Course of Study Bachelor Computer Science	Exercises Statistics WS 2020/21

Sheet IX - Solutions

Hypothesis Testing

1. A tire manufacturer claims that its tires will last no less than an average of 50,000 km before they need to be replaced. A consumer group wishes to challenge this claim.

   (a) Clearly define the parameter of interest in this problem.

   (b) State H_0 and H_1 in terms of this parameter.

   (c) In the context of the problem, state what it means to make a type I and type II error.

   (d) Suppose we set the significance level of the test at 10%, what does this number mean?

   Answer:

   (a) Parameter: μ = the average life span (in km) of this manufacturer's tires.

   (b) $H_o : \mu >= 50, H_1 : \mu < 50$

   (c) Type I error: rejecting the manufacturer's claim when in fact it is true.
   Type II error: not rejecting the manufacturer's claim when in fact it is false.

   (d) A ten percent significance level means that we are setting an upper limit of 10% for the probability of making type I error.

2. Discus the following statement:
   "When test results are significant at the 5-percent level, this means that there is at least a 95% chance of being correct if you reject the null hypothesis."

   Answer: The statement is false. The statement "...there is at least a 95% chance of being correct if you reject the null hypothesis" is equivalent to saying "... there is at least a 95% chance that the null

hypothesis is true". The significance level does not tell us the precise probability that the hypothesis is true: That probability depends on more than just the data - it depends for example on the reputation of the person making the statement and what our general background knowledge suggests.
The significance level (p-value) of the data tells us how likely it would be for us to see such data, if we live in a world where the null hypothesis is defintely true. But there is no direct correspondens between the p-value and the probability that the hypothesis is true.
From Bayes‘ Rule we get:

P(statement is true | we see this data) =
P(we see this data | statement is true) x P(statement is true)
/ P(we see this data)

The last two probabilities both depend on our prior beliefs,

3. A sample of lightbulbs is studied, to test the hypothesis that the mean lifetime of the bulbs is 200 hours. The sample data has a significance level of 1%, i.e. the hypothesis is rejected with significance level of 1%.

   Is the following statement true or false: If the mean lifetime in the population is indeed 200 hours, then a second sample (of the same size, analyzed similarly) has only one chance in a hundred of yielding a sample mean as far from 200 as the first sample mean.

   Answer: True. The significance level of the data (with respect to the null hypothesis) is the probability that, in a world where the null hypothesis is true, if we were to carry out the procedure we just carried out, we would see data at least as contradictory to the null hypothesis as the data we are, in fact, seeing.

4. A vaccine that is currently used to immunize people against a certain infection has an 80% success rate. That is, 80% of individuals who receive this vaccine will develop immunity against the infection. A manufacturer of a new vaccine claims that its vaccine has a higher success rate.

   (a) Define the parameter of interest.

   (b) Suppose in a clinical trial, 200 people received the new vaccine. Of these, 172 became immune to the infection. Based on this, can we say that the new vaccine is indeed more effective than the current one? What is the corresponding Null-Hypothesis? Test at a 5% significance level and state your conclusion in the context of the problem.

(c) In making the above conclusion, which type of error are you risking, type I or type II?

(d) What is the probability of a type II error if the true success rate is 82%? What should be the minimal sample size if the probabiility of the type II error should be less than 5%?

Find the answers by using a normal approximation resp. without a normal approximation.

Answer:

(a) Parameter: p = the proportion of individuals who will develop immunity against the infection after being inoculated with the new vaccine.

(b) Data: $\begin{array}{lcl} n & = & 200 \\ 172 \text{ immune} & \Rightarrow & \hat{p} = \frac{172}{200} = 0.86 \\ \alpha & = & 5\% \end{array}$, $\begin{array}{ll} H_0: & p \leq 0.8 \\ H_1: & p > 0.8 \end{array}$

Test statistic: $Z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$ is approximatively $N(0,1)$-distributed for large n.

From the sample: $z = \frac{0.86-0.8}{\sqrt{\frac{0.8(1-0.8)}{200}}} \approx 2.121$

For 2.121 the normal distribution table gives the value 0.9830, i.e. for a one-sided test the p-value is $p = 1 - 0.9830 = 0.0170$

Decision: Since the p-value 0.0170 is smaller than the given significance level of 0.05, the null hypothesis is rejected. We can conclude that the new vaccine is indeed more effective than the current one.

Alternative solution:

$\alpha = 0.05 \overset{\text{Norm. distribution table}}{\Rightarrow} 1.645$ and $2.121 > 1.645 \Rightarrow H_0$is rejected

(c) Since we reject H_0, we are risking making type I error.

(d) If $p_1 = 0.85$ the random variable $X = n \cdot \hat{p} \sim B(n, p_1)$ and $\hat{p} \sim N(p_1, \frac{p_1(1-p_1)}{n})$ approximately. Thus we get

$$\beta_{exact} = P(n\hat{p} \leq 95\%\text{quantile of } B(n, p_0)) \approx 0.451$$

Using a normal approximation for the test we get

$$\beta_{approx} = P(\hat{p} \leq u_{1-\alpha} \cdot \sqrt{\frac{p_0(1-p_0)}{n}} + p_0) \approx 0.451$$

$$\beta_{approx} \leq 0.05 \quad \Leftrightarrow \quad \Phi(\frac{u_{1-\alpha} \cdot \sqrt{\frac{p_0(1-p_0)}{n}} + p_0 - p_1}{\sqrt{\frac{p1(1-p_1)}{n}}}) \leq 0.05$$

$$\Leftrightarrow \quad \frac{u_{1-\alpha} \cdot \sqrt{\frac{p_0(1-p_0)}{n}} + p_0 - p_1}{\sqrt{\frac{p1(1-p_1)}{n}}} \leq u_\alpha.$$

By $u_\alpha = -u_{1-\alpha}$ we get

$$n \geq \left(\frac{u_{1-\alpha}(\sqrt{p_0(1-p_0)} + \sqrt{p1(1-p_1)})}{p_1 - p_0}\right)^2 \approx 620.28$$

```
#################################################################
# A vaccine that is currently used to immunize people against a
# certain infection has an 80% success rate. That is, 80% of
# individuals who receive this vaccine will develop immunity
# against the infection. A manufacturer of a new vaccine claims
# that its vaccine has a higher success rate.
#################################################################

# parameter of interest: success rate p

# Suppose in a clinical trial, 200 people received the new
# vaccine. Of these, 172 became immune to the infection. Based on
# this, can we say that the new vaccine is indeed more effective
# than the current one?
# What is the corresponding Null-Hypothesis?
# Null Hypothesis H0: p<=p0   against H1: p>p0
p0 <- 0.8
# Test at a 5% significance level and state your conclusion in the
# context of the problem.
n <- 200
p <- 172/n
alpha <- 0.05
# normal approximation
test_statistic <- (p-p0)/sqrt(p0*(1-p0)/n)
# reject if test_statistic > qnorm(1-alpha)
test_statistic > qnorm(1-alpha) # 2.12132 > 1.644854
# reject H0, thus the new vaccine seems to be better than the old one
p_value_app <- 1-pnorm(test_statistic) # 0.01694743
# exact test
binom.test(172,n,alternative = "greater")
p_value <- 1-pbinom(n*p,n,p0)
p_value # = 0.01095077 < 0.05, i.e. rejection

# In making the above conclusion, which type of error are you
# risking, type I or type II?
# Since we reject H0, we are risking making type I error.

# What is the probability of a type II error if the true success
# rate is 85%?
p1 <- 0.85
# beta.approx = P(test_statistic <= qnorm(1-alpa)) <=>
# beta.approx = P(p <= qnorm(1-alpa)*(p0(1-p0)/n)^0.5+p0)
# p ~ N(p1,p1(1-p1)/n) approx.
beta.approx <- pnorm(qnorm(1-alpha)*(p0*(1-p0)/n)^0.5+p0,
                     mean = p1, sd = (p1*(1-p1)/n)^0.5)
# beta.ex = P(np <= 95% quantile of B(n,p0) with n*p ~ B(n,p1)
beta_exact <- pbinom(qbinom(1-alpha,size = n, prob = p0),
                     size = n, prob = p1)

# What should be the minimal sample size if the probability
# of the type II error should be less than 5%?
library(tidyverse)
tibble(
  n = 200:1000,
  b.ex = pbinom(qbinom(1-alpha,size = n, prob = p0),
                size = n, prob = p1),
  b.approx = pnorm(qnorm(1-alpha)*(p0*(1-p0)/n)^0.5+p0,
                   mean = p1, sd = (p1*(1-p1)/n)^0.5)
```

```
) %>% filter(b.approx <= 0.05) %>% filter(n == min(n))

# direct determination of n using a normal approximation
(qnorm(0.95)*(sqrt(p0*(1-p0))+sqrt(p1*(1-p1)))/(p1-p0))^2
```

5. A magician uses a coin. You believe that the coin is biased, but you are not sure if it will come up heads or tails more often. You watch the magician flip the coin and record what percentage of the time the coin comes up heads.

   (a) Is this a one-tailed or two-tailed test?

   (b) Assuming that the coin is fair, what is the probability that out of 30 flips, it would come up one side 23 or more times?

   (c) Can you reject the null hypothesis at the 0.05 level? What about at the 0.01 level?

   Answer: a) two tailed, b) 0.005222879, c) rejection at both levels

   (a) Coin, which we believe is biased. We make a two-sided test with
   $$\begin{cases} H_0: & p = \frac{1}{2} \\ H_1: & p \neq \frac{1}{2} \end{cases}$$

   (b) Assumption: Coin is not biased with $p = \frac{1}{2}$
   Wanted: The probability that for 30 flips, 23 times or more a special side occurs. Let X = Number of heads for 30 flips. Then we have $X \sim B(n = 30, p = \frac{1}{2})$
   We want $P(X \geq 23$ oder $X \leq 7)$ and have
   $P(X = i) = \binom{30}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{30-i} = \binom{30}{i}\left(\frac{1}{2}\right)^{30}$ for $i = 0, 1, 2, \ldots, 30$.
   We become
   $P(X = 0) + \ldots + P(X = 7) + P(X = 23) + \ldots + P(X = 30) \approx 0.005223$
   Reject H_0, if $X \geq o$ or $X \leq u$. Since the p-value $0.005223 \leq \alpha = 0.05\ ;\ 0.01$, we have rejection at both levels.

6. A bag of potato chips of a certain brand has an advertised weight of 250 grams. Actually, the weight (in grams) is a random variable. Suppose that a sample of 81 bags has mean 248 and standard deviation 5. At the 0.05 significance level, conduct the following tests and calculate the p-values.

   (a) $H_0: \mu \geq 250$ versus $H_1: \mu < 250$

   (b) $H_0: \sigma \geq 7$ versus $H_1: \sigma < 7$

Hint: Assume that the data is approximately normally distributed.

Answer: Data:
$$\begin{aligned} n &= 81 \\ \bar{x} &= 248 \\ s &= 5 \\ \alpha &= 0.05 \\ X &= \text{ Weight} \end{aligned}$$

(a) $\begin{array}{ll} H_0: & \mu \geq 250 \\ H_1: & \mu < 250 \end{array}$, Test statistic: $t = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$

From the sample: $t_{beob} = \frac{248-250}{\frac{5}{\sqrt{81}}} = -3.6$

With $\alpha = 0.05$ and a one-sided test, we become the following rejection region from the normal distribution table: $(-\infty, -t_{n-1,1-\alpha}) = (-\infty, -1.6441)$

We have $-3.6 < -1.6441$ i.e. H_0 is rejected. The p-value is given by $P_{H_0}(t \leq t_{beob}) \approx 0.0002750739$.

(b) $\begin{array}{ll} H_0: & \sigma \leq 7 \\ H_1: & \sigma > 7 \end{array}$, Test statistic: $t = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

From the sample: $t_{beob} = \frac{80 \cdot 5^2}{7^2} = 40.81633$

With $\alpha = 0.05$ and a one-sided test, we become the following rejection region from the χ^2-distribution table:
$(0, \chi^2_{n-1,\,\alpha}) = (0,\ \chi^2_{80,\,0.05}) = (0,\ 60.39148)$

We have $40.81633 < 60.39148$ i.e. H_0 is rejected. The p-value is given by $P_{H_0}(t \leq t_{beob}) \approx 8.081861e - 05$.

```
###################################################################
# A bag of potato chips of a certain brand has an advertised weight
# of 250 grams. Actually, the weight (in grams) is a random variable.
# Suppose that a sample of 81 bags has mean 248 and standard
# deviation 5. At the 0.05 significance level, conduct the following
# tests and calculate the p-values.
# a) H_0: mu >= 250 versus H_1: mu < 250
# b) H_0: sigma >= 7 versus H_1: sigma < 7
####################################################################

n <- 81
alpha <- 0.05
mean.sample <- 248
sd.sample <- 5
mean.0 <- 250
sd.0 <- 7

# a) t-test
tstat_a <- (mean.sample-mean.0)*sqrt(n)/sd.sample
tstat_a # -3.6
qt(1-alpha,n-1) # 1.664125
pvalue_a <- pt(tstat_a, df = n-1)
pvalue_a # 0.0002750739

# b)
tstat_b <- (n-1)*sd.sample^2/sd.0^2
tstat_b # 40.81633
qchisq(alpha,n-1) # 60.39148
pvalue_b <- pchisq(tstat_b, df = n-1)
pvalue_b # 8.081861e-05
```

7. The length of a certain machined part is supposed to be 10 centimeters. In fact, due to imperfections in the manufacturing process, the actual length is a random variable. The standard deviation is due to inherent factors in the process, which remain fairly stable over time. From historical data, the standard deviation is known with a high degree of accuracy to be 0.3. The mean, on the other hand, may be set by adjusting various parameters in the process and hence may change to an unknown value fairly frequently. We are interested in testing

$$H_0 : \mu = 10 \quad \text{versus} \quad H_1 : \mu \neq 10$$

   (a) Suppose that a sample of 100 parts has mean 10.1. Perform the test at the 0.1 level of significance.
   (b) Compute the p-value for the data.
   (c) Compute the probability of a type II error β of the test at $\mu = 10.05$.
   (d) Compute the approximate sample size needed for significance level 0.1 and beta = 0.2 when mu = 10.05.
   (e) Plot the probability of a type II error depending on the value of μ for different values of the sample size n = 50, 100, 150, 200, 250.
   (f) Show that H_0 will be not rejected if the sample mean is 10.01. Determine the smallest n that the p-value for a sample with sample mean = 10.01 is less than 0.001. In general by increasing the sample size every small sample mean will become "highly significant - p value < 0.001".

Hint: Assume that the data is approximately normally distributed.
Answer:

(a) Data: $\begin{array}{rcl} n & = & 100 \\ \bar{x} & = & 10.1 \\ \sigma & = & 0.3 \\ \alpha & = & 0.10 \end{array}$, $\begin{array}{ll} H_0 : & \mu = 10 \\ H_1 : & \mu \neq 10 \end{array}$

Test statistic: $T = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

From the sample: $t_{beob} = \frac{10.1-10}{\frac{0.3}{\sqrt{100}}} = 3.33$

With $\alpha = 0.10$ and a two-sided test, we become the following rejection region from the normal distribution table:
$(-\infty, -u_{1-\frac{\alpha}{2}}) \cup (u_{1-\frac{\alpha}{2}}, \infty) = (-\infty, -1.6449) \cup (1.6449, \infty)$
We have $3.33 > 1.6449$ i.e. H_0 is rejected.

(b) $p = P(|T| \geq 3.33) = 2(1 - \Phi(3.33)) = 2(1 - 0.99957) = 0.00086$ with the value $\Phi(3.33) = 0.99957$ from the normal distribution table.

(c) $P_{\mu=10.05}\left(\left|\frac{\bar{X}_n-\mu_0}{\sigma_0/\sqrt{n}}\right| \leq u_{1-\alpha/2}\right) =$

$= P_{\mu=10.05}\left(\mu_0 - \frac{u_{1-\alpha/2}\sigma_0}{\sqrt{n}} \leq \bar{X}_n \leq \mu_0 + \frac{u_{1-\alpha/2}\sigma_0}{\sqrt{n}}\right) =$

$= \Phi\left(\frac{10+\frac{u_{1-\alpha/2}\cdot 0.3}{10}-10.05}{0.3/10}\right) - \Phi\left(\frac{10-\frac{u_{1-\alpha/2}\cdot 0.3}{10}-10.05}{0.3/10}\right) \approx 0.491$ with $\mu_0 = 10, \sigma_0 = 0.3$

```
######################################################################
# The length of a certain machined part is supposed to be 10
# centimeters. In fact, due to imperfections in the manufacturing
# process, the actual length is a random variable. The standard
# deviation is due to inherent factors in the process, which remain
# fairly stable over time. From historical data, the standard
# deviation is known with a high degree of accuracy to be 0.3. The
# mean, on the other hand, may be set by adjusting various parameters
# in the process and hence may change to an unknown value fairly
# frequently. We are interested in testing
# H_0: mu = 10 versus H_1: mu <> 10
# resp.
# H_0: mu <= 10 versus H_1: mu > 10
#
# Hint: Assume that the data is approximately normally distributed.
######################################################################
library(tidyverse)
# H_0: mu = 10 versus H_1: mu <> 10
sigma <- 0.3
hyp.mean <- 10

# a) Suppose that a sample of 100 parts has mean 10.1. Perform the
# test at the 0.1 level of significance.
sample.size <- 100
sample.mean <- 10.1
alpha <- 0.1
# Test statistic
T <- (sample.mean - hyp.mean)*sqrt(sample.size)/sigma
# rejection region: T < lb or T > ub
qnorm(c(alpha/2,1-alpha/2)) # -1.64, 1.64 -> reject H_0

# b) Compute the p-value for the data.
1-pnorm(abs(T)) + pnorm(-abs(T))

# c) Compute the probability of a type II error beta of the
# test at mu =10.05.
beta <- function(mu1,mu0,n,alpha,sigma) {
  # rejection bounds
  lb <- mu0-sigma*qnorm(1-alpha/2)/sqrt(n)
  ub <- mu0+sigma*qnorm(1-alpha/2)/sqrt(n)
  # prob. of type II error
  beta <- pnorm(ub, mean = mu1, sd = sigma/sqrt(n)) -
    pnorm(lb, mean = mu1, sd = sigma/sqrt(n))
  return(beta)
}
beta(10.05, hyp.mean,sample.size,alpha, sigma)

# d) Compute the approximate sample size needed for significance
# level 0.1 and beta = 0.2 when mu = 10.05.
tibble(
  n = 100:500,
  p.II = beta(10.05, hyp.mean,n,alpha, sigma)
) %>% filter(p.II <= 0.2)

# approximated value
((qnorm(1-alpha/2)-qnorm(0.2))*sigma/(10.05-hyp.mean))^2

# e) plot the prob. of a type II error depending on the value of mu for different
# values of the sample size n = 50, 100, 150, 200, 250
plot(x=seq(9.8,10.2,by=0.005),
     y=beta(seq(9.8,10.2,by=0.005), hyp.mean,sample.size,alpha, sigma),
     type="l",
     main = "probability of a type II error",
     sub = "blue n=50, black n=100, red n=150,200,250",
```

```
     xlab = "mu", ylab = "beta")
lines(x=seq(9.8,10.2,by=0.005),
      y=beta(seq(9.8,10.2,by=0.005), hyp.mean,50,alpha, sigma),
      type="l", col="blue")
lines(x=seq(9.8,10.2,by=0.005),
      y=beta(seq(9.8,10.2,by=0.005), hyp.mean,150,alpha, sigma),
           type="l", col="red")
lines(x=seq(9.8,10.2,by=0.005),
      y=beta(seq(9.8,10.2,by=0.005), hyp.mean,200,alpha, sigma),
      type="l", col="red")
lines(x=seq(9.8,10.2,by=0.005),
      y=beta(seq(9.8,10.2,by=0.005), hyp.mean,250,alpha, sigma),
      type="l", col="red")

# f) Show that the H_0 will be not rejected if the sample mean is 10.01. Determine
# the smallest n for the p-value for a sample mean 10.01 is less than 0.001.
# In general by increasing the sample size every small sample mean will become "highly
# significant - p value < 0.001"
pvalue <- function(n,s.mean) {
  # value of the test statistic
  TS <- (s.mean - hyp.mean)*sqrt(n)/sigma
  return(1-pnorm(abs(TS))+pnorm(-abs(TS)))
}
# p values for the sample means 10.1 and 10.01 are 0.0008581207 and 0.7388827
pvalue(sample.size,10.1)
pvalue(sample.size,10.01)
df.pvalue <- tibble(
  n = sample.size:10000,
  p = pvalue(n,10.01)
)
df.pvalue %>% filter(p <= 0.001) # n >= 9745

# exact
(30*qnorm(0.9995))^2
```

8. A coin is tossed 500 times and results in 302 heads. At the 0.05 level, test to see if the coin is unfair.

Answer: Data: $\begin{array}{rcl} n &=& 500 \\ \hat{p} &=& \frac{302}{500} \\ \alpha &=& 5\% \end{array}$, $\begin{array}{ll} H_0: & p = 0.50 \\ H_1: & p \neq 0.50 \end{array}$

Test statistic: $Z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$ is approximately $N(0,1)$-distributed for large n.

From the sample: $z = \frac{\frac{302}{500}-0.5}{\sqrt{\frac{0.5(1-0.5)}{500}}} = 4.6510$

With $\alpha = 0.05$ and a two-sided test, we become the following rejection region from the normal distribution table: $(-\infty, -u_{1-\frac{\alpha}{2}}) \cup (u_{1-\frac{\alpha}{2}}, \infty) = (-\infty, -1.96) \cup (1.96, \infty)$

We have $4.65 > 1.96$ i.e. H_0 is rejected.

```
######################################################################
# A coin is tossed 500 times and results in 302
# heads. At the 0.05 level, test to see if the
# coin is unfair.
#
# file: infstat_testing_coin_tosses.R
#######################################################################
n <- 500
h <- 302
p0 <- 0.5
p <- h/n
alpha <- 0.05

test_statistic <- (p-p0)/(p0*(1-p0)/n)^0.5
test_statistic # 4.651021

lb_rejection_region <- -qnorm(1-alpha/2,0,1)
ub_rejection_region <- qnorm(1-alpha/2,0,1)
```

```
lb_rejection_region; ub_rejection_region
# -1.959964; 1.959964

test_statistic < lb_rejection_region ||
  test_statistic > ub_rejection_region
# TRUE -> reject H0
```

9. The processing time L of a stochastic searching algorithm at a workstation could be assumed to be normally distributed with expectation μ and variance $\sigma^2 = 40 sec^2$. The average of 50 independent measurements of the processing time is $\bar{l} = 121.9$ sec.

   (a) Find a suitable test which guarentees that the expectation is assumed erroneously to be bigger than 120 sec is at most 5%.

   (b) Perform the above test.

   (c) What is the upper bound of $\bar{l}$ in the above test that the null hypothesis will be not rejected?

   (d) What should be the value of the significance level α in the above test that the null hypothesis will be not rejected for the given value of $\bar{l}$?

   (e) Assume that the true expectation μ is bigger than 122 sec. Find the lowest value of the sample size n that in the above test the probability to do not reject the null hypothesis is less than 5%?

   (f) Sketch the the OC-function β of the above for n=25,50, 100.

   Answer: $L \sim N(\mu, 40), \bar{l} = \frac{1}{50}\sum_{i=1}^{50} l_i = 121.9$

   (a) $H_0 : \mu \leq 120 \quad H_1 : \mu > 120$
   with teststatistic $T = \frac{\sqrt{n}}{\sigma}(\bar{L} - \mu_0) \sim N(0, 1)$ and rejection region $T > u_{1-\alpha} = u_{0.95}$

   (b) Plugging in the concrete values we get $T = \frac{\sqrt{50}}{\sqrt{40}}(121.9 - 120) \approx 2.124$. Since $T > u_{0.95} = 1.64$ the null hypothesis must be rejected.

   (c) For $T = u_{0.95}$ we get $\bar{l} = 120 + \frac{u_{0.95}\sqrt{40}}{\sqrt{50}} \approx 121.467$.

   (d) For $T = 2.124 = u_{1-\alpha}$ we get $\Phi(2.124) = 1 - \alpha$, i.e. $\alpha \approx 0.017$.

   (e) Let $T = \frac{\sqrt{n}}{\sqrt{40}}(\bar{L} - 120)$ with $\bar{L} \sim N(\mu, 40/n), \mu > 122$.

$$P(T \leq u_{0.95}) \leq 0.05 \Leftrightarrow P(\bar{L} \leq 120 + \frac{\sqrt{40}}{\sqrt{n}} u_{0.95}) = \Phi\left(\frac{120 + \frac{\sqrt{40}}{\sqrt{n}} u_{0.95} - \mu}{\sqrt{40}/\sqrt{n}}\right) \leq 0.05$$

$$\frac{120 + \frac{\sqrt{40}}{\sqrt{n}} u_{0.95} - \mu}{\sqrt{40}/\sqrt{n}} \leq u_{0.05} = -u_{0.95} \Rightarrow n \geq \left(\frac{2\sqrt{40} u_{0.95}}{\mu - 120}\right)^2$$

The lower bound for n is maximal for $\mu = 122$. Thus $n \geq 108$.

(f) OC function

$$\beta(\mu, n) = P(T \leq u_{0.95}) = P(\bar{L} \leq 120 + \frac{\sqrt{40}}{\sqrt{n}} u_{0.95})$$
$$= \Phi\left(\frac{120 + \frac{\sqrt{40}}{\sqrt{n}} u_{0.95} - \mu}{\sqrt{40}/\sqrt{n}}\right) = \Phi(u_{0.95} - \frac{\sqrt{n}}{\sqrt{40}}(\mu - 120))$$

If $\mu \geq 120$ we get $\beta(\mu, n_1) \leq \beta(\mu, n_2)$ for $n_1 > n_2$. Furthermore we have $\beta(120, n) = \Phi(u_{0.95}) = 0.95$ for every n.

```
#####################################################################
# The processing time L of a stochastic searching algorithm at
# a workstation could be assumed to be normally distributed with
# expectation mu and variance sigma^2 = 40 sec^2. The average of
# 50 independent measurements of the processing time is
# l = 121.9 sec.
#####################################################################

n <- 50
sigma <- 40^0.5
l <- 121.9

# a) Find a suitable test which guarentees that the expectation is
# assumed erroneously to be bigger than 120 sec is at most 5%.
# H0: mu <= 120
l0 <- 120
alpha <- 0.05
test_statistic <- (l-l0)*n^0.5/sigma
ub_rejection_region <- qnorm(1-alpha,0,1)

# b) Perform the above test.
test_statistic; ub_rejection_region
# 2.124265; 1.644854
test_statistic > ub_rejection_region
# TRUE -> rejct H0

# c) What is the upper bound of l in the above test that the null
# hypothesis will be not rejected?
ub_l <- l0+ub_rejection_region*sigma/n^0.5
ub_l # 121.4712

# d) What should be the value of the significance level alpha in the
# above test that the null hypothesis will be not rejected for the
# given value of l?
1-pnorm(test_statistic,0,1) # 0.01682401

# e) Assume that the true expectation mu is bigger than 122 sec.
# Find the lowest value of the sample size n that in the above test
# the probability to do not reject the null hypothesis is less than 5%?
lb_n <- (2*sigma*ub_rejection_region/(122-l0))^2
lb_n # 108.2217

# f) Sketch the the OC-function beta of the above for n=25,50,100.
oc <- function(mu,n) pnorm(
  ub_rejection_region- n^0.5*(mu-l0)/sigma)

seq(from=100,to=110,by=0.5)
library(tidyverse)
data <- tibble(
  mu = seq(from=110,to=130,by=0.25),
  oc_25 = oc(mu,25),
  oc_50 = oc(mu,50),
  oc_100 = oc(mu,100)
)
plot(x=data$mu, y=data$oc_25, col="red",
     type = "l",
     xlab = "mu", ylab = "oc",
     main = "OC-Functions",
     sub="red n=25, black n=50, green n=100")
lines(x=data$mu, y=data$oc_50, col="black")
lines(x=data$mu, y=data$oc_100, col="green")
```