BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

First Semester, 2020-21

MATH F111 : Mathematics-I

Mid Semester Test

Date : 07/01/2021 **Max. Time : 90 Minutes** **Max. Marks: 90**

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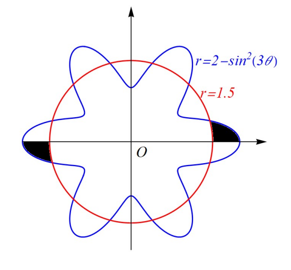
1. (a) Calculate the length of the curve $r = e^{0.3\theta}$, $-\frac{\pi}{3} \leq \theta \leq \frac{10\pi}{3}$ and slope of the tangent at $\theta = \pi$. **[8]**

   (b) Check the symmetry conditions about $x-$axis, $y-$axis, and origin of the curve $r = 2 - \sin^2(3\theta)$. Justify your answer. **[6]**

   (c) Find the area of the shaded region in the following figure: **[8]**



   (d) Convert the following polar equation into Cartesian equation:

$$r^2 \sin 2\theta = 2 \ln r + \ln \cos 2\theta.$$ **[4]**

   (e) Convert the following Cartesian equation into polar equation of the form $r = f(\theta)$ such that $r \geq 0$:

$$x^4 + y^4 + 4y^3 + 4x^2y + 2x^2y^2 - 4x^2 = 0.$$ **[4]**

2. (a) Find $\mathbf{T}$, κ and τ for the space curve $\mathbf{r}(t) = a(1 + \cos(t))\mathbf{i} + a\sin(t)\mathbf{j} + 2a\sin(t/2)\mathbf{k}$, for $0 \leq t \leq \pi$, where $a > 0$ is a constant. **[10]**

   (b) Consider a plane curve given by the polar equation $r = a\sec^3\left(\frac{\theta}{3}\right)$. Let θ varies with time $t \in [0, \infty)$ as $\theta = 3\tan^{-1} t$. Find the arc length $s(t)$ and the curvature $\kappa(t)$. **[8+6]**

   (c) Let

$$f(x,y) = \begin{cases} \cos\left(\dfrac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0), \\ 1, & \text{if } (x,y) = (0,0). \end{cases}$$

   Discuss the continuity of $f(x,y)$ at $(0,0)$. **[6]**

3. (a) The temperature in a region in space is given by $T(x,y,z) = y^2 - 3xyz$. A particle is moving in this region and its position at time t is given by $x = 2e^t$, $y = 3\cos(t)$, $z = -\sin(t)$. Here x, y, z are positions and t denotes time in dimensionless form.

      (i) How fast is the temperature experienced by the particle changing per unit distance when the particle is at point $P(2,3,0)$? **[9]**

      (ii) Use part (i) to evaluate how fast is the temperature experienced by the particle changing per unit time at P? **[3]**

   (b) For the function

$$f(x,y) = \begin{cases} \frac{xy^3}{x+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Check whether $f_{xy}(0,0) = f_{yx}(0,0)$ or not? **[12]**

   (c) Find all the critical points of the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$ **[6]**

*****END*****

Q.1(a) $-\frac{\pi}{3} \leq \theta \leq \frac{10\pi}{3}$

$r = e^{0.3\theta}$

formula of ~~arc~~ length :-

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$\alpha \leq \theta \leq \beta$

$r = e^{0.3\theta}$, $\frac{dr}{d\theta} = 0.3\, e^{0.3\theta}$

$$L = \int_{-\pi/3}^{\frac{10\pi}{3}} \sqrt{e^{0.6\theta} + 0.09\, e^{0.6\theta}}\, d\theta \quad \text{— (2)}$$

$$= \sqrt{1.09} \int_{-\pi/3}^{\frac{10\pi}{3}} e^{0.3\theta}\, d\theta$$

$$= \sqrt{1.09} \left[\frac{e^{0.3\theta}}{0.3}\right]_{-\pi/3}^{10\pi/3}$$

$$\boxed{L = \frac{\sqrt{1.09}}{0.3}\left[e^{0.3\left(\frac{10\pi}{3}\right)} - e^{0.3\left(\frac{-\pi}{3}\right)}\right]} \quad \text{—(2)}$$

$$= \frac{\sqrt{1.09}}{0.3}\left[23.14 - 0.7304\right]$$

$$= \frac{\sqrt{1.09}}{0.3} \times (22.40) = 77.98$$

$$\boxed{L = 77.98}$$

slope of tangent at $\theta = \pi$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

we have, ~~$x = r \sin$~~

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and $r = e^{0.3\theta}$

so,

$$x = e^{0.3\theta}\cos\theta \quad , \quad y = e^{0.3\theta}\sin\theta$$

$$\frac{dx}{d\theta} = 0.3\left(e^{0.3\theta}\right)\cos\theta + (-\sin\theta)\, e^{0.3\theta} \quad \text{— (1)}$$

$$\frac{dy}{d\theta} = 0.3\left(e^{0.3\theta}\right)\sin\theta + \cos\theta\, e^{0.3\theta} \quad \text{— (1)}$$

$$\frac{dy}{dx} = \frac{(0.3\sin\theta + \cos\theta)\,\cancel{e^{0.3\theta}}}{(0.3\cos\theta - \sin\theta)\,\cancel{e^{0.3\theta}}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{(0.3\sin(\pi) + \cos(\pi))}{(0.3\cos(\pi) - \sin(\pi))}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-1}{-0.3} = \frac{1}{0.3} = \frac{10}{3} = 3.33 \quad \text{— (2)}$$

~~$\boxed{\left.\frac{dy}{dx}\right|_{\theta=\pi} = }$~~

Q.1) (b)

~~Sol~~ (a)

Q.1) (b) $r = 2 - \sin^2 3\theta$

About x-axis :-

$r = 2 - \sin^2 3\theta = 2 - \sin^2(3(-\theta))$

Hence, if (r, θ) lies on the graph then $(r, -\theta)$ also lies on the graph.

$\Rightarrow$ symmetric about x-axis. — (2)

About y-axis :-

consider

$2 - \sin^2(3(\pi - \theta)) = 2 - (\sin(3\pi - 3\theta))^2$

$= 2 - \sin^2 3\theta = r$

Hence, if (r, θ) lies on the graph then $(r, \pi - \theta)$ also lies on the graph.

$\Rightarrow$ symmetric about y-axis. — (2)

about origin :-

Consider,

$2 - \sin^2(3(\pi + \theta)) = 2 - (\sin(3\pi + 3\theta))^2$

$= 2 - (\sin(-3\theta))^2 = 2 - \sin^2 3\theta = r$

Hence, if (r, θ) lies then $(r, \pi + \theta)$ also lies on the curve

$\Rightarrow$ symmetric about origin. — (2)

c) Find the area of the region lies outside the circle $r = 1.5$ and inside the curve $r = 2 - \sin^2(3\theta)$.

Point of intersection :-

$1.5 = 2 - \sin^2(3\theta)$

$\sin^2(3\theta) = \frac{1}{2}$; $\sin(3\theta) = \frac{1}{\sqrt{2}}$

$\sin(3\theta) = \sin\left(\frac{\pi}{4}\right)$

$\theta = \frac{\pi}{12}$, ——— (2)

Area of the region lies outside the circle $r = 1.5$ and inside the curve $r = 2 - \sin^2(3\theta)$ is —

$$A = 2 \cdot \left[\int_0^{\pi/12} \frac{1}{2} (2 - \sin^2 3\theta)^2 \, d\theta - \int_0^{\pi/12} \frac{1}{2} (1.5)^2 \, d\theta \right]$$

(2)

$$A = \int_0^{\pi/12} \left[(2 - \sin^2 3\theta)^2 - (1.5)^2 \right] d\theta$$

$$= \int_0^{\pi/12} \left(2 - \left(\frac{1-\cos 6\theta}{2}\right)\right)^2 - \frac{9}{4}\, d\theta$$

$$= \int_0^{\pi/12} \left[\left(\frac{3 + \cos 6\theta}{2} \right)^2 - \frac{9}{4} \right] d\theta$$

$$= \frac{1}{4} \int_0^{\pi/12} (\cos^2 6\theta + 6\cos 6\theta)\, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/12} \left(\frac{1 + \cos 12\theta}{2} + 6\cos 6\theta \right) d\theta$$

$$= \frac{1}{4} \int_0^{\pi/12} \frac{1}{2}\, d\theta + \left(\frac{\sin 12\theta}{24} \right)_0^{\pi/2} + \left(\frac{6\sin 6\theta}{6} \right)_0^{\pi/2}$$

$$= \frac{\pi}{96} + \frac{1}{4} \sin\left(6 \times \frac{\pi}{12}\right)$$

$$\boxed{A = \frac{\pi}{96} + \frac{1}{4}}$$

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Alternative sol^n:-

$$A = \int_0^{\pi/2} \left[(2 - \sin^2 3\theta)^2\, d\theta - (1.5)^2 \right] d\theta$$

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$$A = \int_0^{\pi/12} \left[\left(4 + \sin^4 3\theta - 4 \sin^2 3\theta \right) d\theta - \frac{9}{4} \right] d\theta$$

$$\int_0^{\pi/12} \sin^2 3\theta = \frac{\pi}{24} - \frac{1}{12}$$

$$\int_0^{\pi/12} \sin^4 3\theta = \frac{3\pi - 8}{96}$$

$$A = 4 \times \frac{\pi}{12} + \left(\frac{3\pi - 8}{96} \right) - 4 \left(\frac{\pi - 2}{24} \right) - \frac{\cancel{9}^{3}}{4} \times \frac{\pi}{\cancel{12}_{4}}$$

$$= \left(\frac{\pi}{3} + \frac{\pi}{32} - \frac{\pi}{6} - \frac{3\pi}{16} \right) - \frac{8}{96} + \frac{8}{24}$$

$$\boxed{A = \frac{\pi}{96} + \frac{1}{4}} \text{ ——— } (4)$$

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1) Sol^n :-
(d)
Given :- $r^2 \sin 2\theta = 2\ln(r) + \ln(\cos 2\theta)$

we have :- $r\cos\theta = x$, $r\sin\theta = y$

$$r^2 (2\cos\theta \sin\theta) = \ln r^2 + \ln \cos 2\theta \quad \text{—} \quad (1)$$

[$\sin 2\theta = 2\cos\theta\sin\theta$, $a\ln b = \ln b^a$]

$$\Rightarrow \quad 2(r\cos\theta)(r\sin\theta) = \ln(r^2(\cos^2\theta - \sin^2\theta)) \quad \text{—} (1)$$

[$\ln a + \ln b = \ln(ab)$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$]

$$\Rightarrow \quad 2xy = \ln(x^2 - y^2)$$

— (2)

$$\Rightarrow \quad \boxed{e^{2xy} = x^2 - y^2}$$

→ This is the required equation.

(b) $x^4+y^4+2x^2y^2+4y^3+4x^2y-4x^2=0$

we have,

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

$x^4+y^4+2x^2y^2+4y^3+4x^2y-4x^2=$

$(x^2+y^2)^2+4y(x^2+y^2)-4x^2=0$

$\Rightarrow (r^2)^2+4(r\sin\theta)(r^2)-4r^2\cos^2\theta=0$ — (2)

$\Rightarrow r^2+4r\sin\theta-4(1-\sin^2\theta)=0$

$\Rightarrow r^2+4r\sin\theta+4\sin^2\theta=4$

$\Rightarrow (r+2\sin\theta)^2=4$ — (1)

$[\because r \geq 0]$

$\Rightarrow r+2\sin\theta=2$

$\Rightarrow \boxed{r = 2-2\sin\theta}$ — (1)

2(a) Given eq^n for the space curve is

$$\vec{r}(t) = \langle a(1+\cos t),\ a\sin t,\ 2a\sin t/2 \rangle$$

$$\Rightarrow \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \langle -a\sin t,\ a\cos t,\ a\cos t/2 \rangle \quad [1]$$

and $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$

$$= \langle -a\cos t,\ -a\sin t,\ -\tfrac{1}{2}a\sin t/2 \rangle \quad [1]$$

and $\dddot{r}(t) = \langle a\sin t,\ -a\cos t,\ -\tfrac{1}{4}a\cos t/2 \rangle$ [1]

We have

$$v = |\vec{v}| = a\sqrt{1+\cos^2 t/2}$$

OR

$$= a\sqrt{\frac{3+\cos t}{2}}$$

$$\Rightarrow \hat{T} = \frac{\vec{v}}{|\vec{v}|} =$$

$$\hat{T} = \left\langle \frac{-\sqrt{2}\sin t}{\sqrt{3+\cos t}},\ \frac{\sqrt{2}\cos t}{\sqrt{3+\cos t}},\ \frac{\sqrt{2}\cos t/2}{\sqrt{3+\cos t}} \right\rangle \quad [2]$$

OR

$$\hat{T} = \left\langle \frac{-\sin t}{\sqrt{1+\cos^2 t/2}},\ \frac{\cos t}{\sqrt{1+\cos^2 t/2}},\ \frac{\cos t/2}{\sqrt{1+\cos^2 t/2}} \right\rangle$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\vec{v} \times \vec{a} = \frac{a^2}{4} \left\langle 3\sin t/_2 + \sin^3 t/_2, -\cos^3 t/_2, 1 \right\rangle$$

$$\Rightarrow |\vec{v} \times \vec{a}| = \frac{a^2}{4}\sqrt{26 + 6\cos t} \quad [1]$$

$$, \quad \kappa = \frac{|\vec{v} \times \vec{a}|}{|v|^3} = \frac{1}{a} \frac{\sqrt{13 + 3\cos t}}{(3 + \cos t)^{3/2}}. \quad [2]$$

and finally

$$\tau = \frac{|\vec{v}\,|\vec{a}\,|\,\dddot{r}\,|}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{\frac{3}{4}a^3\cos(t/_2)}{\left(\frac{a^2}{4}\sqrt{26 + 6\cos t}\right)^2} = \frac{6\cos(t/_2)}{a(13 + 3\cos t)} \quad [2]$$

(b) We have the polar eq^n as

$r = a\sec^3(\theta/2)$ where

$\theta = 3\tan^{-1}t \Rightarrow r = a\sqrt{1+t^2}^{\,3}$ [2]

Converting to Cartesian Coordinate the eq^n for the Space Curve is

$\vec{r}(t) = \langle r\cos\theta, r\sin\theta, 0\rangle$

$= a\langle 1-3t^2, t(3-t^2), 0\rangle$ [2]

$\Rightarrow \vec{v} = \dfrac{d\vec{r}}{dt} = a\langle -6t, 3(1-t^2), 0\rangle$

$\Rightarrow |\vec{v}| = 3a(1+t^2)$

$\vec{a} = a\langle -6, -6t, 0\rangle$ [1]

Hence Arc length

$$s = \int_0^t |\vec{v}(t)|\, d\tau = at(3+t^2) \quad [3]$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|v|^3} = \frac{|\langle 0, 0, 18a^2(1+t^2)\rangle|}{\left(3a(1+t^2)\right)^2} \quad [4]$$

$$= \frac{2}{3a(1+t^2)^2} \quad [2]$$

(c) We have

$$f(x,y) = \begin{cases} \cos\left(\dfrac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}\right); & (x,y) \neq (0,0) \\ 1 & ; (x,y) = (0,0) \end{cases}$$

first we show that

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}\right)$$

exist and Equal to 1.

let $\quad g(x,y) = \dfrac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}$

we get

$$|g(x,y)| = \left|\frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}\right|$$

$$= \frac{|x|\sqrt{|y|}}{\sqrt{x^2+y^2}}$$ [1]

$$\leq \frac{\sqrt{x^2+y^2}\,\sqrt{|y|}}{\sqrt{x^2+y^2}}$$ [Since $|x| \leq \sqrt{x^2+y^2}$] [1]

$$= \sqrt{|y|}$$ (for $(x,y) \neq (0,0)$)

Since $\sqrt{\ }$ and $|\ |$ both are continuous fn, then for [1]

$$\lim_{(x,y)\to(0,0)} \sqrt{|y|} = 0$$ [1]

Hence from Sandwitch theorem

$g(x,y) \longrightarrow 0$ as

$(x,y) \to 0$.

Further more $\cos()$ is a continuous fn hence [1]

$$\lim_{(x,y)\to(0,0)} f(x,y) = \cos\left[\lim_{(x,y)\to(0,0)} g(x,y)\right]$$

$$= \cos(0) = 1. = f(0,0) \text{ [1]}$$

Q-3 a) The temperature in a region in space is given by T(x, y, z) = y^2 - 3xyz. A particle is moving in this region and its position at time t is given by x = 2e^t, y = 3Cost, z = - Sint.

a. How fast is the temperature experienced by the particle changing per unit distance when the particle is at the point P(2, 3, 0)?

b. Use part a. to evaluate how fast is the temperature experienced by the particle changing per unit time at P?

Solⁿ :- i) The path of the particle motion is given by

$$r(t) = 2e^t i + 3\cos t\, j - \sin t\, k$$

One can observe that particle is at P(2, 3, 0) for $t = 0$.

Now,

$$v(t) = 2e^t i - 3\sin t\, j - \cos t\, k$$

$$v(0) = 2i - k$$

$$\Rightarrow u = \frac{v}{|v|} = \frac{2i - k}{\sqrt{5}} = \frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} k$$

3 Marks

Here $T(x, y, z) = y^2 - 3xyz$

$$\nabla T(x, y, z) = (-3yz) i + (2y - 3xz) j + (-3xy) k$$

$$\nabla T\Big|_{P(2,3,0)} = 6j - 18k$$

3 Marks

$$D_u T\Big|_{P(2,3,0)} = (6j - 18k) \cdot \left(\frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} k\right)$$

$$= \frac{18}{\sqrt{5}} \text{ unit temp/unit distance}$$ 3 Marks

ii) $$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \nabla T \cdot u$$

$$= (D_u T)\,|u|$$

$$= \frac{18}{\sqrt{5}} \times \sqrt{5}$$

$$= 18 \text{ unit temp/unit time}$$ 3 Marks

Q-3 b) Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

and check whether $f_{xy}(0,0) = f_{yx}(0,0)$ or not?

Sol^n:- $f_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$

$f_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$ — 3 Marks

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{y^3 \Delta x}{\Delta x + y^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y^3}{y^2} = y$$

— 3 marks

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{x(\Delta y)^3}{x + (\Delta y)^2} - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x(\Delta y)^2}{x + (\Delta y)^2}$$ — 3 Marks

$$= 0$$

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$ — 1 Mark

$$= 0$$

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y}$$ — 1 Mark

$$= 1$$

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$. — 1 Mark

Q-3 Find all the critical points of the function

c)
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Sol^n:-

when $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{y(x^2+2y^2) - xy(2x)}{(x^2+2y^2)^2} = \frac{2y^3 - x^2 y}{(x^2+2y^2)^2}$$

& $$\frac{\partial f}{\partial y} = \frac{x(x^2+2y^2) - xy(4y)}{(x^2+2y^2)^2} = \frac{x^3 - 2xy^2}{(x^2+2y^2)^2}$$

considering $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$, we have

$y(2y^2 - x^2) = 0$ & $x^3 - 2xy^2 = 0$

$x^2 = 2y^2$ or $y = 0$ & $x(x^2 - 2y^2) = 0$

$x = 0$ or $x^2 = 2y^2$ — 2 Marks

Here if $x = 0 \Leftrightarrow y = 0$, which is not possible — 1 Mark

$\therefore$ $x = \pm\sqrt{2}\, y$; $(x,y) \neq (0,0)$

$\therefore$ All the points except $(0,0)$ on lines $x = \pm\sqrt{2}\, y$ are critical points of the function. — 1 Mark

Now, when $(x,y) = (0,0)$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

Therefore, $(0,0)$ is also a critical point of a function. — 2 Marks