Course of Study Bachelor Computer Science	Exercises Statistics WS 2020/21

Sheet VI - Solutions

Probability Spaces and Basic Rules

1. Consider a random experiment of tossing two dice. Let A denote the event that the first die score is 1 and B the event that the sum of the scores is 7.

   (a) Give the sample space Ω and find $|\Omega|$.

   (b) Explicitly list the elements of the following events:

   $$A, B, A \cup B, A \cap B, A^c \cap B^c$$

   Answer:

   (a) $\Omega = \{(i,j)|i,j \in \{1,2,3,4,5,6\}\}$, $|\Omega| = 36$

   (b) $A = \{(1,1,),(1,2),(1,3),(1,4),(1,5),(1,6)\}$
   $B = \{(1,6,),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
   $A \cap B = \{(1,6)\}$
   $A \cup B = \{(1,1,),(1,2),(1,3),(1,4),(1,5),(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
   $A^c \cap B^c = \Omega \setminus (A \cup B)$

2. Suppose that A and B are events in an experiment with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cap B) = 1/10$. Express each of the following events verbally and find its probability:

   $$A \setminus B, A \cup B, A^c \cup B^c, A^c \cap B^c, A \cup B^c$$

   Answer:

   (a) $A \setminus B$: The event A but not B occures.
   $P(A \setminus B) = P(A \cap B^C) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$

   (b) $A \cup B$: One or both of the events A and B occure.
   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{10} = \frac{29}{60}$

(c) $A^c \cup B^c$: Both events do not occure.
$P(A^c \cup B^c) = P(A^C \cup B^C) = P((A \cap B)^C) = 1 - P(A \cap B) = 1 - \frac{1}{10} = \frac{9}{10}$

(d) $A^c \cap B^c$: None of the events A and B occure.
$P(A^c \cap B^c) = P(A^C \cap B^C) = P((A \cup B)^C) = 1 - P(A \cup B) = 1 - \frac{29}{60} = \frac{31}{60}$

(e) $A \cup B^c$: only B occures but not A occures $(A \cup B^c = (B \setminus A)^c)$.
$P(A \cup B^c) = 1 - (P(B) - P(A \cap B)) = 1 - \frac{1}{4} + \frac{1}{10} = \frac{17}{20}$

3. Suppose that A, B, and C are events in an experiment with $P(A) = 0.3$, $P(B) = 0.2$, $P(C) = 0.4$, $P(A \cap B) = 0.04$, $P(A \cap C) = 0.1$, $P(B \cap C) = 0.1$, $P(A \cap B \cap C) = 0.01$. Express each of the following events in set notation and find its probability:

   (a) At least one of the three events occurs.

   (b) None of the three events occurs.

   (c) Exactly one of the three events occurs.

   (d) Exactly two of the three events occur.

   Answer:

   (a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.3 + 0.2 + 0.4 - 0.04 - 0.1 - 0.1 + 0.01 = 0.67$

   (b) $P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - 0.67 = 0.33$

   (c) $P((A \cup B \cup C) \setminus ((A \cap B) \cup (A \cap C) \cup (B \cap C))) = P(A \cup B \cup C) - P(A \cap C) - P(A \cap B) - P(B \cap C) + 2P(A \cap B \cap C) = 0.67 - 0.1 - 0.04 - 0.1 + 2 \cdot 0.01 = 0.45$

   (d) $P((A \cap B) \cup (A \cap C) \cup (B \cap C) \setminus (A \cap B \cap C)) = 0.04 + 0.1 + 0.1 - 3 \cdot 0.01 = 0.21$

4. **Urn Models**
   A large number of discrete probability spaces can be traced back to so-called urn models. An urn contains n balls, which do not all have to be different. From these urns r balls are drawn with or without replacement. For the result of the drawing, the order or only the quantity of the drawn balls can be of importance.
   Here an urn with 10 balls is considered. 5 of them are red, 3 balls are blue and 2 balls are green. 3 balls are drawn. The following 4 cases should be distinguished:

I Drawing with replacement with respect to the order

II Drawing with replacement without observing the order

III Drawing without replacement with respect to the order

IV Drawing without replacement without observing the order

Solve the following tasks.

(a) Load the library gtools and inspect the commands combinations() and permutations(). Consider the bags $b_1 = \{a, b, c\}$ and $b_2 = \{a, a, b, c\}$ = and list all combinations and all permutations of order 2 if duplicated elements in the outpot are allowed or not allowed.

(b) Use the function sample() to determine the result of 10 random draws.

(c) Determine a suitable event space Ω and its size to describe the random experiment.
Note that depending on whether the order of the drawn balls is important or not, the result of a drawing is considered as a r-variation or as a r-combination of the n set of balls. With the help of the permutations() and combinations() functions of the R-package gtools, the corresponding r-variations or r-combinations can be determined.

(d) Determine the probabilities of all elementary events in Ω using a Laplace model, i.e. as a determination of the ratio of the number of favorable cases by the number of all cases. The probabilities are first determined by counting methods and then by using the R function permutations().
Hint: To determine the probabilities with R, assume that the n balls are numbered consecutively, i.e. they are distinguishable, and that the order is first observed in a drawing. Every r-variation of the numbers 1 to n is equally probable. Determine the set of all these drawings with permutations(). Then map each such drawing to the corresponding elementary event. By dividing the number of drawings belonging to an elementary event and the number of all drawings, you then obtain the corresponding probabilities.

Answer:

(a) The R function combinations() enumerates the possible combinations of a specified size from the elements of a vector and the

R function permutations() enumerates the possible permutations. In both functions you decide via logical flags whether duplicates should be removed from the source and whether the output may include duplicated values.

(b) Samples of 10 random draws out of the bag $= \{r, r, r, r, r, b, b, b, g, g\}$ are

- without replacement: r, r, r, r, g, b, g, r, b, b
  Mention that the bag is empty after the 10 draws.
- with replacement: b, r, b, g, b, g, r, b, b, b
  Mention that in the sample can occure for example more blue balls than there are in the bag.

(c) Set of elementary events and their probabilities

i. Drawing with replacement with respect to the order

- $|\Omega| = 3^3 = 27$ number of 3-variations of a 3-set with possible repititions
- Every elementary event can be seen as a word of length 3 with 3 possible letters. The possible letters are given by the colours r, b, g of the balls in the urn. To calculate the probability of an elementary event we count how many r, b and g are in the word. If R, B resp. G are the numbers of the red, blue resp. green drawn balls, we get

$$P(R = i, B = j, G = k) = \frac{5^i \cdot 3^j \cdot 2^k}{10^3}$$

with $i + j + k = 3; i, j, k \geq 0$

	event	prob
1	b b b	0.027
2	b b g	0.018
3	b b r	0.045
4	b g b	0.018
5	b g g	0.012
6	b g r	0.030
7	b r b	0.045
8	b r g	0.030
9	b r r	0.075
10	g b b	0.018
11	g b g	0.012
12	g b r	0.030
13	g g b	0.012
14	g g g	0.008
15	g g r	0.020
16	g r b	0.030
17	g r g	0.020
18	g r r	0.050
19	r b b	0.045
20	r b g	0.030
21	r b r	0.075
22	r g b	0.030
23	r g g	0.020
24	r g r	0.050
25	r r b	0.075
26	r r g	0.050
27	r r r	0.125

You can use the R command permutations() to get the elementary events and their probabilities by counting all equally like 3 permutations of numbers 1 to 10. The balls are nump-bererd from 1 to 10 and the balls 1 to 5 are red, 6 to 8 are blue and 9, 10 are green.

ii. Drawing with replacement without respect to the order

- $|\Omega| = \binom{3+3-1}{3} = \binom{5}{3} = 10$ number of 3-multisets from a 3-set
- Every elementary event can be seen as a 3-multiset of the letters r, b, g. Such a 3-multiset is given by a set of words of length 3 with R r's, B b' and G g's. All words of length 3 with the same numbers of r, b and g are equally like. For example

$$\begin{aligned}
P(\{b,b,g\}) &= P(\{R=0,B=2,G=1\}) \\
&= P(\{(b,b,g),(b,g,b),(g,b,b)\}) \\
&= P((b,b,g)) + P((b,g,b)) + P((g,b,b)) \\
&= \frac{3!}{0!2!1!} \cdot P((b,b,g)) \\
&= \frac{3!}{0!2!1!} \cdot \frac{5^0 \cdot 3^2 \cdot 2^1}{10^3} = 0.054
\end{aligned}$$

In general we get

$$P(\{R=i,B=j,G=k\}) = \frac{5^i \cdot 3^j \cdot 2^k}{10^3} \cdot \frac{3!}{i! \cdot j! \cdot k!}$$

with $i+j+k=3; i,j,k \geq 0$

	event	prob
1	b b b	0.027
2	b b g	0.054
3	b b r	0.135
4	b g g	0.036
5	b g r	0.180
6	b r r	0.225
7	g g g	0.008
8	g g r	0.060
9	g r r	0.150
10	r r r	0.125

As in (i) you can generate by applying the R command permutations() all possible ordered sampling results. Every ordered sampling corresponds to an unordered sampling result. Counting all rows with the same unordered sampling result and dividing by total number of ordered sampling results you get the probabilities of the elementary events.

iii. Drawing without replacement with respect to the order

- We must count the number of 3-variations with repititions and subtract the number of impossible events (here: 3 green balls)
$$|\Omega| = 3^3 - 1$$
- Regarding that now the drawn balls are not replaced we get analogousley to the case of drawing with replacement
$$P(R = i, B = j, G = k) = \frac{5^{\underline{i}} \cdot 3^{\underline{j}} \cdot 2^{\underline{k}}}{10^{\underline{3}}}$$
with $i + j + k = 3; i, j, k \geq 0$. Mention that $n^{\underline{m}} = n \cdot \ldots \cdot (n - m + 1)$ is the falling factorial and give the number of m-variations from a set of n distint elements.

	event	prob
1	b b b	0.008
2	b b g	0.017
3	b b r	0.042
4	b g b	0.017
5	b g g	0.008
6	b g r	0.042
7	b r b	0.042
8	b r g	0.042
9	b r r	0.083
10	g b b	0.017
11	g b g	0.008
12	g b r	0.042
13	g g b	0.008
14	g g r	0.014
15	g r b	0.042
16	g r g	0.014
17	g r r	0.056
18	r b b	0.042
19	r b g	0.042
20	r b r	0.083
21	r g b	0.042
22	r g g	0.014
23	r g r	0.056
24	r r b	0.083
25	r r g	0.056
26	r r r	0.083

As in the case drawing with replacement with respect to the order you can apply R to get the probabilities. But set the flag repeats.allowed to FALSE in the R command permutations() to realise drawing without replacement.

iv. Drawing without replacement without respect to the order

- We must count the number of 2-multisets of a 3-set and subtract the number of impossible events (here: 3 green balls)
$$|\Omega| = \binom{3+3-1}{3} - 1 = 9$$
- Regarding that now the drawn balls are not replaced we get analogousley to case drawing with replacement
$$P(\{R = i, B = j, G = k\}) = \frac{\binom{5}{i} \cdot \binom{3}{j} \cdot \binom{2}{k}}{\binom{10}{3}}$$

with $i + j + k = 3; i, j, k \geq 0$

	event	prob
1	b b b	0.008
2	b b g	0.050
3	b b r	0.125
4	b g g	0.025
5	b g r	0.250
6	b r r	0.250
7	g g r	0.042
8	g r r	0.167
9	r r r	0.083

As in the case drawing with replacement without respect to the order you can apply R to get the probabilities. But set the flag repeats.allowed to FALSE in the R command permutations() to realise drawing without replacement.

```
#########################################################
# Urn Models
# Elementary events and their probabilities: Evaluation
# and determination applying the R commands combinations()
# and permutations() in the gtools package.
#
# file: prob_basics_urn_models.R
#########################################################

library(gtools)
library(tidyverse)

# combinations enumerates the possible combinations of a specified
# size from the elements of a vector. permutations enumerates the
# possible permutations.
# Usage
# combinations(n, r, v=1:n, set=TRUE, repeats.allowed=FALSE)
# permutations(n, r, v=1:n, set=TRUE, repeats.allowed=FALSE)
# Arguments
# n size of the source vector
# r      size of the target vectors
# v source vector
# set   = logical flag indicating whether duplicates should be removed
#         from the source vector v.
# repeats.allowed = logical flag indicating whether the constructed
#                   vectors may include duplicated values.
#
# Value: Returns a matrix where each row contains a vector of length r.

# Examples Consider the bags
b1 <-  c("a","b","c")
b2 <- c("a","a","b","c")
# and list all combinations and all permutations of order 2 if duplicated
# elements in the outpot are allowd or not allowed.
combinations(n=3,r=2,v=b1)
combinations(n=3,r=2,v=b1,repeats.allowed = TRUE)

permutations(n=3,r=2,v=b1)
permutations(n=3,r=2,v=b1,repeats.allowed = TRUE)

# n = number of distinct elements of source vector, if set = TRUE!!
combinations(n=3,r=2,v=b2, set = TRUE)
combinations(n=4,r=2,v=b2, set = FALSE)
combinations(n=4,r=2,v=b2, set = FALSE, repeats.allowed = TRUE)

######################################################################
# Bag with 10 balls: 5*red, 3*blue und 2*green
# 3 balls are drawn
bag <- rep(c("r","b","g"),c(5,3,2))
bag

######################################################################
# Use the function sample() to determine the result of 10 random draws.

sample(x=bag, size=10) # without replacement
sample(x=bag, size=10, replace=TRUE) # with replacement

######################################################################
# Drawing with replacement with respect to the order
######################################################################
# Elementary events and their probabilities directly calculated
permutations(n=3, r=3, v=bag, set = TRUE, repeats.allowed = TRUE) %>%
```

```
  as_tibble(.name_repair = "universal") %>%
  # Treatment of problematic column names
  # .name_repair       = "universal": Make the names unique and syntactic
  mutate(
    V1 = ...1, V2 = ...2, V3 = ...3,
    prob = case_when(
      V1 == "r" ~ 0.5,
      V1 == "b" ~ 0.3,
      V1 == "g" ~ 0.2) *
      case_when(
        V2 == "r" ~ 0.5,
        V2 == "b" ~ 0.3,
        V2 == "g" ~ 0.2) *
      case_when(
        V3 == "r" ~ 0.5,
        V3 == "b" ~ 0.3,
        V3 == "g" ~ 0.2
      )
  ) %>% select(V1,V2,V3,prob) %>%
  # sort
  arrange(V1,V2,V3) -> M1
M1

# Elementary events and their probabilities determined by counting all
# equally like 3 permutations of numbers 1 to 10 (balls are numbererd
# from 1 to 10)
permutations(n=10, r=3, v=1:10, set = TRUE, repeats.allowed = TRUE) %>%
  as_tibble() %>%
  # zeilenweises Vorgehen
  rowwise() %>%
  mutate(
    # Übersetzung der 3-Permutation von 1 bis 10 in die Farben der
    # gezogenen Kugeln
    event = paste(c(bag[V1],bag[V2],bag[V3]), collapse = " ")) %>%
  # Zählen der Häufigkeiten der gezogenen Farben
  group_by(event) %>%
  summarise(count = n()) %>%
  # W. = Häufigkeit/Gesamtanzahl
  mutate(prob = count/sum(count)) %>%
  # sortieren
  arrange(event) -> M2

cbind(M1,M2)

# generate a latex table
library(xtable)
M2 %>% select(event, prob) %>% xtable(digits = 3)
M1

########################################################################

########################################################################
# Drawing with replacement without respect to the order
########################################################################
# Elementary events and their probabilities directly calculated
# Multinomial distribution of the number of drawn coloured balls
library(stats)
combinations(n=3, r=3, v=bag, set = TRUE, repeats.allowed = TRUE) %>%
  as_tibble() %>%
  # zeilenweises Vorgehen
  rowwise() %>%
  mutate(
    anz_r = if_else(V1 == "r",1,0) +
      if_else(V2 == "r",1,0) + if_else(V3 == "r",1,0) ,
    anz_b = if_else(V1 == "b",1,0) +
      if_else(V2 == "b",1,0) + if_else(V3 == "b",1,0) ,
    anz_g = if_else(V1 == "g",1,0) +
      if_else(V2 == "g",1,0) + if_else(V3 == "g",1,0) ,
    prob = dmultinom(x = c(anz_r, anz_b, anz_g), size = 3,
                     prob = c(0.5, 0.3, 0.2))
  ) %>%
  select(V1, V2, V3, prob) %>%
  # sortieren
  arrange(V1, V2, V3) -> M2

# Elementary events and their probabilities determined by counting all
# equally like 3 combinations of numbers 1 to 10 (balls are numbererd
# from 1 to 10)
permutations(n=10, r=3, v=1:10, set = FALSE, repeats.allowed = TRUE) %>%
  as_tibble() %>%
  # zeilenweises Vorgehen
  rowwise() %>%
  mutate(
```

```
    # pro Zeile sortieren nach den Farben und Umwandlung des sortierten
    # Vektors in einen String
    event = paste(sort(c(bag[V1],bag[V2],bag[V3])),
      collapse = " ")) %>%
  # Zählen der Häufigkeiten der gezogenen Farbkombinationen
  group_by(event) %>%
  summarise(count = n()) %>%
  # W. = Häufigkeit/Gesamtanzahl
  mutate(prob = count/sum(count)) %>%
  # sortieren
  arrange(event) -> M22

# generate a latex table
M22 %>% select(-count) %>% xtable(digits = 3)

##########################################################################

##########################################################################
# Drawing without replacement with respect to the order
##########################################################################
# Elementary events and their probabilities determined by counting all
# equally like 3 permutations (without repetitions) of numbers 1 to 10
# (balls are numbererd from 1 to 10)
permutations(n=10, r=3, v=bag, set = FALSE, repeats.allowed = FALSE) %>%
  as_tibble() %>%
  # zeilenweises Vorgehen
  rowwise() %>%
  mutate(
    # Übersetzung der 3-Permutation von 1 bis 10 in die Farben der
    # gezogenen Kugeln
    event = paste(c(V1,V2,V3), collapse = " ")) %>%
  group_by(event) %>%
  # Zählen der Häufigkeiten der gezogenen Farben
  summarise(count = n()) %>%
  # W. = Häufigkeit/Gesamtanzahl
  mutate(prob = count/sum(count)) %>% arrange(event) -> M3

# generate a latex table
M3 %>% select(-count) %>% xtable(digits = 3)

# some theoretical results
# P((b,r,b)) = P(b) * P(r|b) * P(b|(b,r)) = 3/10 * 5/9 * 2/8
# P((g,b,b)) = P(g) * P(b|g) * P(b|(g,b)) = 2/10 * 3/9 * 2/8
# ...

# Let (x,y,z) be an elementary event. I's probability depends only on the
# number of drawn coloured balls:
# P((x,y,z)) = P(R=i,B=j,G=k) =
#   fallende.Faktorielle(5,i) * fallende.Faktorielle(3,j) * fallende.Faktorielle(2,k) / (10*9*8)
permutations(n=10, r=3, v=bag, set = FALSE, repeats.allowed = FALSE) %>%
  as_tibble() %>%
  rowwise() %>%
  mutate(
    anz_r = if_else(V1 == "r",1,0) +
      if_else(V2 == "r",1,0) + if_else(V3 == "r",1,0) ,
    anz_b = if_else(V1 == "b",1,0) +
      if_else(V2 == "b",1,0) + if_else(V3 == "b",1,0) ,
    anz_g = if_else(V1 == "g",1,0) +
      if_else(V2 == "g",1,0) + if_else(V3 == "g",1,0)) %>%
  unique() %>%
  mutate(
    prob = (factorial(5)/factorial(5-anz_r)) *
      (factorial(3)/factorial(3-anz_b)) *
      (factorial(2)/factorial(2-anz_g)) / (10*9*8)) %>%
  arrange(V1,V2,V3)
##########################################################################

##########################################################################
# Drawing without replacement without respect to the order
##########################################################################
# Ereignisraum mit theoretischen W.
combinations(n=10, r=3, v=bag, set = FALSE, repeats.allowed = FALSE) %>%
  as_tibble() %>%
#  mutate(
#    V1 = bag[V1],
#    V2 = bag[V2],
#    V3 = bag[V3]) %>%
  unique() %>%
  # Bestimmung der Wahrscheinlichkeiten: zeilenweise werden zunächst die
  # Anzahl der gezogenen Farben bestimmt und anschließend ergeben sich die
  # Wahrscheinlichkeiten durch Anwendung einer verallgemeinert hypergeo-
  # metrischen Verteilung
  rowwise() %>%
```

```
  mutate(
    event = paste(sort(c(V1,V2,V3)), collapse = " "),
    anz_r = if_else(V1 == "r",1,0) +
      if_else(V2 == "r",1,0) + if_else(V3 == "r",1,0) ,
    anz_b = if_else(V1 == "b",1,0) +
      if_else(V2 == "b",1,0) + if_else(V3 == "b",1,0) ,
    anz_g = if_else(V1 == "g",1,0) +
      if_else(V2 == "g",1,0) + if_else(V3 == "g",1,0) ,
    prob = choose(5,anz_r)*choose(3,anz_b)*choose(2,anz_g)/choose(10,3)) %>%
  # sortieren
  arrange(event)

# Bestimmung der Elementarereignisse und ihrer W. durch Zählen aller
# gleichwahrscheinlichen 3-Kombinationen (ohne Wiederholungen) der Zahlen
# 1 bis 10 (Kugeln sind von 1 bis nummeriert)
permutations(n=10, r=3, v=bag, set = FALSE, repeats.allowed = FALSE) %>%
  as_tibble() %>%
  # zeilenweise sortieren und zu einem String zusammenfassen
  rowwise() %>%
  mutate(
    event = paste(sort(c(V1,V2,V3)), collapse = " ")) %>%
  # Zählen der Häufigkeiten der unterscheidlichen 3-Kombinationen
  group_by(event) %>%
  summarise(count = n()) %>%
  mutate(prob = count/sum(count)) %>%
  arrange(event) -> M4

# generate a latex table
M4 %>% select(-count) %>% xtable(digits=3)
```

Independence and Conditional Probabilities

1. Suppose that A and B are events in an experiment with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cap B) = 1/10$. Find each of the following:

$$P(A|B), P(B|A), P(A^c|B), P(B^c|A), P(A^c|B^c)$$

   Answer:

   (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/10}{1/4} = \frac{4}{10} = 0,4$

   (b) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/10}{1/3} = \frac{3}{10} = 0,3$

   (c) $P(A^C|B) = 1 - P(A|B) = 1 - 0,4 = 0,6$

   (d) $P(B^C|A) = 1 - P(B|A) = 1 - 0,3 = 0,7$

   (e) $P(A^C|B^C) = \frac{P(A^C\cap B^C)}{P(B^C)} = \frac{1-P(A\cup B)}{1-P(B)} = \frac{1-(P(A)+P(B)-P(A\cap B))}{1-1/4} = \frac{1-\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{10}\right)}{3/4} = \frac{31}{45} \approx 0,689$

2. In a certain population, 30% of the persons smoke and 8% have a certain type of heart disease. Moreover, 12% of the persons who smoke have the disease.

   (a) What percentage of the population smoke and have the disease?

   (b) What percentage of the population with the disease also smoke?

(c) Are smoking and the disease positively correlated, negatively correlated, or independent?

Answer: S=Smoker and D=Disease, i.e. $P(S) = 0.30$, $P(D) = 0.08$, und $P(D|S) = 0.12$,

(a) $P(S \cap D) = P(S) \cdot P(D|S) = 0.30 \cdot 0.12 = 0.036$

(b) $P(S|D) = \frac{P(S \cap D)}{P(D)} = \frac{0.036}{0.08} = 0.45$

(c) We compare $P(S) \cdot P(D) = 0,30 \cdot 0,08 = 0,024$ and $P(S \cap D) = 0,036$ and get $P(S \cap D) > P(S) \cdot P(D)$ i.e. dependent.

3. Suppose that a bag contains 12 coins: 5 are fair, 4 are biased with probability of heads 1/3 and 3 are two-headed. A coin is chosen at random from the bag and tossed.

   (a) Find the probability that the coin shows head.

   (b) Given that the coin shows head, find the conditional probability of each coin type.

   Answer: We have $n = 12$ coins. 5 of them are fair coins with $P(\text{ Head }) = P(\text{ Tail }) = 0.50$ (=A-coin). 4 are manipulated with $P(\text{ Head }) = 1/3$ and $P(\text{ Tail }) = 2/3$ (=B-coins). The other 3 are manipulated $P(\text{ Head }) = 1$ and $P(\text{ Tail }) = 0$ (=C-coins).

   (a) $P(\text{ Head }) = P(\text{ Head } | \text{ A-coin }) \cdot P(\text{ A-coin}) + P(\text{ Head } | \text{ B-coin }) \cdot P(\text{ B-coin}) + P(\text{ Head } | \text{ C-coin }) \cdot P(\text{ C-coin}) = \frac{1}{2} \cdot \frac{5}{12} + \frac{1}{3} \cdot \frac{4}{12} + 1 \cdot \frac{3}{12} = \frac{41}{72} \approx 0,56944$

   (b) $P(\text{ A-coin } | \text{ Head }) = \frac{P(\text{ A-coin } \cap \text{ Head })}{P(\text{ Head })} = \frac{P(\text{ Head } | \text{ A-coin }) \cdot P(\text{ A-coin })}{P(\text{ Head })} = \frac{\frac{1}{2} \cdot \frac{5}{12}}{\frac{41}{72}} = \frac{15}{41} \approx 0.3658$ and $P(\text{ B-coin } | \text{ Head }) = \frac{P(\text{ B-coin } \cap \text{ Head })}{P(\text{ Head })} = \frac{P(\text{ Head } | \text{ B-coin }) \cdot P(\text{ B-coin })}{P(\text{ Head })} = \frac{\frac{1}{3} \cdot \frac{4}{12}}{\frac{41}{72}} = \frac{8}{41} \approx 0.1951$ and $P(\text{ C-coin } | \text{ Head }) = \frac{P(\text{ C-coin } \cap \text{ Head })}{P(\text{ Head })} = \frac{P(\text{ Head } | \text{ C-coin }) \cdot P(\text{ C-coin })}{P(\text{ Head })} = \frac{1 \cdot \frac{3}{12}}{\frac{41}{72}} = \frac{18}{41} \approx 0.4390$

4. In a computer science course at an university we have the following data over a long time.
   10% of all students have attended the exercises in statistics regularly. 2% of the students who have failed the statistics exam have attended the exercises regularly. 5% of the students who have attended the exercises regularly have failed the statistics exam.

(a) Find the probability to fail the exam in statistics if the exercises in statistics are not attended regularly.

(b) What is the effect of attending the exercises regularly to passing the exam?

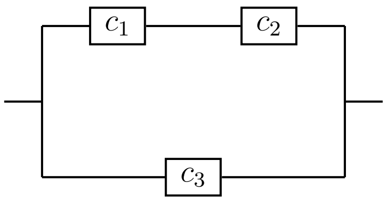
Answer: From the data we could assume for the events A="student attends the exercises regularly", B="student fails exam" the following probabilities

$$P(A) = 0.1, P(A \mid B) = 0.02, P(B \mid A) = 0.05$$

$$\begin{aligned} P(B) &= \frac{P(B \mid A)P(A)}{P(A \mid B)} = \frac{0.05 \cdot 0.2}{0.02} = 0.25 \\ P(B \mid A^c) &= \frac{P(B)P(A^c \mid B)}{1 - P(A^c)} = P(B)\frac{1 - P(A \mid B)}{1 - P(A)} \\ &= 0.25 \cdot \frac{1 - 0.02}{1 - 0.1} \approx 0.272 \end{aligned}$$

Since $P(B \mid A) = 0.05 < P(B \mid A^c) \approx 0.272$ is the probability to fail of an untrained student more than 5 times higher as the probability of a trained student.

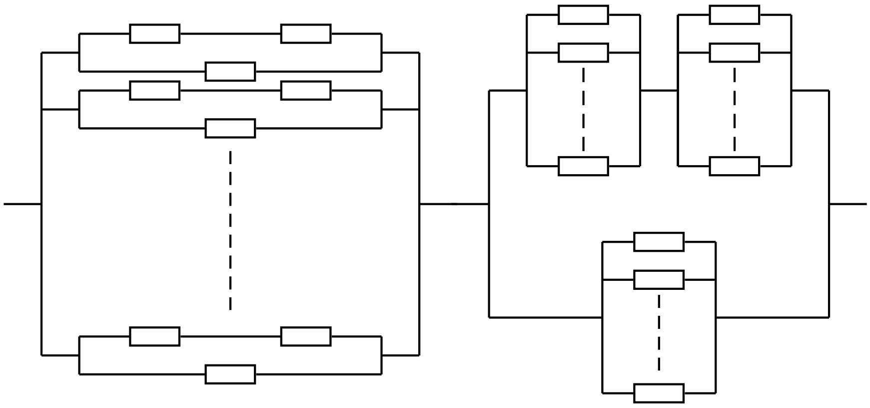
5. A representation of the reliability of a system with 3 components c_1, c_2, c_3 is given by.



(a) Find the structure function of the system.

(b) Let α_i the failure probability of c_i, i=1,2,3. Find the failure probability and the reliability of the system.

(c) Let $\alpha_i = 0.9, i = 1, 2, 3$. Compare the reliability of the following systems:



Answer:

(a) $x_i = \begin{cases} 1 & c_i \text{ is functioning} \\ 0 & c_i \text{ has failed} \end{cases} \Rightarrow$
$\Phi(x_1, x_2, x_3) = 1 - (1 - x_3)(1 - x_1 x_2)$

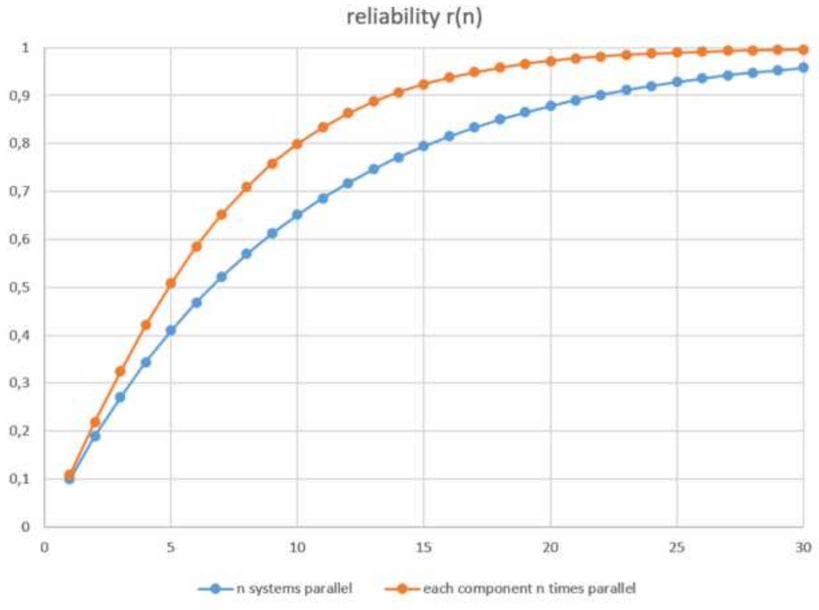
(b) P(system fails)=$\alpha = \alpha_3(1 - (1 - \alpha_1)(1 - \alpha_2))$, reliabilty of the system $r = 1 - \alpha$

(c) A: n parallel systems
With $\alpha_i = 0.9, i = 1, 2, 3$ the failure probability of one system is $\alpha = 0.9(1 - (1 - 0.9)(1 - 0.9)) = 0.891$. $\Rightarrow$

$$r_A(n) = 1 - \alpha^n = 1 - 0.891^n$$

B: each component n times parallel
The failure probability of n parallel components c_1 is $\alpha_n = 0.9^n$ for every i=1,2,3. $\Rightarrow$

$$r_B(n) = 1-\alpha_n(1-(1-\alpha_n)^2) = 1-2\alpha_n^2+\alpha_n^3 = 1-2\cdot 0.9^{2n}+0.9^{3n}$$

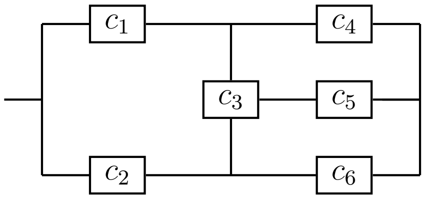


The diagram shows that $r_b(n) \geq r_A(n)$ for $n \leq 30$.

```
#########################################################
# Probabilit: Exercise Reliability
# Greiner / Tinhofer S.52
#
# File: prob_cond_rel_3.R
#
##########################################################

library(tidyverse)

# A representation of the reliability of a system with 3 components
# c1, c2, c3 is given.
# Find the structure function of the system.
# Let alpha_i the failure probability of Ci, i=1,2,3. Find the failure
# probability and the reliability of the system.
# Let alpha_i=0.9, i=1,2,3. Compare the reliability of the the following
# systems:
#  - n parallel system
#- each component n times parallel

# probabilities that component i fails
alpha1 <- 0.9
alpha2 <- 0.9
alpha3 <- 0.9
# probability that the system fails
alpha <- alpha3 * (1-(1-alpha2)*(1-alpha2))
# reliability of the system
r <- 1-alpha

n <- 10
# reliability of the n parallel systems
rnpar <- 1-alpha**n
# reliability of the each component n times parallel system
rncom <- 1-alpha**n*(1-(1-alpha**n)^2)

# all alpha identical = 0.9 !!!
values <- tibble(
  anz = 1:30,
  npar = 1-alpha**anz,
  ncom = 1-alpha**anz*(1-(1-alpha**anz)^2)
)
# display
values

# graphic output
plot(x=values$anz, y=values$npar, type="l",col="blue",
     xlab="no of components", ylab="reliability",
```

```
      main="reliability of different systems",
      sub="n parallel systems = blue, every comp. n times parallel = red")
lines(x=values$anz, y=values$ncom, col="red")
# gather columns npar, ncom
#values <- values %>%
#   gather(npar, ncom, key = "system",value = "reliability")
#ggplot(values) +
#   geom_smooth(mapping = aes(x=anz,y=reliability,group=system,
#                              color=system))
```

6. The reliability of a system with six components is described in the following diagram.



(a) Find the structure function of the system.

(b) Let p_i the probability that component c_i is functioning.

i	1	2	3	4	5	6
p_i	0.8	0.9	0.6	0.6	0.7	0.8

What is the reliability of the system?

Answer:

(a) The minimal path sets of the system are

$$\{c_1, c_4\}, \{c_1, c_3, c_5\}, \{c_1, c_3, c_6\}, \{c_2, c_6\}, \{c_2, c_3, c_5\}, \{c_2, c_3, c_4\} \Rightarrow$$

$$\begin{aligned} \Phi(x) &= 1 - (1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_1x_3x_6) \\ &\quad (1 - x_2x_6)(1 - x_2x_3x_5)(1 - x_2x_3x_4) \end{aligned}$$

(b) Let A3="component c_3 is functioning" and S="System is functioning".

$$\begin{aligned} r &= P(S \mid A3)P(A3) + P(S \mid A3^c)P(A3^c) \\ P(S \mid A3) &= (1 - (1 - p_1)(1 - p_2))(1 - (1 - p_4)(1 - p_5)(1 - p_6)) = 0.95648 \\ P(S \mid A3^c) &= 1 - (1 - p_1p_4)(1 - p_2p_6) = 0.8544 \\ \Rightarrow r &= 0.95648 \cdot 0.6 + 0.8544 \cdot 0.4 = 0.915648 \end{aligned}$$