EFFICIENTLY REALIZABLE DIGITAL FILTER TRANSFER FUNCTIONS

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Abstract. In this paper methods to design IIR digital filters with good attenuation or good attenuation and good group delay characteristics are described. These filters require often significantly less arithmetic operations than conventional efficient IIR and FIR digital filters. For the amplitude approximation problem the passband or stopband is mapped onto the whole unit circle where an equiripple behaviour is generated analytically. The desired stopband or passband behaviour is then obtained by locating the zeros or poles appropriately. For the simultaneous amplitude and group delay approximation problem the equiripple group delay of the all pole section of the filter is first generated on the whole unit circle and then mapped onto the passband. The linear phase FIR section of the filter makes the composite filter fulfill the amplitude specifications. Applications of the methods to some common filter design problems are presented.

Keywords. Digital filtering; approximation theory; optimization; discrete time systems; frequency response.

INTRODUCTION

A normal procedure to design recursive digital filters is to perform the design in the s plane and then transform the result to the z plane. In addition to being indirect, this approach has some obvious disadvantages. Digital filters derived from classical all pole filters have as many zeros at -1 as they have poles. In narrowband filters these zeros have very little effect on the attenuation in the transition and stopbands (Neuvo, 1978), although they increase the complexity of the filter. Furthermore, it has been observed that careful selection of the numerator and denominator orders accompanied with optimized positioning of the poles and zeros leads to effective digital realizations (Martinez, 1978; Saramäki, 1980a, 1981b). In these cases it would be advisable to use a filter design process where the positions and/or numbers of zeros and poles could be determined with the effectiveness of the final digital filter realization in mind.

Naturally, optimization methods can be used to design filters of desired type. However, straightforward optimization often leads to long computations and gives too little insight to the nature of the problem.

A similar situation arises when one wants to design digital filters with good group delay characteristics. The straightforward approach in the IIR case is to use an elliptic filter combined with a delay equalizer. However, this approach is not optimized for the digital realization where the complexity of the filter is largely determined by the number of arithmetic operations.

In this paper we present a unified approach to design filters for both specified amplitude characteristics and specified amplitude and group delay characteristics. The common factor between the squared magnitude and group delay functions of digital filters is that both are rational functions of $z+1/z$ (Saramäki, 1979, 1980c, 1981b). This enables the use of similar transformations

$$w+1/w = [A(z+1/z)+B]/[C(z+1/z)+D] \qquad (1)$$

to map either the passband or stopband of the magnitude squared functions or the passband of the group delay function onto the upper unit semicircle. In this transformed domain it is easy to generate analytically an equal ripple magnitude or group delay function occupying the upper unit semicircle.

In the amplitude approximation case the equal ripple function when mapped appropriately to the z plane results in an equiripple passband or stopband behavior with adjustable zeros or poles, respectively. In the group delay approximation case we obtain an implicit solution for an all pole IIR filter with desired group delay variation and passband edge angles.

In all these three cases we obtain with small computational workload good starting points for further optimization procedures, where the specific aspects of practical digital filter realizations can be taken into account.

Using the analytic method to generate equal ripple passband, we can easily design filters with zeros at specific frequencies, e.g., line frequency and some of its harmonics. We can also design filters having very short (1 or 2 bits) coefficients in the feedforward parts.

Of great importance in narrowband cases are filters where the denominator order is higher than the numerator order. In wideband cases the situation is vice versa. We describe shortly an effective optimization procedure that can be used to find the optimal transfer functions and compare the resulting filters with elliptic ones.

Filters requiring good attenuation and group delay characteristics are designed in two parts. The all pole IIR section is designed to have an equiripple group delay in the passband. The linear phase FIR section is then designed to make the overall filter meet the amplitude specifications. Careful optimization of the whole filter leads to an optimal solution with often a negligible variation in the group delay combined with good attenuation properties. The resulting filters are compared with optimal FIR and delay equalized elliptic designs.

ANALYTIC METHODS FOR AMPLITUDE AND PHASE APPROXIMATIONS

Frequency Transformations

The magnitude squared function $H(z)H(1/z)$ of a general transfer function

$$H(z) = \frac{\sum_{i=0}^{m} c_i z^{-i}}{\sum_{i=0}^{n} d_i z^{-i}} = z^{n-m} \frac{\prod_{i=1}^{m} (z-a_i)}{\prod_{i=1}^{n} (z-b_i)} \tag{2}$$

can be written as

$$H(z)H(1/z) = P(z+1/z) \tag{3}$$

where

$$P(z+1/z) = E \frac{\prod_{i=1}^{m} \left[(z+1/z) - (a_i+1/a_i)\right]}{\prod_{i=1}^{n} \left[(z+1/z) - (b_i+1/b_i)\right]} \tag{4}$$

From (3) and (4) we observe that $H(z)H(1/z)$ is a function $P(z+1/z)$ of $z+1/z$. The necessary conditions for $P(z+1/z)$ to correspond to the magnitude squared function of a stable filter are (Saramäki, 1981b)

1) $P(z+1/z)$ is a real rational polynomial
2) $P(z+1/z)$ is finite (has no poles) and nonnegative (has no zeroes of odd multiplicity) on the unit circle.

According to 1) all values of $P(z+1/z)$ on the entire z plane are completely determined by its values on the domain $|z|\leq 1$, $\mathrm{Im}\{z\}\geq 0$.

A linear transformation

$$\frac{z+1/z-z_1-1/z_1}{z+1/z-z_2-1/z_2} \cdot \frac{z_3+1/z_3-z_2-1/z_2}{z_3+1/z_3-z_1-1/z_1} = \frac{w+1/w-w_1-1/w_1}{w+1/w-w_2-1/w_2} \cdot \frac{w_3+1/w_3-w_2-1/w_2}{w_3+1/w_3-w_1-1/w_1} \tag{5}$$

between $z+1/z$ and $w+1/w$ transforms the domains $|z| \leq 1$, $\mathrm{Im}\{z\} \geq 0$ and $|w| \leq 1$, $\mathrm{Im}\{w\} \geq 0$ into a one-to-one correspondence (Saramäki, 1981b). In addition, it transforms a real rational polynomial $P(z+1/z)$ into a real rational polynomial $Q(w+1/w)$ and vice versa.

By selecting z_i, w_i, $i=1,2,3$ to lie on the upper unit semicircle or on the real axis inside the unit circle, (5) maps the boundary of $|z| \leq 1$, $\mathrm{Im}\{z\} \geq 0$ to be the boundary of $|w| \leq 1$, $\mathrm{Im}\{w\} \geq 0$ in such a way that the points z_1, z_2, z_3 are carried to the points w_1, w_2, w_3, respectively, and vice versa.

Of special importance are transformations mapping the unit circle interval $z = e^{j\omega}$, $\omega_1 \leq \omega \leq \omega_2$ (passband or stopband of the filter) onto the upper unit semicircle $w = e^{j\Omega}$, $0 \leq \Omega \leq \pi$. By setting $z_1 = e^{j\omega_1}$, $w_1 = 1$, $z_2 = e^{j\omega_2}$, $w_2 = -1$, $z_3 = w_3 = 0$, we obtain

$$w + 1/w = A(z+1/z) + B \tag{6}$$

with

$$A = 2/(\cos\omega_1-\cos\omega_2) \tag{7}$$

$$B = -2(\cos\omega_1+\cos\omega_2)/(\cos\omega_1-\cos\omega_2) \tag{8}$$

When the stopbands (passbands) $z = e^{j\omega}$, $0 \leq \omega \leq \overline{\omega}_1$, and $z = e^{j\omega}$, $\overline{\omega}_2 \leq \omega \leq \pi$, are specified they are transformed, respectively, onto real axis intervals $\zeta_3 \leq \zeta \leq \zeta_4$ and $\zeta_1 \leq \zeta \leq \zeta_2$ inside the unit circle (see Fig. 1). The real axis inside the unit circle, in turn, is mapped onto the interval $\zeta_2 \leq \zeta \leq \zeta_3$.

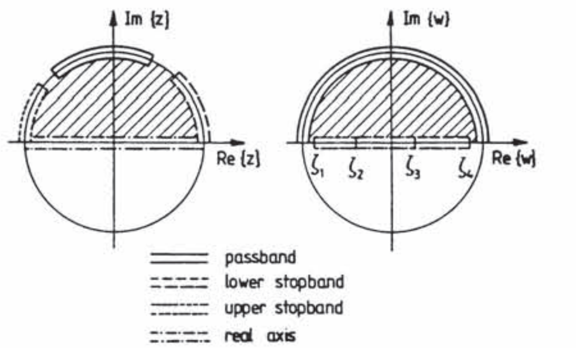

Fig. 1. Relation between the z and w planes under transformation (6).

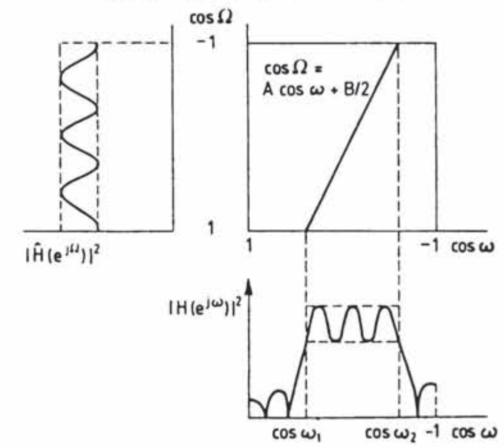

Fig. 2. Transformation (9) between the frequency variables ω and Ω.

$P(z+1/z)$ obtained by transforming the magnitude squared function $\hat{H}(w)\hat{H}(1/w)$ to the z plane via (6) corresponds to the magnitude squared function of a stable filter if it satisfies the condition 2). This implies that $\hat{H}(w)\hat{H}(1/w)$

has no pole or no zero of odd multiplicity in $[-1,\zeta_2)$ or $(\zeta_3,1]$.

On the unit circles, $z = e^{j\omega}$, $w = e^{j\Omega}$, (6) takes the form

$$\cos\Omega = A\cos\omega + B/2 \quad (9)$$

As seen from Fig. 2, the transformation (6) or (9) converts the Chebyschev-type behaviour of $|\hat{H}(e^{j\Omega})|^2$ attained in the interval $0 \le \Omega \le \pi$ onto the interval $\omega_1 \le \omega \le \omega_2$ and only distorts the frequency axis. In the intervals $0 \le \omega \le \omega_1$ and $\omega_2 \le \omega \le \pi$ $|H(e^{j\omega})|^2$ takes the corresponding values of $\hat{H}(w)\hat{H}(1/w)$ in the real axis intervals $[\zeta_3,1]$ and $[-1,\zeta_2]$.

Equiripple Amplitude

We consider an all-pass function of the form

$$F(w) = \prod_{i=1}^{n}\left(\frac{1-w_i w}{w-w_i}\right) = \prod_{i=1}^{n} w\left(\frac{1/w-w_i}{w-w_i}\right) \quad (10)$$

where $|w_i| < 1$ and w_i's are real or occur in complex conjugate pairs. When Ω ranges from 0 to π, the phase $f(\Omega)$ of $F(w)$ varies from 0 to $-n\pi$ (Saramäki, 1979, 1981b). By virtue of this result, the following rational polynomial of $w + 1/w$

$$G(w+1/w) = \frac{1}{2}\left[F(w)+1/F(w)\right] \quad (11)$$

takes on the unit circle, $w = e^{j\Omega}$, the form

$$G(w+1/w) = \cos\{f(\Omega)\} \quad (12)$$

This function exhibits the desired maximum number (n+1) of alternating extrema ±1 on $w = e^{j\Omega}$, $0 \le \Omega \le \pi$. Its value at $w=1$ ($\Omega=0$) is +1.

In view of the properties of $G(w+1/w)$, the magnitude squared function $\hat{H}(w)\hat{H}(1/w)$ having arbitrary zeros α_i and $1/\alpha_i$ and taking on n+1 alternating extrema $1\pm\hat{\delta}_1$ on $w = e^{j\Omega}$, $0 \le \Omega \le \pi$, is readily constructed as follows

$$\hat{H}(w)\hat{H}(1/w) = \frac{2(1-\hat{\delta}_1^2)}{2 + \hat{\delta}_1\left[F(w)+1/F(w)\right]} \quad (13)$$

where

$$F(w) = \prod_{i=1}^{n}\left(\frac{1-\alpha_i w}{w-\alpha_i}\right) \quad (14)$$

with $|\alpha_i|<1$. The value of $\hat{H}(w)\hat{H}(1/w)$ at $w=1$ is $1-\hat{\delta}_1$.

The function $\hat{H}(w)\hat{H}(1/w)$ having arbitrary poles β_i and $1/\beta_i$ and taking on m+1 alternating extrema $\hat{\delta}_2$ and 0 on $w = e^{j\Omega}$, $0 \le \Omega \le \pi$, takes, in turn, the form

$$\hat{H}(w)\hat{H}(1/w) = \frac{\hat{\delta}_2}{2}\left\{1 + \frac{1}{2}\left[F(w)+1/F(w)\right]\right\} \quad (15)$$

where

$$F(w) = \prod_{i=1}^{m}\left(\frac{1-\beta_i w}{w-\beta_i}\right) \quad (16)$$

with $|\beta_i|<1$. The value of $\hat{H}(w)\hat{H}(1/w)$ at $w=1$ is $\hat{\delta}_2$.

In the case of filters with equiripple passband the n zeros α_i of $\hat{H}(w)\hat{H}(1/w)$ as given by (13), (14) are adjustable. This enables the design of filters of the form (2) with $m \le n$ (n-m zeros at the origin), m prescribed zeros a_i, and Chebyschev passband $\omega_1 \le \omega \le \omega_2$. First, the zeros a_i are mapped to the w plane zeros α_i inside the unit circle via (6), (7), (8). The n-m zeros at the origin are mapped to the origin, i.e., $\alpha_{m+1} = \dots = \alpha_n = 0$. Then, by transforming the resulting $\hat{H}(w)\hat{H}(1/w)$ back to the realization plane we obtain the desired magnitude squared function $H(z)H(1/z)$.

In the case of filters with equiripple stopband, in turn, the m poles β_i of $\hat{H}(w)\hat{H}(1/w)$ as given by (15), (16) are adjustable. In a similar manner as we used zeros α_i in synthesizing equiripple passband filters, the adjustable poles β_i enable the design of equiripple stopband filters of the form (2) with $m \ge n$ (m-n poles at the origin), n prescribed poles b_i, and Chebyschev stopband $\omega_1 \le \omega \le \omega_2$.

To design lowpass filters with equiripple passband $0 \le \omega \le \omega_c$, we set $\omega_1 = \omega_c$ and $\omega_2 = 0$. In the highpass case with passband $\omega_c \le \omega \le \pi$ we set $\omega_1 = \omega_c$ and $\omega_2 = \pi$. Since $|\hat{H}(1)|^2 = 1 - \hat{\delta}_1$ and $\omega = \omega_1$ is mapped to $\Omega = 0$ (see Fig. 2), these selections ensure that $|H(e^{j\omega_c})|^2 = 1 - \hat{\delta}_1$. For the same reason, to guarantee that $|H(e^{j\omega})|^2$ takes the value $\hat{\delta}_2$ at the stopband edge angle $\omega = \omega_s$, we set $\omega_1 = \omega_s$ in lowpass and highpass cases with equiripple stopband.

Stopband and Passband Optimization

When all the zeros of the filter with equiripple passband are not specified, the adjustable zeros can be used to maximize the minimum difference Δ between the loss $L(\omega)$ of the filter and the specified loss $L_s(\omega)$ in the stopband(s). The stated problem can be solved conveniently in the w plane where the loss of the filter can be written, according to (13), (14), in the form

$$\hat{L}(w) = 10\log_{10}\left\{\frac{2+\hat{\delta}_1\left[F(w)+1/F(w)\right]}{2(1-\hat{\delta}_1^2)}\right\} \quad (17)$$

with

$$F(w) = \prod_{i=1}^{m_a}\left(\frac{1-\alpha_{ai}w}{w-\alpha_{ai}}\right)^2 \prod_{i=1}^{n-2m_a}\left(\frac{1-\alpha_{fi}w}{w-\alpha_{fi}}\right) \quad (18)$$

Here α_{ai}'s correspond to the adjustable zero pairs on the unit circle (or at the same or mirror image points on the real axis) which are mapped to the same real axis point inside the unit circle. α_{fi}'s are the transformed fixed zeros.

Since Δ takes its maximum when the difference between $\hat{L}(w)$ and the transformed specified loss $\hat{L}_s(w)$ has the value Δ at m_a+1 points in the transformed stopband(s), the optimization of the locations of α_i's is conveniently performed using a Remez-type procedure (Saramäki, 1981b) having similarities to those (Orchard, 1968; Temes, 1968) used in designing analog filters.

When all the poles of the filter with equiripple stopband are not specified, the adjustable poles can be utilized for achieving a Chebyschev approximation to a given passband response $P(\omega)$. When the stopband ripple $\hat{\delta}_2$ is specified, the approximation problem is to find $\hat{\delta}_1$ and the adjustable poles β_{ai} in such a way that the difference between

$$\hat{H}(w)\hat{H}(1/w) = \frac{\hat{\delta}_2}{2}\{1 + \frac{1}{2}[\prod_{i=1}^{n_a}(\frac{1-\beta_{ai}w}{w-\beta_{ai}})\prod_{i=1}^{m-n_a}(\frac{1-\beta_{fi}w}{w-\beta_{fi}}) + \prod_{i=1}^{n_a}(\frac{w-\beta_{ai}}{1-\beta_{ai}w})\prod_{i=1}^{m-n_a}(\frac{w-\beta_{fi}}{1-\beta_{fi}w})]\} \quad (19)$$

and the transformed passband response $\hat{P}(w)$ takes on the values $\pm\hat{\delta}_1$ at n_a+1 points in the transformed passband(s). When $\hat{\delta}_1$ is specified, the problem is to find $\hat{\delta}_2$ and the adjustable poles in such a way that the desired oscillations occur.

Both previous approximation problems can be solved conveniently by appropriately modifying (Saramäki, 1981b) the conventional Remez multiple exchange algorithm applied to rational approximations.

Equiripple Group Delay

The group delay of an all-pole filter of the form (2) with m=0 can be written as (Saramäki, 1981b)

$$\tau_g(z) = -\frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n}\frac{(b_i-1/b_i)}{(z+1/z)-(b_i+1/b_i)}$$

$$= \tau_0 - \varepsilon Q(z) \quad (20)$$

From (20) we observe that $\tau_g(z)$ and $Q(z)$ are rational polynomials of $z+1/z$ whose numerator and denominator are related to each other via the pole locations. This relation implies that the residues of $Q(z)$ at the poles $z=b_i$ are $-b_i/2\varepsilon$ and $Q(\infty) = (\tau_0+n/2)/\varepsilon$

The desired equiripple nature of $\tau_g(e^{j\omega})$ within the extrema $\tau_0\pm\varepsilon$ in the passband $\omega_1 \le \omega \le \omega_2$ implies that $Q(e^{j\omega})$ has the maximum number $(n+1)$ of alternating extrema ±1 in the passband. Since $Q(z)$ is a rational polynomial of $z+1/z$ and the rational polynomial of $w+1/w$

$$\hat{Q}(w) = \frac{1}{2}[\prod_{i=1}^{n}(\frac{1-\beta_i w}{w-\beta_i}) + \prod_{i=1}^{n}(\frac{w-\beta_i}{1-\beta_i w})] \quad (21)$$

has the admissable equiripple nature on $w=e^{j\Omega}$, $0 \le \Omega \le \pi$ (Eqs. (11), (10)), the desired $Q(z)$ can be found conveniently by mapping it to the w plane via (6),(7),(8). By equating the residues of the transformed $Q(z)$ and $\hat{Q}(w)$ at the transformed poles $w = \beta_i$ and using the condition $Q(\infty) = \hat{Q}(\infty) = (\tau_0+n/2)/\varepsilon$, we obtain n+1 equations (Saramäki, 1980c, 1981b) relating τ_0, ε and the transformed poles β_i. By fixing either τ_0 or ε, this system of equations can be solved for the remaining n+1 unknowns.

FILTERS WITH SPECIFIED AMPLITUDE PERFORMANCE

Stopband Optimization for Interference Rejection

To exemplify the use of the iterative procedure constructed for finding equiripple passband filters with fixed and adjustable zeros, we have designed a tenth order bandpass filter whose sampling rate is 400 Hz and whose passband ripple is 0.5 dB. The cutoff frequencies of the filter are 68 Hz and 78 Hz and the filter must have a zero at the power line frequency (50 Hz) and its second harmonic. Two zeros of the filter are at the origin. The two movable zero pairs are adjusted so that the minimum attenuation in the stopbands 0 Hz $\le \omega \le$ 60 Hz and 86 Hz $\le \omega \le$ 200 Hz takes its maximum. The optimized amplitude characteristic is shown in Fig. 3. We note that the use of one additional adjustable zero pair, instead of two zeros at the origin, improves the stopband attenuation only less than 0.1 dB.

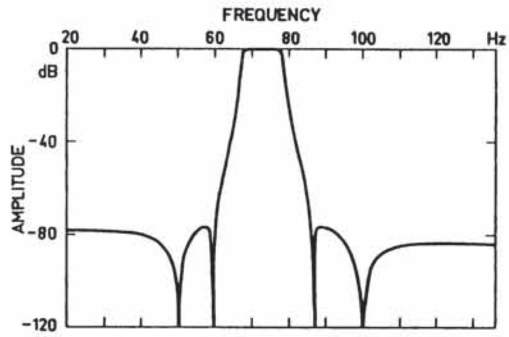

Fig. 3. Application 1: Transfer function for a 10th -order filter with fixed and adjusted zeros.

Quantization of Numerator Coefficients

A second application is shown in Fig. 4. Here an elliptic filter with 1 dB passband ripple, 48 dB stopband attenuation, and cutoff frequency $\omega_c = 0.25\pi$ was used as a model (curve a). Equiripple filters were generated for the same passband specifications and with zeroes located at the angles θ obtained with 2-bit (curve b) and 1-bit (curve c) representation of the 2nd -order numerator coefficients $-2\cos\theta$. This quantization yields a simpler coefficients representation at the cost of somewhat reduced stopband attenuation. We note that no multiplications are required in implementing the numerator of the curve (c) filter with cascaded 2nd order sections.

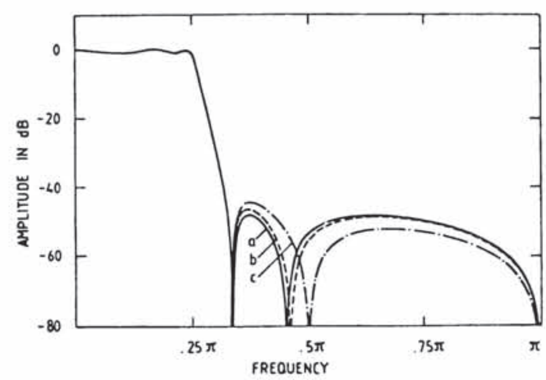

Fig. 4. Application 2: $\omega_c=.25\pi$. Transfer functions for 5th -order filters with (a) elliptic characteristic, (b) 2-bit quantization of $-2\cos\omega_i$, (c) 1-bit quantization of $-2\cos\omega_i$.

Optimal Selections of Numerator and Denominator Orders

To illustrate the efficiency of filters with denominator order n higher than numerator order m in narrowband applications, we have compared filters whose squared magnitude takes on n+1 alternating extrema $1 \pm \hat{\delta}_1$ in the passband $0 \leq \omega \leq \omega_c$ and m+1 alternating extrema $\hat{\delta}_2$ and 0 in the stopband $\omega_s \leq \omega \leq \pi$. The filters with $m < n$ are designed by fixing n-m zeros at the origin and one zero at z=-1 if m is odd and by adjusting the remaining zeros of the filter with equiripple passband so that the desired stopband behavior is achieved. When $n > m$, the m poles of the filter with equiripple stopband are adjusted so that the desired passband oscillations around 1 is attained. Fig. 5 shows the minimum stopband attenuations $-10 \log_{10} \hat{\delta}_2$ as functions of the transition width $\omega_s - \omega_c$ for filters with $\hat{\delta}_1 = 0.1$, $\omega_c = 0.025\pi$ and requiring eight multipliers in cascade realization: (a) n=5, m=5; (b) n=7, m=1; (c) n=6, m=3; (d) n=4, m=7. It is assumed that the zero pair on the unit circle requires one multiplier, while the implementation of the zero at z=-1 requires no multiplier. As seen from Fig. 5, the attenuation of the filter with n=6, m=3 (curve c) is from 10 dB to 20 dB better than that of the elliptic one.

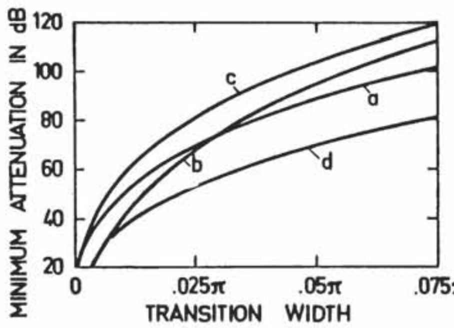

Fig. 5. $\omega_c = 0.025\pi$, $\hat{\delta}_1 = 0.1$. Comparison between filters with (a) n=m=5, (b) n=7, m=1, (c) n=6, m=3, (d) n=4, m=7.

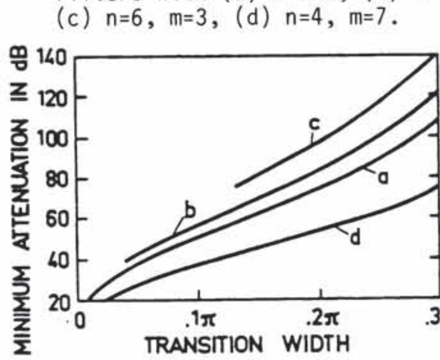

Fig. 6. $\omega_c = 0.6\pi$, $\hat{\delta}_1 = 0.1$. Comparison between filters with (a) n=m=5, (b) n=4, m=6, (c) n=4, m=7, (d) n=6, m=3.

Fig. 6, in turn, illustrates the efficiency of wideband filters with $m > n$. It compares filters with $\hat{\delta}_1 = 0.1$, $\omega_c = 0.6\,\pi$ and requiring eight multipliers: (a) n=5, m=5; (b) n=4, m=6; (c) n=4, m=7; (d) n=6, m=3. The filter with n=4, m=7 is the best one, when $\omega_s - \omega_c > 0.13\,\pi$. When $\omega_s - \omega_c < 0.13\,\pi$, no desired equiripple solution exists for n=4, m=7. The filter with n=4, m=6 has the highest attenuation, when $0.13\,\pi < \omega_s - \omega_c < .04\,\pi$, and, finally, when $\omega_s - \omega_c < 0.04\,\pi$, the elliptic filter is the best one.

FILTERS WITH SPECIFIED AMPLITUDE AND GROUP DELAY PERFORMANCES

Statement of the Problem

Here the group delay of the all-pole IIR part $1/D(z)$ of

$$H(z) = \frac{C(z)}{D(z)} = \sum_{i=0}^{m} c_i z^{-i} \Big/ \sum_{i=0}^{n} d_i z^{-i} \qquad (22)$$

is designed, using the method presented previously, to exhibit an equiripple variation in the passband $\omega_{c1} \leq \omega \leq \omega_{c2}$ with maximum deviation ε. If $|H(e^{j\omega})|$ is desired to approximate 1 in the passband with tolerance δ_1 and 0 in the stopbands $0 \leq \omega \leq \omega_{s1}$ and $\omega_{s2} \leq \omega \leq \pi$ with tolerance δ_2, the requirements for the linear phase FIR part $C(z)$ become

$$|D(e^{j\omega})|(1-\delta_1) \leq |C(e^{j\omega})| \leq |D(e^{j\omega})|(1+\delta_1)$$

in the passband and (23)

$$0 \leq |C(e^{j\omega})| \leq |D(e^{j\omega})|\,\delta_2 \qquad (24)$$

in the stopbands. In the lowpass case $\omega_{c1} = 0$ and the lower stopband is absent, while in the highpass case $\omega_{c2} = \pi$ and the upper stopband is absent.

Above equations specify the parameters for the FIR filter design program (McClellan, 1973). The final parameter to be determined is the minimum order $M(n,\varepsilon)$ of the FIR part to meet the amplitude requirements. The resulting filter requires

$$R(n,\varepsilon) = [(M(n,\varepsilon)+2)/2] + n \qquad (25)$$

where $[\cdot]$ stands for integer part of, multipliers when realized as a cascade of the all-pole IIR component and the linear phase FIR filter.

The aim of this section is to consider the solutions to the following optimization problem: given the passband and the stopband(s), the amplitude requirements, and the maximum allowable deviation ε_{max} in the group delay response, minimize $R(n,\varepsilon)$ with respect to n and ε.

Comparisons Between Different Filter Types

Using the optimization procedure given in (Saramäki, 1980c, 1981b), we have compared 9 lowpass filters with cutoff frequencies varing from 0.025 π to 0.1 π. Table 1 gives the filter specifications and compares optimum hybrid filters with maximally flat group delay (Saramäki, 1981a, 1981b), optimum hybrid filters with equiripple group delay, optimum linear phase FIR filters, and delay equalized elliptic filters. The delay equalized elliptic filters are designed to have as many multipliers as the hybrid filters with flat delay. For the hybrid filter with equiripple delay ε_{max} and the optimum values of n, ε, M, and R are given, while for the corresponding filter with flat delay,

TABLE 1 Lowpass Filter Design

Filter Specifications				Hybrid Filter with Max. Flat Group Delay					Hybrid Filter with Equiripple Group Delay					FIR Filter		Delay Equalized Elliptic Filter			
ω_c/π	ω_s/π	δ_1	δ_2	n_{opt}	$T_{0_{opt}}$	ε_m	$\hat{M}$	$\hat{R}$	ε_{max}	n_{opt}	ε_{opt}	$\hat{M}$	$\hat{R}$	M_{FIR}	R_{FIR}	N_{el}	N_{eq}	ε_{el}	R_{el}
.1	.2	.075	.0075	9	18.4	0.70	9	14	.1	8	.021	6	12	33	17	4	7	1.79	14
.05	.1	.075	.0075	10	45.4	1.19	6	14	.1	8	.042	6	12	64	32	4	7	3.59	14
.025	.05	.075	.0075	10	95.5	2.34	6	14	.1	8	.082	6	12	129	65	4	7	7.23	14
.1	.2	.025	.0025	13	25.0	0.21	10	19	.001	12	.000087	8	17	42	22	5	11	0.38	19
.05	.1	.025	.0025	13	55.8	0.38	10	19	.001	12	.00015	8	17	84	43	5	11	0.77	19
.025	.05	.025	.0025	13	59.0	0.76	10	19	.001	12	.00030	8	17	167	84	5	11	1.46	19
.1	.15	.075	.0075	14	28.1	0.32	12	21	.001	12	.00042	10	18	63	32	5	13	3.08	21
.05	.075	.075	.0075	14	62.7	0.62	12	21	.001	12	.00080	10	18	125	63	5	13	6.13	21
.025	.0375	.075	.0075	14	132.1	1.21	12	21	.001	12	.001	10	18	227	114	5	13	12.17	21

instead of ε, the optimum maximum value T_0 of the delay and the resulting maximum deviation ε_m of the group delay from the average value in the passband are given. The data for the FIR filter include the order M_{FIR} and the number R_{FIR} of multiplications. The data for the delay equalized elliptic filter include the order N_{el} of the elliptic filter, the order of the delay equalizer N_{eq}, and the maximum deviation ε_{el} of the group delay from the average value in the passband. In computing the required number of multiplications of elliptic filters it is assumed that the zero at $z=-1$ is implemented without multipliers.

As seen from Table 1, the new hybrid filters require in narrowband applications significantly less multiplications per sample than equivalent FIR filter designs at the expense of a small variation in the group delay response. The variation in the equiripple case is extremely small. When the hybrid filters are compared with equalized elliptic filters, it is observed that these filters provide a considerably smaller variation in the group delay.

It is interesting to note from the table that when the passband and the transition band are halved, the order of the FIR filter becomes roughly double, while the number of multiplications for hybrid filters remains about the same. The same nature can be seen from Fig. 7, where the required number of multiplications

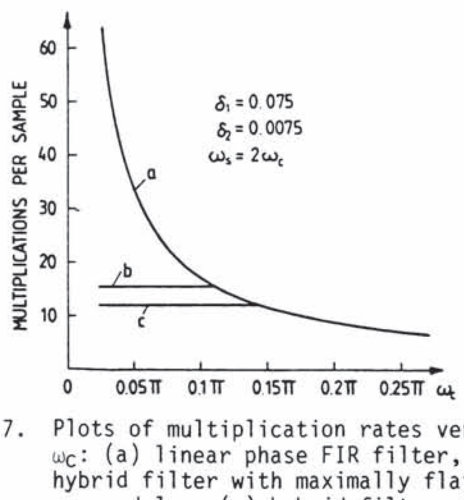

Fig. 7. Plots of multiplication rates versus ω_c: (a) linear phase FIR filter, (b) hybrid filter with maximally flat group delay, (c) hybrid filter with equiripple group delay.

is given for hybrid filters and for linear phase FIR filters as functions of the passband edge angle ω_c, when the filter specifications are: $\delta_1=0.075$, $\delta_2=0.0075$, $\omega_s=2\omega_c$. The hybrid filter with equiripple delay requires less multipliers than the FIR filter when $\omega_c<0.14\pi$.

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