Comparing two conjugate multivariate stochastic volatility models

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Same marginals, different models

The models we worked with are awkward to write, but the idea is fairly simple

- Comparing two time-series models, we learned that two models can have the same marginal likelihoods, but be pretty different!
- I didn't think this was possible

Why is this interesting?

- Marginal likelihoods are often used for comparing models
- If you have models M_0 and M_1 with marginal likelihoods m_0 and m_1, the Bayes factor of M_0 to M_1 is
$$B_{01} = m_0/m_1$$
- Posterior probabilities of models depend on the data only through B_{01}
- If $B_{01} = 1$, $\mathbb{P}(M_1 \mid \text{data}) = \mathbb{P}(M_1)$

My talk in a slide

Two conjugate (read *convenient*) Bayesian models for time-varying covariance matrices can have identical marginal likelihoods but different posterior distributions over covariance matrices

We can't use posterior probabilities of hypotheses to distinguish these models

Can we use other metrics to compare them?

Wishart Distribution

Our models use extensively Wishart distributions

Let's define the distribution quickly

Wishart distribution

Let $A \in \mathcal{S}_{++}^q$. Then, $A \sim \text{Wishart}_q(h, S)$ if its probability density function is (up to constants)

$$p(A) \propto |S|^{-h/2} \, |A|^{(h-q-1)/2} \operatorname{etr}\left\{-\frac{1}{2}S^{-1}A\right\},$$

where $h > q - 1$

$$\mathbb{E}(A) = hS$$

- h: "degree of freedom"
- S: scale matrix

Working model

$\boldsymbol{y}_t$ is a q-dimensional vector

$$\boldsymbol{y}_t \mid \Phi_t \overset{\text{ind.}}{\sim} N(\mathbf{0}_q, \Phi_t^{-1})$$

How can we model the dynamics of precision Φ_t?

- Decomposition of Φ_t + univariate SV

  Lopes, McCulloch and Tsay (2010)

- Factor SV — Pitt and Shephard (1999), Kastner (2019)

- Matrix exponential — Ishihara, Omori and Asai (2016)

Great! But MCMC required, so costly (and boring to implement)

Two conjugate models for Φ_t

1. Uhlig-extended model (UE) $\nu > 0$ and $\lambda \in (0, 1)$
2. Beta-Bartlett model (BB): parameters $\nu_0 > 0$ and $\beta \in (0, 1)$, and $\lambda \in (0, 1)$

Parameters in these models are usually set by maximizing marginal likelihoods (if we put priors on them, we'd need MCMC)

Throughout, I'll use the notation

$$\mathcal{D}_t = \{y_1, y_2, \ldots, y_t\}$$

Uhlig-extended(ν, λ)

- Prior at t: $\Phi_t^{\mathrm{UE}} | \mathcal{D}_{t-1} \sim W_q(\nu, (\lambda V_{t-1})^{-1})$
- Posterior at t: $\Phi_t^{\mathrm{UE}} | \mathcal{D}_t \sim W_q(\nu + 1, V_t^{-1})$
  $V_t = \lambda V_{t-1} + \boldsymbol{y}_t \boldsymbol{y}_t'$
- Forecast at t: $\boldsymbol{y}_t | \mathcal{D}_{t-1} \sim$ Student's t
  (d.f. $= \nu$, scale$= (\lambda V_{t-1})^{-1}$)

easy sequential updating

Beta-Bartlett(ν_0, β, λ)

- Prior at t: $\Phi_t^{\mathrm{BB}} | \mathcal{D}_{t-1} \sim W_q(\beta\nu_{t-1}, (\lambda V_{t-1})^{-1})$
- Posterior at t: $\Phi_t^{\mathrm{BB}} | \mathcal{D}_t \sim W_q(\nu_t, V_t^{-1})$

  $V_t = \lambda V_{t-1} + \boldsymbol{y}_t \boldsymbol{y}_t'$ and $\nu_t = \beta\nu_{t-1} + 1$
- Forecast at t: $\boldsymbol{y}_t | \mathcal{D}_{t-1} \sim$ Student's t

  (d.f. $= \beta\nu_{t-1}$, scale$= (\lambda V_{t-1})^{-1}$)

Comparing models

- UE: $\Phi_t^{\mathrm{UE}} | \mathcal{D}_t \sim W_q(\nu + 1, V_t^{-1})$
- BB: $\Phi_t^{\mathrm{BB}} | \mathcal{D}_t \sim W_q(\nu_t, V_t^{-1})$, $\nu_t = \beta \nu_{t-1} + 1$
- The marginals are the same if

$$\beta = \frac{\nu}{\nu + 1},\ \nu_0 = \nu + 1,\ \lambda = \lambda$$

Same, but not equivalent

- If the hyperparameters are set so that they have the same marginals, the models are not identical

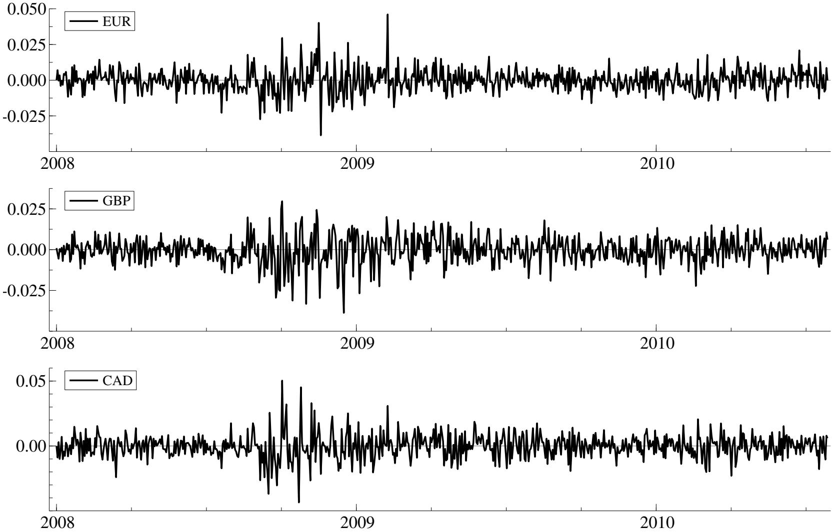
Retrospective posteriors

Sampling from the posterior

$$p(\Phi_{1:T}|\mathcal{D}_T) = p(\Phi_T|\mathcal{D}_T) \prod_{t=1}^{T-1} p(\Phi_t|\Phi_{t+1}, \mathcal{D}_T)$$

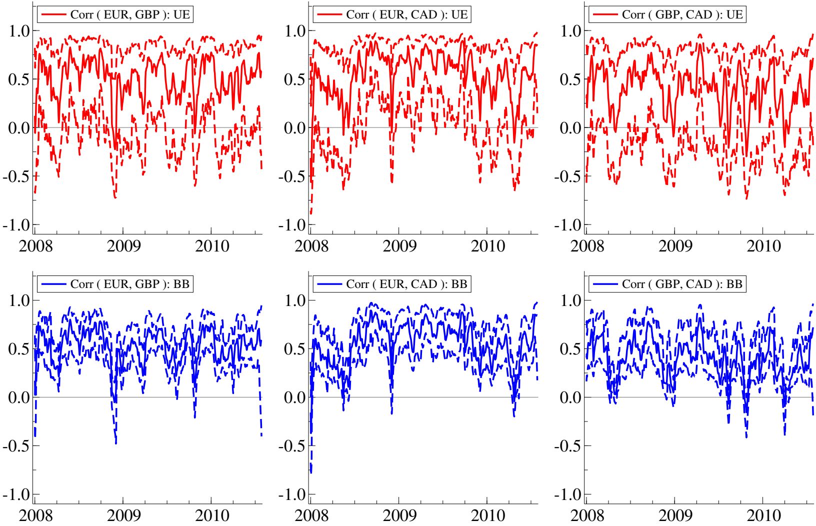
- UE: $\Phi_t^{\mathrm{UE}} = \lambda \Phi_{t+1} + W_q(1, V_t^{-1})$
- BB (new! our contribution): Given Φ_{t+1}^{BB}
  - Decompose Φ_{t+1}^{BB} as $\Phi_{t+1}^{\mathrm{BB}} = (U_{t+1}^* P_t)' U_{t+1}^* P_t / \lambda$
  - Sample $(u_{iit}^*)^2 = (u_{ii,t+1}^*)^2 + \chi^2_{(1-\beta)\nu_t}$ and set $u_{ijt}^* = u_{ij,t+1}^*$
  - Then, $\Phi_t^{\mathrm{BB}} = (U_t^* P_t)' U_t^* P_t$

Data: returns from financial assets



Dynamic correlations

$$\boldsymbol{y}_t = (\mathrm{EUR}_t, \mathrm{GBP}_t, \mathrm{CAD}_t)' \text{ in USD. } (q = 3)$$



How can we compare the models?

- Marginal likelihoods and related metrics can't be used for comparing the models
- <u>Alternatives:</u> Posterior likelihood ratios, mixture models

Posterior likelihoods

- Plug-in posterior draws into your likelihoods
- Take draws $\Phi_{1:T}^{*(i)}$ from $\Phi_{1:T}|\mathcal{D}_T$ and plot

$$\{p(\boldsymbol{y}_{1:T}|\Phi_{1:T}^{*(1)}), \ldots, p(\boldsymbol{y}_{1:T}|\Phi_{1:T}^{*(M)})\}$$

- Do this for both models and compare them

Aitkin (1991)

Mixture estimation model

Fit mixture model

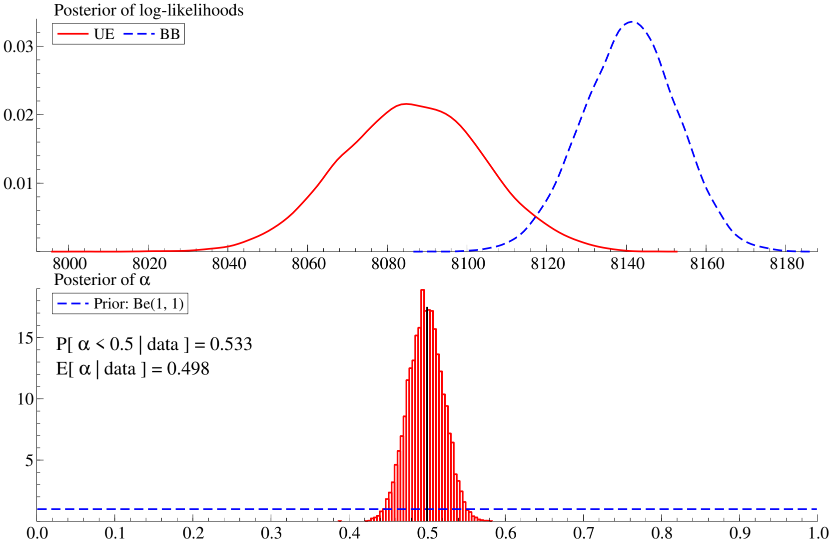
$$
\begin{aligned}
&p_{\mathrm{mix}}(\boldsymbol{y}_t|\alpha, \Phi_t^{\mathrm{UE}}, \Phi_t^{\mathrm{BB}}) \\
&\ = \alpha\ p_{\mathrm{UE}}(\boldsymbol{y}_t|\Phi_t^{\mathrm{UE}}) + (1-\alpha)\ p_{\mathrm{BB}}(\boldsymbol{y}_t|\Phi_t^{\mathrm{BB}}) \\
&\alpha \sim \mathrm{Beta}(a, b)
\end{aligned}
$$

Use posterior on α to compare models

- $\alpha < 1/2$: Support for BB
- $\alpha > 1/2$: Support for UE

Kamary, Mengersen, Roberts and Rousseau (2014)

Post. Likelihoods vs $p(\alpha|\mathcal{D}_T)$



Summary

- Learned that marginal likelihoods can be the same and yet have pretty different models
- It would be really interesting to find more examples of this, and understand when it happens more generally

That's it! Thanks

Matrix Beta distribution

Let $A_1 \sim \text{Wishart}_q(n_1, \Sigma^{-1})$ and $A_2 \sim \text{Wishart}_q(n_2, \Sigma^{-1})$ be independent.

$\Sigma \in \mathcal{S}^q_{++}$, $n_2 > q - 1$ and either $n_1 < q$ is an integer or $n_1 > q - 1$ is real-valued.

Let $T = \text{uchol}(A_1 + A_2)$ and $B = (T^{-1})' A_1 T^{-1}$.

Then, $B \sim \text{MatrixBeta}_q(n_1/2, n_2/2)$.