| Course of Study Bachelor Computer Science | Exercises Statistics WS 2023/24 |
|---|---|
| Sheet V - Solutions | |

# Probability Spaces and Basic Rules

1. Consider a random experiment of tossing two dice. Let $A$ denote the event that the first die score is 1 and $B$ the event that the sum of the scores is 7.

   (a) Give the sample space $\Omega$ and find $|\Omega|$.

   (b) Explicitly list the elements of the following events:

   $$A, B, A \cup B, A \cap B, A^c \cap B^c$$

   **Answer:**

   (a) $\Omega = \{(i, j) | i, j \in \{1, 2, 3, 4, 5, 6\}\}$, $|\Omega| = 36$

   (b) $A = \{(1, 1, ), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$
   $B = \{(1, 6, ), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
   $A \cap B = \{(1, 6)\}$
   $A \cup B = \{(1, 1, ), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
   $A^c \cap B^c = \Omega \setminus (A \cup B)$

```
##################################################################
# Rolling 2 dice: sample space and events
#
# file: prob_basics_rolling_2_dices.R
##################################################################
library(tidyverse)
# create sample space
Omega <- expand.grid(x=1:6, y=1:6) %>% as_tibble()
Omega
# A = first die = 1, B = sum of the scores is 6
A <- Omega %>% filter(x==1)
A
B <- Omega %>% filter(x+y==6)
B
# A and B
intersect(A,B)
# A or B
union(A,B)
# not A and not B
intersect(setdiff(Omega,A), setdiff(Omega,B))
setdiff(Omega, union(A,B))
```

2. Suppose that $A$ and $B$ are events in an experiment with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cap B) = 1/10$. Express each of the following events verbally and find its probability:

$$A \setminus B, A \cup B, A^c \cup B^c, A^c \cap B^c, A \cup B^c$$

**Answer:**

(a) $A \setminus B$: The event A but not B occures.
$P(A \setminus B) = P(A \cap B^C) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$

(b) $A \cup B$: One or both of the events A and B occure.
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{10} = \frac{29}{60}$

(c) $A^c \cup B^c$: Both events do not occure.
$P(A^c \cup B^c) = P(A^C \cup B^C) = P((A \cap B)^C) = 1 - P(A \cap B) = 1 - \frac{1}{10} = \frac{9}{10}$

(d) $A^c \cap B^c$: None of the events A and B occure.
$P(A^c \cap B^c) = P(A^C \cap B^C) = P((A \cup B)^C) = 1 - P(A \cup B) = 1 - \frac{29}{60} = \frac{31}{60}$

(e) $A \cup B^c$: only B does not occure ($A \cup B^c = (B \setminus A)^c$).
$P(A \cup B^c) = 1 - (P(B) - P(A \cap B)) = 1 - \frac{1}{4} + \frac{1}{10} = \frac{17}{20}$

3. Suppose that $A$, $B$, and $C$ are events in an experiment with $P(A) = 0.3$, $P(B) = 0.2$, $P(C) = 0.4$, $P(A \cap B) = 0.04$, $P(A \cap C) = 0.1$, $P(B \cap C) = 0.1$, $P(A \cap B \cap C) = 0.01$. Express each of the following events in set notation and find its probability:

(a) At least one of the three events occurs.

(b) None of the three events occurs.

(c) Exactly one of the three events occurs.

(d) Exactly two of the three events occur.

**Answer:**

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.3 + 0.2 + 0.4 - 0.04 - 0.1 - 0.1 + 0.01 = 0.67$

(b) $P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - 0.67 = 0.33$

(c) $P((A \cup B \cup C) \setminus ((A \cap B) \cup (A \cap C) \cup (B \cap C))) = P(A \cup B \cup C) - P(A \cap C) - P(A \cap B) - P(B \cap C) + 2P(A \cap B \cap C) = 0.67 - 0.1 - 0.04 - 0.1 + 2 \cdot 0.01 = 0.45$

(d) $P((A \cap B) \cup (A \cap C) \cup (B \cap C) \setminus (A \cap B \cap C)) = 0.04 + 0.1 + 0.1 - 3 \cdot 0.01 = 0.21$

4. **Urn Models**

A large number of discrete probability spaces can be traced back to so-called urn models. An urn contains n balls, which do not all have to be different. From these urns r balls are drawn with or without replacement. For the result of the drawing, the order or only the quantity of the drawn balls can be of importance.
Here an urn with 10 balls is considered. 5 of them are red, 3 balls are blue and 2 balls are green. 3 balls are drawn. The following 4 cases should be distinguished:

I Drawing with replacement with respect to the order

II Drawing with replacement without observing the order

III Drawing without replacement with respect to the order

IV Drawing without replacement without observing the order

Solve the following tasks.

(a) Determine a suitable event space $\Omega$ and its size to describe the random experiment.

(b) Determine the probabilities of all elementary events in $\Omega$ using a Laplace model, i.e. as a determination of the ratio of the number of favorable cases by the number of all cases. The probabilities are first determined by counting methods and then by using the R function permutations().
**Hint:** To determine the probabilities with R, assume that the n balls are numbered consecutively, i.e. they are distinguishable, and that the order is first observed in a drawing. Every r-variation of the numbers 1 to n is equally probable. Determine the set of all these drawings with permutations(). Then map each such drawing to the corresponding elementary event. By dividing the number of drawings belonging to an elementary event and the number of all drawings, you can obtain the corresponding probabilities.

**Answer:**

(a) Set of elementary events and their probabilities

i. Drawing with replacement with respect to the order

- $|\Omega| = 3^3 = 27$ number of 3-variations of a 3-set with possible repititions
- Every elementary event can be seen as a word of length 3 with 3 possible letters. The possible letters are given by the colours r, b, g of the balls in the urn. To calculate the probability of an elementary event we count how many r, b and g are in the word. If R, B resp. G are the numbers of the red, blue resp. green drawn balls, we get

$$P(R = i, B = j, G = k) = \frac{5^i \cdot 3^j \cdot 2^k}{10^3}$$

with $i + j + k = 3; i, j, k \geq 0$

| | event | prob |
|---|---|---|
| 1 | b b b | 0.027 |
| 2 | b b g | 0.018 |
| 3 | b b r | 0.045 |
| 4 | b g b | 0.018 |
| 5 | b g g | 0.012 |
| 6 | b g r | 0.030 |
| 7 | b r b | 0.045 |
| 8 | b r g | 0.030 |
| 9 | b r r | 0.075 |
| 10 | g b b | 0.018 |
| 11 | g b g | 0.012 |
| 12 | g b r | 0.030 |
| 13 | g g b | 0.012 |
| 14 | g g g | 0.008 |
| 15 | g g r | 0.020 |
| 16 | g r b | 0.030 |
| 17 | g r g | 0.020 |
| 18 | g r r | 0.050 |
| 19 | r b b | 0.045 |
| 20 | r b g | 0.030 |
| 21 | r b r | 0.075 |
| 22 | r g b | 0.030 |
| 23 | r g g | 0.020 |
| 24 | r g r | 0.050 |
| 25 | r r b | 0.075 |
| 26 | r r g | 0.050 |
| 27 | r r r | 0.125 |

ii. Drawing with replacement without respect to the order

- $|\Omega| = \binom{3+3-1}{3} = \binom{5}{3} = 10$ number of 3-multisets from a 3-set
- Every elementary event can be seen as a 3-multiset of the letters r, b, g. Such a 3-multiset is given by a set of words of length 3 with R r's, B b' and G g's. All words of length 3 with the same numbers of r, b and g are equally like. For example

$$\begin{aligned} P(\{b, b, g\}) &= P(\{R = 0, B = 2, G = 1\}) \\ &= P(\{(b, b, g), (b, g, b), (g, b, b)\}) \\ &= P((b, b, g)) + P((b, g, b)) + P((g, b, b)) \\ &= \frac{3!}{0!2!1!} \cdot P((b, b, g)) \\ &= \frac{3!}{0!2!1!} \cdot \frac{5^0 \cdot 3^2 \cdot 2^1}{10^3} = 0.054 \end{aligned}$$

In general we get

$$P(\{R=i, B=j, G=k\}) = \frac{5^i \cdot 3^j \cdot 2^k}{10^3} \cdot \frac{3!}{i! \cdot j! \cdot k!}$$

with $i+j+k=3; i,j,k \geq 0$

| | event | prob |
|---|---|---|
| 1 | b b b | 0.027 |
| 2 | b b g | 0.054 |
| 3 | b b r | 0.135 |
| 4 | b g g | 0.036 |
| 5 | b g r | 0.180 |
| 6 | b r r | 0.225 |
| 7 | g g g | 0.008 |
| 8 | g g r | 0.060 |
| 9 | g r r | 0.150 |
| 10 | r r r | 0.125 |

iii. Drawing without replacement with respect to the order

- We must count the number of 3-variations with repitions and subtract the number of impossible events (here: 3 green balls)

$$|\Omega| = 3^3 - 1$$

- Regarding that now the drawn balls are not replaced we get analogousley to the case of drawing with replacement

$$P(R=i, B=j, G=k) = \frac{5^{\underline{i}} \cdot 3^{\underline{j}} \cdot 2^{\underline{k}}}{10^{\underline{3}}}$$

with $i+j+k=3; i,j,k \geq 0$. Mention that $n^{\underline{m}} = n \cdot \ldots \cdot (n-m+1)$ is the falling factorial and give the number of m-variations from a set of n distint elements.

| | event | prob |
|---|---|---|
| 1 | b b b | 0.008 |
| 2 | b b g | 0.017 |
| 3 | b b r | 0.042 |
| 4 | b g b | 0.017 |
| 5 | b g g | 0.008 |
| 6 | b g r | 0.042 |
| 7 | b r b | 0.042 |
| 8 | b r g | 0.042 |
| 9 | b r r | 0.083 |
| 10 | g b b | 0.017 |
| 11 | g b g | 0.008 |
| 12 | g b r | 0.042 |
| 13 | g g b | 0.008 |
| 14 | g g r | 0.014 |
| 15 | g r b | 0.042 |
| 16 | g r g | 0.014 |
| 17 | g r r | 0.056 |
| 18 | r b b | 0.042 |
| 19 | r b g | 0.042 |
| 20 | r b r | 0.083 |
| 21 | r g b | 0.042 |
| 22 | r g g | 0.014 |
| 23 | r g r | 0.056 |
| 24 | r r b | 0.083 |
| 25 | r r g | 0.056 |
| 26 | r r r | 0.083 |

iv. Drawing without replacement without respect to the order

- Regarding that now the drawn balls are not replaced we get analogousley to case drawing with replacement

$$P(\{R = i, B = j, G = k\}) = \frac{\binom{5}{i} \cdot \binom{3}{j} \cdot \binom{2}{k}}{\binom{10}{3}}$$

with $i + j + k = 3; i, j, k \geq 0$

| | event | prob |
|---|---|---|
| 1 | b b b | 0.008 |
| 2 | b b g | 0.050 |
| 3 | b b r | 0.125 |
| 4 | b g g | 0.025 |
| 5 | b g r | 0.250 |
| 6 | b r r | 0.250 |
| 7 | g g r | 0.042 |
| 8 | g r r | 0.167 |
| 9 | r r r | 0.083 |

```
##################################################################
# Urn Models
# Elementary events and their probabilities: direct
# determination
#
# file: prob_basics_urn_models_simple_version_without_gtools.R
##################################################################
# A large number of discrete probability spaces can be traced back to
# so-called urn models. An urn contains n balls, which do not all have
# to be different. From these urns r balls are drawn with or without
# replacement. For the result of the drawing, the order or only
# the quantity of the drawn balls can be of importance.
# Here an urn with 10 balls is considered. 5 of them are red, 3 balls
# are blue and 2 balls are green. 3 balls are drawn. The following 4
# cases should be distinguished:
# I Drawing with replacement with respect to the order
# II Drawing with replacement without observing the order
# III Drawing without replacement with respect to the order
# IV Drawing without replacement without observing the order
# a) Determine suitable event spaces and its sizes to describe the random
# experiment.
# b) Determine the probabilities of all elementary events in $\Omega$
# using a Laplace model, i.e. as a determination of the ratio of the
# number of favorable cases by the number of all cases.
# Hint: To determine the probabilities with R, assume that the
# n balls are numbered consecutively, i.e. they are distinguishable,
# and that the order is first observed in a drawing. Every
# r-variation of the numbers 1 to n is equally probable. Determine
# the set of all these drawings and map each drawing to the corresponding
# elementary event. By dividing the number of drawings belonging to an
# elementary event and the number of all drawings, you can obtain the
# corresponding probabilities.

library(tidyverse)

# Bag with 10 balls: 5*red, 3*blue und 2*green
bag <- rep(c("r","b","g"),c(5,3,2))
bag

# I: sampling r=3 balls with replacement regarding the order
sample.with.repl.ordered <- tibble(
  # construct all possible r-variations of 1,2,...,10
  x.1 = rep(1:100, each=100, length.out=10^3),
  x.2 = rep(1:10, each=10, length.out=10^3),
  x.3 = rep(1:10,length.out = 10^3)
) %>%
  # the following operations must be performed rowwise
  rowwise() %>%
  mutate(
    # maps the numbered balls to the corresponding colors
    event = c(bag[x.1],bag[x.2],bag[x.3]) %>% paste(collapse = "")
  ) %>%
  # regard only the events
  select(event) %>%
  # count the number of elementary events; all are equally like
  group_by(event) %>%
  mutate(
    prob = n()/10^3
```

```
  ) %>%
  # remove duplicates
  unique()

# II: sampling 3 balls with replacement regarding the order
sample.with.repl.unordered <- tibble(
  # construct all possible r-variations
  x.1 = rep(1:100, each=100, length.out=10^3),
  x.2 = rep(1:10, each=10, length.out=10^3),
  x.3 = rep(1:10,length.out = 10^3)
) %>%
  # the following operations must be performed rowwise
  rowwise() %>%
  mutate(
    # maps the numbered balls to the corresponding colors and consider only the
    # set of drawan colors
    event = c(bag[x.1],bag[x.2],bag[x.3]) %>% sort() %>% paste(collapse = "")
  ) %>%
  # regard only the events
  select(event) %>%
  # count the number of elementary events; all are equally like
  group_by(event) %>%
  mutate(
    prob = n()/10^3
  ) %>%
  # remove duplicates
  unique()

# III: sampling r=3 balls without replacement und regard the order
sample.without.repl <- matrix(data=0, ncol = 3, nrow = 10*9*8)
# construct all possible r-variations
l <- 1
for (i in 1:10) {
  for (j in setdiff(1:10,i)) {
    for (k in setdiff(1:10,c(i,j))) {
      sample.without.repl[l,] <- c(i,j,k)
      l <- l+1
    }
  }
}

sample.without.repl %>%
  # convert to a tibble
  as_tibble() %>%
  # the following operations must be performed rowwise
  rowwise() %>%
  mutate(
    # maps the numbered balls to the corresponding colors
    event = c(bag[V1],bag[V2],bag[V3]) %>% paste(collapse = "")
    ) %>%
  # consider only the elementary events
  select(event) %>%
  # counts the number of elementary events (all are equally like)
  group_by(event) %>%
  mutate(
    prob = n()/(10*9*8)
  ) %>%
  unique() -> sample.without.repl.ordered

# IV: sampling 3 balls without replacement without regarding the order
sample.without.repl %>%
  # convert to a tibble
  as_tibble() %>%
  # the following operations must be performed rowwise
  rowwise() %>%
  mutate(
    # maps the numbered balls to the corresponding colors and consider only the
    # set of drawan colors
    event = c(bag[V1],bag[V2],bag[V3]) %>% sort() %>% paste(collapse = "")
  ) %>%
  # consider only the elementary events
  select(event) %>%
  # counts the number of elementary events (all are equally like)
  group_by(event) %>%
  mutate(
    prob = n()/(10*9*8)
  ) %>%
  unique() -> sample.without.repl.unorderd
```

# Independence and Conditional Probabilities

1. Suppose that $A$ and $B$ are events in an experiment with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cap B) = 1/10$. Find each of the following:

$$P(A|B), P(B|A), P(A^c|B), P(B^c|A), P(A^c|B^c)$$

**Answer:**

(a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/10}{1/4} = \frac{4}{10} = 0,4$

(b) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/10}{1/3} = \frac{3}{10} = 0,3$

(c) $P(A^C|B) = 1 - P(A|B) = 1 - 0,4 = 0,6$

(d) $P(B^C|A) = 1 - P(B|A) = 1 - 0,3 = 0,7$

(e) $P(A^C|B^C) = \frac{P(A^C\cap B^C)}{P(B^C)} = \frac{1-P(A\cup B)}{1-P(B)} = \frac{1-(P(A)+P(B)-P(A\cap B))}{1-1/4} = \frac{1-\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{10}\right)}{3/4} = \frac{31}{45} \approx 0,689$

2. In a certain population, 30% of the persons smoke and 8% have a certain type of heart disease. Moreover, 12% of the persons who smoke have the disease.

(a) What percentage of the population smoke and have the disease?

(b) What percentage of the population with the disease also smoke?

(c) Are smoking and the disease positively correlated, negatively correlated, or independent?

**Answer:** S=Smoker and D=Disease, i.e. $P(S) = 0.30, \quad P(D) = 0.08, \quad \text{und } P(D|S) = 0.12$,

(a) $P(S \cap D) = P(S) \cdot P(D|S) = 0.30 \cdot 0.12 = 0.036$

(b) $P(S|D) = \frac{P(S\cap D)}{P(D)} = \frac{0.036}{0.08} = 0.45$

(c) We compare $P(S) \cdot P(D) = 0,30 \cdot 0,08 = 0,024$ and $P(S \cap D) = 0,036$ and get $P(S \cap D) > P(S) \cdot P(D)$ i.e. dependent.

3. Suppose that a bag contains 12 coins: 5 are fair, 4 are biased with probability of heads 1/3 and 3 are two-headed. A coin is chosen at random from the bag and tossed.

(a) Find the probability that the coin shows head.

(b) Given that the coin shows head, find the conditional probability of each coin type.

**Answer:** We have $n = 12$ coins. A-coins: 5 fair coins with $P(\text{ Head }) = P(\text{ Tail }) = 0.50$. B-coins: 4 manipulated coins with $P(\text{ Head }) = 1/3$ and $P(\text{ Tail }) = 2/3)$. C-coins: 3 manipulated coins with $P(\text{ Head }) = 1$ and $P(\text{ Tail }) = 0$.

(a) $P(\text{ Head }) = P(\text{ Head } \mid \text{A-coin }) \cdot P(\text{ A-coin}) + P(\text{ Head } \mid \text{B-coin }) \cdot P(\text{ B-coin}) + P(\text{ Head } \mid \text{C-coin }) \cdot P(\text{ C-coin}) = \frac{1}{2} \cdot \frac{5}{12} + \frac{1}{3} \cdot \frac{4}{12} + 1 \cdot \frac{3}{12} = \frac{41}{72} \approx 0,56944$

(b) $P(\text{ A-coin } \mid \text{Head }) = \frac{P(\text{ A-coin} \cap \text{Head })}{P(\text{ Head })} = \frac{P(\text{ Head } \mid \text{A-coin }) \cdot P(\text{ A-coin })}{P(\text{ Head })} = \frac{\frac{1}{2} \cdot \frac{5}{12}}{\frac{41}{72}} = \frac{15}{41} \approx 0.3658$ and $P(\text{ B-coin } \mid \text{Head }) = \frac{P(\text{ B-coin} \cap \text{Head })}{P(\text{ Head })} = \frac{P(\text{ Head } \mid \text{B-coin }) \cdot P(\text{ B-coin })}{P(\text{ Head })} = \frac{\frac{1}{3} \cdot \frac{4}{12}}{\frac{41}{72}} = \frac{8}{41} \approx 0.1951$ and $P(\text{ C-coin } \mid \text{Head }) = \frac{P(\text{ C-coin} \cap \text{Head })}{P(\text{ Head })} = \frac{P(\text{ Head } \mid \text{C-coin }) \cdot P(\text{ C-coin })}{P(\text{ Head })} = \frac{1 \cdot \frac{3}{12}}{\frac{41}{72}} = \frac{18}{41} \approx 0.4390$

4. Suppose we know the accuracy rates of the test for both the positive case (positive result when the patient has HIV) and negative case (result when the patient doesn't have HIV). These are referred to as test sensitivity and specificity, respectively. Let "D" be the event that the patient has HIV, and let "+" indicate a positive test result and "-" a negative.

   (a) Describe test sensitivity and specifity by the above notation.

   (b) Suppose a person gets a positive test result. Express the probability that he really has HIV by the above notation.

   (c) Let the disease prevalence be .001, test sensitivity be 99.7% and test specificity be 98.5%. The probability that a person has the disease given his positive test result, i.e. $P(D|+)$. This quantity is called the positive predictive value. Similarly, $P(D^c|-)$, is called the negative predictive value, the probability that a patient does not have the disease given a negative test result. Apply Bayes Rule to evaluate both values.

   **Answer:**

   (a) sensitivity: $P(+|D)$, specificity: $P(-|D^c)$

   (b) $P(D|+)$

(c) Bayes Rule:

$$P(D|+) = \frac{P(D)\cdot P(+|D)}{P(D)\cdot P(+|D)+P(D^c)\cdot P(+|D^c)}$$
$$P(D^c|-) = \frac{P(D^c)\cdot P(-|D^c)}{P(D^c)\cdot P(-|D^c)+P(D)\cdot P(-|D)}$$

With $P(D) = 0.001, P(+|D) = 0.997, P(-|D^c) = 0.985$ we get

$$P(D|+) = \frac{0.001\cdot 0.997}{0.001\cdot 0.997+(1-0.001)\cdot(1-0.985)} = 0.06238268$$
$$P(D^c|-) = \frac{(1-0.001)\cdot 0.985}{(1-0.001)\cdot 0.985+0.001\cdot(1-0.997)} = 0.999997$$

5. In a computer science course at an university we have the following data over a long time.
10% of all students have attended the exercises in statistics regularly. 2% of the students who have failed the statistics exam have attended the exercises regularly. 5% of the students who have attended the exercises regularly have failed the statistics exam.

   (a) Find the probability to fail the exam in statistics if the exercises in statistics are not attended regularly.

   (b) What is the effect of attending the exercises regularly to passing the exam?

   **Answer:** From the data we could assume for the events A="student attends the exercises regularly", B="student fails exam" the following probabilities

$$P(A) = 0.1, P(A\,|\,B) = 0.02, P(B\,|\,A) = 0.05$$

$$P(B) = \frac{P(B\,|A)P(A)}{P(A\,|\,B)} = \frac{0.05\cdot 0.2}{0.02} = 0.25$$
$$P(B\,|\,A^c) = \frac{P(B)P(A^c\,|\,B)}{1-P(A^c)} = P(B)\frac{1-P(A\,|\,B)}{1-P(A)}$$
$$= 0.25\cdot\frac{1-0.02}{1-0.1} \approx 0.272$$

   Since $P(B\,|\,A) = 0.05 < P(B\,|\,A^c) \approx 0.272$ is the probability to fail of an untrained student more than 5 times higher as the probability of a trained student.

6. Gamblers Ruin
Consider two gamblers whose capitals sum to 7 dollar, so that as soon as one has all seven dollars the other is ruined and the game stops. Plays form independent trials with even chances for winning and losing. Let $X[n]$ be the capital of the first gambler at the end of the nth play.

   (a) Determine the probabilities of the values of $X[3]$ if $X[1] = 3$.
   (b) Simulate the game for every possible value of $X[1]$ the values of $X[10]$.
   (c) Describe the probabilities $P(X[2] = j \mid X[1] = j)$ by a 8x8 matrix $P$.
   (d) Show that $P^n$ contains the probabilities $P(X[n] = j \mid X[1] = i)$.
   (e) Evaluate $P^{10}$ and compare the values of the matrix with the relative frequencies of $X[10]$ given by 100 simulations.
   (f) Estimate the probabilites of the stop of the game by evaluating $P^{50}$.

**Answer:**

(a)
- The possible values of $X[2]$ are 2 and 4. If $X[1] = 3$ we get $P(X[2] = 2|X[1] = 3) = 0.5 = P(X[2] = 4|X[1] = 3) = 0.5$
- $P(X[3] = 1|X[2] = 2) = 0.5 = P(X[3] = 3|X[2] = 2) = 0.5$, $P(X[3] = 3|X[2] = 4) = 0.5 = P(X[3] = 5|X[2] = 4) = 0.5$, other values of $X[3]$ are not possible:

$$P(X[3] = j|X[1] = 3) = \begin{cases} 0.25 & j = 1 \\ 0.5 & j = 3 \\ 0.25 & j = 5 \\ 0 & \text{else} \end{cases}$$

c) $i, j \in \{0, 1, ..7\}$

$$(P_{i,j}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

d) We prove the statement for n=2. Applying complete induction the statement can be proven analogously.

$$
\begin{aligned}
P(X[3]=k|X[1]=i) &= \sum_j P(X[3]=k, X[2]=j|X[1]=i) \\
&= \sum_j \frac{P(X[3]=k, X[2]=j, X[1]=i)}{P(X[1]=i)} \\
&= \sum_j \frac{P(X[3]=k|X[2]=j, X[1]=i)P(X[2]=j, X[1]=i)}{P(X[1]=i)} \\
&= \sum_j P(X[3]=k|X[2]=j, X[1]=i)\frac{P(X[2]=j, X[1]=i)}{P(X[1]=i)} \\
&= \sum_j P(X[3]=k|X[2]=j)P(X[2]=j|X[1]=i) \\
&= \sum_j P_{jk}P_{ij} = \sum_j P_{ij}P_{jk} = (P^2)_{i,k}
\end{aligned}
$$

Since $X[3]$ only depends on the value of $X[2]$ we get $P(X[3]=k|X[2]=j, X[1]=i) = P(X[3]=k|X[2]=j)$

```
################################################################################
# Consider two gamblers whose capitals sum to 7 dollar, so that as soon as one
# has all seven dollars the other is ruined and the game stops.  Plays form
# independent trials with even chances for winning and losing. Let X[n] be the
# capital of the first gambler at the end of the nth play.
# a) Determine the probabilities of the values of X[3] if X[1]=3.
# b) Simulate for every possible initial value of X[1] the values of X[10].
# c) Describe the probabilities P(X[2]=j | X[1]=j) by 8x8 matrix P.
# d) Show that P^n contains the probabilities P(X[n]=j | X[1]=i).
# e) Evaluate P^10 and compare it with the relative frequencies of X[10] given
#    by 100 simulations.
# f) Estimate the probabilites of the stop of the by evaluating P^50
#
# file: prob_gamblers_ruin.R
################################################################################
library(tidyverse)

# a)
# X[1]                         3
#                0.5|                    | 0.5
# X[2]              2                    4
#           0.5|        |0.5     0.5|       |0.5
# X[3]            1        3          3       5
#              0.25     0.25        0.25    0.25

# c) transition matrix
P <- matrix(data = 0, ncol = 8, nrow = 8, byrow = FALSE)
P[1,1] <- 1
P[8,8] <- 1
for (i in 2:7) {
  P[i,i-1] <- 0.5
  P[i,i+1] <- 0.5
}
P

# c)  Simulation
n.sim <- 10
sim.games <- matrix(data = 0, nrow = 6, ncol = n.sim, byrow = TRUE)
sim.games[,1] <- 1:6
# store whole games
for (j in 1:6) {
  for (i in 2:n.sim) {
    sim.games[j,i] <- case_when(
```

```
      sim.games[j,i-1]==0 ~ 0,
      (sim.games[j,i-1] < 7) & (sim.games[j,i-1] > 0) ~
        sample(c(sim.games[j,i-1]-1,sim.games[j,i-1]+1), size = 1, replace = FALSE),
      sim.games[j,i-1]==7 ~ 7
    )
  }
}
colnames(sim.games) <- rep("value.",n.sim) %>% paste(1:n.sim, sep="")
rownames(sim.games) <- rep("init.",6) %>% paste(1:6, sep="")
sim.games

# d) P( X[10] = j | X[1]=i ) = P^n_(i,j) Chapman-Kolmogoroff equation

# e)
P.10 <- P
for (i in 2:10) {
  P.10 <- P.10 %*% P
}
colnames(P.10) <- rep("X[10]=",8) %>% paste(0:7, sep="")
rownames(P.10) <- rep("X[1]=",8) %>% paste(0:7, sep="")
P.10 %>% round(digits=3)
# X[10]=0 X[10]=1 X[10]=2 X[10]=3 X[10]=4 X[10]=5 X[10]=6 X[10]=7
# X[1]=0   1.000   0.000   0.000   0.000   0.000   0.000   0.000   0.000
# X[1]=1   0.754   0.041   0.000   0.087   0.000   0.064   0.000   0.054
# X[1]=2   0.549   0.000   0.128   0.000   0.151   0.000   0.064   0.107
# X[1]=3   0.344   0.087   0.000   0.192   0.000   0.151   0.000   0.226
# X[1]=4   0.226   0.000   0.151   0.000   0.192   0.000   0.087   0.344
# X[1]=5   0.107   0.064   0.000   0.151   0.000   0.128   0.000   0.549
# X[1]=6   0.054   0.000   0.064   0.000   0.087   0.000   0.041   0.754
# X[1]=7   0.000   0.000   0.000   0.000   0.000   0.000   0.000   1.000

# store X[10] in 100 simulations
final.values <- matrix(data = 0, ncol = 6, nrow = 100, byrow = TRUE)
for (k in 1:100) {
  for (j in 1:6) {
    g <- rep(0,n.sim)
    g[1] <- j
    for (i in 2:n.sim) {
      g[i] <- case_when(
        g[i-1]==0 ~ 0,
        (g[i-1] < 7) & (g[i-1] > 0) ~
          sample(c(g[i-1]-1,g[i-1]+1), size = 1, replace = FALSE),
        g[i-1]==7 ~ 7
      )
    }
    final.values[k,j] <- g[n.sim]
  }
}
final.values

# estimation the ratios of the final values after n.sim games
tibble(value = 0:7) %>%
  left_join(final.values %>% as_tibble() %>% count(V1) %>%
              rename(value = V1, init.1 = n), by = "value") %>%
  left_join(final.values %>% as_tibble() %>% count(V2) %>%
              rename(value = V2, init.2 = n), by = "value") %>%
  left_join(final.values %>% as_tibble() %>% count(V3) %>%
              rename(value = V3, init.3 = n), by = "value") %>%
  left_join(final.values %>% as_tibble() %>% count(V4) %>%
              rename(value = V4, init.4 = n), by = "value") %>%
  left_join(final.values %>% as_tibble() %>% count(V5) %>%
              rename(value = V5, init.5 = n), by = "value") %>%
  left_join(final.values %>% as_tibble() %>% count(V6) %>%
              rename(value = V6, init.6 = n), by = "value") %>%
  # exchange NA by 0 and divide by 50
  mutate(
    init.1 = replace_na(init.1, 0)/100,
    init.2 = replace_na(init.2, 0)/100,
    init.3 = replace_na(init.3, 0)/100,
    init.4 = replace_na(init.4, 0)/100,
    init.5 = replace_na(init.5, 0)/100,
    init.6 = replace_na(init.6, 0)/100) -> estimated.ratios
estimated.ratios %>% as.matrix() %>% t() -> M
M <- M[2:7,]
colnames(M) <-  rep("X[10]=",8) %>% paste(0:7, sep="")
rownames(M) <- rep("X[1]=",6) %>% paste(1:6, sep="")
M
# A tibble: 8 x 7
# value init.1 init.2 init.3 init.4 init.5 init.6
# <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>
# 1     0  0.76   0.46   0.3    0.18   0.11   0.03
# 2     1  0      0.07   0      0.12   0      0.04
# 3     2  0.1    0      0.17   0      0.08   0
```

```
# 4    3   0     0.19   0     0.2    0     0.08
# 5    4   0.08  0      0.22  0      0.2   0
# 6    5   0     0.14   0     0.21   0     0.07
# 7    6   0.04  0      0.1   0      0.08  0
# 8    7   0.02  0.14   0.21  0.29   0.53  0.78

# f)
n <- 50
P.50 <- P
for (i in 2:50) {
  P.50 <- P.50 %*% P
}
colnames(P.50) <- rep("X[50]=",8) %>% paste(0:7, sep="")
rownames(P.50) <- rep("X[1]=",8) %>% paste(0:7, sep="")
# aproximation of a stop with value=0 or value=7
P.50[2:7,c(1,8)]
#          X[50]=0   X[50]=7
# X[1]=1 0.8555890 0.1414571
# X[1]=2 0.7117630 0.2829143
# X[1]=3 0.5679370 0.4254256
# X[1]=4 0.4254256 0.5679370
# X[1]=5 0.2829143 0.7117630
# X[1]=6 0.1414571 0.8555890

# exact values
# transition probabilities from the transient states 1 to 6 to the
# absorbing states 0 and 7

# probabilities of a transition from a transient state to another transient state
Q <- P[2:7,2:7]
# probabilities of a transition from a transient state in an absorbing state
B <- P[2:7,c(1,8)]
# B.n = (I+Q+Q^2+...+QÂ°(n-1))*B, B.n[i,j] is the probanility that starting from
# a transient state i (1,...,6) to enter an absorbing state j (0,7) at or before
# the n-th step
# G=(I+Q+Q^2+...)*B = (I-Q)^-1 * B, G[i,j] is the probability of ever reaching
# an absorbing state j from a transient state i

I <- matrix(data=0, nrow = 6, ncol = 6, byrow = TRUE)
for (i in 1:6) I[i,i] <- 1
# solve(I-Q) is the inverse of (I-Q)
solve(I-Q) %*% B -> G
colnames(G) <- c("value=0","value=7")
rownames(G) <- rep("start with",6) %>% paste(1:6)
G
#                 value=0   value=7
# start with 1 0.8571429 0.1428571
# start with 2 0.7142857 0.2857143
# start with 3 0.5714286 0.4285714
# start with 4 0.4285714 0.5714286
# start with 5 0.2857143 0.7142857
# start with 6 0.1428571 0.8571429
```