

Designs with two factors

Last time: Completely randomized designs

- ▶ Two equivalent parametrizations:

1. Group means parametrization:

$$Y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{\text{iid}}{\sim} N(0, \sigma^2),$$

2. Sum-to-zero parametrization:

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \sum_{i=1}^a \tau_i = 0, \quad \varepsilon_{ij} \stackrel{\text{iid}}{\sim} N(0, \sigma^2),$$

- ▶ $i \in \{1, \dots, a\}$ indexes group membership
- ▶ $j \in \{1, \dots, n_i\}$ indexes observations within a group

Types of factors: treatments vs blocks

Factors (variables) can be of two types

- ▶ **Treatments:** We are interested in how they affect the response. We can control how we assign them to the experimental units.
- ▶ **Blocks** We are not interested in how they affect the response, but suspect that they affect the response. We can control them when we're performing the experiments, but often can't control them outside of the experimental design.

Additive vs interaction models

- ▶ **Additive model:** The effects of the factors on the response don't depend on levels of other factors
- ▶ **Interaction model:** The effects of the factors on the response can depend on levels of other factors

Example: Painting line

- ▶ An engineer working at a painting line wants to know if different types of paints have different average drying times
- ▶ They will perform the experiments on four different days
- ▶ The engineers think that the weather may affect the outcome: that is, the day may affect the outcome
- ▶ The treatment is the type of paint, the block is the day

Balanced complete randomized block designs with replicates

- ▶ We assign experimental units randomly to blocks and treatments
- ▶ The block has b levels
- ▶ The treatment has a levels
- ▶ We collect r observations for each combination of block and treatment: there are r replicates
- ▶ Total sample size is $N = rba$

Terminology

- ▶ The design is *balanced* because each combination of block and treatment is observed the same number of times: r
- ▶ The design is *complete* because all the combinations of block and treatment are observed
- ▶ The term “balanced complete randomized block designs” usually refers to the case $r = 1$, which is the most common
- ▶ Designs with $r > 1$ are often referred to as “balanced complete randomized block designs with replicates”

Additive vs interaction models

- ▶ Additive models assume that the treatment effect does not depend on the block (or the other treatment, if there are two treatments)
- ▶ Block designs are often additive; this assumption comes from prior knowledge of the experimenters
- ▶ If $r = 1$, we can only fit an additive model: we don't have enough data to estimate the interaction and we can't properly test if it's needed
- ▶ If $r > 1$ we can fit interaction terms

Example: Painting line

- ▶ Day (block): $b = 4$ days
- ▶ Type of paint (treatment): $a = 3$ types
- ▶ $r = 1$ replicate
- ▶ $N = rba = 12$ observations

	Day 1	Day 2	Day 3	Day 4
Type A	2.60	2.10	3.5	2.65
Type B	3.25	2.65	3.8	3.50
Type C	2.60	2.05	3.1	2.62

Additive model

$$Y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}, \quad \sum_{i=1}^a \tau_i = \sum_{j=1}^b \beta_j = 0, \quad \varepsilon_{ijk} \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$$

- ▶ $i \in \{1, \dots, a\}$ indexes treatments, $j \in \{1, \dots, b\}$ indexes blocks, and $k \in \{1, \dots, r\}$ indexes replicates
- ▶ As before, we assume that the errors are independent and normal, and that the variance σ^2 doesn't depend on the treatment or block
- ▶ We also assume an additive relationship between treatment and block

Additivity

- ▶ Treatment effect doesn't depend on blocks; block effect doesn't depend on treatment
- ▶ **Example:** $a = 2$ treatments and $b = 2$ blocks, table with expected values

	Block 1	Block 2
Treat 1	$\mu + \tau_1 + \beta_1$	$\mu + \tau_1 + \beta_1$
Treat 2	$\mu + \tau_2 + \beta_2$	$\mu + \tau_2 + \beta_2$

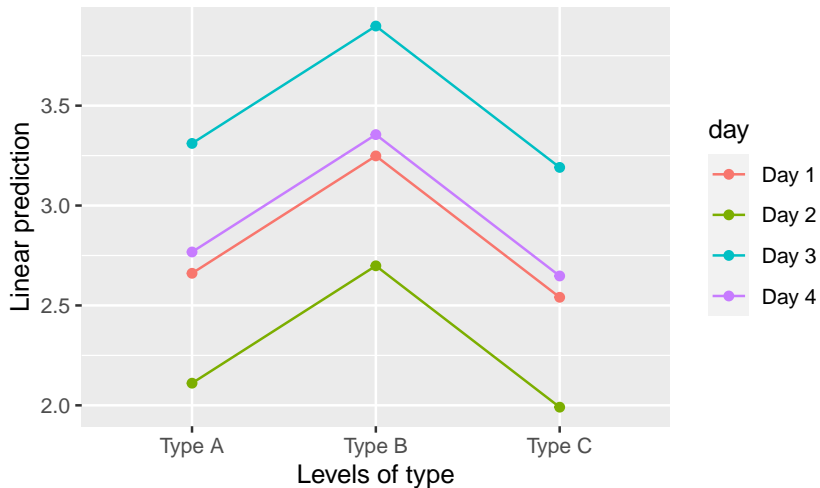
Average treatment effect in block 1: $\mathbb{E}(y_{11k} - y_{21k'}) = \tau_1 - \tau_2$

Average treatment effect in block 2: $\mathbb{E}(y_{12k} - y_{22k'}) = \tau_1 - \tau_2$

Average block effect in treat 1: $\mathbb{E}(y_{11k} - y_{12k'}) = \beta_1 - \beta_2$

Average block effect in treat 2: $\mathbb{E}(y_{21k} - y_{22k'}) = \beta_1 - \beta_2$

Example: estimated effects with additive model



Estimating parameters: Additive design

► Let

$$\text{total mean} = \bar{Y} = \frac{1}{N} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r Y_{ijk}$$

$$\text{treat } i \text{ mean} = \bar{Y}_{i.} = \frac{1}{br} \sum_{j=1}^b \sum_{k=1}^r Y_{ijk}$$

$$\text{block } j \text{ mean} = \bar{Y}_{.j} = \frac{1}{ar} \sum_{i=1}^a \sum_{k=1}^r Y_{ijk}$$

► Point estimates

$$\hat{\mu} = \bar{Y}, \quad \hat{\tau}_i = \bar{Y}_{i.} - \bar{Y}, \quad \hat{\beta}_j = \bar{Y}_{.j} - \bar{Y},$$

and

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (Y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j)^2$$

Example: Reading in data

```
mat = matrix(c(2.60, 2.10, 3.50, 2.65,  
              3.25, 2.65, 3.80, 3.5,  
              2.60, 2.05, 3.10, 2.62),  
            nrow = 3,  
            ncol = 4,  
            byrow = TRUE)  
rownames(mat) = c("Type A", "Type B", "Type C")  
colnames(mat) = c("Day 1", "Day 2", "Day 3", "Day 4")
```

Example: Converting into convenient (long) format

```
library(tidyverse)
df = as.data.frame(mat)
df$type = rownames(df)
df = df %>% pivot_longer(!type,
                        names_to = "day",
                        values_to = "outcome")
df$type = factor(df$type); df$day = factor(df$day)
```

Example: model with sum-to-zero constraint

```
# sum-to-zero
options(contrasts = c("contr.sum", "contr.poly"))
# fit model
mod = aov(outcome ~ day + type, data = df)
# point estimates
dummy.coef(mod)
```

Full coefficients are

##

(Intercept): 2.868333

## day:	Day 1	Day 2	Day 3	Day 4
##	-0.05166667	-0.60166667	0.59833333	0.05500000

## type:	Type A	Type B	Type C
##	-0.1558333	0.4316667	-0.2758333

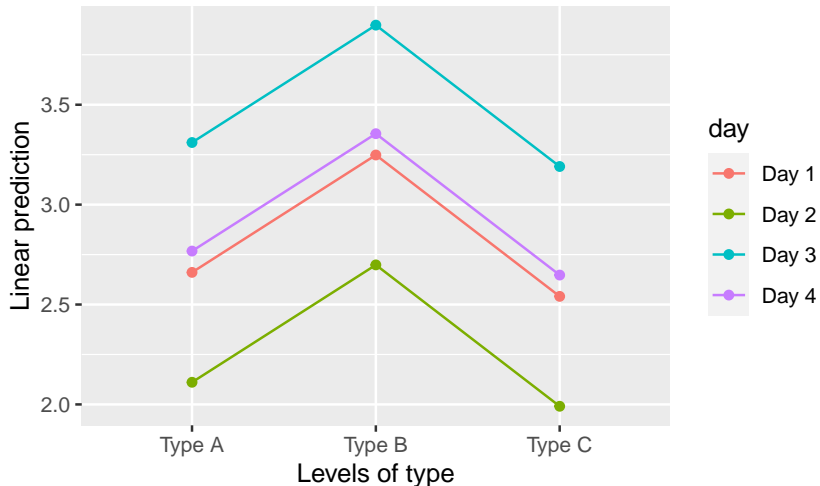
Example: 95% confidence intervals

##	2.5 %	97.5 %
## (Intercept)	2.7760320	2.96063468
## day1	-0.2115373	0.10820396
## day2	-0.7615373	-0.44179604
## day3	0.4384627	0.75820396
## type1	-0.2863672	-0.02529951
## type2	0.3011328	0.56220049

No intervals for $\tau_a = -\sum_{i=1}^{a-1} \tau_i$ and $\beta_b = -\sum_{j=1}^{b-1} \beta_j$

Effect plots

```
library(emmeans)  
emmip(mod, day ~ type)
```



Global F tests

Now we have two F tests

- ▶ One for τ_i

$$H_{0,\tau} : \tau_i = 0, \text{ for all } i \in \{1, \dots, a\}$$

$$H_{1,\tau} : \text{at least one } \tau_i \neq 0$$

- ▶ Another one for β_j :

$$H_{0,\beta} : \beta_j = 0, \text{ for all } j \in \{1, \dots, b\}$$

$$H_{1,\beta} : \text{at least one } \beta_j \neq 0.$$

Sums of squares

$$SS_{\text{total}} = SS_{\text{treat}} + SS_{\text{block}} + SS_{\text{error}},$$

where

$$SS_{\text{total}} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (\bar{Y}_{ijk} - \bar{Y})^2$$

$$SS_{\text{treat}} = br \sum_{i=1}^a (\bar{Y}_{i.} - \bar{Y})^2$$

$$SS_{\text{block}} = ar \sum_{j=1}^b (\bar{Y}_{.j} - \bar{Y})^2$$

$$SS_{\text{error}} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (Y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j)^2$$

Under $H_{0,\tau}$, $\bar{Y}_{i.} \approx \bar{Y}$ and SS_{treat} will be small; same story with $H_{0,\beta}$ and SS_{block}

ANOVA table

There are two F statistics: one for the treatment and another one for the block

	df	SS	MS	F	p
Treat	$a - 1$	SS_{treat}	$MS_{\text{treat}} = \frac{SS_{\text{treat}}}{a-1}$	$f_{\tau,\text{obs}} = \frac{MS_{\text{treat}}}{MS_{\text{error}}}$	p_{τ}
Block	$b - 1$	SS_{block}	$MS_{\text{block}} = \frac{SS_{\text{block}}}{b-1}$	$f_{\beta,\text{obs}} = \frac{MS_{\text{block}}}{MS_{\text{error}}}$	p_{β}
Error	ν	SS_{error}	$MS_{\text{error}} = \frac{SS_{\text{error}}}{\nu}$		

where $\nu = N - a - b + 1$,

$$p_{\tau} = P(F_{a-1,\nu} > f_{\tau,\text{obs}}), \quad p_{\beta} = P(F_{b-1,\nu} > f_{\beta,\text{obs}}),$$

and $F_{\alpha,\nu}$ is notation for the F -distribution with α and ν degrees of freedom.

Example: ANOVA table

```
summary(mod)
```

```
##              Df Sum Sq Mean Sq F value    Pr(>F)
## day           3  2.1771   0.7257    42.50 0.000195 ***
## type          2  1.1468   0.5734    33.58 0.000552 ***
## Residuals     6  0.1024   0.0171
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

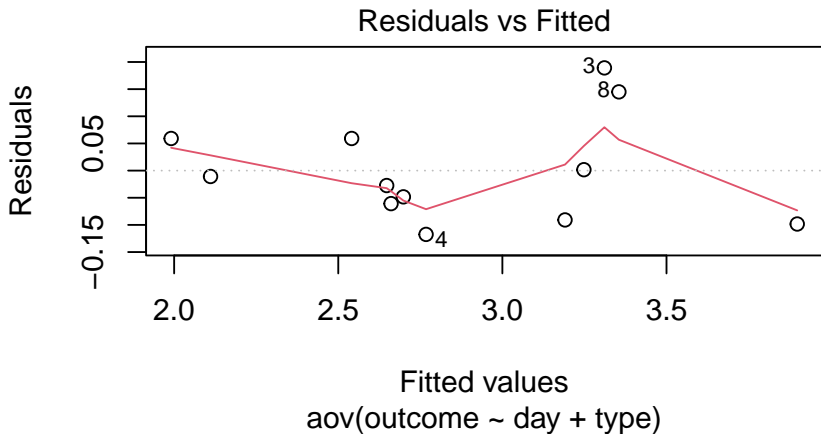
Block and treatment effect are significant at the $\alpha = 0.05$ significance level

Checking model assumptions

- ▶ We can check the hypothesis of equality of variances by looking at a plot of fitted values vs residuals: if the assumption is satisfied, the variance on the y -axis shouldn't depend much on where we are on the x -axis
- ▶ As before, we can check normality by looking at a qq-plot of the residuals
- ▶ Checking independence is hard unless we have a variable that allows us to check for space- or time-dependence, for example
- ▶ We assume that the block effect is additive. If $r = 1$, this assumption can't be checked. If $r > 1$, we can check whether an interaction term is needed or not.

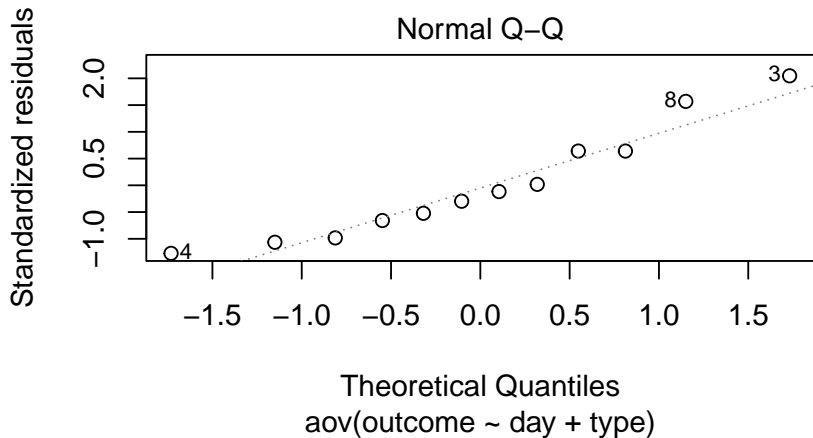
Example: checking equality of variances

```
plot(mod, which = 1)
```



Example: checking normality of residuals

```
plot(mod, which = 2)
```



Pairwise comparisons

- ▶ The global F test tells us that there are differences between treatments, but doesn't tell us where the differences are
- ▶ We can do pairwise tests to compare treatments; however, we have to be careful and control the type I error rate appropriately
- ▶ We can do that with TukeyHSD, as we did with completely randomized designs

Example: TukeyHSD

```
TukeyHSD(mod, which = "type")
```

```
##    Tukey multiple comparisons of means
##      95% family-wise confidence level
##
## Fit: aov(formula = outcome ~ day + type, data = df)
##
## $type
##              diff              lwr              upr              p adj
## Type B-Type A  0.5875  0.3039958  0.8710042  0.0017250
## Type C-Type A -0.1200 -0.4035042  0.1635042  0.4459693
## Type C-Type B -0.7075 -0.9910042 -0.4239958  0.0006345
```

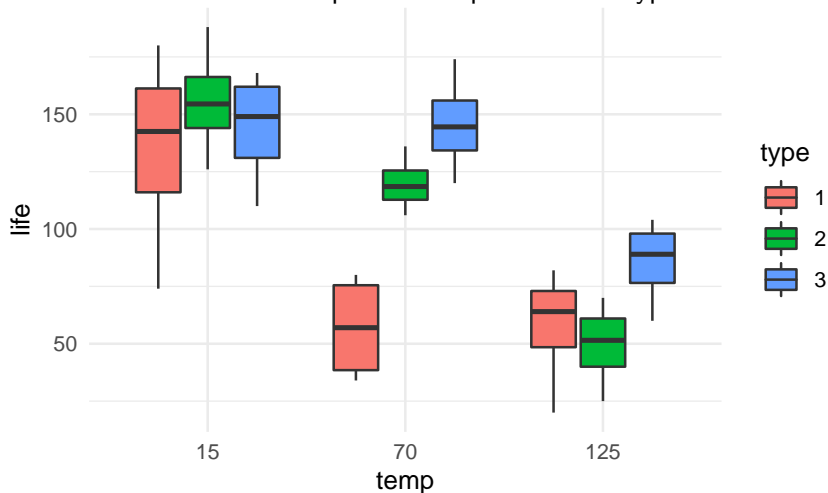
Significant differences between types A and B and types B and C
at the $\alpha = 0.05$ significance level

Example: Batteries (Montgomery)

- ▶ A company is producing batteries that need to work at different temperatures and for different types of devices
- ▶ Battery life is suspected to depend on the temperature and the material that the device is made of (type A, B, or C)
- ▶ The company wants to know how temperature and the device affect battery life; they also want to know if there is an interaction between temperature and type of device
- ▶ There are 36 observations: **3 replicates** for each combination of type and temperature. Therefore, we can estimate an interaction.
- ▶ If there is an interaction, it means that the effect of temperature depends on the type of device

Plotting the data

Does the effect of the temperature depend on the type of device?



Model with interaction

$$Y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk}, \quad \varepsilon_{ijk} \stackrel{\text{iid}}{\sim} N(0, \sigma^2),$$

with sum-to-zero constraints

$$\sum_{i=1}^a \tau_i = \sum_{j=1}^b \gamma_j = 0, \quad \sum_{i=1}^a (\tau\gamma)_{ij} = \sum_{j=1}^b (\tau\gamma)_{ij} = 0$$

- ▶ $i \in \{1, \dots, a\}$ indexes the first treatment, $j \in \{1, \dots, b\}$ indexes the second treatment, and $k \in \{2, \dots, r\}$ indexes replicates (important: $r > 1$)
- ▶ The errors are independent and normal, the variance σ^2 doesn't depend on the treatment
- ▶ $(\tau\gamma)_{ij}$ is the interaction term

Point estimation

Let

$$\begin{aligned}\bar{Y} &= \frac{1}{N} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r Y_{ijk}, & \bar{Y}_{ij} &= \frac{1}{r} \sum_{k=1}^r Y_{ijk} \\ \bar{Y}_{i.} &= \frac{1}{br} \sum_{j=1}^b \sum_{k=1}^r Y_{ijk}, & \bar{Y}_{.j} &= \frac{1}{ar} \sum_{i=1}^a \sum_{k=1}^r Y_{ijk}\end{aligned}$$

Then,

$$\begin{aligned}\hat{\mu} &= \bar{Y}, & (\widehat{\tau\gamma})_{ij} &= \bar{Y}_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j \\ \hat{\tau}_i &= \bar{Y}_{i.} - \bar{Y}, & \hat{\gamma}_j &= \bar{Y}_{.j} - \bar{Y}\end{aligned}$$

and

$$\widehat{\sigma^2} = \frac{1}{N-1} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (Y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j - (\widehat{\tau\gamma})_{ij})^2$$

Example: point estimation

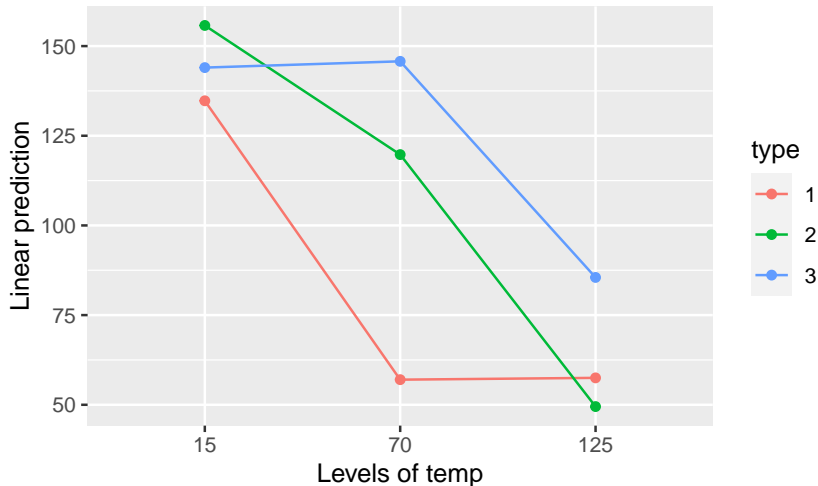
Here's how to find estimates and confidence intervals in R

```
battery = read.csv("https://vicpena.github.io/doe/battery.csv")
battery$type = factor(battery$type)
battery$temp = factor(battery$temp)
options(contrasts = c("contr.sum", "contr.poly"))
mod = aov(life ~ type*temp, data = battery)
dummy.coef(mod)
confint(mod)
```

Not printing the results here for concreteness

Example: Interaction plot

```
emmip(mod, type ~ temp)
```



Global F tests

We have three F tests

- ▶ One for τ_i

$$H_{0,\tau} : \tau_i = 0, \text{ for all } i$$

$$H_{1,\tau} : \text{at least one } \tau_i \neq 0$$

- ▶ One for γ_j

$$H_{0,\gamma} : \gamma_j = 0, \text{ for all } j$$

$$H_{1,\gamma} : \text{at least one } \gamma_j \neq 0$$

- ▶ And another one for $(\tau\gamma)_{ij}$

$$H_{0,\tau\gamma} : (\tau\gamma)_{ij} = 0, \text{ for all } i, j$$

$$H_{1,\tau\gamma} : \text{at least one } (\tau\gamma)_{ij} \neq 0$$

Sums of squares

$$SS_{\text{total}} = SS_{\tau} + SS_{\gamma} + SS_{\tau\gamma} + SS_{\text{error}},$$

where

$$SS_{\text{total}} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (\bar{Y}_{.j} - \bar{Y})^2$$

$$SS_{\tau} = br \sum_{i=1}^a (\bar{Y}_{i.} - \bar{Y})^2$$

$$SS_{\gamma} = ar \sum_{j=1}^b (\bar{Y}_{.j} - \bar{Y})^2$$

$$SS_{\tau\gamma} = r \sum_{i=1}^a \sum_{j=1}^b (\bar{Y}_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y})^2$$

$$SS_{\text{error}} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (Y_{ijk} - \bar{Y}_{ij})^2$$

ANOVA table

	df	SS	MS	F	p
τ	$a - 1$	SS_{τ}	$MS_{\tau} = \frac{SS_{\tau}}{a-1}$	$f_{\tau, \text{obs}} = \frac{MS_{\tau}}{MS_{\text{error}}}$	p_{τ}
γ	$b - 1$	SS_{γ}	$MS_{\gamma} = \frac{SS_{\gamma}}{b-1}$	$f_{\gamma, \text{obs}} = \frac{MS_{\gamma}}{MS_{\text{error}}}$	p_{γ}
$(\tau\gamma)$	$(a - 1)(b - 1)$	$SS_{\tau\gamma}$	$MS_{\tau\gamma} = \frac{SS_{\tau\gamma}}{(a-1)(b-1)}$	$f_{\tau\gamma, \text{obs}} = \frac{MS_{\tau\gamma}}{MS_{\text{error}}}$	$p_{\tau\gamma}$
Error	ν	SS_{error}	$MS_{\text{error}} = \frac{SS_{\text{error}}}{\nu}$		

Example: ANOVA table

```
summary(mod)
```

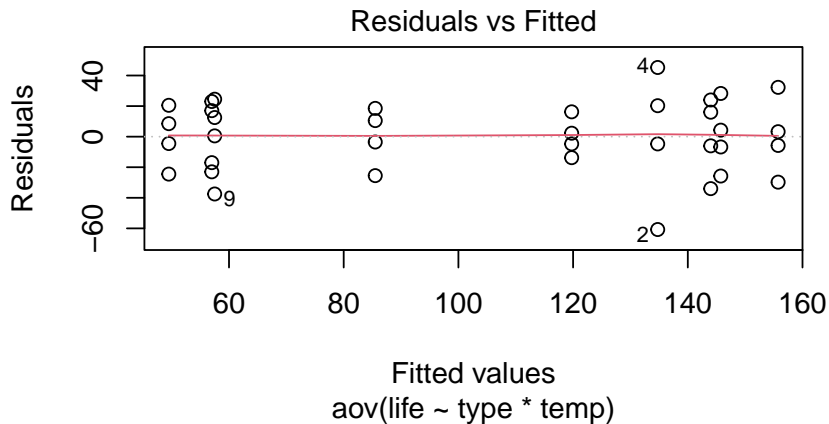
	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
## type	2	10728	5364	7.960	0.00192	**
## temp	2	39115	19557	29.020	1.88e-07	***
## type:temp	4	9670	2417	3.587	0.01803	*
## Residuals	27	18196	674			
## ---						
## Signif. codes:	0	'***'	0.001	'**'	0.01	'*' 0.05 '.' 0.1

Significant treatment effects and interaction at $\alpha = 0.05$.

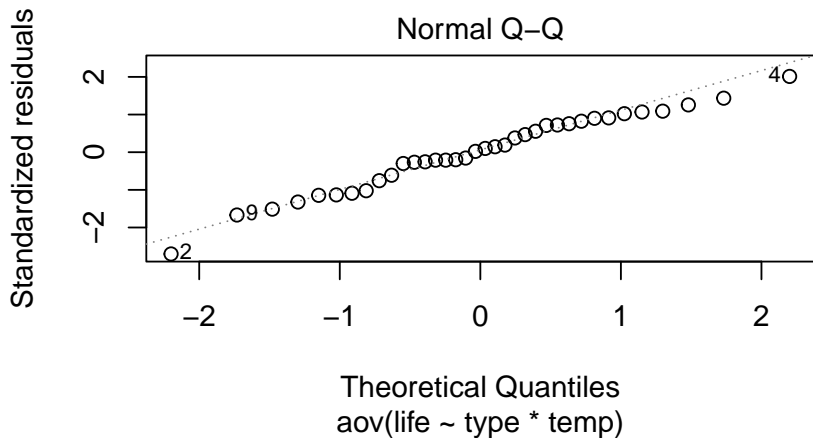
Assumptions

- ▶ We still have assumptions of independence, normality, and equality of variances
- ▶ We can check them as we did with the blocked design
- ▶ Our model has an interaction: if it isn't significant, we can drop it from our model and fit an additive model

Example: checking equality of variances



Example: checking normality



Pairwise comparisons

- ▶ As before, global tests don't tell us where the differences are
- ▶ For that, we can still use TukeyHSD

Conclusions

- ▶ Factors can be treatments or blocks
- ▶ Models with two factors can be either additive or have an interaction; for the latter, we need $r > 1$
- ▶ Models with a treatment and a block are often assumed to be additive, but they don't have to be: if you have $r > 1$, you can test whether there is a significant interaction