\oplus W$

Since for any $\bar{v}\in V$, there exists $\bar{u}\in U$ and $\bar{w}\in W$ such that $\bar{v}=\bar{u}+\bar{w}$

$$V=U+W$$

Next to prove, $U\cap W=\{\bar{0}\}$

For this let $U\cap W$ consisting a non zero vector say $\bar{v}_1$

Then $\bar{v}_1=\bar{0}+\bar{v}_1=\bar{v}_1+\bar{0}=\frac{1}{2}\bar{v}_1+\frac{1}{2}\bar{v}_1=\frac{1}{3}\bar{v}_1+\frac{2}{3}\bar{v}_1=\dots$

i.e., $\bar{v}_1$ can be expressed in different ways, which is a contradiction to for any $\bar{v}\in V$, $\exists$ unique $\bar{u}\in U$ and $\bar{w}\in W$ such that

$$\bar{v}=\bar{u}+\bar{w}$$

So, our assumption $U\cap W$ consisting a non-zero vector is wrong.

$$\therefore U\cap W=\{\bar{0}\}$$

Hence $V=U\oplus W$

$\therefore$ The condition is sufficient

** Example:-

3) Give any example for sum but not direct sum

Let $U = \{a\bar{i}+c\bar{k} : a, c \in R\}$ and

$W = \{b\bar{j}+c\bar{k} : b, c \in R\}$

Then U and W are subspaces of R^3 and a vector in $U+W$ is of the form

$$(a\bar{i}+c_1\bar{k})+(b\bar{j}+c_2\bar{k}) = a\bar{i}+b\bar{j}+c_1\bar{k}+c_2\bar{k}$$

$$= a\bar{i}+b\bar{j}+(c_1+c_2)\bar{k}$$

$$= a\bar{i}+b\bar{j}+c\bar{k} \text{ where } c = c_1+c_2$$

i.e., $U+W = R^3$

Since for $\bar{k} \in U \cap W$ is not zero vector, because

$$\bar{k} = \tfrac{1}{2}\bar{k}+\tfrac{1}{2}\bar{k} = \tfrac{1}{3}\bar{k}+\tfrac{2}{3}\bar{k}$$

$\therefore U \oplus W \neq R^3$

Bases:-

Definition:- Let V be a vector space and let $\{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m\}$ be a set of vectors in V, then a vector $\bar{y}$ in V of the form

$$\bar{y} = a_1\bar{x}_1 + a_2\bar{x}_2 + \ldots a_m\bar{x}_m$$

is called the linear combination of the vectors $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m$ where $a_1, a_2, \ldots a_m$ are scalars.

Definition:-

A set of vectors $\{\bar{x}_1, \bar{x}_2, \dots \bar{x}_m\}$ in a vector space V is said to be linearly independent if the vector equation, called the linear dependence of $\bar{x}_i$'s,

$$c_1\bar{x}_1 + c_2\bar{x}_2 + \dots c_m\bar{x}_m = \bar{0}$$

has only the trivial solution $c_1=0, c_2=0, \dots c_m=0$. Otherwise it is said to be linearly dependent

Example :-

1. $(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

2. $(1,2,3) = 1(1,0,0) + 2(0,1,0) + (-3)(0,0,1)$

3. Express the given functions as a linear combination of functions in the given set Q

i) $P(x) = -1-3x+3x^2$ & $Q(x) = \{P_1(x), P_2(x), P_3(x)\}$ where

$P_1(x) = 1+2x+x^2$, $P_2(x) = 2+5x$, $P_3(x) = 3+8x-2x^2$

ii) $P(x) = -2-4x+x^2$ & $Q = \{P_1(x), P_2(x), P_3(x), P_4(x)\}$ where

$P_1(x) = 1+2x^2+x^3$, $P_2(x) = 1+x+2x^3$, $P_3(x) = -1-3x-4x^3$,

$P_4(x) = 1+2x-x^2+x^3$

Consider $P(x) = a_1P_1(x) + a_2P_2(x) + a_3P_3(x)$

$$\Rightarrow -1-3x+3x^2 = a_1\{1+2x+x^2\} + a_2\{2+5x\} +$$
$$a_3\{3+8x-2x^2\}$$

$$-1-3x+3x^2 = a_1 + a_1 2x + a_1x^2 + 2a_2 + 5a_2x + 3a_3 +$$
$$8a_3x - 2a_3x^2$$

$$-1-3x+3x^2 = (a_1+2a_2+3a_3) + (2a_1+5a_2+8a_3)x +$$
$$(a_1-2a_3)x^2$$

Compare like terms on both sides, we get

$$a_1+2a_2+3a_3=-1 \quad \text{—} (1)$$

$$2a_1+5a_2+8a_3=-3 \quad \text{—} (2)$$

$$a_1-2a_3=3 \quad \text{—} (3)$$

$$(1)\times 5 \Rightarrow 5a_1+10a_2+15a_3=-5$$

$$(2)\times 2 \Rightarrow 4a_1+10a_2+16a_3=-6$$

$$a_1 \quad -a_3=1 \quad \text{—} (4)$$

$$(3) \Rightarrow a_1-2a_3=3$$

$$(4) \Rightarrow a_1-a_3=1$$

$$-a_3=2$$

$\boxed{a_3=-2}$ in equ (3)

$$a_1+4=3$$

$$\boxed{a_1=-1}$$

$a_1=-1, a_3=-2$ in equ (1)

$$-1+2a_2-6=-1$$

$$2a_2-7=-1$$

$$2a_2=6$$

$$\boxed{a_2=3}$$

$$a_1=-1,\ a_2=3,\ a_3=-2$$

$$-1-3x+3x^2=-1\{1+2x+x^2\}+3\{2+5x\}-2\{3+8x-2x^2\}$$

1. Is $\{\cos^2 x, \sin^2 x, 1, e^x\}$ linearly independent in the vector space $C(\mathbb{R})$?

$$\begin{vmatrix} \cos^2 x & \sin^2 x & 1 & e^x \\ -\sin 2x & \sin 2x & 0 & e^x \\ -2\cos 2x & 2\cos 2x & 0 & e^x \\ 4\sin 2x & -4\sin 2x & 0 & e^x \end{vmatrix} \quad C_1 \to C_1 + C_2$$

$$= \begin{vmatrix} 1 & \sin^2 x & 1 & e^x \\ 0 & \sin 2x & 0 & e^x \\ 0 & 2\cos 2x & 0 & e^x \\ 0 & -4\sin 2x & 0 & e^x \end{vmatrix}$$

$= 0$

$\therefore \{\cos^2 x, \sin^2 x, 1, e^x\}$ is linearly dependent.

2. Verify the given sets of functions are linearly independent in the vector space $C[-\pi, \pi]$

i) $\{1, x, x^2, x^3, x^4\}$

ii) $\{1, e^x, e^{2x}, e^{3x}\}$

iii) $\{\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4\}$ where $\bar{V}_1 = (1,1,2,4)$, $\bar{V}_2 = (2,-1,-5,2)$, $\bar{V}_3 = (1,-1,-4,0)$, $\bar{V}_4 = (2,1,1,6)$

iv) $\{x, \cos x, \sin x\}$ L.I

v) $\{x, e^x, e^{-x}\}$ L.I

vi) $\{x|x|, x^2\}$ L.D

vii) $\{(1,2,3), (3,2,1)\}$

ii) $a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3 + a_5 x^4 = \bar{0}$

$a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3 + a_5 x^4 = 0.1 + 0.x + 0.x^2 + 0.x^3 + 0.x^4$

$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0$. $\therefore$ It is linearly independent

iii) Let $a_1\bar{V_1} + a_2\bar{V_2} + a_3\bar{V_3} + a_4\bar{V_4} = \bar{0}$

Then $a_1(1,1,2,4) + a_2(2,-1,-5,2) + a_3(1,-1,-4,0) + a_4(2,1,1,6) = (0,0,0,0)$

$\Rightarrow (a_1, a_1, 2a_1, 4a_1) + (2a_2, -a_2, -5a_2, 2a_2) + (a_3, -a_3, -4a_3, 0) + (2a_4, a_4, a_4, 6a_4) = (0,0,0,0)$

$\Rightarrow (a_1 + 2a_2 + a_3 + 2a_4, a_1 - a_2 - a_3 + a_4, 2a_1 - 5a_2 - 4a_3 + a_4, 4a_1 + 2a_2 + 0 + 6a_4) = (0,0,0,0)$

$\Rightarrow a_1 + 2a_2 + a_3 + 2a_4 = 0$ —— ①

$a_1 - a_2 - a_3 + a_4 = 0$ —— ②

$2a_1 - 5a_2 - 4a_3 + a_4 = 0$ —— ③

$4a_1 + 2a_2 + 6a_4 = 0$ —— ④

① $\Rightarrow a_1 + 2a_2 + \cancel{a_3} + 2a_4 = 0$

② $\Rightarrow a_1 - a_2 - \cancel{a_3} + a_4 = 0$

$2a_1 + a_2 + 3a_4 = 0$ —— ⑤

④ $\Rightarrow 4a_1 + \cancel{2a_2} + 6a_4 = 0$

⑤ $\times 2 \Rightarrow 4a_1 + 2a_2 + 6a_4 = 0$

$8a_1 + 12a_4 = 0$ —— ⑥

① $\times 2 \Rightarrow 2a_1 + 2a_2 + 2a_3 + 4a_4 = 0$

③ $\Rightarrow 2a_1 - 5a_2 - 4a_3 + a_4 = 0$

$\quad - \quad + \quad + \quad -$

$7a_2 + 6a_3 + 3a_4 = 0$ —— ⑥

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad R_4 \to R_4 - 2R_2 \quad \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ It has one zero row it is linearly dependent.

$$① \Rightarrow a_1 + 2a_2 + a_3 + 2a_4 = 0$$

$$② \Rightarrow a_1 - a_2 - a_3 + a_4 = 0$$

$$2a_1 + a_2 + 3a_4 = 0 \quad —— ⑤$$

$$④ \Rightarrow 4a_1 + 2a_2 + 6a_4 = 0$$

$$⑤ \times 2 \Rightarrow \underline{4a_1 + 2a_2 + 6a_4 = 0}$$

$$8a_1$$

Since the two equations give answer zero it is linearly dependent

(4) $\rightarrow$ $-a_2 + 3a_3 = 3$

$$\frac{\begin{array}{r} -a_2 + 3a_3 = 3 \\ a_2 + 2a_3 = 7 \end{array}}{5a_3 = 10}$$

$\boxed{a_3 = 2}$

(4) $\rightarrow$ $-a_2 + 3(2) = 3$

$-a_2 + 6 = 3$

$-a_2 = 3 - 6$

$\boxed{a_2 = 3}$

(1) $\rightarrow$ $a_1 + 3 + 2(2) = 1$

$a_1 + 7 = 1$

$\boxed{a_1 = -6}$

$\alpha = (-6)e_1 + (3)e_2 + (2)e_3$

3. Express $\alpha = (1,-2,5)$ as a linear combination of the vectors $e_1 = (1,1,1)$, $e_2 = (1,2,3)$, $e_3 = (2,-1,1)$

Let $\alpha = a_1e_1 + a_2e_2 + a_3e_3$

Then $(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$

$= (a_1, a_1, a_1) + (a_2, 2a_2, 3a_2) + (2a_3, -a_3, a_3)$

$(1,-2,5) = (a_1+a_2+2a_3), (a_1+2a_2-a_3), (a_1+3a_2+a_3)$

$\Rightarrow a_1+a_2+2a_3 = 1$ —— ①

$a_1+2a_2-a_3 = -2$ —— ②

$a_1+3a_2+a_3 = 5$ —— ③

① $\Rightarrow a_1+a_2+2a_3 = 1$

② $\Rightarrow a_1+2a_2-a_3 = -2$

$\quad - \quad - \quad + \quad (+)$

$-a_2+3a_3 = 3$ —— ④

① $\Rightarrow a_1+a_2+2a_3 = 1$

③ $\Rightarrow a_1+3a_2+a_3 = 5$

$\quad - \quad - \quad - \quad -$

$-2a_2+a_3 = -4$ —— ⑤

④ $\Rightarrow -a_2+a_3 = 3$

⑤ $\Rightarrow -2a_2+a_3 = -4$

$(+) \quad (-) \quad (+)$

$a_2 = 7$

② $\Rightarrow a_1+2a_2-a_3 = -2$

③ $\Rightarrow a_1+3a_2+a_3 = 5$

$\quad - \quad - \quad - \quad -$

$-a_2-2a_2 = -7$ —— ⑤

4. Is the vector $\alpha = (2,-5,3)$ can be expressed as a linear combination of the vectors $e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$, $e_3 = (1,-5,-7)$

Let $\alpha = a_1e_1 + a_2e_2 + a_3e_3$

Then $(2,-5,3) = a_1(1,-3,2) + a_2(2,-4,-1) + a_3(1,-5,-7)$

$= (a_1, -3a_1, 2a_1) + (2a_2, -4a_2, -a_2) + (a_3, -5a_3, -7a_3)$

$(2,-5,3) = (a_1+2a_2+a_3), (-3a_1-4a_2-5a_3), (2a_1-a_2-7a_3)$

$a_1 + 2a_2 + a_3 = 2$

$-3a_1 - 4a_2 - 5a_3 = -5$

$2a_1 - a_2 - 7a_3 = 3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & -7 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & -9 & -1 \end{bmatrix} R_3 \to 2R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -28 & 3 \end{bmatrix}$$

$\Rightarrow a_1 + 2a_2 + a_3 = 2$ —— ①

$2a_2 - 2a_3 = 1$ —— ②

$-28a_3 = 3$ —— ③

③ $\Rightarrow \boxed{a_3 = \frac{-3}{28}}$

② $\Rightarrow 2a_2 - \cancel{2}\left(\frac{-3}{\cancel{28}_{14}}\right) = 1$

$$2a_2 + \frac{3}{14} = 1$$ $1 - \frac{3}{14}$

$$2a_2 = \frac{11}{14}$$

$$\boxed{a_2 = \frac{11}{28}}$$

$$① \Rightarrow \quad a_1 + 2\left(\frac{11}{28}\right) - \frac{3}{28} = 2$$

$$a_1 + \frac{22}{28} - \frac{3}{28} = 2$$ $2 - \frac{19}{28}$

$$a_1 + \frac{19}{28} = 2$$ $56 - 19$

$$\boxed{a_1 = \frac{37}{28}}$$ 37

$$\alpha = \left(\frac{37}{28}\right)e_1 + \left(\frac{11}{28}\right)e_2 + \left(\frac{-3}{28}\right)e_3$$

NOTE :-

i) A set of vectors is linearly dependent if and only if atleast one of the vectors in the set can be written as a linear combination of the others.

ii) The non zero rows of a matrix in row-echelon form are linearly independent, and so are the columns that containing leading 1s.

iii) If $n > m$, any set of n vectors in m-space $\mathbb{R}^m$ is linearly dependent.

iv) If U is the reduced row-echelon form of A, then the columns of A are linearly independent if and only if the columns of A are linearly independent.

Span:- Let $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m$ be vectors in a vector space V. Then the set $W = \{a_1\bar{x}_1 + a_2\bar{x}_2 + \ldots + a_m\bar{x}_m\}$ of all linear combinations of $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m$ is a subspace of V. It is called the subspace of V spanned by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ or $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m$ span the subspace W.

Basis :-

Let V be a vector space. A basis for V is a set of linearly independent vectors that spans V.

For example,

the set $\{\bar{\epsilon}_1, \bar{\epsilon}_2, \ldots \bar{\epsilon}_n\}$ is a standard basis for the n-space $\mathbb{R}^n$. $L(S) = V$

3M- It will be dependent

Example :-

8M 1. Verify the set of vectors (1,1,0) (0,-1,1) and (1,0,1) is a basis or not for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$= 1(-1) - 1(-1)$

$= -1 + 1$

$= 0$

$\therefore$ The given vectors are linearly dependent & hence it does not form a basis for $\mathbb{R}^3$.

2. Verify the set of vectors $\{(1,1,1),(0,1,1),(0,0,1)\}$ are form a basis or not for $\mathbb{R}^3$

Let $S=\{(1,1,1),(0,1,1),(0,0,1)\}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$\{(1,1,0),(0,-1,1),(1,0,1)\}$

$=1(1-0)$

$=1\neq 0$

$\therefore$ S is linearly independent

Let $(x,y,z)\in\mathbb{R}^3$

Then $(x,y,z)=a(1,1,1)+b(0,1,1)+c(0,0,1)$

$=(a,a,a)+(0,b,b)+(0,0,c)$

$=(a,a+b,a+b+c)$

$\Rightarrow a=x,\ a+b=y,\ a+b+c=z$

$b=y-a \qquad c=z-a-b$

$b=y-x \qquad =z-x-(y-x)$

$=z-x-y+x$

$c=z-y$

$\therefore (x,y,z)=x(1,1,1)+(y-x)(0,1,1)+(z-y)(0,0,1)$

$L(S)=V$

Hence S is a basis for $\mathbb{R}^3$

3. Show that the vectors $\bar{V}_1=(1,2,1)$, $\bar{V}_2=(2,9,0)$ & $\bar{V}_3=(3,3,4)$ in the 3-space $\mathbb{R}^3$ form a basis

Let $S = \{(1,2,1), (2,9,0), (3,3,4)\}$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix} = 1(36-0) - 2(8-0) + 1(6-27)$$

$$= 36 - 16 - 21$$

$$= -1 \neq 0$$

$\therefore$ S is linearly independent.

Let $(x,y,z) \in \mathbb{R}^3$

Then $(x,y,z) = V_1(1,2,1) + V_2(2,9,0) + V_3(3,3,4)$

$$= (V_1, 2V_1, V_1) + (2V_2, 9V_2, 0) + (3V_3, 3V_3, 4V_3)$$

$$= (V_1 + 2V_2 + 3V_3,\ 2V_1 + 9V_2 + 3V_3,\ V_1 + 4V_3)$$

$V_1 + 2V_2 + 3V_3 = x$ —(1) $\quad 2V_1 + 9V_2 + 3V_3 = y$ —(2) $\quad V_1 + 4V_3 = z$ —(3)

$(1)\times 9 \Rightarrow 9V_1 + \cancel{18V_2} + 27V_3 = 9x$

$(2)\times 2 \Rightarrow 4V_1 + \cancel{18V_2} + 6V_3 = 2y$

$\quad (-) \quad (-) \quad (-) \quad (-)$

$5V_1 \quad + 21V_3 = 9x - 2y$ —(4)

$(3)\times 5 \Rightarrow \cancel{5V_1} + 20V_3 = 5z$

$(4) \Rightarrow 5V_1 + 21V_3 = 9x - 2y$

$\quad (-) \quad (-) \quad (-)$

$-V_3 = 5z - 9x + 2y$

$$\boxed{V_3 = 9x - 2y - 5z}$$

③ $\Rightarrow \quad V_1 + 36x - 8y - 20z = z$

$$\boxed{V_1 = -36x + 8y + 21z}$$

① $\Rightarrow \quad -36x + 8y + 21z + 2V_2 + 27x - 6y - 15z = x$

$$-9x + 2y + 6z + 2V_2 = x$$

$$2V_2 = 10x - 2y - 6z$$

$$\boxed{V_2 = 5x - y - 3z}$$

$$(x, y, z) = (-36x + 8y + 21z)(1,2,1) + (5x - y - 3z)(2,9,0) + (9x - 2y - 5z)(3,3,4)$$

$L(S) = V$

$\therefore$ S is a basis for $\mathbb{R}^3$

4. Let w_1 and w_2 be two subspace for $\mathbb{R}^4$ given by $w_1 = \{(a,b,c,d) : b - 2c + d = 0\}$,
$w_2 = \{(a,b,c,d) : a = d,\ b = 2c\}$ then find the basis for w_1, w_2 & $w_1 \cap w_2$

$$w_1 = \{(a,b,c,d) : b - 2c + d = 0\}$$

$$= \{(a,b,c,d) : b = 2c - d\}$$

$$= \{(a, 2c-d, c, d)\}$$

$$= \{a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)\}$$

Hence $\{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$ is a basis for w_1.

$W_2 = \{(a,b,c,d) : a=d,\ b=2c\}$

$= \{(d, 2c, c, d)\}$

$= \{c(0,2,1,0) + d(1,0,0,1)\}$

Hence $\{(1,0,0,1)\ (0,2,1,0)\}$ is a basis of W_2

$W_1 \cap W_2 \quad (0,2,1,0)$

5 Show that the set $\{1, x, x^2, \dots x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\leq n$ with real coefficients

$a_0.1 + a_1.x + a_2.x^2 + \dots + a_n x^n = \bar{0}$

$\Rightarrow a_0.1 + a_1.x + a_2.x^2 + \dots + a_n x^n = 0.1 + 0.x + 0.x^2 + \dots + 0.x^n$

$\Rightarrow a_0 = 0,\ a_1 = 0,\ a_2 = 0, \dots a_n = 0$

$\therefore \{1, x, x^2, \dots x^n\}$ is linearly independent

Then $(x, y, z) \in$

Consider $1.x^2 - 2.x^{10} \in P_n(\mathbb{R})$

Then $1.x^2 - 2.x^{10} = 0.1 + 0.x + 1.x^2 + 0x^3 + \dots +$
$(-2)x^{10} + 0.x^{11} + \dots + 0.x^n$

i.e Every element of $P_n(\mathbb{R})$ can be expressed as a linear combination of $\{1, x, x^2, \dots x^n\}$

Hence $\{1, x, x^2, \dots x^n\}$ is a basis for $P_n(\mathbb{R})$.

Example: In the 3-space $\mathbb{R}^3$, let W be the set of all vectors (x_1, x_2, x_3) that satisfies the equation $x_1 - x_2 - x_3 = 0$. Prove that W is a subspace of $\mathbb{R}^3$. Find a basis for the subspace W.

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

It is standard element

$\therefore (x_1, x_2, x_3)$ is linearly independent.

Basis:-

$$W = \{(x_1, x_2, x_3) : x_1 - x_2 - x_3 = 0\}$$

$$= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\}$$

$$= \{(x_2 + x_3, x_2, x_3)\}$$

$$W = \{x_2(1,1,0) + x_3(1,0,1)\}$$

Hence $\{(1,1,0)\ (1,0,1)\}$ is a basis of W

Subspace:-

$$W = \{(x_1, x_2, x_3)\} : x_1 - x_2 - x_3 = 0\}$$

Let $\alpha, \beta \in W$

$\alpha = (x_1, x_2, x_3)$ $\beta = (y_1, y_2, y_3)$

where $x_1 - x_2 - x_3 = 0$

$y_1 - y_2 - y_3 = 0$

$$\alpha + K\beta = (x_1 + Ky_1, x_2 + Ky_2, x_3 + Ky_3)$$

Now $(x_1 + Ky_1) - (x_2 + Ky_2) - (x_3 + Ky_3) = (x_1 - x_2 - x_3) + K(y_1 - y_2 - y_3)$

$= 0 + K(0)$

$= 0$

$\therefore \alpha + K\beta \in W$

W is a subspace

Dimension :

Definition:

The dimension of vector space V is the number, say n, of vectors in a basis for V, denoted by $\dim V = n$ when V has a basis of a ~~∅~~ finite no. of vectors, V is said to be <u>finite</u> <u>dimensional</u>

Note :-

i) If V has only the zero vector i.e., $V = \{\bar{0}\}$ then $\dim V = 0$

ii) If $V = \mathbb{R}^n$, then the standard basis $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ for V implies $\dim \mathbb{R}^n = n$

iii) If $V = P_n(\mathbb{R})$ of all polynomials of degree $\leq n$, then $\dim P_n(\mathbb{R}) = n+1$, since $\{1, x, x^2, \dots, x^n\}$ is a basis for V

iv) If $V = M_{m\times n}(\mathbb{R})$ of all $m\times n$ matrices, then $\dim (M_{m\times n}(\mathbb{R})) = m\cdot n$

v) If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, the V is that finite dimensional i.e, $\dim (C(\mathbb{R})) = \infty$

vi) Let $V = M_{n\times n}(\mathbb{R})$ of all ~~m~~ $n\times n$ matrices, then

a) the dimension of the subspace of all $n\times n$ diagonal matrices whose traces are zero is $n-1$

b) the dimension of the subspace of all $n\times n$ symmetric matrices $= \frac{n(n+1)}{2}$

c) the dimension of the subspace of all $n\times n$ skew-symmetric matrix $= \frac{n(n-1)}{2}$

Note:- v(i) Let V be a finite dimensional vector space,

i) any linearly independent set in V can be extended to a basis by adding more vectors if necessary

ii) Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.

Note:- Let V be a vector space of dimension n, then

i) any set of n vectors that spans V is a basis

ii) any set of n linearly independent vectors is a basis for V

Example:-

1. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\bar{x}_1 = (1,-2,5,-3)$, $\bar{x}_2 = (0,1,1,4)$, $\bar{x}_3 = (1,0,1,0)$. Find a basis for W and extend it to a basis of $\mathbb{R}^4$.

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & +2 & -4 & 3 \end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -5 \end{bmatrix} R_3 \Rightarrow -R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

The above matrix have three non-zero row vectors are clearly linearly independent

$\therefore \{(1,-2,5,-3),(0,1,1,4),(1,0,1,0)\}$ is a basis for W

To extend it to a basis of $\mathbb{R}^4$, just add any non-zero vector of the form $\bar{x}_4 = (0,0,0,t)$

2 Let V_1 & V_2 be the subspaces of $\mathbb{R}^4$ generated by $\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ respectively. Find the dimension of i) V_1 ii) V_2 iii) V_1+V_2 iv) $V_1 \cap V_2$

$V_1 = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$

$V_2 = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$

V_1:-

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Dimension of $V_1 = 2$

V_2 :-

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & +1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \Rightarrow R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow -R_3 \\ R_3 \Rightarrow -R_3/2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dimension of $V_2 = 2$

$V_1 + V_2$:-

$$V_1 + V_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1 \\ R_6 \to R_6 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_5 \to R_5 - R_4 \\ R_6 \to R_6/2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \quad R_6 \to R_6 - R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~dim(V+W)~~

$\dim(V+W) = 3$

iv) $\dim(V+W) + \dim(V \cap W) = \dim(V) + \dim(W)$

$3 + \dim(V \cap W) = 2 + 2$

$\dim(V \cap W) = 4 - 3$

$\dim(V \cap W) = 1$

Every polynomial is of degree $<n$ is a vector space over R

Note: $(R,+)$ is a vector space

Proof:- Let $P(x)$ or $P = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$ be the polynomial of degree $<n$

where $a_0, a_1, a_2, \dots a_{n-1} \in R$

i) Closure axiom:-

Let $P = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$

$q = b_0 + b_1 x + b_2 x^2 + \dots + b_{n-1} x^{n-1} \in V$

$p+q = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_{n-1}+b_{n-1})x^{n-1} \in V$

closure axiom is true.

ii) Associative axiom:-

Associative axiom is always true over R

iii) Identity axiom:-

$P = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} \in V$

$\exists e = 0 + 0x + 0x^2 + \dots + 0x^{n-1} \in V$

$\therefore p + e = P$

$\therefore$ Identity axiom is true

iv) Inverse axiom:-

Let $P = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} \in V$

$\exists P^{-1} = a_0^{-1} + a_1^{-1} x + \dots + a_{n-1}^{-1} x^{n-1}$

Such that $P + P^{-1} = e \in V$

$\therefore$ Inverse axiom is true

v) Commutative axiom:

Let $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$q = b_0 + b_1 x + \dots + b_{n-1} x^{n-1} \in V$

$\therefore p + q = q + p$ is true.

Commutative axiom is true.

$\therefore (V, +)$ is an abelian group.

vi) $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$q = b_0 + b_1 x + \dots + b_{n-1} x^{n-1} \in V$

$\alpha \in F$ (field of scalar)

$$\alpha(p+q) = \alpha((a_0+b_0) + (a_1+b_1)x + \dots + (a_{n-1}+b_{n-1})x^{n-1})$$

$$= \alpha(a_0+b_0) + \alpha(a_1+b_1)x + \dots + \alpha(a_{n-1}+b_{n-1})x^{n-1}$$

$\therefore \alpha(p+q) = \alpha p + \alpha q$ is true

vii) $(\alpha+\beta)p$

$p = a_0 + a_1 x + \dots a_{n-1} x^{n-1} \in V$

$\alpha, \beta \in F$

$$(\alpha+\beta)p = (\alpha+\beta)(a_0 + a_1 x + \dots + a_{n-1}x^{n-1})$$

$$= a_0(\alpha+\beta) + a_1(\alpha+\beta)x + \dots + a_{n-1}(\alpha+\beta)x^{n-1}$$

$= ~~a_0\alpha + a_0\beta + a_1~~$

$$= a_0\alpha + a_1\alpha x + \dots a_{n-1}\alpha x^{n-1}$$
$$+ \beta a_0 + \beta a_1 x + \dots + \beta a_{n-1} x^{n-1} \in V$$

$\therefore (\alpha+\beta)p = \alpha p+\beta p$ is true

vii) $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$\alpha, \beta \in F$

$(\alpha\beta)p = \alpha\beta(a_0 + a_1 x + \dots + a_{n-1}x^{n-1})$

$= \alpha(\beta p)$

$\therefore (\alpha\beta)p = \alpha(\beta p)$ is true

ix) $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$1 \in F$

$1 \cdot p = 1(a_0 + a_1 x + \dots + a_{n-1} x^{n-1})$

$1 \cdot p = p$ is true

Conclusion:-

Every polynomial is of degree $<n$ is a vector space

Row and column spaces :-

Let A be an $m \times n$ matrix with row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$ and column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$

i) The row space of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$, denoted by $R(A)$

ii) The column space of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$ denoted by $C(A)$.

iii) The solution set of the homogeneous equation $Ax=0$ is called <u>null space</u> of A, denoted by $N(A)$

*** 1. Find the bases for the row, column and null space of A, where $A = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix}$ $R_2 \to R_2 + 2R_1$

4×5 $R_4 \to R_4 - 3R_1$

$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{pmatrix}$ $R_3 \Rightarrow R_3 - 3R_2$

$R_4 \Rightarrow R_4/-13$

$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $R_2 \Rightarrow R_2/-1$

$R_3 \Rightarrow R_3/-5$

$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $R_1 \to R_1 - 2R_2$

$\sim \begin{pmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $R_1 \to R_1 - 8R_3$

$R_2 \to R_2 + 3R_3$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & & c_3 & & c_5 \end{array}$$

Since the non-zero row vectors of U are

$\bar{V}_1 = (1,0,2,0,1)$ $\bar{V}_2 = (0,1,-1,0,1)$

$\bar{V}_3 = (0,0,0,1,1)$ are linearly independent

and they form a basis for the row space

Basis for $R(A) = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$

Basis for $N(A)$;-

For this, we solve the homogenious

solution system $UX = 0$, we have

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$\Rightarrow x_1 + 2x_3 + x_5 = 0$

$x_2 - x_3 + x_5 = 0$

$x_4 + x_5 = 0$

choose $x_3 = s$ and $x_5 = t$

Take the first

Then $x_4 = -t$

$x_2 = s - t$

$x_1 = -2s - t$

$x_1 = -2s - t = -2s - t$

$x_2 = s - t = s - t$

$x_3 = s = s + 0t$

$x_4 = -t = 0.s + (-1)t$

$x_5 = t = 0.s + t$

Since any solution x is a linear combination of u_s and u_t.

$\therefore$ The set $\{u_s, u_t\}$ is a basis to the null space

$\therefore$ Basis of $N(A) = \{u_s, u_t\} = \{(-2,1,1,0,0), (-1,-1,0,-1,1)\}$

Basis for $C(A)$:

Let C_1, C_2, C_3, C_4 & C_5 denotes the column vectors of A. Since these column vectors of A can span $C(A)$, we only need to discard some of the columns that can be expressed as a linear combination of others.

$AX = 0$

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

By taking $x = u_s = (-2,1,1,0,0)$ then

$-2C_1 + C_2 + C_3 = 0$ —— ①

Similarly consider $x = u_t = (-1,-1,0,-1,1)$ then

$-C_1 - C_2 - C_4 + C_5 = 0$ —— ②

From eqn ①, we have $C_3 = 2C_1 - C_2$

From eqn ③, we have $C_5 = C_1 + C_2 + C_4$

Hence the column vectors C_3 & C_5 corresponding to the free variable in $AX=0$ can be expressed as

$C_3 = 2C_1 - C_2$

$C_5 = C_1 + C_2 + C_4$

i.e., the column vectors C_3, C_5 of A are linearly dependent

Hence $\underline{\{C_1, C_2, C_4\}}$ spans the column space $C(A)$

i.e., $C(A) = \{(1,-2,0,3), (2,-5,-3,6), (2,-1,4,-7)\}$

*** 8M

2. Find the bases for the row, column & null space for each of the following matrices

i) $A = \begin{pmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{pmatrix}$

ii) $A = \begin{pmatrix} 0 & 2 & 1 & -5 \\ 1 & 1 & -2 & 2 \\ 1 & 5 & 0 & 0 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{pmatrix}$

iv) $A = \begin{pmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix}$

v) $A = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}$

iv) $$A = \begin{pmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 - 2R_1 \\ R_5 \to R_5 - 3R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 2 & -2 & -4 & 7 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 + R_2 \\ R_5 \to R_5 + 2R_2 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 / -2 \\ R_4 \to R_4 / 2 \\ R_5 \to R_5 / 9 \end{array}$$

$$\sim \begin{pmatrix} 1 & +1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} R_5 \to R_5 - R_4$$

$$\sim \begin{pmatrix} 1 & +1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_1 \to R_1 \mp R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$1 - \frac{1}{2}$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 \to R_2 - \frac{1}{2} R_4 \\ R_3 \to R_3 - \frac{1}{2} R_4 \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the non-zero rows vectors of U are

$\bar{v}_1 = (1, 0, 0, 5, 0)$ $\bar{v}_2 = (0, 1, 0, -3, 0)$

$\bar{V}_3 = (0,0,1,-1,0)$ $\bar{V}_4 = (0,0,0,0,1)$ are linearly independent and they form a basis for row space

$\therefore$ Basis for $R(A) = \{\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4\}$

<u>Basis for N(A):-</u>

$$UX = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$\Rightarrow$ $x_1 + 5x_4 = 0$ —— (1)

$x_2 - 3x_4 = 0$ —— (2)

$x_3 - x_4 = 0$ —— (3)

$x_5 = 0$

$x_4 = t$ (1) $\rightarrow$ $x_1 = -5t$

$x_2 = 3t$

$x_3 = t$

$x_4 = t$, $x_5 = 0$

$\therefore$ Solution X is n_t

$\therefore$ The set $\{n_t\}$ is a basis for null space

$\therefore$ Basis for $N(A) = \{n_s, n_t\} = \{-5, 3, 1, 1, 0\}$

<u>Basis for C(A):-</u>

For this, we consider

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

By taking $x = u_4 = (-5, 3, 1, 1, 0)$ then

$$-5c_1 + 3c_2 + c_3 + c_4 = 0$$

$$c_4 = 5c_1 - 3c_2 - c_3$$

i.e., the column vector c_4 of A is linearly dependent.

Hence $\{c_1, c_2, c_3, c_5\}$ spans the column space $C(A)$

i.e., $C(A) = \{(0,1,2,0,3), (1,1,1,0,5), (-1,-1,-1,-2,-5), (1,1,3,1,10)\}$

3. Find a basis for the null space & column space of

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of $R(A) = \{(1,0,0,2,2), (0,1,0,-1,3), (0,0,1,4,-1)\}$

<u>Basis of $N(A)$</u>:-

For this, we consider $U\bar{x} = \bar{0}$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_4 + 2x_5 = 0$

$x_2 - x_4 + 3x_5 = 0$

$x_3 + 4x_4 - x_5 = 0$

Consider $x_4 = s$, $x_5 = t$

$x_1 = -2s - 2t$

$x_2 = s - 3t$

$x_3 = -4s + t$, $x_4 = s$, $x_5 = t$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-2t \\ s-3t \\ -4s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$= su_3 + tu_4$

where $n_s = (-2, 1, -4, 1, 0)$.

$n_t = (-2, -3, 1, 0, 1)$

$\therefore$ Basis of $N(A) = \{n_s, n_t\}$

$= \{(-2, 1, -4, 1, 0)\ (-2, 3, 1, 0, 1)\}$

$\therefore$ $\dim N(A) = 2 =$ nullity of A

<u>Basis of $C(A)$</u> :-

For this, we consider

$$[c_1\ c_2\ c_3\ c_4\ c_5]\bar{x} = \bar{0}$$

$$\Rightarrow [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0} \quad \text{——— (1)}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_s$

$$\text{Then } [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow -2c_1 + c_2 - 4c_3 + c_4 = 0 \quad \text{——— (2)}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_t$

$$\text{Then } [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow -2c_1 - 3c_2 + c_3 + c_5 = 0 \quad \text{——— (3)}$$

From eqn (2), we have

$$c_4 = 2c_1 - c_2 + 4c_3$$

From eqn (3), we have

$C_5 = 2C_1 + 3C_2 - C_3$

Hence basis of $C(A) = \{C_1, C_2, C_3\}$

$\therefore \dim C(A) = 3$

Note:-

i) The dimension of the null space of A is called the nullity of A

ii) ~~&~~ The row vectors of A are just the column vectors of its transpose A^T and the column vectors of A are the row vectors of A^T, the row (column) space of A is just the column (row) space of A^T

i.e., $R(A) = C(A^T)$ and $C(A) = R(A^T)$

Example:-

*** Find basis for $R(A)$ and $N(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Also find a basis for $C(A)$ by finding a basis for $R(A^T)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3/5 \\ R_4 \Rightarrow R_4 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \Rightarrow R_2/-1 \\ \cancel{R_3 \Rightarrow R_3/-2} \\ R_4 \Rightarrow R_4/2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 9 & 4 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \Rightarrow R_1 + 2R_2 \\ R_3 \Leftrightarrow R_4 \end{matrix}$$

$$\sim \cancel{\begin{bmatrix} 1 & 0 & 6 & 4 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 9 & 4 & 0 \end{bmatrix}} \quad \cancel{R_1 \Rightarrow R_1 -}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 9 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3 - 5R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3/-6$$

$$\frac{-15}{9}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \Rightarrow R_1 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & 4 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 - 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $R(A) = \{(1,0,0,-2,3), (0,1,0,-1,0), (0,0,1,1,0)\}$

$\dim R(A) = 3$

Basis for $N(A)$:-

For this, we consider $Ux = \bar{0}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_4 + 3x_5 = 0$$
$$x_2 - x_4 = 0$$
$$x_3 + x_4 = 0$$

$x_4 = t$

$x_5 = s$

$$x_3 = -t$$
$$x_2 = t$$
$$x_1 = 2t - 3s$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2t-3s \\ t \\ -t \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} +2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= s u_s + t u_t$$

where $u_s = (-3,0,0,0,1)$

$u_t = (2,1,-1,1,0)$

$\therefore$ Basis of $N(A) = \{u_s, u_t\}$ where

$u_s = (2,1,-1,1,0)$, $u_t = (-3,0,0,0,1)$

dim $N(A) = 2 =$ nullity of A

<u>Basis of C(A)</u>:-

For this, we consider

$[c_1\ c_2\ c_3\ c_4\ c_5]\,\bar{x} = \bar{0}$

$$\Rightarrow [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0} \quad \text{—— ①}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = u_s$

$$\text{Then } [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$2c_1 + c_2 - c_3 + c_4 = 0 \quad \text{—— ①}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = u_t$

Then $\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$

$-3C_1 + C_5 = 0$ —— (3)

$C_5 = 3C_1$

$C_4 = -2C_1 - C_2 + C_3$

Hence basis of $C(A) = \{C_1, C_2, C_3\}$

$\therefore \dim C(A) = 3$

Note:-

i) $\dim R(A) = \dim C(A)$

ii) $\dim N(A) = \dim N(U)$

= The number of free variables in $U\bar{x} = \bar{0}$

iii) $\dim R(A) = \dim R(U)$

= The number of non-zero rows of U

= The maximal number of linearly independent row vector of A

= The number of free variables in $U\bar{x} = \bar{0}$

= The maximal number of linearly independent column vector of A

= $\dim C(A)$

Definition :-

For an $m \times n$ matrix A, the <u>rank of A</u> is defined to be the dimension of the row space (or column space) and is denoted by rank A.

Note :-

i) If A is an $m \times n$ matrix, then ~~rank (A) ≤~~ $\text{rank}(A) \le \min\{m, n\}$

ii) For any $m \times n$ matrix A,

$$\dim R(A) + \dim N(A) = \text{rank}(A) + \text{nullity}(A) = n$$ n of column / variables

$$\dim C(A) + \dim N(A^T) = \text{rank}(A) + \text{nullity}(A^T) = m$$

<Rank - nullity theorem>

*** Find the nullity and the rank of each of the following matrices

i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}$ $\begin{array}{l} R_2 \Rightarrow R_2 + R_1 \\ R_3 \Rightarrow R_3 - R_1 \\ R_4 \Rightarrow R_4 - R_1 \end{array}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & -9 & 1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix}$ $\begin{array}{l} R_3 \Rightarrow R_3 + 3R_2 \\ R_4 \Rightarrow R_4 + 2R_2 \end{array}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ $\begin{array}{l} R_3 \Rightarrow R_3/4 \\ R_4 \Rightarrow R_4/4 \end{array}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $R_4 \Rightarrow R_4 - R_3$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{l} R_1 \Rightarrow R_1 - R_3 \\ R_2 \Rightarrow R_2 - R_3 \end{array}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

ii)

$\therefore$ Basis of $R(A) = \{(1,2,0,2,0), (0,0,1,3,0), (0,0,0,0,1)\}$

$\therefore$ dim $R(A) =$ rank $(A) = 3$

Since rank(A) + nullity $(A) = 5$

$\therefore$ nullity$(A) = 5 - 3 = 2$

ii) $$A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 + R_1 \end{array}$$

$-\frac{14}{9}$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & -3 & -3 & -5 \\ 0 & 1 & 1 & 2 \end{bmatrix} \; R_3 \Rightarrow 3R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; R_2 \Rightarrow R_2/-3$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 1 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; R_1 \Rightarrow R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \Rightarrow R_1 - 2R_3 \\ R_2 \Rightarrow R_2 - 5/3 R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Basis of $R(A) = \{(1,0,-2,0),(0,1,1,0),$
$(0,0,0,1)\}$

$\dim R(A) = \text{rank}(A) = 3$

$\text{rank}(A) + \text{nullity of }(A) = 4$

$\text{nullity}(A) = 4-3 = 1$

(ii) $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 3 & -6 \end{bmatrix}$ $R_3 \to R_3 - 3R_2$

$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2/-1$

$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Basis of $R(A) = \{(1,0,3,-2)(0,1,-1,2)\}$

$\dim R(A) = \text{rank}(A) = 2$

$\therefore$ rank (A) + nullity (A) = 4

$\therefore$ nullity (A) = 4 - 2

= 2

24 Theorem :-

Let A be an n×n square matrix, then A is invertible if and only if rank(A) = n

Proof :-

Let A be an n×n square matrix

Then to prove that A is invertible $\Leftrightarrow$ rank (A) = n

A is invertible $\Leftrightarrow$ A is non singular

$\Leftrightarrow |A| \neq 0$

$\Leftrightarrow$ All these row vectors (n) are linearly independent

$\Leftrightarrow$ set of all these vectors (n) forms a basis for row space of A

$\Leftrightarrow$ number of elements in the basis = dim R(A)

$\Leftrightarrow$ rank(A) = n.

Note :

i) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

ii) Let A be an invertible square matrix, then for any matrix B, rank (AB) = rank (B) = rank (BA)

Basis for subspaces :-

Let $\alpha = \{\bar{v}_1, \bar{v}_2 \ldots \bar{v}_k\}$ & $\beta = \{\bar{w}_1, \bar{w}_2 \ldots \bar{w}_l\}$ be bases for V and W respectively.

Let Q be the $n \times (k+l)$ matrix whose columns are these basis vectors

i.e, $Q = \{\bar{v}_1\ \bar{v}_2 \ldots \bar{v}_k\ \bar{w}_1\ \bar{w}_2 \ldots \bar{w}_l\}$

****

Theorem :-

Let V and W be two subspaces of $\mathbb{R}^n$, and Q be the matrix whose columns are these basis vectors of V and W.

Then i) $C(Q) = V+W$, so that a basis for the column space $C(Q)$ is a basis for $V+W$

ii) $N(Q)$ can be identified with $V \cap W$ so that $\dim(V \cap W) = \dim N(Q)$

Proof :-

i) It is clear that $C(Q) = V+W$

ii) Let $x = (a_1, \ldots, a_k; b_1, \ldots, b_l) \in N(Q) \subseteq \mathbb{R}^{k+l}$ then

$$Q(x) = a_1 v_1 + \ldots + a_k v_k + b_1 w_1 + \ldots + b_l w_l = 0$$

from which we get

$$a_1 v_1 + \ldots + a_k v_k = -(b_1 w_1 + \ldots + b_l w_l)$$

if we set

$$y = a_1 v_1 + \ldots + a_k v_k$$
$$= -(b_1 w_1 + \ldots + b_l w_l)$$

then $y \in V \cap W$ since the first right hand side $a_1v_1 + \dots + a_kv_k$ is in V as a linear combination of the basis vectors in α and the second right hand side $-(b_1w_1 + \dots b_\ell w_\ell)$ is in W as a linear combination of the basis vectors in β that is to each $x \in N(Q)$, there corresponds a vector y in $V \cap W$.

On the other hand, if $y \in V \cap W$, then y can be written in two linear combinations by the bases for V and W separately as

$$y = a_1v_1 + \dots + a_kv_k \in V$$
$$y = b_1w_1 + \dots + b_\ell w_\ell \in W$$

for some $a_1, \dots, a_k$ and $b_1, \dots, b_\ell$. Let $x = (a_1, \dots, a_k, -b_1, \dots, -b_\ell) \in \mathbb{R}^{k+\ell}$ then if it is quite clear that $Qx = 0$ i.e, $x \in N(Q)$ therefore, the correspondence of x in $N(Q) \subseteq \mathbb{R}^{k+\ell}$ to a vector y in $V \cap W \subseteq \mathbb{R}^n$ gives us a one to one correspondence between the sets $N(Q)$ and $V \cap W$

Moreover, if x_i, $i = 1, 2$ correspond to y_i, then one can easily check that $x_1 + x_2$ correspond to $y_1 + y_2$ and kx_1 corresponds to ky_1 this means that the two vector spaces $N(Q)$ and $V \cap W$ can be identified as vector spaces.

In particular, for a basis for $N(Q)$, the corresponding set in $V \cap W$ is a basis for $V \cap W$ that is if the set of vectors

$$\left\{\begin{array}{l} x_1 = (a_{11}, \ldots, a_{1K}, b_{11}, \ldots, b_{1\ell}) \\ \vdots \\ x_s = (a_{s1}, \ldots, a_{sK}, b_{s1}, \ldots, b_{s\ell}) \end{array}\right.$$

is a basis for $N(Q)$, then the set of vectors

$$\left\{\begin{array}{l} y_1 = a_{11}V_1 + \ldots + a_{1K}V_K \\ \vdots \\ y_s = a_{s1}V_1 + \ldots + a_{sK}V_K \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} y_1 = -(b_{11}W_1 + \ldots + b_{1\ell}W_\ell) \\ \vdots \\ y_s = -(b_{s1}W_1 + \ldots + b_{s\ell}W_\ell) \end{array}\right.$$

is a basis for $V \cap W$, and vice versa. this implies that $\dim N(Q) = \dim(V \cap W)$

(i) Find an interpolating polynomial for $(0,3), (1,0), (-1,2), (3,6)$

(ii) Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$

(iii) Let $f(x) = \sin x$ that at $x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$, the values of f are $y = 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0$. Find the polynomial $p(x)$ of degree ≤ 4 that passes through these five points

Let $y = f(x) = a + bx + cx^2 + dx^3 + ex^4$

Since $x = 0$, $y = 0$

$\therefore 0 = a$ —— (1)

$x = \frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}}$

$$\frac{1}{2} = a + \frac{b\pi}{4} + c\frac{\pi^2}{16} + d\frac{\pi^3}{64} + \frac{e\pi^4}{256} \quad \text{——— (2)}$$

$x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2} = a + \frac{b\pi}{3} + \frac{c\pi^2}{9} + d\frac{\pi^3}{27} + \frac{e\pi^4}{81} \quad \text{——— (3)}$$

$x = \frac{3\pi}{4}$, $y = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = a + \frac{3\pi b}{4} + \frac{9\pi^2 c}{16} + \frac{27 d\pi^3}{64} + \frac{81 e\pi^4}{256} \quad \text{——— (4)}$$

$x = \pi$, $y = 0$

$$\therefore 0 = a + b\pi + c\pi^2 + d\pi^3 + e\pi^4 \quad \text{——— (5)}$$

$$(2) \Rightarrow a + \frac{b\pi}{4} + \frac{c\pi^2}{16} + d\frac{\pi^3}{64} + e\frac{\pi^4}{256} = \frac{1}{2}$$

$$(3) \Rightarrow a + \frac{b\pi}{3} + \frac{c\pi^2}{9} + d\frac{\pi^3}{27} + e\frac{\pi^4}{81} = \frac{\sqrt{3}}{2}$$

$$- \quad - \quad - \quad - \quad - \quad -$$

$$-\frac{\pi}{12}b - \frac{7\pi^2}{144}c - \frac{37\pi^3 d}{1728} -$$

Basis (1)

$\therefore 1(-1) - 1(-1)$

$= -1 + 1$

$= 0$

(2)

$1(1) - 1(-\cdot) + 1(\cdot$

$= 1 \neq 0$