LISTA DE GA - 6

1. a) $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \|(1, 1, 1)\|$

$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$ u.c

b) $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$

$\|\vec{u}\| = \sqrt{3^2 + 0^2 + 4^2}$

$= \sqrt{9 + 0 + 16} = \sqrt{25} = 5$ //

c) $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$

$\|\vec{u}\| = \sqrt{(-1)^2 + 1^2 + 0^2}$

$= \sqrt{1 + 1 + 0} = \sqrt{2}$ //

d) $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$

$\|\vec{u}\| = \sqrt{4^2 + 3^2 + (-1)^2}$

$= \sqrt{16 + 9 + 1} = \sqrt{26}$ //

2. a) Para ser uma base ortonormal é necessário que seja ortogonal e que cada vetor da base é unitário

- O conjunto $E = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ são unitários como dito no enunciado + $\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1$
- $\vec{e}_1 \cdot \vec{e}_2 = 0$ pois $1 \neq 2$ $\quad \vec{e}_3 \cdot \vec{e}_1 = 0$ pois $3 \neq 1$

$\vec{e}_2 \cdot \vec{e}_3 = 0$ pois $2 \neq 3$

b) $\vec{e}_1 = \overrightarrow{DH}$ $\quad \vec{e}_2 = \overrightarrow{DC}$ $\quad \vec{e}_3 = \overrightarrow{DA}$

$\vec{u} = \overrightarrow{CD} + \overrightarrow{CB}$ $\quad \vec{u} = -\vec{e}_2 + CB$

$CB = DA = \vec{e}_3$ $\quad \vec{u} = -\vec{e}_2 + \vec{e}_3$ //

$\vec{v} = \overrightarrow{DC} + \overrightarrow{CB}$ $\quad CB = DA = \vec{e}_3$

$\vec{v} = \vec{e}_2 + \vec{e}_3$ //

$\vec{w} = \overrightarrow{GC} = -\vec{e}_1$ //

$\vec{u} = (0, -1, 1)_E$

$\vec{v} = (0, 1, 1)_E$

$\vec{w} = (-1, 0, 0)_E$ //

c) Para ser ortonormal é necessário ser ortogonal e normalizado

- Normalização

$\vec{u} = (0, -1, 1)$

$\|\vec{u}\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$

$\vec{f_1} = \dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{(0, -1, 1)}{\sqrt{2}} = \left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$\vec{v} = (0, 1, 1)$

$\|\vec{v}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

$\vec{f_2} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{(0, 1, 1)}{\sqrt{2}} = \left(0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$\vec{w} = (-1, 0, 0)$

$\|\vec{w}\| = \sqrt{0^2 + 0^2 + (-1)^2}$

$f_3 = 1$

- Ortogonalidade

$\vec{f_1} \cdot \vec{f_2} =$

$\left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) \cdot \left(0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \left(0, -\dfrac{1}{2}, \dfrac{1}{2}\right) = 0 - \dfrac{1}{2} + \dfrac{1}{2} = 0$

$\vec{f_2} \cdot \vec{f_3} =$

$\left(0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)(-1, 0, 0) = (0, 0, 0) = 0$

$\vec{f_1} \cdot \vec{f_3} =$

$\left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)(-1, 0, 0) = (0, 0, 0) = 0$

∴ Todos são ortogonais e como os versores resultam, naturalmente em 1 e $\vec{f_3} = 1$ é uma base ortonormal

d)

$\vec{e_1} = \left(0 + \left(-\dfrac{1}{\sqrt{2}}\right) + \dfrac{1}{\sqrt{2}}\right)$ $\quad \vec{e_2} = \left(0 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right)$ $\quad \vec{e_3} = (-1 + 0 + 0)$

$F \rightarrow E$

$$\begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = M$$

$E \rightarrow F$

$$M = \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \ell_2 / \frac{1}{\sqrt{2}} \\ \ell_3 / \frac{1}{\sqrt{2}} \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & \sqrt{2} & 0 \\ 1 & 1 & 0 & 0 & 0 & \sqrt{2} \end{array}\right] \begin{array}{l} \ell_1 \leftrightarrow \ell_2 \\ \ell_2 \leftrightarrow \ell_3 \\ \ell_3 \leftarrow \ell_1 \end{array} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & \sqrt{2} & 0 \\ 1 & 1 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \ell_2 \leftarrow \ell_2 + \ell_1 \\ \ell_3 \leftarrow \ell_3(-1) \end{array} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 2 & 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \ell_1 \leftarrow \ell_1 - \ell_2/2 \rightarrow \begin{bmatrix} -1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 2 & 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{array}{l} \ell_1 \leftarrow \ell_1(-1) \\ \ell_2 \leftarrow \ell_2/2 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

Para que M seja ortogonal é necessario que:

$M \cdot M^t = Id$

$M \cdot M^t$

$$\begin{array}{c|c} & \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Como isso se concretiza a matriz M é ortogonal

e)

$\vec{HB}$ na base E

$\vec{HB} = \vec{HD} + \vec{DC} + \vec{CB}$

$= -\vec{e_1} + \vec{e_2} + \vec{e_3}$

$\vec{HB} : (-1, 1, 1)$

$$\vec{HB_F} = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ 0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ 1 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$\vec{HB_F} = (0, \sqrt{2}, 1)$

3.

a) $\vec{AB} = B - A = (5, 1, -3) - (2, 4, 3) = (3, -3, -6)$ $\quad \vec{BC} = C - B = (0, -3, 1) - (5, 1, -3) = (-5, -4, 4)$

$\vec{CA} = A - C = (2, 4, 3) - (0, -3, 1) = (2, 7, 2)$

b) $\|\vec{AB}\| = \sqrt{(3^2 + (-3)^2 + (-6)^2} = \sqrt{9 + 9 + 36} = \sqrt{54} = 3\sqrt{6}$ $\quad \|\vec{CA}\| = \sqrt{2^2 + 7^2 + 2^2} = \sqrt{57}$

$\|\vec{BC}\| = \sqrt{(-5)^2 + (-4)^2 + 4^2} = \sqrt{25 + 16 + 16} = \sqrt{57}$

- Logo, como 2 lados são iguais e um é diferente é um triangulo iscoceles

c) Ponto médio de $\vec{AB}$

$M = \left(\frac{5+2}{2}, \frac{1+4}{2}, \frac{-3+3}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}, 0\right)$

$\vec{CM} = M - C = \left(\frac{7}{2}, \frac{5}{2}, 0\right) - (0, -3, 1) = \left(\frac{7}{2}, \frac{11}{2}, -1\right)$

- Ortogonalidade

$\vec{AB} \cdot \vec{CM} = (3, -3, -6)\left(\frac{7}{2}, \frac{11}{2}, -1\right)$

$= \left(\frac{21}{2} - \frac{33}{2} + 6\right) = 0$

$\therefore$ Como o produto escalar deles é 0, eles são perpendiculares.

d) 

$\cos\alpha = \frac{\vec{CA} \cdot \vec{BC}}{\|\vec{CA}\| \|\vec{BC}\|} = \frac{(2, 7, 2)(-5, -4, 4)}{\sqrt{57} \cdot \sqrt{57}} = \frac{-10 - 28 + 8}{57} = -\frac{30}{57}$

$B\hat{C}A = \arccos -\frac{30}{57}$

e) A soma de $\vec{AB} + \vec{BC} + \vec{CA} = 0$ pois eles não são paralelos, logo, com L.I e porque eles partem e terminam no mesmo ponto.

4.

a) $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$

$|\vec{u}\cdot\vec{v}| = |\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta|$

$= \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta|$

Note que $0 \leq |\cos\theta| \leq 1 \leftrightarrow \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta| \leq \|\vec{u}\|\,\|\vec{v}\|$

$\therefore |\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|_{/\!/}$

b) $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$

Usando o item a), temos:

$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\| \rightarrow 2(\vec{u}\cdot\vec{v}) \leq 2\|\vec{u}\|\,\|\vec{v}\|$

$\|\vec{u}+\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\cdot\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2$

$\|\vec{u}\cdot\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

c) $4\vec{u}\cdot\vec{v} = \|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2$

$= (\|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2) - (\|\vec{u}\|^2 - (2\cdot\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2)$

$= \cancel{\|\vec{u}\|^2} + 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2} - \cancel{\|\vec{u}\|^2} + 2\vec{u}\cdot\vec{v} - \cancel{\|\vec{v}\|^2}$

$= 4\vec{u}\cdot\vec{v}_{/\!/}$

5. a) $\vec{u} = (1, 0, 1)$ $\vec{v} = (-2, 10, 2)$

$$\cos\theta = \frac{(1, 0, 1) \cdot (-2, 10, 2)}{\sqrt{1^2 + 0^2 + 1^2} \cdot \sqrt{(-2)^2 + 10^2 + 2^2}}$$

$$= \frac{(-2 + 0 + 2)}{\sqrt{2} \cdot \sqrt{108}} = 0$$

$\cos\theta = 0$ $\quad \pi — 180°$

$\theta = 90°$ $\quad x — 90°$

$$x = \frac{\pi}{2} \text{rad}$$

b) $\vec{u} = (-1, 1, 1)$ $\vec{v} = (1, 1, 1)$

$$\cos\theta = \frac{(-1, 1, 1) \cdot (1, 1, 1)}{\sqrt{(-1)^2 + 1^2 + 1^2} \cdot \sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{(-1 + 1 + 1)}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$$

$\cos\theta = \frac{1}{3}$ $\quad \pi — 180°$

$\quad Y — \frac{1}{3}$

$$Y = \arccos\left(\frac{1}{3}\right) \cong 1{,}231 \text{ rad}$$

c) $\vec{u} = (3, 3, 0)$ $\vec{v} = (2, 1, -2)$

$$\cos\theta = \frac{(3, 3, 0) \cdot (2, 1, -2)}{\sqrt{3^2 + 3^2 + 0^2} \cdot \sqrt{2^2 + 1^2 + (-2)^2}}$$

$$= \frac{(6 + 3 + 0)}{\sqrt{18} \cdot \sqrt{9}} = \frac{9}{\sqrt{162}} = \frac{9}{9\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = 45°$ $\quad \pi — 180°$

$\quad h — 45°$

$$h = \frac{\pi}{4} \text{rad}$$

d) $\vec{u} = (\sqrt{3}, 1, 0)$ $\vec{v} = (\sqrt{3}, 1, 2\sqrt{3})$

$$\cos\theta = \frac{(\sqrt{3}, 1, 0) \cdot (\sqrt{3}, 1, 2\sqrt{3})}{\sqrt{(\sqrt{3})^2 + 1^2 + 0^2} \cdot \sqrt{\sqrt{3}^2 + 1^2 + (2\sqrt{3})^2}}$$

$$= \frac{(3 + 1 + 0)}{\sqrt{4} \cdot \sqrt{16}} = \frac{4}{\sqrt{64}} = \frac{4}{8} = \frac{1}{2}$$

$\cos\theta = \frac{1}{2}$ $\quad \pi — 180°$ $\quad x = \frac{\pi}{6} \text{rad}$

$\theta = 30°$ $\quad z — 30°$

6. a) $\vec{u} \cdot \vec{v} = 0$

$\vec{u} = (x + 1, 1, 2)$ $\vec{v} = (x - 1, -1, -2)$

$(x + 1, 1, 2)(x - 1, -1, -2) = 0$

$x^2 - 1 + 0 - 4 = 0$

$x^2 = 5$

$x = \sqrt{5}$

b) $\vec{u} \cdot \vec{v} = 0$

$\vec{u} = (x, x, 4)$ $\vec{v} = (4, x, 1)$

$(x, x, 4)(4, x, 1) = 0$

$4x + x^2 + 4 = 0$

$x^2 + 4x + 4 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 4^2 - 4 \cdot 1 \cdot 4$

$\Delta = 8$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{2}$$

$$x_1 = \frac{-4 + 2\sqrt{2}}{2} = -2 + \sqrt{2}$$

$$x_2 = \frac{-4 - 2\sqrt{2}}{2} = -2 - \sqrt{2}$$

7.

a) Sabendo que $\vec{v} . \vec{w} = \vec{u}$ pois $\vec{v}$ e $\vec{w}$ são ortogonais a $\vec{u}$

$$\vec{v} . \vec{w} = \begin{vmatrix} i & j & k \\ 4 & -1 & 5 \\ 1 & -2 & 3 \end{vmatrix} = -3i + 5j - 8k - (-k - 10i + 12j)$$

$$= -3i + 5j - 8k + k + 10i - 12j$$

$$= 7i - 7j - 7k$$

$\vec{u} = (7, -7, -7)$

$\vec{u} \cdot (1, 1, 1) = -1$

$t \cdot (7, -7, -7)(1, 1, 1) = -1$

$7t - 7t - 7t = -1$

$-7t = -1$

$t = \frac{1}{7}$

$\therefore$ Então o vetor $\vec{u}$ para se encaixar na especificação ficará $\vec{u} = (1, -1, -1)$

b)

$\vec{u} = \vec{v} \cdot \vec{w} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 2 & -4 & 6 \end{vmatrix} = 18i - 2j - 8k - (6k + 4i + 12j)$

$= 18i - 2j - 8k - 6k - 4i - 12j$

$= 14i - 14j - 14k$

$\vec{u} = (14, -14, -14)$

$= (1, -1, -1)$

$\|\vec{u}\| = \sqrt{(-1)^2 + (-1)^2 + 1^2}$

$= \sqrt{3}$

porém como é especificado que $\|\vec{u}\| = 3\sqrt{3}$

$\vec{u} = 3(1, -1, -1) = (3, -3, -3)$

$\therefore$ Logo, $\vec{u} = (3, -3, -3)$

Para que um angulo seja agudo

$\cos > 0$ e $\cos \neq 1$

$\vec{u} = (3, -3, -3)$ $\vec{i} = (1, 0, 0)$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{i}}{\|\vec{u}\| \|\vec{i}\|} = \dfrac{(3, -3, -3)(1, 0, 0)}{3\sqrt{3} \cdot 1}$

$\|\vec{i}\| = \sqrt{1^2 + 0^2 + 0^2}$

$= \sqrt{1} = 1$

$= \dfrac{(3 + 0 + 0)}{3\sqrt{3}} = \dfrac{3}{3\sqrt{3}}$

$= \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} > 0$

c) Sabendo que $\dfrac{\pi}{4} = 45°$

$\cos 45° = \dfrac{\sqrt{2}}{2}$ $\quad \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

$\dfrac{\sqrt{2}}{2} = \dfrac{\vec{u} \cdot \vec{v}}{\sqrt{5} \cdot 1} \Rightarrow \vec{u} \cdot \vec{v} = \dfrac{\sqrt{10}}{2}$

$\|\vec{u}\|^2 = \vec{u}$

$(\sqrt{5})^2 = 5 = \vec{u}$ $\quad 1^2 = 1 = \vec{v}$

$\|\vec{u} + \vec{v}\| = \sqrt{\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u} \cdot \vec{v})}$

$= \sqrt{5 + 1 + 2\left(\dfrac{\sqrt{10}}{2}\right)}$

$= \sqrt{6 + \sqrt{10}}$ //

$\|\vec{u} - \vec{v}\| = \sqrt{\|\vec{u}\|^2 + \|\vec{v}\| - 2(\vec{u} \cdot \vec{v})}$

$= \sqrt{5 + 1 - 2\left(\dfrac{\sqrt{10}}{2}\right)} = \sqrt{6 - \sqrt{10}}$ //

$(\vec{u} + \vec{v})(\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2 = 5 - 1 = 4$ //

$\cos\theta = \dfrac{(\vec{u} + \vec{v})(\vec{u} - \vec{v})}{\|\vec{u} + \vec{v}\| \|\vec{u} - \vec{v}\|}$

$= \dfrac{4}{(\sqrt{6 + \sqrt{10}})(\sqrt{6 - \sqrt{10}})} =$

$= \dfrac{4}{\sqrt{26}} = \dfrac{4\sqrt{26}}{26} = \dfrac{2\sqrt{26}}{13}$ //

$\theta = \arccos\left(\dfrac{2\sqrt{26}}{13}\right)$ //

8.

a) $\text{Proj}_{\vec{u}}^{\vec{v}} = \left(\vec{v} \cdot \dfrac{\vec{u}}{\|\vec{u}\|}\right) \dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{(\vec{v} \cdot \vec{u})\vec{u}}{\|\vec{u}\|^2} =$

$\dfrac{(1, -1, 2) \cdot (3, -1, 1)\vec{u}}{\left(\sqrt{3^2 + (-1)^2 + 1^2}\right)^2} = \dfrac{(3 + 1 + 2)\vec{u}}{9 + 1 + 1} =$

$= \dfrac{6}{11}(3, -1, 1) = \left(\dfrac{18}{11}, -\dfrac{6}{11}, \dfrac{6}{11}\right)$

b) $\text{Proj}_{\vec{u}}^{\vec{v}} = \dfrac{(\vec{v} \cdot \vec{u})\vec{u}}{\|\vec{u}\|^2}$

$= \dfrac{(1, 3, 5)(-3, 1, 0)\vec{u}}{(-3)^2 + 1^2 + 0^2} = \dfrac{-3 + 3 + 0}{9 + 1}\vec{u}$

$= \dfrac{0}{10}(-3, 1, 0) = (0, 0, 0)$

c) $\text{Proj}_{\vec{u}}^{\vec{v}} = \dfrac{(\vec{v} \cdot \vec{u})\vec{u}}{\|\vec{u}\|^2}$

$= \dfrac{(-1, 1, 1)(-2, 1, 2)\vec{u}}{(-2)^2 + 1^2 + 2^2}$

$= \dfrac{2 + 1 + 2}{4 + 1 + 4}\vec{u} = \dfrac{5}{9}(-2, 1, 2) =$

$= \left(-\dfrac{10}{9}, \dfrac{5}{9}, \dfrac{10}{9}\right)$ //

d) $\text{Proj}_{\vec{u}}^{\vec{v}} = \dfrac{(\vec{v} \cdot \vec{u})\vec{u}}{\|\vec{u}\|^2} = \dfrac{(1, 2, 4)(-2, -4, -8)\vec{u}}{(-2)^2 + (-4)^2 + (-8)^2}$

$= \dfrac{(-2 - 8 - 32)\vec{u}}{4 + 16 + 64} = -\dfrac{42}{84}\vec{u} = -\dfrac{1}{2}(-2, -4, -8) =$

$= (1, 2, 4)$ //

9.

a) $\vec{v} = (3, -6, 0)$ $\quad \vec{u} = (2, -2, 1)$

$\text{Proj}_{\vec{u}}^{\vec{v}} = \dfrac{(\vec{v} \cdot \vec{u})\vec{u}}{\|\vec{u}\|^2} = \dfrac{(3, -6, 0)(2, -2, 1)\vec{u}}{2^2 + (-2)^2 + 1^2}$

$= \dfrac{(6 + 12 + 0)\vec{u}}{4 + 4 + 1} = \dfrac{18}{9}\vec{u} = 2(2, -2, 1)$

$= (4, -4, 2)$ //

$\text{Proj}_{\vec{v}}^{\vec{u}} = \dfrac{(\vec{u} \cdot \vec{v})\vec{v}}{\|\vec{v}\|^2} = \dfrac{(2, -2, 1)(3, -6, 0)\vec{v}}{3^2 + (-6)^2 + 0^2}$

$= \dfrac{(6 + 12 + 0)}{9 + 36}\vec{v} = \dfrac{18}{45}\vec{v} = \dfrac{6}{15}\vec{v} =$

$= \dfrac{6}{15}(3, -6, 0) = \left(\dfrac{18}{15}, -\dfrac{36}{15}, 0\right)$

$= \left(\dfrac{6}{5}, -\dfrac{12}{5}, 0\right)$ //

b) Como $\vec{p}$ é paralelo a $\vec{u}$

$\text{Proj}_{\vec{u}}^{\vec{v}} = \vec{p}$

Logo, $\vec{p} = (4, -4, 2)$ como resolvido anteriormente. //

$\vec{v} = \vec{p} + \vec{q}$

$(3, -6, 0) = (4, -4, 2) + \vec{q}$

$\vec{q} = (3 - 4, -6 + 4, 0 - 2)$

$= (-1, -2, -2)$ //

Verificação de ortogonalidade de $\vec{q}$ com $\vec{u}$:

$\vec{q} \cdot \vec{u} = 0$

$(-1, -2, -2)(2, -2, 1) = (-2 + 4 - 2) = 0$ //

LISTA DE GA - 6

c) Área $= \|\vec{u}\| \cdot h$

$h = \|\vec{q}\|$

Área $: \|\vec{u}\| \cdot \|\vec{q}\|$

$= \sqrt{2^2+(-2)^2+1^2} \cdot \sqrt{(-1)^2+(-2)^2+(-2)^2}$

$= \sqrt{9} \cdot \sqrt{9}$

$= 9$

10.a)

$\vec{u} \times \vec{v} =$

$= \begin{vmatrix} i & j & k \\ 3 & 3 & 0 \\ 5 & 4 & 0 \end{vmatrix} = i(3\cdot0)+j(5\cdot0)+k(3\cdot4)-(k(3\cdot5)+i(4\cdot0)+j(3\cdot0)$

$= i(3\cdot0-4\cdot0)+j(5\cdot0-3\cdot0)+k(3\cdot4-3\cdot5)$

$= 0+0+(-3)$

$\vec{u} \times \vec{v} = (0,0,-3)$

$\|\vec{u} \times \vec{v}\| =$

$\sqrt{0^2+0^2+(-3)^2} = \sqrt{9} = 3$

b) $\vec{u} = (7,0,-5)$ $\vec{v} = (1,2,-1)$

$\begin{vmatrix} i & j & k \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix} = i(-1\cdot0)+j(-5\cdot1)+k(7\cdot2)-(k(1\cdot0)+i(-5\cdot2)+j(-1\cdot7))$

$= i(0+10)+j(-5+7)+k(14-0)$

$= i10 - j35 = k14$

$\vec{u} \times \vec{v} = (10, 2, 14)$

$\|\vec{u} \times \vec{v}\| = \sqrt{10^2+2^2+14^2} = \sqrt{100+4+196}$

$= \sqrt{300} = 10\sqrt{3}$

c) $\vec{u} = (1,-3,1)$ $\vec{v} = (1,1,4)$

$\begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = i(-3\cdot4)+j(1\cdot1)+k(1\cdot1)-(k(-3\cdot1)+i(1\cdot1)$
$+j(4\cdot1))$

$= i(-13)+j(3)+k(4)$

$\vec{u} \times \vec{v} = (-13, -3, 4)$

$\|\vec{u} \times \vec{v}\| = \sqrt{(-13)^2+3^2+4^2} = \sqrt{169+9+16} = \sqrt{194}$

d) $\vec{u} = (2,1,2)$ $\vec{v} = (4,2,4)$

$\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{vmatrix} = i(4)+j(8)+k(4)-(k(4)+i(4)+j(8))$

$= i0+j(0)+k.0$

$\vec{u} \times \vec{v} = (0,0,0)$

$\|\vec{u} \times \vec{v}\| = \sqrt{0^2+0^2+0^2} = \sqrt{0} = 0$

11. a) $\vec{u} \times \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

$\|\vec{u} \times \vec{v}\|^2 = \|u\|^2 \cdot \|\vec{v}\|^2 \cdot \operatorname{sen}^2\theta$

$\|\vec{u} \times \|\vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \operatorname{sen}\theta$

Sabendo disso:

$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

$\operatorname{sen}^2\theta + \cos^2\theta = 1$

$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 (1-\cos^2\theta)$

$= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

b) $\|\vec{u} \times \vec{v}\|$

$\vec{u} \cdot \vec{v} = 3$ $\|\vec{u}\| = 1$ $\|\vec{v}\| = 5$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\|$

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \operatorname{sen}\theta$

$(\vec{u} \cdot \vec{v})^2 + \|\vec{u} \times \vec{v}\|^2 = (\|\vec{u}\| \|\vec{v}\|)^2 \cdot \cos^2\theta + (\|\vec{u}\| \|\vec{v}\|)^2 \cdot \operatorname{sen}^2\theta$

$(\vec{u} \cdot \vec{v})^2 + \|\vec{u} \times \vec{v}\|^2 = (\|\vec{u}\| \|\vec{v}\|)^2 (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1})$

$(\vec{u} \cdot \vec{v})^2 + \|\vec{u} \times \vec{v}\|^2 = (\|\vec{u}\| \|\vec{v}\|)^2$

$3^2 + \|\vec{u} \times \vec{v}\|^2 = (1 \cdot 5)^2$

$\|\vec{u} \times \vec{v}\|^2 = 25 - 9 = 16$

$\|\vec{u} \times \vec{v}\| = \sqrt{16} = 4$

c) Como é um triangulo equilátero $\vec{AB} = \vec{AC} = \ell$

$\|\vec{AB} \times \vec{AC}\| = \|\vec{AB}\| \cdot \|\vec{AC}\| \cdot \operatorname{sen}\theta$

$\|\vec{AB}\| = \|\vec{AC}\| = \|\vec{BC}\| = \ell$

$\|\vec{AB} \times \vec{AC}\| = \ell \cdot \ell \cdot \operatorname{sen}60$

$= \dfrac{\ell^2 \cdot \sqrt{3}}{2}$

12.

a) $\begin{cases} \vec{x}\cdot(2\vec{i}+3\vec{j}+4\vec{k})=9 \\ \vec{x}\times(-\vec{i}+\vec{j}-\vec{k})=-2\vec{i}+2\vec{k} \end{cases}$

$\vec{x}=a\vec{i}+b\vec{j}+c\vec{k}$

• Produto escalar:

$(a\vec{i}+b\vec{j}+c\vec{k})(2\vec{i}+3\vec{j}+4\vec{k})=9$

$2ai+3bj+4ck=9$

$2a+3b+4c=9$

• Produto vetorial:

$(a\vec{i}+b\vec{j}+c\vec{k})\times(-\vec{i}+\vec{j}-\vec{k})=-2\vec{i}+2\vec{k}$

$\begin{vmatrix} i & j & k \\ a & b & c \\ -1 & 1 & -1 \end{vmatrix} = -bi-cj+ak-(-bk+ci-aj)=$
$= i(-b-c)-j(c-a)+k(a+b)=-2i+2k+0j$

$(-b-c,\ -c+a,\ a+b)=(-2,+2)$

$a+b=0 \quad -b-c=-2 \quad -c+a=2$

$\begin{cases} a+b=0 & a=-b \\ -b-c=-2 \rightarrow & a-c=-2 \\ a-c=2 \rightarrow & a-c=2 \end{cases}$

$2=-2$

sistema sem solução

∴ Não existe vetor $\vec{x}$ que satisfaça ambas condições.

b) $\begin{cases} \vec{x}\times(1,0,1)=2(1,1,-1) \\ \|\vec{x}\|=\sqrt{6} \end{cases}$

• Produto vetorial:

$(a,b,c)\times(1,0,1)=(2,2,-2)$

$\begin{vmatrix} i & j & k \\ a & b & c \\ 1 & 0 & 1 \end{vmatrix} = bi+cj+ak\cdot 0-(bk+0\cdot ci+aj)$
$= i(b-0)+j(c-a)+k(0-b)=(2,2,-2)$

$\begin{cases} b=2 \\ c-a=2 \\ -b=-2 \end{cases} \quad b=2 \quad c=a+2$

• Norma:

$\|\vec{x}\|=\sqrt{a^2+b^2+c^2}=\sqrt{6}$

$=\sqrt{a^2+2^2+(a+2)^2}=\sqrt{6}$

$=\sqrt{a^2+4+a^2+4a+4}=\sqrt{6}$

$=2a^2+4a+8=6$

$=2a^2+4a+2=0$

$=a^2+2a+1=0$

$S=-2 \quad P=1$

$a=-1$

$b=2$

$c=-1+2=1$

$\vec{x}=(-1,2,1)$

c) $\|\vec{x}\|=\sqrt{3}$

$\sqrt{a^2+b^2+c^2}=\sqrt{3}$

$a^2+b^2+c^2=3$

∴ Como $a=b=c$, logo

$a=b=c=1$ ou -1

$\vec{u}=(-3,0,3)$

$\vec{x}\cdot\vec{u}=0$

$(a,b,c)(-3,0,3)=0$

$-3a+0+3c=0$

$-3a+3c=0$

$\vec{v}=(2,-2,0)$

$\vec{x}\cdot\vec{v}=0$

$(a,b,c)(2,-2,0)=0$

$2a-2b+0=0$

$2a-2b=0$

$\begin{cases} 2a-2b=0 & 2a=2b \rightarrow a=b \\ -3a+3c=0 & -3a=-3c \rightarrow a=c \end{cases}$

$a=b=c$

• Ângulo obtuso

Para um angulo ser obtuso ($90°<\theta<180°$), logo $\cos<0$

$\cos\theta=\dfrac{\vec{x}\cdot\vec{j}}{\|\vec{x}\|\|\vec{j}\|}=\dfrac{\vec{x}\cdot(0,1,0)}{\sqrt{3}\cdot 1}=\dfrac{b}{\sqrt{3}}$

Para $\cos\theta=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$ $\quad\cos\theta>0$

Para $\cos\theta=\dfrac{-1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3}$ $\quad\cos\theta<0$

$\cos\theta=-\dfrac{\sqrt{3}}{3}$

$\theta=\arccos\left(-\dfrac{\sqrt{3}}{3}\right)\cong 125{,}3°$

∴ $a=b=c=-1$

$\vec{x}=(-1,-1,-1)$

13. a)

$\vec{AD}=\vec{D}-\vec{A}$

$=(5,3,3)-(3,2,-1)$

$=(2,1,4)=\vec{AD}$

$\vec{AB}=(1,1,-1)$

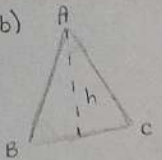
Area $=\|\vec{AB}\times\vec{AD}\|$

$\vec{AB}\times\vec{AD}=\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 4i-2j+k-(2k-1i+4j)$
$=5i-6j-k$

$\|\vec{AB}\times\vec{AD}\|=\sqrt{5^2+(-6)^2+(-1)^2}$

$=\sqrt{25+36+1}$

$=\sqrt{62}$

b) 

$\text{Área}_\Delta = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$

$= \frac{1}{2}\|(3,3,-1)\|$

$= \frac{1}{2}\cdot\sqrt{3^2+3^2+(-1)^2}$

$= \frac{1}{2}\sqrt{19} = \frac{\sqrt{19}}{2}$

$\|\vec{AB}\times\vec{AC}\| = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 3i + 0j - k + (0k + 0\cdot i - 3j)$

$= 3i + 3j - k$

$= (3,3,-1)$

• Altura:

$\|\vec{AB}\| = \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$ $\qquad \|\vec{BC}\| = \sqrt{1^2+0^2+3^2} = \sqrt{10}$

$\vec{BC} = -\vec{AB} + \vec{AC}$

$= -(-1,1,0)+(0,1,3)$

$= (1,-1,0)+(0,1,3)$

$\vec{BC} = (1,0,3)$

$\text{Área}_\Delta = \frac{1}{2}\cdot\text{base}\cdot\text{altura}$

$\dfrac{\frac{\sqrt{19}}{2}}{\frac{1}{2}\cdot\sqrt{10}} = h \qquad h = \dfrac{\frac{\sqrt{19}}{2}}{\frac{\sqrt{10}}{2}} = \dfrac{\sqrt{19}}{\sqrt{10}} = \sqrt{\dfrac{19}{10}}$

14. a) • $\vec{u}\cdot(\vec{v}\times\vec{w})$

$\vec{u} = (x_1, y_1, z_1)$

$\vec{v} = (x_2, y_2, z_2)$

$\vec{w} = (x_3, y_3, z_3)$

• $\vec{v}\times\vec{w} =$

$\begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = i(y_2\cdot z_3) + j(z_2\cdot x_3) + k(x_2\cdot y_3) - (k(y_2\cdot x_3) + i(z_2\cdot y_3) + j(z_3\cdot x_2))$

$= i(y_2z_3 - z_2\cdot y_3) + j(z_2x_3 - z_3x_2) + k(x_2y_3 - y_2x_3)$

$= (y_2z_3 - z_2y_3,\ z_2x_3 - z_3x_2,\ x_2y_3 - y_2x_3)$

$(x_1, y_1, z_1)(y_2z_3 - z_2y_3,\ z_2x_3 - z_3x_2,\ x_2y_3 - y_2x_3)$

$= (x_1y_2z_3 - x_1y_3z_2 + x_3y_1z_2 - x_2y_1z_3 + x_1x_2y_3 - x_1x_3y_2)$

• $(\vec{u}\times\vec{v})\cdot\vec{w}$

$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = i(y_1z_2) + j(z_1x_2) + k(x_1y_2) - (k(y_1x_2) + i(z_1y_2) + j(z_2x_1))$

$= i(y_1z_2 - z_1y_2) + j(z_1x_2 - z_2x_1) + k(x_1y_2 - y_1x_2)$

$= (y_1z_2 - z_1y_2,\ z_1x_2 - z_2x_1,\ x_1y_2 - y_1x_2)$

$(x_3, y_3, z_3)(y_1z_2 - z_1y_2,\ z_1x_2 - z_2x_1,\ x_1y_2 - y_1x_2) =$

$= (x_3y_1z_2 - x_3y_2z_1 + x_2y_3z_1 - x_1y_3z_2 + x_1y_2z_3 - x_2y_1z_3)$

∴ Logo, reordenando os termos $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\vec{w}$

• $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1y_2z_3 + x_3y_1z_2 + x_2y_3z_1 - (x_3y_2z_1 + x_1y_3z_2 + x_2y_1z_3)$

$= x_1y_2z_3 + x_3y_1z_2 + x_2y_3z_1 - x_3y_2z_1 - x_1y_3z_2 - x_2y_1z_3$

∴ Assim, reordenando os termos $\vec{u}(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\vec{w} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

b)

$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ -1 & 2 & 0 \end{vmatrix} = 1\cdot1\cdot0 + 3\cdot(-2)(-1) + 2\cdot0\cdot2 - (2\cdot1\cdot(-1) + 1\cdot(-2)\cdot2 + 0\cdot0\cdot3)$

$= 0 + 6 + 0 + 2 + 4 - 0 = 12 = [\vec{u}, \vec{v}, \vec{w}]$

• Nos determinantes mudar a posição de uma coluna, deixa o determinante negativo

$[\vec{u}, \vec{w}, \vec{v}] = -12$

$[\vec{v}, 2\vec{w}, \vec{u}] = 2[\vec{v}, \vec{w}, \vec{u}] = 2\cdot12 = 24$

$[\vec{u}, 3\vec{v} - 2\vec{u}, \vec{w} + 3\vec{u}]$

$\rightarrow 3\vec{v} - 2\vec{u} = 3(0,1,-2) - 2(1,3,2)$

$= (0,3,-6) - (2,6,4)$

$= (-2,-3,-10)$

$\rightarrow \vec{w} + 3\vec{u} = (-1,2,0) + 3(1,3,2)$

$= (-1,2,0) + (3,9,6)$

$= (2,11,6)$

$\begin{vmatrix} 1 & 3 & 2 \\ -2 & -3 & -10 \\ 2 & 11 & 6 \end{vmatrix} = -18 - 60 - 44 - (-12 - 110 - 36)$

$= -122 + 158 = 36$

$[\vec{u}, 3\vec{v} - 2\vec{u}, \vec{w} + 3\vec{u}] = 36$

15. a) $Area_{ABCD} = \|\vec{AB} \times \vec{AD}\|$

$\vec{AD} = \vec{AF} + \vec{FD}$ $\quad\quad$ $\vec{FD} = \vec{EA} = -\vec{BE} - \vec{AB}$

$\vec{AD} = \vec{AF} - \vec{AB} - \vec{BE}$

$\vec{AD} = (3,5,6) - (1,0,1) - (2,2,2)$

$\vec{AD}: (0,3,3)$

$Area_{ABCD}:$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 3 & 3 \end{vmatrix} = i\cdot 0 + j\cdot 0 + k\cdot 3 - (k\cdot 0 + i\cdot 3 + j\cdot 3)$$

$= -3i - 3j + 3k$

$(-3, -3, 3)$

$\|(-3,-3,3)\| = \sqrt{(-3)^2 + (-3)^2 + 3^2} = 3\sqrt{3}$

$\therefore Area_{ABCD} = 3\sqrt{3}$

b) $Volume = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$= |(\vec{AD} \times \vec{AB}) \cdot AE|$

$\vec{AE}: \vec{AB} + \vec{BE}$

$\vec{AE} = (1,0,1) + (2,2,2)$

$= (3,2,3)$

$$\vec{AD} \times \vec{AB} = \begin{vmatrix} i & j & k \\ 0 & 3 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 3i + 3j + 0\cdot k - (3k + 0\cdot i + 0\cdot j)$$

$= 3i + 3j - 3k$

$= (3, 3, -3) = (1, 1, -1)$

$Vol = |(3,3,-3)(3,2,3)| = |(9+6-9)| = |6|$

$= 6$

c) $Volume = \text{área base} \cdot \text{altura}$

$6 = 3\sqrt{3} \cdot \text{altura}$

$\text{altura} = \dfrac{6}{3\sqrt{3}} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

d) $Volume = \dfrac{1}{6} |(\vec{AB} \times \vec{AD}) \cdot \vec{AE}|$

Usando o resultado do item b), então:

$Volume = \dfrac{1}{6} \cdot 6 = \dfrac{6}{6} = 1$

e) Base:

$\vec{DE} = -\vec{AD} + \vec{AE}$

$\vec{DB} = -\vec{AD} + \vec{AB}$

$\vec{DE} = -(0,3,3) + (3,2,3)$

$= (0,-3,-3) + (3,2,3)$

$= (3,-1,0)$

$\vec{DB} = -(0,3,3) + (1,0,1)$

$= (0,-3,-3) + (1,0,1)$

$= (1,-3,-2)$

Area Base: $\dfrac{1}{2} \|\vec{DB} \times \vec{DE}\|$

$$\begin{vmatrix} i & j & k \\ 3 & -1 & 0 \\ 1 & -3 & -2 \end{vmatrix} = 2i + 0j - 9k - (-1k + 0i - 6j)$$

$= 2i - 6j - 8k$

$(2, -6, -8)$

$\sqrt{2^2 + (-6)^2 + (-8)^2} = \sqrt{4+36+64} = \sqrt{104}$

Área da base $= \dfrac{\sqrt{104}}{2} = \dfrac{2\sqrt{26}}{2} = \sqrt{26}$

Usando o volume calculado no item d)

$Volume = \dfrac{1}{6} \cdot \text{base} \cdot \text{altura}$

$1 = \dfrac{1}{3} \cdot \sqrt{26} \cdot \text{altura}$

$\text{altura} = \dfrac{3}{\sqrt{26}} = \dfrac{3\sqrt{26}}{26}$