BITS Pilani Basics Engineering Math

1) Complex Numbers & Quadratic Equations

Complex Number : $x + iy$ or $a + ib$

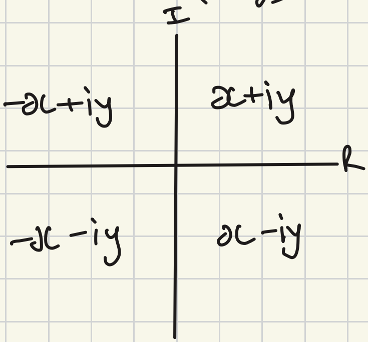
$\to$ real part

$\to$ imaginary part

where $i = \sqrt{-1}$ or $i^2 = -1$

Argand Plane

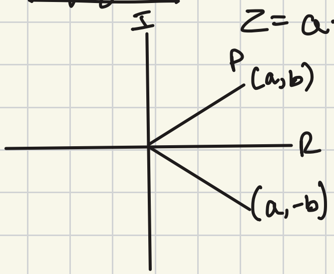
Complex Number $Z = (x, y)$ from $x + iy$



y = imaginary axis

x = real axis

Complex Conjugate



$Z = a + ib$, $\bar{Z} = a - ib$

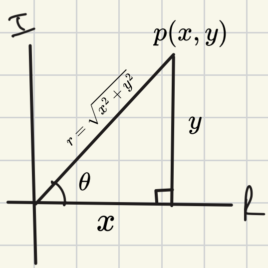
$\hookrightarrow$ conjugate of z

Polar Form

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}(y/x)$$

$\to$ argument



$$z = x + iy$$

$$z = r(\cos\theta + i\sin\theta)$$

$$x = \cos\theta$$

$$y = \sin\theta$$

$$r = \sqrt{x^2 + y^2} \quad |z| = \sqrt{x^2 + y^2}$$

modulus / absolute value

Algebra of Complex Numbers

Equality = $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

so if $z_1 = z_2$ then $x_1 = x_2$ and $y_1 = y_2$

Addition = $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

so if $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication = $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

so if $z_1 \times z_2 =$

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + \overset{-1 = i^2}{i^2 y_1 y_2}$$

so, $\underbrace{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}_{\text{final answer}}$

Division = $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

so if $z_1 / z_2 =$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \boxed{\frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2)^2 - (y_2)^2}}$$

Final answer

Addition

$z_1 = 2-4i \quad z_2 = -2+5i$

$z_1 + z_2 = 2-4i -2+5i = (2-2) + i(-4+5) = i$

so, $z_1 + z_2 = i$

Subtraction

$z_1 = -2+3i \quad z_2 = 4+5i$

$z_1 - z_2 = -2+3i-(4+5i) = -2+3i-4-5i$

$= -6-2i$

so ans $= -6-2i$

Multiplication

$z_1 = 2+3i \quad z_2 = 3-2i$

$z_1 \times z_2 = 2+3i(3-2i) = 6-4i+9i-6i^2$

$= 6-4i+9i+6$

$= 12-5i$

Division

$z_1 = 2+3i \quad z_2 = 3-4i$

$$\frac{z_1}{z_2} = \frac{2+3i}{3-4i} \times \frac{(3+4i)}{(3+4i)} = \frac{6+9i+8i-12}{9-16i^2} = \frac{-6+17i}{25}$$

$$= \boxed{\frac{-6}{25} + \frac{17}{25}i}$$

Quadratic Elements

Discriminant = $b^2 - 4ac$

$b^2 - 4ac < 0$ Then No Solutions

$b^2 - 4ac = 0$ Then One Solution

$b^2 - 4ac > 0$ Then 2 solutions

Complex Solutions to Quadratic Equations appear as conjugate pairs

Types of Equations

★ Number of roots same as degree

Expression: $x^2 + 2x + 5$ or $xy + 2x + 3y + 6$

Polynomial of degree 'n': $a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$

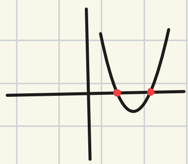
Equation: If an expression is equal to 0 it is an equation

Quadratic Equation: Polynomial with degree '2'

Cubic Equation: Polynomial with degree '3'

Quadratic Equations

Root: The value of x which satisfies a polynomial equation.



• – Roots

Factorising Example

$x^2 - 3x + 2 = 0$ $\quad a \times c = 2$

$ax^2 + bx + c = 0$ $\quad -2 \times -1 = 2$

$-2 - 1 = -3$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2)$

$(x-1)(x-2) = 0$ $\quad x = 2, 1$

$2x^2 - x - 1 = 0$ $\quad -2$

$2x^2 - 2x + x - 1 = 0$

$2x(x-1)\ 1(x-1) = (2x+1)(x-1) = 0$

Roots of Quadratic Equation using Quadratic Formula

Quadratic Formula = $\dfrac{-b \pm \sqrt{\boxed{b^2 - 4ac}}}{2a}$ → Discriminant

$b^2 - 4ac < 0$ = No real roots
$b^2 - 4ac = 0$ = One real root
$b^2 - 4ac > 0$ = Two real roots

Roots of a Quadratic Equation → $ax^2 + bx + c = 0$

1) If α and β are the roots

then $(\alpha + \beta)$ = sum of roots $= -b$

$(\alpha\beta)$ = product of roots $= c$

2) If $f(x) = 0$ has roots α, β

then $f(-x) = 0$ has roots $-\alpha, -\beta$

then $f(x - \alpha) = 0$ has roots $\alpha + \alpha, \beta + \alpha$

then $f(x + \alpha) = 0$ has roots $\alpha - \alpha, \beta - \alpha$

then $f(\sqrt{x}) = 0$ has roots α^2, β^2

then $f(1/x) = 0$ has roots $1/\alpha, 1/\beta$

Question Example: Find equation with roots $2+3i$, $2-3i$

$(x-2-3i)(x-2+3i) = x^2 - 2x - \cancel{3ix} - 2x + 4 + \cancel{6i} + \cancel{3ix} - \cancel{6i} - 9i^2$
$= x^2 - 4x + 4 + 9$
$= \underline{\underline{x^2 - 4x + 13}}$

or

$-b =$ sum of α, β so

$2 + \cancel{3i} + 2 - \cancel{3i} = 4 \times -1 = -4$ $\quad c = (2+3i)(2-3i) = \underline{\underline{13}}$

2) Matrices

- Used for solving system of linear equations
- Statistics
- Cryptography

Matrix : A rectangular arrangement of mn numbers in m rows and n columns, enclosed in [ ] or ( ).

Order of matrix is denoted by $m \times n$, read as $\boxed{\text{m by n}}$.

Example : $\begin{bmatrix} 2 & 4 & 5 \\ 1 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ is a matrix of order 3×3 with 9 elements

3 rows and 3 columns

Elements are identified using a_{ij} where i = row, j = column

So, $a_{12} = 4$ $\quad a_{22} = 0$

$a_{13} = 5$ $\quad a_{21} = 1$

rows columns

A matrix of order $m \times n$ =

goes first goes second

Type of Matrices

- Row Matrix : A matrix having only one row it is of order $1 \times n$ where $n \geq 1$ Ex: $\begin{bmatrix} 1 & 2 & -3 & 6 \end{bmatrix} \longrightarrow$ order 1×4

- Column Matrix : A matrix having only one column it is of order $m \times 1$ where $m \geq 1$ Ex:

$$\begin{bmatrix} 0 \\ 8 \\ 1 \\ 6 \end{bmatrix} \longrightarrow \text{order } 4 \times 1$$

- Zero or Null Matrix : A matrix in which every element is zero. denoted by O. Ex:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \text{order } 4 \times 2$$

- Diagonal Matrix : A square matrix in which each every non-diagonal element is zero Ex:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \text{order } 3 \times 3$$

- Scalar Matrix : A diagonal matrix in which all diagonal elements are the same. Ex:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \text{order } 3 \times 3$$

- Unit or Identity Matrix : A scalar matrix in which all diagonal elements are 1 (unity). Denoted I_n

Ex:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{order } 2$$

- Singular matrixes: Have a determinant of zero and nonsingular don't.

- **Upper Triangular Matrix**: A square matrix in which every element below the diagonal is 0. Matrix $A = [a_{ij}]_{m \times n}$ is upper triangular if $a_{ij} = 0$ for all $i > j$.

Ex:

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{bmatrix} \to \text{diagonal}$$

- **Lower Triangular Matrix**: A square matrix in which every element above the diagonal is zero. Matrix $A = [a_{ij}]_{m \times n}$ is lower triangular if $a_{ij} = 0$ for all $i < j$. Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 1 & 9 \end{bmatrix} \to \text{diagonal}$$

- **Triangular Matrix**: A square matrix that is either upper or lower triangular, that is a triangular matrix.

- **Transpose of a Matrix**: The matrix obtained by interchanging rows and columns of matrix. It is denoted by A' or A^T. If a matrix is of order $m \times n$, then the order of A^T is $n \times m$.

Ex:

$A = \begin{bmatrix} -2 & 5 \\ 6 & -1 \\ 4 & 3 \end{bmatrix}$ is of order 3×2 then $A^T = \begin{bmatrix} -2 & 6 & 4 \\ 5 & -1 & 3 \end{bmatrix}$ is of order 2×3

→ , → → Columns becomes rows

- **Symmetric Matrix**: A square matrix $[a_{ij}]_{n \times n}$ in which $a_{ij} = a_{ji}$ for all i and j.

$$\star \; A = A^T$$

Ex:

$$A = \begin{array}{c|c} & \begin{array}{ccc} j_1 & j_2 & j_3 \end{array} \\ \begin{array}{c} i_1 \\ i_2 \\ i_3 \end{array} & \begin{bmatrix} 2 & 4 & -7 \\ 4 & 5 & -1 \\ -7 & -1 & -3 \end{bmatrix} \end{array} \to \text{order } 3 \times 3$$

$a_{12} = 4$ and $a_{21} = 4$

$a_{31} = -7$ and $a_{13} = -7$

$a_{32} = -1$ and $a_{23} = -1$

- Skew Symmetric Matrix: A square symmetric matrix $[a_{ij}]_{n\times n}$ in which $a_{ij} = -a_{ji}$ for all i and j, is called a skew symmetric matrix. Ex:

$$A = \begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & 5 \\ 7 & -5 & 0 \end{bmatrix} \rightarrow \text{order } 3\times 3$$

★ $A = -A^T$

$a_{12} = 4$ and $a_{21} = -4$

$a_{13} = -7$ and $a_{31} = 7$

$a_{23} = 5$ and $a_{32} = -5$

★ In a skew symmetric matrix all diagonals are zero.

- Determinant of a matrix: A determinant is a real number associated with a matrix. Only SQUARE matrixes have a determinant. Denoted by eithe det (A) or |A|

★ A matrix is said to be a singular matrix if its det is 0 or else it is called non singular matrix

Finding the determinant of a Matrix

2x 2 Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = 12 - (-2) = 14$ So $ad - bc$ $\quad 3\times 4 - (-1\times 2) = 12 - (-2) = \underline{\underline{14}}$

3x3 Matrix

Formula:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

★ The basic methodology is that you choose the first row and each element from that row the row and column that element is in must be omitted.

3x3 Matrix Example For Determinant

$$\begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} - (-4)\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$3[-1+6] - (-4)(-1+4) + 5(3-2)$$

$$= 15 + 12 + 5 = 32$$

$$\begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & -1 \end{bmatrix} = 1\begin{bmatrix} -4 & 5 \\ 3 & -1 \end{bmatrix} - 1\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}$$

$$= 1(4-15) - 1[-3-10] + 4[9+8]$$

$$= -11 + 13 + 68 = 70$$

Algebra of Matrices

- Equality of Matrices: Two matrices are equal if their orders are the same and all corresponding elements are the same.

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$ A and B are equal

Every element is multiplied by k

- Multiplication of matrix by a scalar: If A is a matrix and it is multiplied by a scalar k, then the matrix obtained is kA.

If $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$

- Addition or Subtraction of matrices: You can add or subtract two matrices if they have the same dimensions like 3x2 or 3x3 just they have to be having same dimensions.

The addition and subtraction operations are carried out on corresponding elements of each of the matrix. Ex: $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 0 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ -1 & 3 & 2 \end{bmatrix}$

- Matrix Multiplication: Two matrices can be multiplied only if the number of columns in the first matrix equals the number of rows in the second matrix.

If A has order 2x3 then it can only be multiplied with a matrix of order 3x2 or 3x4...

Ex: $A = \begin{bmatrix} \boxed{1 \quad -1 \quad 2} \\ \boxed{0 \quad -1 \quad 3} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 0 & -1 \end{bmatrix}$

$$AB = \begin{matrix} \boxed{1+2+0} & \boxed{3+0+-2} \\ \boxed{0+2+0} & \boxed{0+0-3} \end{matrix}$$

$$AB = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$$

$A = m \times n$
$B = p \times q$

A·B only when $n = p$

Properties of Matrices

Subtraction is not commutative

- Matrix Addition is commutative $A + B = B + A$
- Matrix Addition is associative $A + (B + C) = (A + B) + C$
- Matrix multiplication is not commutative

Even if this occurs on the rare occasion where both combos can be multiplied the resultant matrix won't be the same

$$A_{2\times4} \quad B_{4\times3} \xrightarrow{\text{Result}} \boxed{2\times3}$$

$$\underbrace{B_{4\times3} \quad A_{2\times4}}_{\text{Not Same}} \xrightarrow{\text{Result}} \text{No Result}$$

- Matrix multiplication is associative $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
- If a Matrix is multiplied with an identity matrix the product will be the matrix itself.

$$AI = IA = A \longrightarrow \text{Identity property of a Matrix.}$$

Inverse exists only for non singular matrixes

AND Inverse is always unique

Inverse of a square matrix by adjoint method

★ If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that $AB = BA = I$, then B is called inverse matrix of A and B would be denoted as A^{-1} and A is said to be the invertible. $AA^{-1} = A^{-1}A = I$ → Inverse property of matrix

Minor : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Ex : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Minor of 1 = $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$ $= 45 - 48$ $= -3$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Minor of 9 = $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ $= 5 - 8$ $= -3$

★ Minor of element in a 2×2 matrix is the element that is left after ruling out the row and column of the element.

Co-Factor : Co-factor of an element a_{ij} of a determinant is given by $(-1)^{i+j} \times M_{ij}$, where M_{ij} is minor of the element a_{ij}. Denoted by A_{ij}.

Ex : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ a_{11}

Minor of 1 = $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$ $= 45 - 48$ $= -3$

So co-factor of 1 is $= (-1)^{1+1} \times -3 = \underline{\underline{-3}}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Minor of 9 = $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ $= 5 - 8$ $= -3$

So co-factor of 9 is $= (-1)^{3+3} \times -3 = \underline{\underline{-3}}$

Adjoint of a Square matrix : The adjoint of a matrix $A = [a_{ij}]_{m \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{m \times n}$ where A_{ij} is the co-factor of the element

a_{ij} of A, for all i and j. where $i, j = 1, 2 \ldots . m$.

Adjoint of matrix A is denoted as adj A.

So $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ a matrix made of all the co-factors of its elements is transposed. Then it forms an adjoint of the square matrix.

Co-Factor matrix

$$\begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Formula

$$\boxed{(-1)^{i+j} \times M_{ij}}$$

$a_{11}) \ (-1)^2 \times (-3) = -3$

$a_{12}) \ (-1)^3 \times (-6) = 6$

$a_{13}) \ (-1)^4 \times (-3) = -3$

$a_{21}) \ (-1)^3 \times (-6) = 6$

$a_{22}) \ (-1)^4 \times (-12) = -12$

$a_{23}) \ (-1)^5 \times (-6) = 6$

$a_{31}) \ (-1)^4 \times (-3) = -3$

$a_{32}) \ (-1)^5 \times (-6) = 6$

$a_{33}) \ (-1)^6 \times (-3) = -3$

Transposing it will give the adjoint

↳ Here it ends up the same

$$\begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

adj A

Final Part

Using the adjoint to find the inverse of a matrix. Use the given formula.

$$A^{-1} = \frac{adj A}{|A|}$$

Inverse of a matrix exists only when a matrix is non-singular.

Because non singular matrixes have determinants that are non zero

Determinant of matrix

Inverse of a Matrix Example

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Minors = a_{11}) $4-(1) = 3$
a_{12}) $-2-(-1) = -1$
a_{13}) $1-2 = -1$
a_{21}) $-2--1 = -1$
a_{22}) $4-1 = 3$
a_{23}) $-2-(-1) = -1$
a_{31}) $1-2 = -1$
a_{32}) $-2--1 = -1$
a_{33}) $4-1 = 3$

Co-factor matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Co-factors $(-1)^{i+j} \times M_{ij}$

a_{11}) $(-1)^2 \times 3 = 3$
a_{12}) $(-1)^3 \times -1 = 1$
a_{13}) $(-1)^4 \times -1 = -1$
a_{21}) $(-1)^3 \times -1 = 1$
a_{22}) $(-1)^4 \times 3 = 3$
a_{23}) $(-1)^5 \times -1 = 1$
a_{31}) $(-1)^4 \times -1 = -1$
a_{32}) $(-1)^5 \times -1 = 1$
a_{33}) $(-1)^6 \times 3 = 3$

$$\text{Adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Determinant of A

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A| = 2(4-1) - (-1)(-2+1) + 1(1-2)$$
$$= 6 + -1 + -1$$
$$= \underline{4}$$

Inverse / A^{-1}

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$\hookrightarrow$ det A

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

System of Equations Using Matrices

$$\left.\begin{aligned} 2x+3y+4&=0 \\ 3x+5y+7&=0 \end{aligned}\right\} \longrightarrow \text{System of equations (Linear)}$$

$$\left.\begin{aligned} 2x^2+3xy+4&=0 \\ 3x^2+4y^2+7&=0 \end{aligned}\right\} \longrightarrow \text{System of equations (non-linear)}$$

Linear equations

Only linear in this course

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \text{Linear}$$

Homogenous and non-homogenous

Both linear Systems

$$\underbrace{\begin{aligned} 2x+3y &= 5 \\ 3x+7y &= 8 \end{aligned}}_{\text{Non-Homogenous}} \qquad \underbrace{\begin{aligned} 2x+3y &= 0 \\ 3x+7y &= 0 \end{aligned}}_{\text{Homogenous}}$$

☆ If the system of equations involves constants like 5 or 8 then they are non-homogenous. But if they are equated to 0 they are homogenous.

System of Equations – Matrix form

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \text{Non-homogenous as equated to a constant}$$

As a matrix

Represented in the form $AX=B$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Example 1

$2x + 3y + 4z = 8$

$3x + 4y + 2z = 9$

$3x + 2y + 5z = 10$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$$

$AX = B$ } Linear and non-homogeneous

Example 1

$2x + 3y + 4z = 0$

$3x + 2y + 2z = 0$

$4x + 1y + 3z = 0$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = 0$ } Linear and homogeneous

Solution of System of Equations

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

★ The solution for this given system of equations exists only if A is non-singular which means that the det $A \neq 0$ or $|A| \neq 0$. The matrix must have an inverse.

$AX = B$

$A^{-1} \times AX = A^{-1}B$

$(A^{-1} \times A)X = A^{-1}B$

$IX = A^{-1}B \longrightarrow IX = X$ ★ When an identity matrix is multiplied with the matrix the matrix remains same.

$X = A^{-1}B$

→ Basically $X = \frac{B}{A} = A^{-1}B$

★ $A^{-1} \times A = I$ → Identity matrix

Solution of System of Equations Example 1

$$x - y + z = 4$$

$$x + y + z = 2$$

$$2x + y - 3z = 0$$

$$\begin{array}{ccccc} 3\times3 & & 3\times1 & & \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 22 & 1 & -3 \end{bmatrix} & & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \\ \text{1st} & & \text{2nd} & & \end{array}$$

no: of columns = no: of rows

Solution yes or no

$$\det|A| = 1(-3-1) + 1(-3-2) + 1(1-2) = -4-5-1$$
$$= \underline{\underline{-10}} \text{ so yes}$$

adj A =

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

Minors

$a_{11} = -3-1 = -4$ $\quad a_{21} = 3-1 = 2$ $\quad a_{31} = -1-1 = -2$

$a_{12} = -3-2 = -5$ $\quad a_{22} = -3-2 = -5$ $\quad a_{32} = 1-1 = 0$

$a_{13} = 1-2 = -1$ $\quad a_{23} = 1--2 = 3$ $\quad a_{33} = 1+1 = 2$

Co-factors

$a_{11} = (-1)^2 \times -4 = -4$ $\quad a_{21} = (-1)^3 \times 2 = -2$ $\quad a_{31} = (-1)^4 \times -2 = -2$

$a_{12} = (-1)^3 \times -5 = 5$ $\quad a_{22} = (-1)^4 \times -5 = -5$ $\quad a_{32} = (-1)^5 \times 0 = 0$

$a_{13} = (-1)^4 \times -1 = -1$ $\quad a_{23} = (-1)^5 \times 3 = -3$ $\quad a_{33} = (-1)^6 \times 2 = 2$

$$\begin{bmatrix} -4 & 5 & -1 \\ -2 & -5 & -3 \\ -2 & 0 & 2 \end{bmatrix} \xrightarrow{\text{Transposed}} \begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \text{adj} A$$

$$A^{-1} = \frac{-1}{10} \begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$x = A^{-1}B$$

$$X = \frac{-1}{10} \underset{3\times3}{\begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}} \times \underset{3\times1}{\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}} = \begin{matrix} -16-4+0 \\ 20-10+0 \\ -4-6+0 \end{matrix} = \begin{bmatrix} -20 \\ 10 \\ -10 \end{bmatrix} \times \frac{-1}{10}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 2 \\ -1 \\ 1 \end{matrix} \quad \boxed{\begin{matrix} x = 2 \\ y = -1 \\ z = 1 \end{matrix}} \checkmark$$

Rank of a Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}_{3\times 3}$$

$$|A| = 1(2-3) - 2(4-3) + 3(2-1)$$
$$= -1 - 2 + 3 = 0$$

Rank $\neq 3$ as the det A is 0

Rank : Order of the largest non zero minor

Case for 3x3 but basically for any Start from largest and keep doing until you get non zero.

Minors of the Matrix

$a_{11} = 2-3 = -1$ $\quad a_{21} = 4-3 = 1$ $\quad a_{31} = 6-3 = 3$

$a_{12} = 4-3 = 1$ $\quad a_{22} = 2-3 = -1$ $\quad a_{32} = 3-6 = -3$

$a_{13} = 2-1 = 1$ $\quad a_{23} = 1-2 = -1$ $\quad a_{33} = 1-4 = -3$

✱ Check for the largest minor first which is the determinant if that $=0$ then next minors those also $=0$ then rank $=1$.

So for a 2x2 sub matrix there are non zero determinant / minor so Rank = 2

$$\boxed{\text{So Rank of } A = 2}$$

If all minors / determinants of the 2x2 sub matrixes end up to be zero then the Rank will become 1.

So if there is a 3x3 matrix the largest rank it could possibly have is 3 but isn't necessary that it have rank 3 it can be lesser also.

If the matrix is 3x2 or 2x2 largest possible rank is 2.

If the matrix is 2x1 or 1x1 or 3x1 largest possible rank is 1.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

$|A| \neq 0$ $\quad$ Rank(A) $= 3$

Rank$(I_3) = 3$

$$\boxed{\text{Rank of } I_n = n}$$

Rank of a Matrix Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

3×4 so largest possible rank is 3.

1st minor (order 3): $1(12-4) - 2(9-2) + 3(6-4)$
$= 8 - 14 + 6 = 0$ ×

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

2nd minor (order 3): $2(8-15) - 3(16-10) + 4(12-4)$
$= -14 - 18 + 32 = 0$ ×

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

3rd minor (order 3): $1(8-15) - 3(12-5) + 4(9-2)$
$= -7 - 21 + 28 = 0$ ×

So Rank 3 not possible

So now from each of these 3 matrices if even 1 has a non zero minor then rank becomes 2.

Ex: Minor $a_{11}(1) = 12 - 4 = 8$ so $\boxed{\text{Rank of this matrix is 2}}$ ✓
Because $8 \neq 0$

Methods to Find Rank without need to calculate Minors

1) Echelon Form

2) Normal Form

$$\begin{matrix} 3 & 0 & -3 \\ 6 & 3 & -6 \end{matrix} + \begin{matrix} -2 & 6 & 4 \\ 0 & 2 & -4 \end{matrix} = \begin{matrix} 1 & 6 & 1 \\ 6 & 5 & -10 \end{matrix}$$

$2 \times \cancel{2}$ $\quad \cancel{2} \times 2$

$2 \times 2 =$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \quad \begin{matrix} 0+2 & 3-4 \\ 0-1 & 6+2 \end{matrix} = \begin{matrix} 2 & -1 \\ -1 & 8 \end{matrix}$$

$$\frac{\text{adj } A}{|A|} = \quad 5 - 6 = -1 = |A|$$

$\rightarrow$ Transpose first

$$\begin{aligned} a_{11} &= 5 \times (-1)^2 = 5 \\ a_{12} &= 2 \times (-1)^3 = -2 \\ a_{21} &= 3 \times (-1)^3 = -3 \\ a_{22} &= 1 \times (-1)^4 = 1 \end{aligned} \quad \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} \times -1$$

$$= \begin{matrix} -5 & 2 \\ 3 & -1 \end{matrix}$$

$$\begin{matrix} 5 & -3 \\ -2 & 1 \end{matrix} \times -1 = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$2(-2+0) - 0 + 1(0-3)$$

$$= -4 + -3 = -7$$

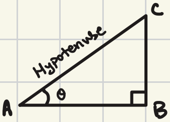
$$\begin{matrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad 3(1 \times 0$$

$$1(45-48) - 2(36-42) + 3(32-35)$$

$$-3 - 12 + 15 = 0$$

$$5 - 8 = -3$$

3) Trigonometry



$\sin A = \frac{opp}{hyp} = \frac{BC}{AC}$ $\cos A = \frac{adj}{hyp} = \frac{AB}{AC}$ $\tan A = \frac{opp}{adj} = \frac{\sin A}{\cos A} = \frac{BC}{AB}$

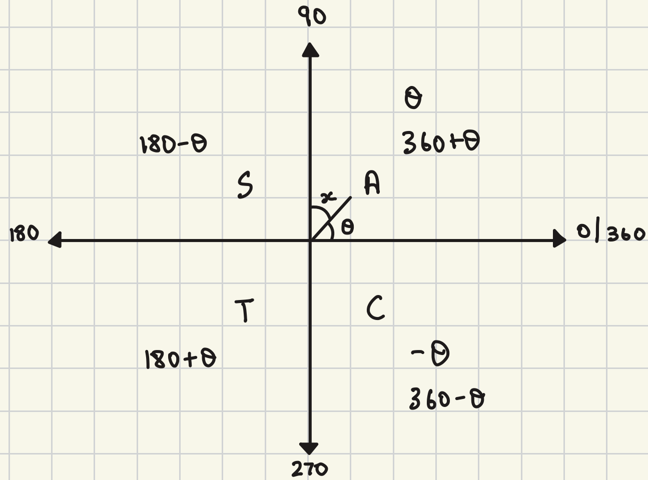
$\operatorname{cosec} A = \frac{hyp}{opp} = \frac{1}{\sin A} = \frac{AC}{BC}$ $\sec A = \frac{hyp}{adj} = \frac{1}{\cos A} = \frac{AC}{AB}$ $\cot A = \frac{adj}{opp} = \frac{\cos A}{\sin A} = \frac{AB}{BC} = \frac{1}{\tan A}$

Degree & Radians

$\boxed{\pi = 180^\circ}$

Standard Angles

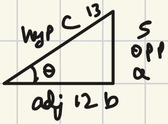
	0	30	45	60	90
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—
cosec	—	2	$\sqrt{2}$	$2/\sqrt{3}$	1
sec	1	$2/\sqrt{3}$	$\sqrt{2}$	2	—
cot	—	$\sqrt{3}$	1	$1/\sqrt{3}$	0



Example 1

$\sec\theta = 13/12$ Find all other ratios

$\cos\theta = 12/13$



$a^2 + b^2 = c^2$

$a^2 = c^2 - b^2$

$a = \sqrt{13^2 - 12^2}$

$a = 5$

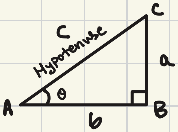
$\sin\theta = 5/13$

$\operatorname{cosec}\theta = 13/5$

$\tan\theta = 5/12$

$\cot\theta = 12/5$

Trigonometric Identities



$\sin A = \frac{BC}{AC}$

$\cos A = \frac{AB}{AC}$

$\tan A = \frac{BC}{AB}$

$AB = b$

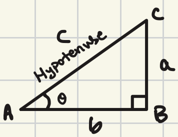
$BC = a$

$AC = c$

$a^2 + b^2 = c^2$

$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \quad = \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = 1$

$$\boxed{\sin^2 A + \cos^2 A = 1}$$



$\operatorname{cosec} A = \frac{AC}{BC}$

$\sec A = \frac{AC}{AB}$

$\cot A = \frac{AB}{BC}$

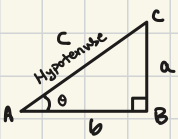
$AB = b$

$BC = a$

$AC = c$

$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2} \quad \left(\frac{b}{b}\right)^2 = \left(\frac{c}{b}\right)^2 - \left(\frac{a}{b}\right)^2$

$$\boxed{\sec^2 A - \tan^2 A = 1}$$



$\operatorname{cosec} A = \frac{AC}{BC}$

$\sec A = \frac{AC}{AB}$

$\cot A = \frac{AB}{BC}$

$AB = b$

$BC = a$

$AC = c$

$a^2 + b^2 = c^2$

$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2} \quad \left(\frac{a}{a}\right) = \left(\frac{c}{a}\right)^2 - \left(\frac{b}{a}\right)^2$

$$\boxed{\cos ec^2 A - \cot^2 \theta = 1}$$

$$\boxed{\sin^2 A + \cos^2 A = 1}$$

$$\boxed{\sec^2 A - \tan^2 A = 1}$$

$$\boxed{\cos ec^2 A - \cot^2 \theta = 1}$$

All Trigonometric Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1 \qquad 1 + \tan^2\theta \equiv \sec^2\theta \qquad \cot^2\theta + 1 \equiv \csc^2\theta$$

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta \equiv 2\cos^2\theta - 1$$
$$\cos 2\theta \equiv 1 - 2\sin^2\theta$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan(A-B) \equiv \frac{\tan A + \tan B}{1 + \tan A\tan B}$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Example 1

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{1}{\sec\theta - \tan\theta}$$ using $1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - \tan^2\theta = 1$

$$\frac{\frac{\sin\theta}{\cos\theta} - 1 + \sec\theta}{\frac{\sin\theta}{\cos\theta} + 1 - \sec\theta} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta} \times \frac{\tan\theta - \sec\theta}{\tan\theta - \sec\theta}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{(\tan\theta - \sec\theta + 1)(\tan\theta - \sec\theta)}$$

$$= \frac{(-1) - (\tan\theta - \sec\theta)}{(\tan\theta - \sec\theta + 1)(\tan\theta - \sec\theta)} = \frac{-1(\cancel{1 + \tan\theta - \sec\theta})}{(\cancel{\tan\theta - \sec\theta + 1})(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta} \times \frac{-1}{-1} = \frac{1}{\sec\theta - \tan\theta}$$

Example 2

If $\boxed{\cos\theta = -3/5}$ and θ lies in the third quadrant, find other ratios. → only tan +

$\sin^2\theta + \cos^2\theta = 1$

$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

$\sin\theta = \pm\sqrt{1 - (-3/5)^2}$

$\boxed{\sin\theta = -4/5}$

$1 + \tan^2\theta = \sec^2\theta$

$\boxed{\tan\theta = \frac{-4}{\cancel{5}} \times \frac{\cancel{5}}{-3} = \frac{4}{3}}$

$\sec\theta = \pm\sqrt{1 + (-4/3)^2}$

$\boxed{\sec\theta = -5/3}$

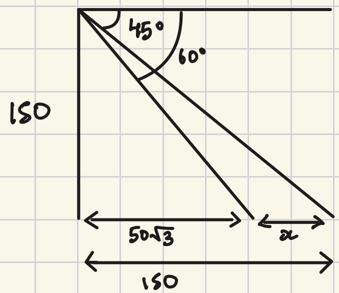
$\boxed{\operatorname{cosec}\theta = \frac{-5}{4}}$

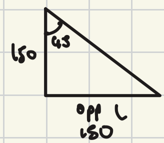
$\boxed{\cot\theta = 3/4}$

Applications of Trigonometry

1) Angle of elevation and depression

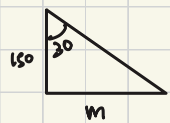
Example 1





$\tan 45 = \frac{l}{150}$

$l = 150$

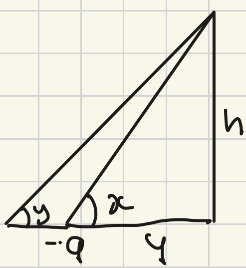


$\tan 30 = \frac{m}{150}$

$m = 50\sqrt{3}$

$150 - 50\sqrt{3} = \underline{\underline{63.4}}$

☆ Important



$x + y = 90$

$\tan x = \frac{h}{4}$

$\tan y = \frac{h}{9}$

$\frac{\tan x}{\tan y} = \frac{\cancel{h}}{4} \times \frac{9}{\cancel{h}}$

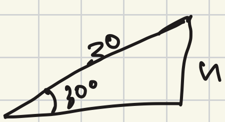
$\tan x = \frac{h}{4}$

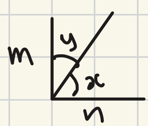
$\tan x = \cot y$

so $\tan x = \frac{1}{\tan y}$

$= \frac{h}{4} = \frac{9}{h}$

$h^2 = 36 \quad \underline{\underline{h = 6}}$





$x + y = 90$

$$\boxed{\tan x = \frac{m}{n} \quad \tan y = \frac{n}{m} \quad \text{so essentially } \tan x = \cot y}$$

4) Differential Calculus and Its Applications

Functions

1) Graphs
2) equation
3) Numerical table

A function from a set D to set Y is a rule that assigns a unique value $f(x)$ in Y to each x in D.

Domain — All possible input values

Codomain

Types of Functions

Let f be a function defined on an interval I and let x_1 and x_2 two distinct points in I.

Def(Increasing on I) : If $f(x_2) \geq f(x_1)$ whenever $x_1 < x_2$

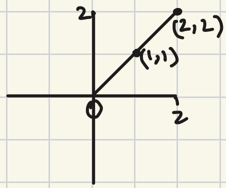
Def(Increasing on I) : If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$

Strictly Increasing

Def(Decreasing on I) : If $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$

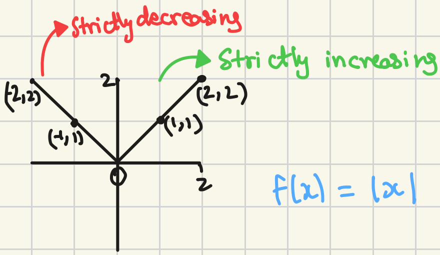
Def(Decreasing on I) : If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$

Strictly Decreasing



$1<2$ and $2>1$ so strictly increasing

$f(x) = x$



$f(x) = |x|$

Even Function

$f(-x) = f(x)$

Example 1: $f(x) = \cos x$
$f(-x) = \cos(-x)$
$\hookrightarrow \cos(x)$ so $f(-x) = \cos x$

Example 2: $f(x) = x^2$
$f(-x) = (-x)^2$
$\hookrightarrow x^2$ so $f(-x) = x^2$

Odd Function

$f(-x) = -f(x)$

Example 1: $f(x) = x^3$
$f(-x) = (-x)^3$
$\hookrightarrow -x^3$ so $f(-x) = -x^3 = -f(x)$

One-One/Injective Function

If the images of distinct element of X under f are distinct for every $x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So if a particular output is shared by two domains those two domains must be the same.

For each input there is a unique output.

Onto/Surjective Function

If every element of Y is the image of some element of X under f.

Every element of the range / codomain set must have a corresponding input or domain value. The range / codomain is mapped to atleast one element in the domain.

Bijective Function

If the function follows the rule of an onto and one-one function then it is bijective.

So that means each codomain has only one corresponding domain but for sure has one and cannot have no inputs.

Example 1:

$f : N \to N$ defined by $f(x) = 2x$

Because function is only for natural numbers it can't be an onto function as some codomains such as 1 whose input would be 0.5 so this function is only one-one and not onto so not bijective.

Real not natural

Example 2:

$f : R \to R$ defined by $f(x) = 2x$

This function is both onto and one-one so it can be bijective as it allows real numbers so there aren't any codomains which don't have an input.

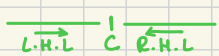
Limits of a function

The limits can approach from either the left or the right side

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

x approaches c from left as well as right

★ If the limits approaching from both sides is the same then the limit exists.



Limit of a Function

- Describe the way a function behaves
- Limits arise when finding the instoneous rate of change of a function or a tangent line to a curve.

Let f and g be two functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$
Then:

i) $\lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$

ii) $\lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$

iii) $\lim_{x \to c} f(x) \times \lim_{x \to c} g(x) = L \times M$

iv) $\lim_{x \to c} f(x) \div \lim_{x \to c} g(x) = L \div M$

v) $\lim_{x \to c} [f(x)]^n = L^n$

vi) $\lim_{x \to c} [f(x)]^{1/n} = L^{1/n}$

vii) Limit of $\frac{\sin x}{x} = 1$

Example 1: $\lim_{x \to a} (a_0 + a_1 x + a_2 x^2 + \dots a_n x^n)$

$$\underset{f_0}{\downarrow} \quad \underset{f_1}{\downarrow} \quad \underset{f_2}{\downarrow} \quad \underset{f_n}{\downarrow}$$

$$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + \lim_{x \to a} a_n x^n$$

$$= a_0 1 + a_1 a + a_2 a^2 + \dots a_n a^n$$

Example 2: $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

$$= \frac{\lim_{x \to c} x^4 + x^2 - 1}{\lim_{x \to c} x^2 + 5}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

Example 3: $\lim_{x \to c} \sqrt{4x^2 - 3}$

$$= (4x^2 - 3)^{1/2}$$

$$= (4c^2 - 3)^{1/2}$$

$$= \sqrt{4c^2 - 3}$$

Sandwich Theorem

Let f, g, and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For a real number c

If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$

Example : $\cos x < \frac{\sin x}{x} < 1$ for $0 < |x| < \frac{\pi}{2}$

Limit of $\cos x$ as $x \to 0$ is 1

Limit of constant as $x \to 0$ $= 1$

So limit for both $\cos x$ and 1 is 1 that means according to the Sandwich theorem limit of $\frac{\sin x}{x} = 1$

Indeterminate Forms

$\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{0}{0}$ cannot be determined. Such forms are indeterminate forms.

Other indeterminate forms $\frac{\infty}{\infty}$, ∞, $-\infty$

If $g(x)$ and $h(x)$ are polynomials. $\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$.

If $h(a) = 0$. There are two cases

Case 1: When $g(a) \neq 0$

Limit does not exist

Case 2 : When $g(a) = 0$

$$g(x) = (x-a)^k g_1(x) \text{ and } h(x) = (x-a)^l h_1(x)$$

If $k > l$

$$\frac{g(x)}{h(x)} = \frac{(x-a)^k g_1(x)}{(x-a)^l h_1(x)} = (x-a)^{k-l} \frac{g_1(x)}{h_1(x)}$$

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} (x-a)^{k-l} \times \frac{g_1(x)}{h_1(x)} = 0$$

If $k < l$

$$\frac{1}{(x-a)^{l-k}} \times \text{ limit doesn't exist}$$

Example 1 : $\lim_{x \to 1} \frac{x^2 + x - 2}{x-1} = \frac{1+1-2}{1-1} = \frac{0}{0}$ form

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}} = \lim_{x \to 1} (x+2) = 3$$

$1+2 = 3$ so limit $= 3$

Example 2 : $\lim_{x \to 2} \frac{x^3 - 4x^2 + 4x}{x^2 - 4} = \frac{\cancel{8}-\cancel{16}+\cancel{8}}{4-4} = \frac{0}{0}$ form

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x-2)(x)}{(x+2)\cancel{(x-2)}} = \lim_{x \to 2} \frac{x(x-2)}{x+2} = \lim_{x \to 2} \frac{x(x-2)}{x+2}$$

$$\frac{2(0)}{4} = 0$$

So limit $= 0$

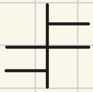
$$\begin{array}{r|l} & x^2 - 2x \\ \hline x-2 & x^3 - 4x^2 + 4x \\ & -x^3 + 2x^2 \\ \hline & 0 \; -2x^2 + 4x \\ & +2x^2 - 4x \\ \hline & 0 \end{array}$$

$(x-2)(x^2 - 2x)$

$(x-2)(x)(x-2)$

Limit Examples

Example 1 : $f(x) = \begin{Bmatrix} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{Bmatrix}$ Find $\lim\limits_{x \to 0} f(x)$



$$L.H.L = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

for $x < 0$ x = -ve number to make -ve +ve you take $-x$ = +ve so $|x| = -x$

$-(-x) = +$

$$R.H.L = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$RHL \neq LHL$$

So limit doesn't exist

Example 2: $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

$$= \frac{\sqrt{1} - 1}{0} = \frac{0}{0} \text{ form}$$

$$= \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \frac{1 + x - 1}{x\sqrt{1+x} + x} = \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)}$$

$$= \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2} \text{ so limit is } \frac{1}{2}$$

Example 3: $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = \dfrac{1 - \cos(0)}{0} = \dfrac{0}{0}$ form

$$= \lim_{x \to 0} \frac{1 - \cos x}{x} =$$

$\cos 2x = 1 - 2\sin^2 x$

$\cos x = 1 - 2\sin^2 x/2$

$$\lim_{x \to 0} \frac{2\sin^2 x/2}{x} = 2 \lim_{x \to 0} \frac{\sin^2 x/2}{x} = \cancel{2} \lim_{x \to 0} \frac{\sin x/2}{\cancel{2}\,\frac{x}{2}} \times \lim_{x \to 0} \sin x/2$$

same as $\dfrac{\sin x}{x}$

$$= 1 \times 0 = 0$$

Example 4: $f(x) = \begin{cases} a+bx & x<1 \\ 4 & x=1 \\ b-ax & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) = f(1)$

What are possible values of a and b.

$f(1) = 4$

$\lim_{x\to 1^-} a+bx$ → LHL

$\lim_{x\to 1^+} b-ax$ → RHL

$\lim_{x\to 1^-} a+bx = a+b$

$\lim_{x\to 1^+} b-a = b-a$

$a+b=4$

$b-a=4$

$\cancel{a}+b = 4$

$\cancel{-a}+b = 4$

$2b = 8$

$\underline{\underline{b=4 \quad a=0}}$

<u>Continuity</u>

f(x) is defined on $[a,b]$, then for f to be continuous it must be continuous at every point in $[a,b]$ including the end points a, b.

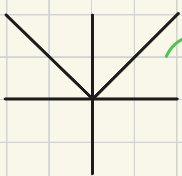
<u>Continuity</u> → If a function f(x) is differentiable at a point its also continuous at that point. But all continuous aren't differentiable

Let f(x) be real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous if $\lim_{x\to c} f(x) = f(c)$

If f is not continous at $x=c$ then f is discontinuous at c

☆ A function is set to be continuous if it is continuous at every point in the domain of f.

Example 1: $f(x) = |x|$ at $x=0$



Continuous

Example 2: $f(x) = \begin{cases} x^3+3 & x\neq 0 \\ 1 & x=0 \end{cases}$ at $x=0$

Not continuous

Suppose f and g be two real functions continuous at a real number c.

i) f+g limit exists as it is $\lim_{x \to c} (f+g) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

Continuous

$= f(c) + g(c)$

$= (f+g)(c)$

ii) f-g limit exists as it is $\lim_{x \to c} (f-g) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

Continuous

$= f(c) - g(c)$

$= (f-g)(c)$

iii) fg limit exists as it is $\lim_{x \to c} (fg) = \lim_{x \to c} f(x) \times \lim_{x \to c} g(x)$

Continuous

$= f(c) \times g(c)$

$= (fg)(c)$

iii) $\frac{f}{g}$ limit exists as it is $\lim_{x \to c} \left(\frac{f}{g}\right) = \lim_{x \to c} f(x) \div \lim_{x \to c} g(x)$

Continuous as long as $g(c) \neq 0$

$= f(c) \div g(c)$

$= \left(\frac{f}{g}\right)(c)$

Example 1: $f(x) = \begin{cases} 5 & x \leq 2 \\ ax+b & 2 < x < 10 \\ 21 & x \geq 10 \end{cases}$ is continuous find a and b

$ax + b = 5$

$2a + b = 5$

$$\begin{array}{r} 10a + \cancel{b} = 21 \\ -2a \cancel{+b} = -5 \\ \hline 8a = 16 \end{array}$$

$a = 2$

$b = 1$

Suppose f and g are real valued functions such that $f(g(c))$ is defined at c. If g is continuous at c and if f is continuous at c then $f(g(c))$ is continuous at c.

Example 1: $f(x) = \sin(x^2)$ is continuous

$$g(x) = \sin x \quad \text{and} \quad h(x) = x^2$$

continuous continuous

$$g(h(x)) = f(x) = \sin(x^2)$$

so continuous

Example 2: $f(x) = |1 - x + |x||$

$$g(x) = |x| \quad \text{and} \quad h(x) = 1 - x + |x|$$

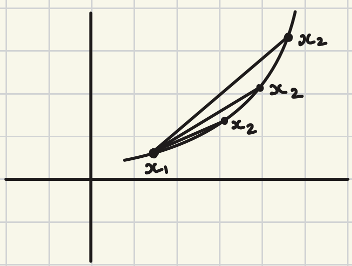
continuous continuous

$$g(h(x)) = f(x) = |1 - x + |x||$$

so continuous

Differentiation : The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + n) - f(x_1)}{n}, n \neq 0$$



$$x_2 = x_1 + n$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

can also be used

Derivative of a Function : Derivative of a function $f(x)$ at a point a denoted by $f'(a)$ or $\frac{df}{dx}(a)$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$ provided the limit exists.

Derivative Formula : $y = x^n$ $\quad \frac{dy}{dx} = nx^{n-1}$

Example 1: $f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= 2x + h$$

$$= 2x + 0$$

$$\frac{df}{dx} = 2x$$

<u>Deriving the differentiation Equation</u>

$$\frac{d}{dx}(x^n)$$

Formula being used

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

✫ Assumption is that n is a +ve integer

$$= \lim_{h \to 0} \frac{\cancel{{}^nC_0 x^n} + {}^nC_1 x^{n-1} h + h^2 P(x,h) - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}\cancel{h} + \cancel{h^2} p(x,h)}{\cancel{h}}$$

$$= \lim_{h \to 0} nx^{n-1} + 0 \times P(x,0)$$

$$= nx^{n-1}$$

Example 1: $\frac{d}{dx}(\sqrt{x})$

$$\sqrt{x} = x^{1/2} = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

Example 2: $\frac{d}{dx} \sin x$ $\quad \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

$$= \lim_{x\to 0} \frac{\sin \overset{a}{(x+h)} - \sin \overset{b}{x}}{h} = \frac{2\sin\left(\frac{x+h-\cancel{x}}{2}\right)\cos\left(\frac{x+h+x}{2}\right)}{h}$$

Limit of $\frac{\sin h}{h} = 1$

$$= \lim_{x\to 0} \frac{2x \sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h} = \cancel{2} \times \frac{1}{\cancel{2}} \frac{\sin\left(\frac{h}{2}\right)}{h \times \frac{1}{2}} \times \cos\left(\frac{2x+h}{2}\right)$$

$$= 1 \times \cos\left(\frac{2x+0}{2}\right) = \underline{\underline{\cos x}}$$

Example 2: $\frac{d}{dx} f(x) = |x|$ at $x = 0$

$$\lim_{x\to 0} = \lim \frac{|x| - 0}{x} = \lim_{x\to 0} \frac{|x|}{x}$$

if $x < 0$ then $-x$

LHL $= -1$

and RHL $= 1$ So limit doesn't exist hence $|x|$ cannot be differentiated

<u>Higher</u> <u>order</u> <u>Derivatives</u>

1st order $= \frac{dy}{dx}$

2nd order $= \frac{d^2 y}{dx^2}$

3r order $= \frac{d^3 y}{dx^3}$

Derivatives

1) $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = u' + v'$$

2) $y = uv$

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = u'v + uv'$$

3) $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

Proof for 2

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$uv = \lim_{h\to 0} \frac{(uv)(x+h) - (uv)(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)\,v(x+h) - u(x)\,v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)\,v(x+h) - u(x)\,v(x+h) + u(x)\,v(x+h) - u(x)\,v(x)}{h}$$

$$= \lim_{h\to 0} \underbrace{v(x+h)}_{v}\underbrace{\boxed{\left(\frac{u(x+h) - u(x)}{h}\right)}}_{\frac{du}{dx}} + \underbrace{u(x)}_{u}\underbrace{\boxed{\left(\frac{v(x+h) - v(x)}{h}\right)}}_{\frac{dv}{dx}}$$

$$= \frac{du}{dx}v + u\frac{dv}{dx}$$

Proof for 3

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$\frac{u}{v} = \lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)\ v(x) - u(x)\,v(x+h) - u(x)v(x) + u(x)\,v(x)}{h \times v(x+h) \times v(x)}$$

$$= \lim_{h\to 0}\frac{v(x)\big(u(x+h) - u(x)\big) - u(x)\big(v(x+h) - v(x)\big)}{h \times v(x+h) \times v(x)}$$

$$= \lim_{h\to 0}\ \underset{\substack{\downarrow\\ v}}{v(x)}\ \underset{\substack{\downarrow\\ \frac{du}{dx}}}{\boxed{\frac{\big(u(x+h) - u(x)\big)}{h}}} - \underset{\substack{\downarrow\\ u}}{u(x)}\ \underset{\substack{\downarrow\\ \frac{dv}{dx}}}{\boxed{\frac{\big(v(x+h) - v(x)\big)}{h}}} \times \underset{\substack{\downarrow\\ \frac{1}{v^2}}}{\frac{1}{v(x+h)\,v(x)}}$$

$$= \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

Proof for 1

$$u+v = \lim_{h\to 0}\frac{(u+v)(x+h) \quad (u+v)(x)}{h}$$

$$= \underset{\substack{\downarrow\\ \frac{du}{dx}}}{\frac{u(x+h) + u(x)}{h}} + \underset{\substack{\downarrow\\ \frac{dv}{dx}}}{\frac{v(x+h) + v(x)}{h}}$$

$$= \frac{du}{dx} + \frac{dv}{dx}$$

Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composition $(f \circ g)x = f(g(x))$ is differentiable at x.

☆ Basically if $g(x)$ is differentiable at x and if $f(x)$ is differentiable at the point $u = g(x)$ where u is basically the output of $g(x)$ when you plug x in then $f(g(x))$ is differentiable at x.

So derivative of composite function

$$(f \circ g)'(x) = f'(g(x)) \times g'(x)$$

$y = f(u) \qquad u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{dx} \times \frac{du}{dx} = f'(x) \times g'(x)$$

Example : $f(x) = \sin(x^2)$

$= \cos(x^2) \times 2x$

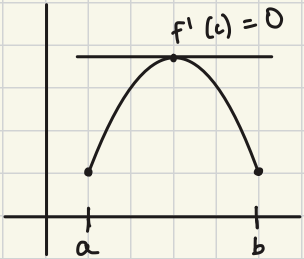
$= 2x \cos(x^2)$

$y = (u)^n \qquad \underline{\underline{n \neq 1}}$

$$\frac{dy}{dx} = nu^{n-1} \times \frac{du}{dx}$$

Rolle's Theorem (Part of Mean Value Theorem)

$f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=f(b)$, where a and b are some real numbers. Then there exists some c in (a,b) such that $f'(c)=0$.



Example 1:

$f(x) = x^2 + 2 \quad [-2,2]$

$f'(x) = 2x$

$f(-2) = 4+2 = 6$

$f(2) = 4+2 = 6$

$2x = 0$

$x = 0$

so $c = 0$

at $c=0$

$\underline{\underline{f'(x) = 0}}$

Example 2: If Rolle's Theorem holds true for the function $f(x) = x^4 + ax^3 + bx$ in $[-1,1]$ and $f'(1/2) = 0$ then $ab = ?$

$f'(x) = 4x^3 + 3ax^2 + b$

$f(-1) = f(1)$

$1 - a - b = 1 + a + b$

$2a + \cancel{2b} = 0$

$-\frac{3}{4}a - \cancel{2b} = -1$

$1/2\, a = -1$

$\underline{\underline{a = -2}}$

$f'(1/2) = 4 \times \frac{1}{8} + 3a \times \frac{1}{4} + b$

$= \frac{1}{2} + \frac{3}{4}a + b$

$-4 + 2b = 0$

$2b = 4$

$\underline{\underline{b = 2}}$

$\underline{ab = -4}$

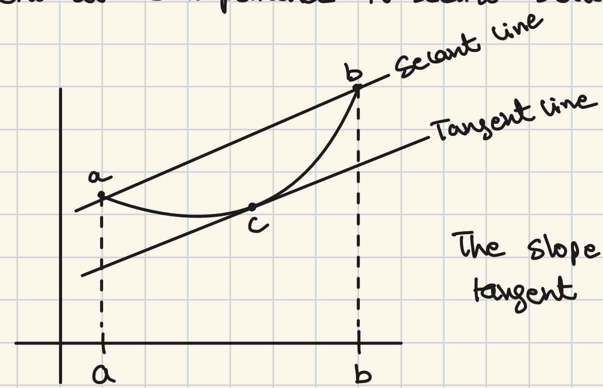
Mean Value Theorem

$f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

There is point c such that slope of the tangent at $(c, f(c))$ is same as slope of the secant between $(a, f(a))$, $(b, f(b))$

Tangent at c is parallel to secant between $(a, f(a))$, $(b, f(b))$.



The slope of secant line = slope of tangent line

Example 1: Verify mean value theorem in $f(x) = x^2$ for $[2,4]$

$(x_1, y_1) = (2,4)$

$(x_2, y_2) = (4,16)$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16-4}{4-2} = \frac{12}{2} = 6$$

$f'(x) = 2x$

$2x = 6$

$\underline{\underline{x = 3}}$

So at $x = 3$ $y = 9$

$2 < 3 < 4$ ✓

The slope of tangent is equal to slope between $(2,4)$ and $(4,16)$

1st Derivative Test

Differentiation can be used to identify whether a function is increasing or decreasing.

Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing at x_0, if there exists an open interval I containing x_0 such that f is increasing, decreasing respectively in I.

Example 1: $f(x) = 7x-3$

$$f(x) = 7$$

$7>0$ so always increasing

Let f be continuous on $[a,b]$ and differentiable on the open interval (a,b) Then:

f is increasing in $[a,b]$ if $\frac{dy}{dx} > 0$ for $x \in [a,b]$

f is decreasing in $[a,b]$ if $\frac{dy}{dx} < 0$ for $x \in [a,b]$

f is constant in $[a,b]$ if $\frac{dy}{dx} = 0$ for $x \in [a,b]$

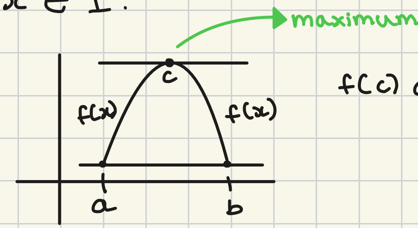
2nd Derivative Test

Differentiation can be used to find maximum and minimum of a function.

Let f be a function defined in an interval Then f is said to have

1) a maximum value in I, if there is a point c in I such that $f(c) > f(x)\ \forall x \in I$.

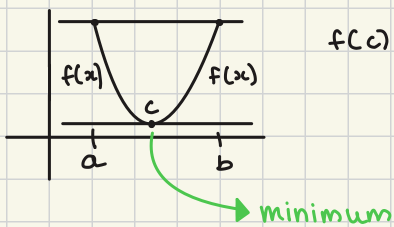
$$\frac{d^2y}{dx^2} < 0$$



f(c) always > f(x)
↳ for all x

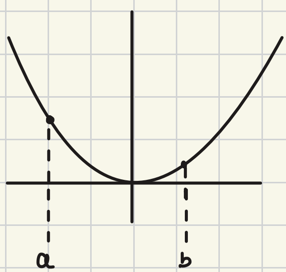
2) a minimum value in I, if there exists a point c in I such that $f(c) < f(x)\ \forall x \in I$.

$$\frac{d^2y}{dx^2} > 0$$



f(c) always < f(x)
↳ for all x

☆ Every continuous function on a closed and bounded interval has a maximum and a minimum.



a = maximum
0 = minimum
in a closed interval

Let f be a real valued function and let c be an interior point in the domain of f. Then:

1) c is called a point of local maxima if there is a number $h > 0$ $\ni$ $f(c) \geq f(x)$ for all $x \in (c-h, c+h)$, $x \neq c$. $f(c)$ is local maximum value of f.

2) c is called a point of local minima if there is a number $h > 0$ $\ni$ $f(c) \leq f(x)$ for all $x \in (c-h, c+h)$, $x \neq c$. $f(c)$ is local minimum value of f.

Special case

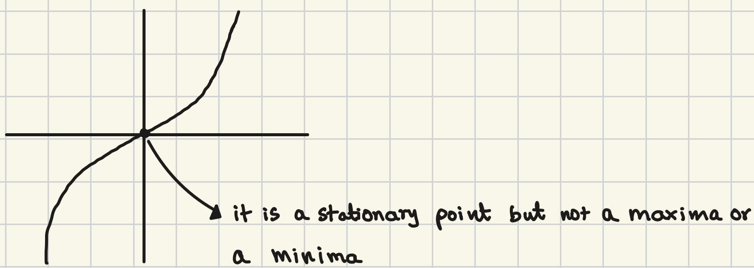
Let f be a function defined on an open interval I. Let $c \in I$ be any point. If f has a local maxima and or local minima at $x = c$ then $f'(c) = 0$ or f is not differentiable at c. Converse need not be true. So even if the $f'(c) = 0$ it need not be a minima or maxima it can just be a stationary point.

Example 1: $f(x) = x^3$

$f'(x) = 3x^2$

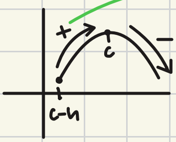
$0 = 3x^2$

$\underline{\underline{x = 0}}$



1st Derivative Test

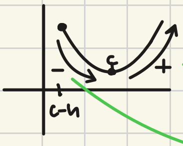
1) Local maxima at c if $f'(x)$ changes sign from positive to negative $+ \longrightarrow -$ as x increases through c i.e $f'(x) > 0 \; \forall x \in (c-h, c)$ and $f'(x) < 0 \; \forall x \in (c, c+h)$



$\frac{dy}{dx} > 0$

$\frac{dy}{dx} < 0$

2) Local minima at c if $f'(x)$ changes sign from negative to positive $- \longrightarrow +$ as x increases through c i.e $f'(x) < 0 \; \forall x \in (c-h, c)$ and $f'(x) > 0 \; \forall x \in (c, c+h)$



$\frac{dy}{dx} > 0$

$\frac{dy}{dx} < 0$

3) If $f'(x)$ does not changes sign as x increases through c is neither a point of local maxima nor a point of local minima (point of inflection or a stationary point)

Special Case in Previous Page

☆ Not all stationary points are minima or maxima but all minima or maxima are stationary points.

2nd Derivative Test

Let f be a function defined on an open interval I. let $c \in I$ be any point.

1) f has a local maxima at $x=c$ if $f'(c)=0$ and $f''(c)<0$

2) f has a local minima at $x=c$ if $f'(c)=0$ and $f''(c)>0$

3) Test fails if $f'(c)=0$ and $f''(c)=0$

Example 1: $y = 3x^4 + 4x^3 - 12x^2 + 12$

$\frac{dy}{dx} = 12x^3 + 12x^2 - 24x$ $\qquad$ $\frac{d^2y}{dx^2} = 36x^2 + 24x - 24$

$0 = 12x^3 + 12x^2 - 24x$

$x = -2, 0, 1 \longrightarrow$ So three stationary points

1st derivative Test (-2)

$f'(-3) = -288$ and $f'(-1) = 24$ $\qquad - \longrightarrow +$ so minimum

1st derivative test (0)

$f'(-1) = 24$ $\quad$ $f'(0.5) = -7.5$ $\quad + \longrightarrow -$ so maximum

2nd derivative test (0)
$f''(0) = -24$ $\qquad -24 < 0$ so maximum

2nd derivative test (1)

$f''(1) = 36$ $\qquad 36 > 0$ so minimum

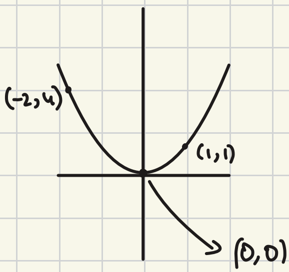
Absolute Maximum and Minimum in $[a, b]$

1) Find the critical points in the interior and find function value at those points. $[a, b]$ and $c \longrightarrow$ Stationary points

2) Find the function value at the end points of the interval.
$f(a)$ and $f(b)$ and $f(c)$

3) Find the maximum and minimum values of the function among the function values obtained in step 1 and step 2.

Example 1: $f(x) = x^2 \quad [-2, 1]$



$f'(x) = 2x$

$= \quad 0 = 2x$

$\underline{\underline{x = 0}} \qquad f(0) = 0$

Absolute maximum $= (-2, 4)$

Absolute minimum $= (0, 0)$

5) Integral Calculus and its Applications

Used to calculate the area bounded by the graph of the function.

indefinite integral

When a derivative of a function is given, integration is used to obtain the original function.

The functions that could possibly have given function as a derivative are called antiderivatives or primitive of the function

Lets say a function is $f(x)$ but it is a derivative of a function to find that function we use integration and that function is called the antiderivative of $f(x)$.

$y = x^n + c$

$f(x) = \frac{dy}{dx} = nx^{n-1} + 0$

So $\int \frac{dy}{dx} \, dx = \frac{x^{n+1}}{n} + c$

can only be found if a point is known

Always add integration constant

$y = \boxed{\sin x}$

$\frac{dy}{dx} = \cos x$

$y = \boxed{\sin x + c}$

$\frac{dy}{dx} = \cos x$

Antiderivative is not unique so always assume to add c and c can end up being 0.

$y = x^n \qquad \frac{dy}{dx} = nx^{n-1} \qquad \int \frac{dy}{dx} \, dx = y + c = x^n + c$

$f(x) \qquad \frac{df}{dx} \qquad \int \frac{df}{dx} \, dx = f(x) + c$

Properties of Integration

1) $$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

2) $$\int kf(x)dx = k\int f(x)dx \qquad k \in R$$

3) $$\int (k_1f_1+k_2f_2+k_nf_n)dx = k_1\int f_1dx + k_2\int f_2dx + kn\int f_ndx$$

Example 1 : $$\int \frac{x^3-1}{x^2}\,dx = \int \frac{x^3}{x^2} - \int \frac{1}{x^2}\,dx$$

$$= \int x\,dx - \int \frac{1}{x^2}\,dx = \frac{x^2}{2} - \int x^{-2}\,dx = \frac{x^2}{2} + x^{-1} + c$$

$$= \frac{x^2}{2} + \frac{1}{x} + c$$

Example 2: $$\int (x^{2/3}+1)\,dx = x^{5/3} \times \frac{3}{5} + x + c$$

$$= \frac{3}{5}x^{5/3} + x + c$$

Integration by Substitution

$$\int f(x)\,dx$$

$$x = g(t)$$

$$\frac{dx}{dt} = g'(t)$$

$$dx = g'(t)\,dt$$

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

Example 1: $\displaystyle\int \frac{\tan^4\sqrt{x}\ \sec^2\sqrt{x}}{\sqrt{x}}\,dx$

$$\tan\sqrt{x} = u$$

$$\tan x^{1/2} = u$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$$

$$du = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

$$2du = \frac{\sec^2\sqrt{x}}{\sqrt{x}}$$

$$\int u^4 \times 2du = 2\int u^4\,du$$

$$= 2 \times \frac{u^5}{5} = \frac{2u}{5} = \frac{2\tan\sqrt{x}}{5} + c$$

$$= \frac{2}{5}\tan\sqrt{x} + c$$

Example 2: $\displaystyle\int \tan x\,dx$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x\,dx$$

$$-du = \sin x\,dx$$

$$= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln|\cos x| + c$$

Example 3: $\int \frac{1}{1+\tan x}\,dx = \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx = \int \frac{1}{\frac{\cos x+\sin x}{\cos x}}\,dx$

$= \int \frac{\cos x}{\cos x+\sin x}\,dx$

$u = \sin x + \cos x$

$\frac{du}{} = \cos x - \sin x \, dx$

use $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$

$2\cos x = \cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x$

$\cos x = \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2}$

So $= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x}$

$= \frac{1}{2}\int \left(\frac{\cos x + \sin x}{\cos x + \sin x}\right) dx + \frac{1}{2}\int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) dx$

$= \frac{1}{2}\left(\int 1\,dx + \int \frac{1}{u}\,du\right) = \frac{1}{2}\left(x + \ln|\cos x + \sin x| + c\right.$

$= \frac{1}{2}\left(x + \ln|\cos x + \sin x| + c\right)$

Partial Fractions

Consider $\frac{P(x)}{Q(x)}$ $P(x)$ and $Q(x)$ are polynomials in x, $Q(x) \neq 0$

Degree of $P(x)$ is less than degree of $Q(x)$ if it isn't use polynomial division to make it a mixed fraction.

1) $\frac{px+q}{(x-a)(x-b)}$ $a \neq b \rightarrow px+q = \frac{A}{(x-a)} + \frac{B}{(x-b)} \rightarrow px+q = A(x-b) + B(x-a)$

2) $\frac{px+q}{(x-a)^2} \rightarrow px+q = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} \rightarrow px+q = A(x-a)^2 + B(x-a)$

3) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} \rightarrow px^2+qx+r = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} \rightarrow$

$$px^2+qx+r = A(x-c)(x-b) + B(x-c)(x-a) + C(x-a)(x-b)$$

4) $\frac{px^2+qx+r}{(x-b)(x-a)^2} \rightarrow px^2+qx+r = \frac{A}{(x-b)} + \frac{B}{(x-a)} + \frac{C}{(x-a)^2} \rightarrow$

$$px^2+qx+r = A(x-a)^2 + B(x-a)(x-b) + C(x-b)$$

5) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} \rightarrow px^2+qx+r = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)} \rightarrow$

$$px^2+qx+r = A(x^2+bx+c) + (Bx+C)(x-a)$$

Example 1:

For the repeating Factor you don't need to multiply the numerator with the single linear factor of the repeating one

$$\frac{2x+1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$2x+1 = A(x+2)^2 + B(x+2)(x+1) + C(x+1)$$

Example 2:

$$\frac{2x+1}{(x+1)(x^3)} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

$$2x+1 = A(x^3) + B(x^2)(x+1) + C(x)(x+1) + D(x+1)$$

Proof:

$$\frac{Ax + B(x+1)}{(x+1)x} + \frac{C}{x^2} = \frac{Ax^3 + B(x+1)(x^2) + C(x+1)x}{(x+1)(x^3)}$$

$$= \frac{Ax^3 + B(x+1)(x^2) + C(x+1)x}{(x+1)(x^3)} + \frac{D}{x^3}$$

$$= \frac{Ax^3 + B(x+1)(x^5) + C(x+1)x^4 + D(x+1)(x^3)}{x^6}$$

$$= \frac{\cancel{x^3}(A + B(x+1)x^2 + C(x+1)x + D(x+1))}{(x+1)(\cancel{x^6})x^3}$$

Example 3 :

$$\frac{x^2 + 4x - 3}{(x-1)(x^2+2x+1)} = \frac{A}{(x-1)} + \frac{Bx+c}{x^2+2x+1}$$

$$x^2 + 4x - 3 = A(x^2+2x+1) + (Bx+c)(x-1)$$

☆ So you can use partial fractions to express equations and then apply integration.

<u>Integration By Parts</u>

$$\int uv'dx = uv - \int u'vdx \quad \text{or} \quad \int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx$$

$$\int f(x)g(x)dx = f(x) \times \int g(x)dx - \int f'(x) \times [\int g(x)dx]dx$$

Example 1 :

$u = x \qquad v' = \cos x$

$u' = 1 \qquad v = \sin x$

$$\int \overset{u}{x}\,\overset{v'}{\cos x}\,dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

Example 2 :

$$\int \operatorname{Log} x\,dx = \int \overset{v'}{1} \times \overset{u}{\operatorname{Log} x}\,dx$$

$u = \operatorname{Log} x \qquad v' = 1$

$u' = \frac{1}{x} \qquad v = x$

$x/x = 1$

$$= x\operatorname{Log} x - \int \frac{x}{x}dx$$

$$= x\operatorname{Log} x - x + c$$

Example 3:

$$\int \overset{u}{e^x} \overset{v'}{\sin x}\, dx = \qquad u = e^x \quad u' = e^x$$
$$v' = \sin x \quad v = -\cos x$$

$$= -e^x \cos x + \int \overset{u}{e^x} \overset{v'}{\cos x}\, dx \qquad u = e^x \quad v' = \cos x$$
$$u' = e^x \quad v = \sin x$$

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x$$

$$= \frac{1}{2} e^x (\sin x - \cos x) + c$$

Example 4: (Used substitution)

$$\int x^2 e^{x^3}\, dx = \qquad u = x^3$$
$$\frac{du}{dx} = 3x^2$$

$$= \frac{1}{3} \int e^u\, du \qquad du = 3x^2\, dx$$

$$= \frac{1}{3} e^u + c \qquad \frac{du}{3} = x^2 dx$$

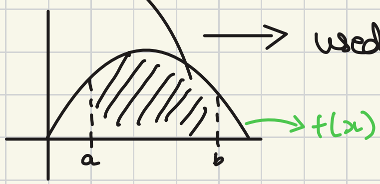
$$= \frac{1}{3} e^{x^3} + c$$

Definite Integrals

$$\int_a^b f(x)\,dx$$

$a =$ lower limit

$b =$ upper limit



$\longrightarrow$ used to find area under a function

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of $f(x)$ then

$$\int_a^b f(x) = F(b) - F(a)$$

Example 1:

$$\int_2^3 x^2\,dx = \frac{x^3}{3}$$

$$\frac{3^{\cancel{3}}}{\cancel{3}} - \frac{8}{3} = 9 - \frac{8}{3} = \frac{19}{3}$$

Example 2:

$$\int_0^{\pi/4} \sin^3 2t \cos 2t\,dt$$

$\sin 2t = 2\sin t \cos t$

$\sin 4t = 2\sin 2t \cos 2t$

$\sin 2t = u$

$\frac{du}{dt} = 2\cos 2t$

$\frac{1}{2} du = \cos 2t\,dt$

$$\frac{1}{2} \times \int u^3\,du = \frac{u^4}{4}$$

$\sin 2t = u$

$u_1 = \sin 0$ $\quad$ $u_2 = \sin(\pi/4 \times 2)$

$u_1 = 0$ $\quad$ $u_2 = 1$

$$= \left(\frac{1^4}{4} - 0\right) \times \frac{1}{2}$$

$$= \boxed{\frac{1}{8}}$$

Properties

1) $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

2) $$\int_a^{-b} f(x)\,dx = -\int_b^a f(x)\,dx$$

3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4) $$\int_a^a f(x)\,dx = 0$$

5) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

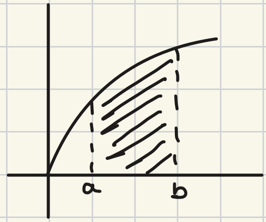
6) $$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } \boxed{f(-x) = f(x)}$$ even function

$$\int_{-a}^a f(x)\,dx = 0 \quad \text{if } \boxed{f(-x) = -f(x)}$$ odd function

Area under the Curve

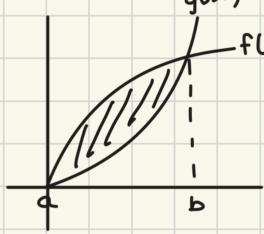
$y = f(x)$

$$\int_a^b y \, dx = \text{Area} = \int_a^b f(x) \, dx$$



Area under Two curves

$$\text{Shaded Area} = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$



Example 1: Find the area bounded by $|x| + |y| = 1$

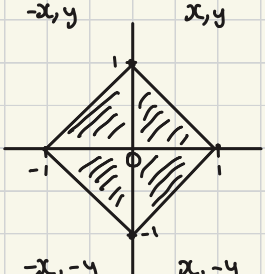
$-x, y$ $\quad$ x, y

$-x + y = 1$

$y = x + 1$

$x + y = 1$

$y = -x + 1$



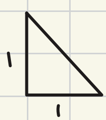
$-x - y = 1$

$y = -x - 1$

$x - y = 1$

$y = x - 1$

$-x, -y$ $\quad$ $x, -y$



$$y = -x + 1$$

$$\int_0^1 y \, dx = \int_0^1 \frac{-x^2}{2} + x \, dx$$

$$= \frac{-1}{2} + 1 = 1/2 \times 4$$

$$= \boxed{2}$$

6) Ordinary Differential Functions

Notation : $\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$, $\frac{d^3y}{dx^3} = y'''$ . . .

derivative term

$$x\frac{dy}{dx} + y = 0$$

Independent   dependant on x

☆ Differential Equation is an equation involving the derivatives of the dependent variable wrt to independent variable(s)

Types of Differential Equations

1) Ordinary differential Equation : Equation involving only derivatives of the dependent variable w.r.t only one independent variable.

Example 1: $\frac{dy}{dx} + p(x)y = Q(x)$ Example 2: $\frac{d^2y}{dx^2} + y = f(x)$

2) Partial differential Equation : Equation involving derivatives w.r.t more than one independent variables.

Independent variables

Example 1 : $\boxed{x, y}$ $u = u(x,y)$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \ , \ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Order

Order of the highest order derivative of the dependent variable w.r.t to the independent variable involved in given differential equation.

$$\frac{dy}{dx} = e^{x} \qquad \text{1st order}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad \text{2nd order}$$

$$\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = \qquad \text{3rd order}$$

Degree

* The differential equation must be a polynomial equation in derivatives y', y'', y''' etc.

Degree : The highest power of the highest order derivative involved in given differential equation.

$$\left(\boxed{\frac{d^3y}{dx^3}}\right)^1 + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \qquad \text{so degree} = 1$$

Highest order

not a polynomial with equation in derivatives

$$\frac{dy}{dx} + \sin\boxed{\left(\frac{dy}{dx}\right)} = 0$$

So no degree

Solution of Differential Equation

Consider $\frac{dy}{dx} = x$

Look for $\boxed{y = \varphi(x)}$ such that LHS = RHS

Form of Solution

Example 1: $\frac{dy}{dx} = x$

$= dy = x \, dx$

$= \int dy = \int x \, dx$

$= \int 1 \, dy = \int x \, dx$

$= y = \frac{x^2}{2} + C$ Can be found out by having a point in the function

Example 2: Verify that $y = e^{-x}$ is a solution of $\frac{d^2y}{dx^2} - y = 0$

$y' = \frac{dy}{dx} = -e^{-x}$ and $y'' = \frac{d^2y}{dx^2} = --e^{-x} = +e^{-x}$

$e^{-x} - y = 0$

$y = e^{-x}$ so yes it is a solution

Key Points:

1) Existence and Uniqueness of solution of DE is important
2) We may not find closed form solution $y = \varphi(x)$ for a given DE
3) We will see solution techniques for some ODE.

Variable Seperable Method

General form of first degree ODE is $\frac{dy}{dx} = F(x,y)$

If $F(x,y) = g(x)\, h(y)$

$= \frac{dy}{dx} = g(x)\, h(y)$

$= \frac{1}{h(y)} dy = g(x)\, dx$

$= \int \frac{1}{h(y)} dy = \int g(x)\, dx$

$= H(y) = G(x) + c$

Example 1: $\frac{dy}{dx} = \frac{x+1}{2-y}$, $y \neq 2$

$= (2-y)\, dy = (x+1)\, dx$

$= \int 2-y\, dy = \int x+1\, dx$

$= 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C$

Homogenous Function

Consider $F_1(x,y) = y^2 + 2xy$, $F_2(x,y) = \cos\left(\frac{y}{x}\right)$, $F_3(x,y) = \sin x + \cos y$

Homogenous degree 2

$F_1(\lambda x, \lambda y) = \lambda^2 y^2 + \lambda^2 2xy = \lambda^2(y^2 + 2xy) = \boxed{\lambda^2 \times F_1(x,y)}$

$F_2(\lambda x, \lambda y) = \cos\left(\frac{\lambda x}{\lambda y}\right) = \cos \frac{y}{x} = \boxed{\lambda^0 F_2(x,y)}$ → Homogenous degree 0

$F_3(\lambda x, \lambda y) = \boxed{\sin \lambda x + \cos \lambda y}$ $\quad F_3(\lambda x, \lambda y) \neq \lambda^n F_3(x,y)$

Non homogenous

Homogenous Function : $F(x,y)$ is said to be homogenous function of degree n if $F(\mu x, \mu y) = \mu^n F(x,y)$ for any non-zero constant μ.

$$F_1(x,y) = y^2 + 2xy$$
$$= x^2\left[\frac{y^2}{x^2} + \frac{2y}{x}\right]$$
$$= x^2 G\left(\frac{y}{x}\right)$$

$$F_1(x,y) = y^2 + 2xy$$
$$= y^2\left(1 + \frac{2x}{y}\right)$$
$$= y^2 H\left(\frac{x}{y}\right)$$

If $F(x,y)$ is a homogenous function of degree n then

$$F(x,y) = x^n g(y/x) \quad \text{or} \quad F(x,y) = y^n\left(\frac{x}{y}\right)$$

Homogenous Equation

A DE of the form $\frac{dy}{dx} = F(x,y)$ is said to be homogenous if $F(x,y)$ is a homogenous function of degree 0. $n=0$

To solve $\frac{dy}{dx} = F(x,y) = x^0 g\left(\frac{y}{x}\right)$ Let $v = \frac{y}{x}$

$y = \overset{u}{v}\overset{v}{x}$

$u = v \qquad v = x$

$u' = \frac{dv}{dx} \qquad v' = 1$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$v + x\frac{dv}{dx} = g(v)$$

$$= x\frac{dv}{dx} = g(v) - v$$

$$= \boxed{\frac{1}{g(v)-v}\,dv = \frac{1}{x}\,dx}$$

$$= \int \frac{1}{g(v)-v}\,dv = \int \frac{1}{x}\,dx$$

Example 1: Solve $(x^2-y^2)\,dx + xy\,dy = 0$

$$xy\,dy = (y^2-x^2)\,dx$$

$$= \frac{dy}{dx} = \frac{y^2-x^2}{xy} = F(x,y)$$

Verify whether homogeneous equation or not

$$F(\mu x, \mu y) = \frac{\mu^2 y^2 - \mu^2 x^2}{\mu^2 xy} = \frac{\mu^2}{\mu^2}\left(\frac{y^2-x^2}{xy}\right)$$

☆ Yes it is homogenous $= \mu^0\left(\frac{y^2-x^2}{xy}\right)$

$V = \frac{y}{x}$ $\quad F(\mu x, \mu y) = g(y/x)$ or $h(x/y)$

$$\frac{dy}{dx} = g(v) = \frac{y^2-x^2}{xy}$$

$y = vx$

$v = \frac{y}{x}$

$v^2 = \frac{y^2}{x^2}$

$$\frac{dy}{dx} = \overset{u}{v}\overset{v}{x}$$

$u = v \qquad v = x$

$u' = \frac{dv}{dx} \qquad v' = 1$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \qquad \frac{dy}{dx} = g(v)$$

$$g(v) = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\int \frac{1}{g(v)-v}\,dv = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{v^2\cancel{x^2}-x^2}{v\cancel{x^2}} - v}\,dv = \frac{1}{x}\,dx$$

$$= \int \frac{1}{\frac{\cancel{v^2}-1}{v} \cancel{-v^2}}\,dv = \ln x + c$$

$$= \int -v\,dv = \ln x + c$$

$$= -\frac{v^2}{2} = \ln x + c$$

$$= \boxed{\frac{-y^2}{2x^2} = \ln x + c}$$

Non Homogeneous Equations

Can be converted into variable seperable or homogeneous form to be solved.

$$\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f}$$

Case 1 : If $ae \neq bd$, then constants h, k can be chosen such a way that substitution $x = z - h$, $y = w - k$ reduces the given equation to homogeneous form.

$$x = z - h \quad \text{and} \quad y = w - k$$

$$\frac{ax + by + c}{dx + ey + f} = \frac{a(z-h) + b(w-k) + c}{d(z-h) + e(w-k) + f}$$

$$= \frac{az - ah + bw - bk + c}{dz - dh + ew - ek + f}$$

$$= \frac{az + bw + \boxed{c - ah - bk}}{dz + ew + \boxed{f - dh - ek}}$$

When these two are equal to 0 then homogenous form is achieved.

$$c - ah - bk = 0 \quad \text{and} \quad f - dh - ek = 0$$

$$c = ah + bk \qquad f = dh + ek$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

$$\frac{dz}{dw} = \frac{az + bw}{dz + ew}$$

This will become homogeneous and can be solved that way.

Case 2 : If $ae = bd$

$$\frac{a}{d} = \frac{b}{e} = t \qquad a = td \,, \quad b = et$$

$$\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f} = \frac{tdx + ety + c}{dx + ey + f}$$

$$= \frac{t\left[dx + ey\right] + c}{dx + ey + f}$$

So $\underbrace{dx + ey}_{\text{constant}} = v$

$$\frac{dv}{dx} = d \times 1 + e\frac{dy}{dx}$$

$$= \frac{dv}{dx} = d + e\frac{dy}{dx}$$

$$= \frac{dy}{dx} = \left(\frac{dv}{dx} - d\right)\frac{1}{e}$$

So $\frac{1}{e}\left(\frac{dv}{dx} - d\right) = \frac{tv + c}{v + f}$

$$\frac{dv}{dx} - d = \frac{etv + ec}{v + f}$$

$$\boxed{\frac{dv}{dx} = \frac{bv + ec}{v + f} + d}$$

Exact and Non-exact differential Equation

Total differential of a function $f(x,y)$ is defined by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \forall (x,y) \in D$$

$M(x,y)\,dx + N(x,y)\,dy$ is called an exact differential in a domain D if there exists a function $f(x,y)$ such that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{y\,\partial} dy = M\,dx + N\,dy$$

That is, there exists a function $f(x,y)$ such that $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$

Then, $M(x,y)\,dx + N(x,y)\,dy = 0$ is called exact differential equations.

Therefore there exists a function $f(x,y)$ such that $df = 0$

$\rightarrow f(x,y) = C$ is solution

Necessary and sufficient condition for exact equation

$M(x,y)\,dx + N(x,y)\,dx = 0$ is exact only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \forall (x,y) \in D$$

Example 1: $y^2 dx + 2xy\,dy = 0$

$M = y^2 \qquad N = 2xy$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ so it is an exact equation

Example 2: $y\,dx + 2x\,dy = 0$

$M = y \qquad N = 2x$

$\dfrac{\partial M}{\partial y} = 1 \qquad \dfrac{\partial N}{\partial x} = 2 \qquad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ so not an exact equation

Solving exact Equations

Example 1 : $e^y\,dx + (xe^y + 2y)\,dy = 0$

$M = e^y$ $\qquad$ $N = xe^y + 2y \longrightarrow$ considered constant

$\dfrac{\partial M}{\partial y} = e^y$ $\qquad$ $\dfrac{\partial N}{\partial x} = e^y + 0$

If exact then there exists a function $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = Mdx + Ndy$

$M = \dfrac{\partial f}{\partial x}$ and $N = \dfrac{\partial f}{\partial y}$

$\dfrac{\partial f}{\partial x} = e^y$ and $\dfrac{\partial f}{\partial y} = xe^y + 2y$

$\displaystyle\int \frac{\partial f}{\partial x}\,dx = \int e^y \times 1\,dx \longrightarrow$ when integrating in terms of x y variable is constant so the integration constant is a function in terms of y.

$= f(x,y) = xe^y + h(y)$

Kept as constant as now in terms of y

$\dfrac{\partial f}{\partial y} = xe^y + \dfrac{dh}{dy} = N$

$\dfrac{dh}{dy} = 2y$

$xe^y + \dfrac{dh}{dy} = xe^y + 2y$ $\qquad$ $h(y) = y^2 + c$

$f(x,y) = xe^y + y^2$

Solution = $\boxed{xe^y + y^2 = c_1}$

Non exact equation - Integrating Factor

Example 1: $y\,dx + (x^2y - x)\,dx = 0$

$M = y$ $\quad N = x^2y - x$ $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so not exact

$\frac{\partial M}{\partial y} = 1$ $\quad \frac{\partial N}{x} = 2xy - 1$

Integrating Factor $= \frac{1}{x^2}$ $\quad$ so, $\frac{y}{x^2}\,dx + y - \frac{1}{x}\,dy = 0$

$M = \frac{y}{x^2}$ $\quad \frac{\partial M}{\partial y} = \frac{1}{x^2}$ $\quad N = y - \frac{1}{x}$ $\quad \frac{\partial N}{\partial x} = 0 - \left(\frac{-1}{x^2}\right) = \frac{1}{x^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then solve as an exact equation

Then $\quad df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M\,dx + N\,dy$

$M = \frac{\partial f}{\partial x}$ $\quad \frac{\partial f}{\partial x} = \frac{y}{x^2}$ $\quad \int \frac{\partial f}{\partial x}\,dx = \int \frac{y}{x^2}$

$N = \frac{\partial f}{\partial y}$ $\quad \frac{\partial f}{\partial y} = y - \frac{1}{x}$

$= f(x,y) = -\frac{y}{x} + h(y)$

$\frac{\partial f}{\partial y} = 0 + \frac{dh}{dy}$

$\frac{dh}{dy} = y - \frac{1}{x}$

$h(y) = \frac{y^2}{2} + 0$

$$\boxed{\frac{-y}{x} + \frac{y^2}{2} = C}$$

First Order Linear Differential Equations

$$\frac{dy}{dx} + p(x)y = q(x)$$

If $q(x) = 0$

$$\frac{dy}{dx} = -p(x)y$$ Can be solved using variable seperable method

If $q(x) \neq 0$

$$\frac{dy}{dx} = q(x) - p(x)y$$ Non homogeneous

Solving

Multiply with integrating factor $\mu(x)$ on both sides

$$\mu \frac{dy}{dx} + \mu\, p(x)y = \mu\, q(x)$$

$\mu(x)$ is such that $\mu \frac{dy}{dx} + \mu p(x) y = \frac{d}{dx}(\mu y)$

$\overset{u}{\mu}\overset{v}{y}$ $\quad u = \mu \quad v = y$

$u' = \frac{d\mu}{dx} \quad v' = \frac{dy}{dx}$

$$\cancel{\mu \frac{dy}{dx}} + \mu p(x)\cancel{y} = \frac{d\mu}{dx}\cancel{y} + \cancel{\mu \frac{dy}{dx}}$$

$$\frac{d\mu}{dx} = \mu\, p(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int p(x)\, dx$$

$$= \ln \mu = \int p(x)\, dx + C$$ not required $C = 0$

$$\boxed{\mu = e^{\int p(x)\, dx}}$$

$$\mu = e^{\int p(x)\,dx}$$

$$\mu \frac{dy}{dx} + \mu p(x) y = \mu q(x)$$

$$\boxed{\frac{d}{dx}\left[\mu y\right]} = \mu\, q(x)$$

This basically conveys that that term is the derivative of μy so when you $\int \frac{d}{dx} \mu y \, dx = \underline{\underline{\mu y}}$

$$\mu y = \int \mu\, q(x) + c$$

$$\boxed{\begin{gathered} y = \frac{1}{\mu}\left(\int \mu\, q(x) + c\right) \\ y = e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx} \times q(x) + c\right) \end{gathered}}$$

Example 1: $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$

$$p(x) = \frac{2x+1}{x}$$

$$\int p(x)\,dx = \int 2 + \frac{1}{x}\,dx =$$

$$= 2x + \ln x$$

$$\mu = e^{2x+\ln x}$$

$$\boxed{\mu = x e^{2x}}$$

$$\frac{d}{dx}\left[x e^{2x} y\right] = x \cancel{e^{2x}} \times \cancel{e^{-2x}}$$

$$\int \frac{d}{dx}\left[x e^{2x} y\right] = \int x\,dx$$

$$x e^{2x} y = \frac{x^2}{2} + c$$

$$y = \frac{x^2}{2x e^{2x}} + \frac{c}{x e^{2x}}$$

$$\boxed{y = \frac{x}{2e^{2x}} + \frac{c}{x e^{2x}}}$$