STUDENT ARTICLE

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Building and Solving Differential Equations Using Electronic Circuits

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Introduction

Ordinary Differential Equations (ODEs) create a relationship between a function of one variable and its derivatives. Circuits can be used to build and solve these ODEs. Such circuits have historically been called analog computers. Primarily used to calculate projectile motion during WWII, these analog computers now find their home in control systems. This article will go into detail on how circuits capture differential behavior, how to turn a differential equation into a circuit, and how to solve differential equations using circuits.

Building Blocks

Kirchhoff's circuit laws provide the first building block for modeling differential equations using circuits. These two laws relate current and voltage to the graph-like structure of a circuit, with current being a quality of edges and voltage a quality of vertices. Kirchhoff's Voltage Law says voltage sums to zero in a cycle (i.e., an ordered collection of vertices beginning and ending with the same vertex), while Kirchhoff's Current Law says the weighted sum of current relative to a vertex is zero (i.e., the sum of incoming current equals the sum of outgoing current). In engineering parlance, vertices are the components of a circuit and nodes are the places where two or more components meet. Kirchhoff's circuit laws therefore allow us to construct differential equations by inspecting cycles and nodes.

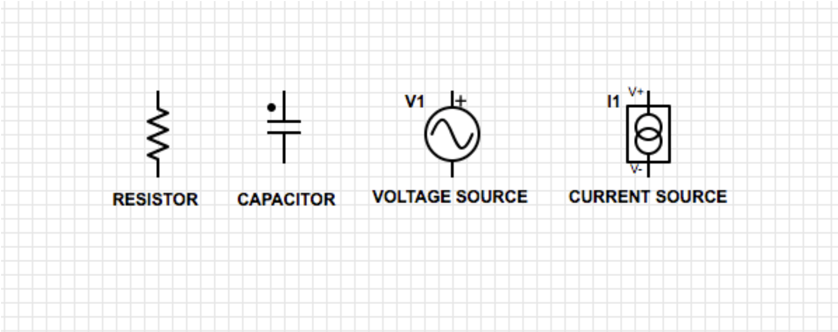

Figure 1: Linear Circuit Components

The next step is characterizing the electric components shown in Figure 1. Current through a component can be related to the voltage across it. For a resistor with constant resistance R, the relationship between voltage and current is described by Ohm's Law:

$$V = RI \tag{1}$$

Current through a component can also be related to the change in voltage across it. For a capacitor with constant capacitance C, the relationship is described by

$$I = C\frac{dV_C}{dt} \tag{2}$$

Here, V_C is the voltage across the capacitor. Sources, the remaining components, either add voltage across a node or specify a current.

Building Differential Equations

Consider the following series circuit, so called because it consists of a single cycle.

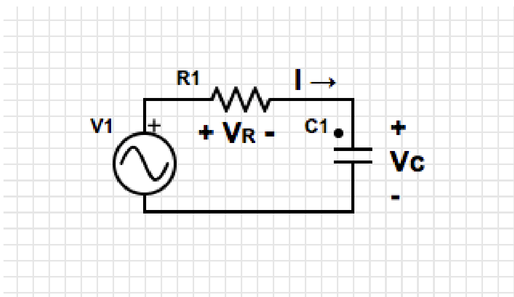

Figure 2: RC Circuit

Kirchhoff's Voltage Law and Ohm's Law together give

$$0 = V_R + V_C - V_1 = RI + V_C - V_1 \tag{3}$$

The current is constant throughout the series circuit because each node is the junction of only two vertices. Using Equation (2) to write I in terms of V_C and solving for V_1 gives the first-order linear non-homogeneous differential equation

$$V_1 = RC\frac{dV_C}{dt} + V_C \tag{4}$$

We can perform a similar procedure to find a second-order linear non-homogeneous equation characterizing the circuit shown below.

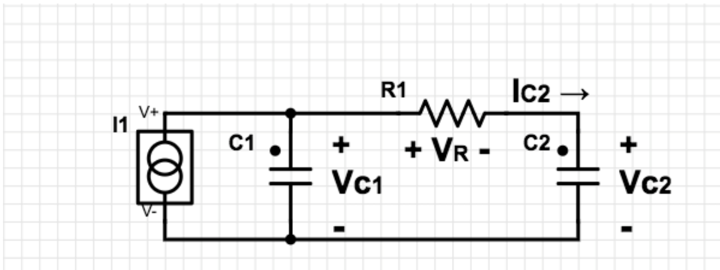

Figure 3: Second Order RC Circuit

Kirchhoff's Current Law and Equation (2) together give

$$I_1 = I_{C_1} + I_{C_2} = C_1\frac{dV_{C_1}}{dt} + C_2\frac{dV_{C_2}}{dt} \tag{5}$$

Kirchhoff's Voltage Law and Ohm's Law together give

$$V_{C_1} = V_R + V_{C_2} = RC_2\frac{dV_{C_2}}{dt} + V_{C_2} \tag{6}$$

Substituting this into Equation (5) and collecting like terms gives

$$I_1 = C_1\frac{d}{dt}\left[RC_2\frac{dV_{C_2}}{dt} + V_{C_2}\right] + C_2\frac{dV_{C_2}}{dt} = RC_1C_2\frac{d^2V_{C_2}}{dt^2} + (C_1 + C_2)\frac{dV_{C_2}}{dt} \tag{7}$$

We now have two examples of recreating differential equations using circuits, and thus two cases in which we can approximate solutions by taking measurements. Namely, we can use an oscilloscope to read the relevant capacitor voltages and thereby measure solutions to Equation (4) and Equation (7). Unfortunately, this approach has two key issues.

1. Linear electric circuits will not have outputs that rise above the input – i.e., the waveform will be chopped off above the maximum of the non-homogeneous side of the equation.

2. Such circuits are often difficult to synthesize even if they are easy to analyze.

Fortunately, there is an electronic component known as an Operational Amplifier that allows us to address these issues.

Operational Amplifiers

Consider the following schematic.

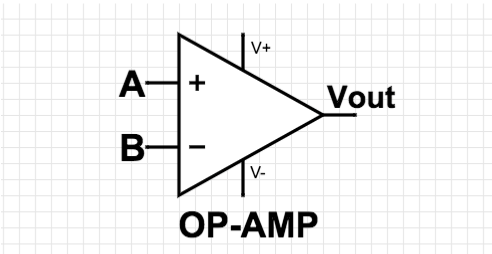

Figure 4: Operational Amplifier with Supply Lines

Ideally, Operational Amplifiers (Op-Amps for short) keep the same voltage at nodes A and B shown in Figure 4. This is done using feedback from V_{out}. V_+ and V_- are the power supply lines, which limit V_{out} via

$$V_- \leq V_{out} \leq V_+ \tag{8}$$

Staying within these limits is important for minimizing distortion. For the sake of simplicity, we will largely ignore supply lines when building equations. One big advantage of Op-Amps is that they allow us to neatly package the available operations of sums, coefficients, and derivatives – the building blocks of linear differential equations. This is illustrated by using Kirchhoff's Current Law at node A of the figures below (note that node A is not explicitly marked but can be identified by comparison with Figure 4).

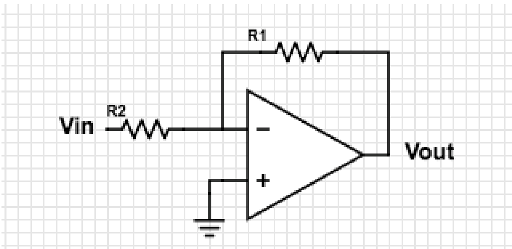

Figure 5: Coefficient Circuit

$$V_{out} = -\frac{R_1}{R_2} V_{in} \tag{9}$$

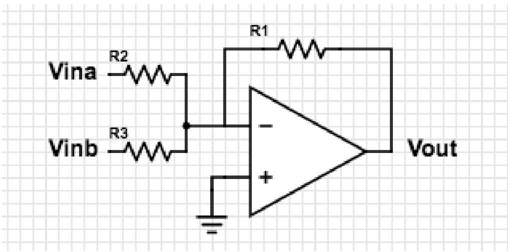

Figure 6: Summing Circuit

$$V_{out} = -\frac{R_1}{R_2}V_{ina} - \frac{R_1}{R_3}V_{inb} \tag{10}$$

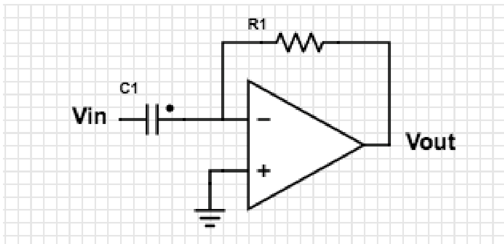

Figure 7: Differentiating Circuit

$$V_{out} = -RC\frac{d}{dt}V_{in} \tag{11}$$

Selecting appropriate resistances and capacitances therefore allows us to control any coefficients that appear. To simplify the process of building equations, we will treat these circuits like black boxes and adopt the notational shorthand shown in the following figure. Note that each operational block changes the sign of its input, which is why such circuits are called Inverting Amplifiers.

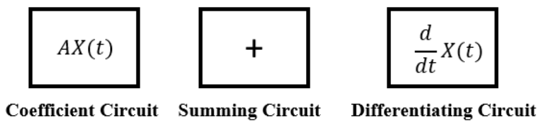

Figure 8: Circuit Shorthand

The symbol A in the Coefficient Circuit represents a choice of coefficient, which should be clear in context. Because we are dealing with an Inverting Amplifier, the output is $-A$ times the input. Now, let's put these tools to use. Consider the following differential equation.

$$V_{in} = B\frac{d}{dt}V_{out} + V_{out} \tag{12}$$

Here, V_{in} is the input voltage, V_{out} is the desired output voltage, and B is a constant. A first attempt at representing this equation is as follows:

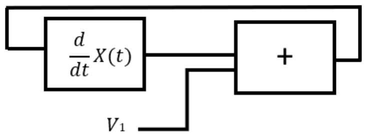

Figure 9: ODE Block Representation 1

When reading these sorts of diagrams, note that all inputs enter the left of a block and all outputs exit the right of a block. The result of such an analysis is

$$V_{out} = -V_{in} + \frac{d}{dt}V_{out} \tag{13}$$

A slight modification to account for signs and the coefficient on the derivative yields the desired equation.

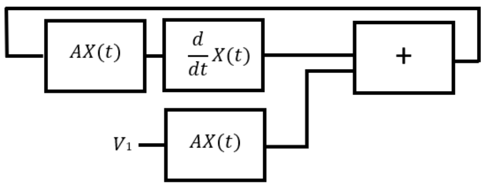

Figure 10: ODE Block Representation 2

To close, try writing out the equation associated to the following diagram. You will see just how much Op-Amps simplify the design process

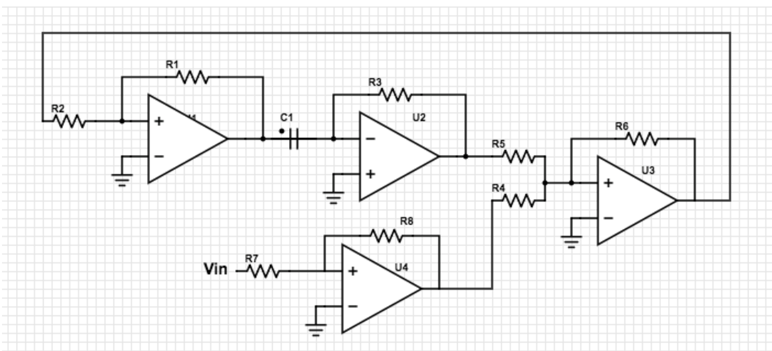

Figure 11: Differential Equation Solver Implemented with Operational Amplifiers