Crteria for Bayesian hypothesis testing in two-sample problems

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Introduction to Bayesian testing

Bayesian testing

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- To much dismay of scientists (and our students), ***p*-values aren't probabilities of hypotheses** being true given the data.
- Bayesian testing allows us to find them by
  1. Specifying (prior) **probabilities** to anything we don't know, including competing hypotheses.
  2. When we gather data, we condition on it and find probabilities of hypotheses given the data.

Problem solved?

Not really...

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- Bayesian answers in testing can depend strongly on prior probability specifications.
- If we're not careful, our Bayes decisions can be catastrophic.
- As a consequence, there is a vast literature discussing prior choice in testing, dating since (at least) Jeffreys' work at the beginning of the 20th century.

How to choose priors?

Strategy:

- List desirable properties that Bayes decisions should satisfy, usually by taking limits or looking at extreme cases.
- Determine which priors satisfy them, and which don't.

For example, we arguably want...

- **Consistency:** If one of our hypotheses is true and we have infinite data, the posterior probability of the truth should converge to 1.

For other criteria, see Bayarri et al. (2012).

Today

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- I'll propose 2 new properties that (I believe) two-sample tests should satisfy.
- Alternatives to limit consistency, which was proposed recently.
- Application: so far, normal means; there is much yet to be explored!

Limit consistency

- **Limit consistency** was introduced in Chapter 6 of Ly (2017).
- **Context:** Testing if rates of two homogeneous Poisson processes are equal (H_0) or different (H_1).
- **Idea:** If there is infinite data for one of the groups but the data for the other group is fixed, there is no decisive evidence in favor of H_0 or H_1.

Setup & notation

- Independent samples from groups A and B:

$$y_{iA} \overset{\text{iid}}{\sim} P_{\theta_A}, \; y_{jB} \overset{\text{iid}}{\sim} P_{\theta_B},$$

  where the parameters θ_A and θ_B are unknown.
- **Test:** $H_0 : P_{\theta_A} = P_{\theta_B}$ against $H_1 : P_{\theta_A} \neq P_{\theta_B}$.
- In Bayesian hypothesis tests, **all unknowns are modeled probabilistically**.
- We don't know which hypothesis is true and we don't know the parameters, so we specify **priors** on them.

Bayes factor

- Let the conditional densities of the data $\mathcal{D}$ given $\theta_{A,B}$ under H_0 and H_1 be $p(\mathcal{D} \mid \theta_{A,B}, H_0)$ and $p(\mathcal{D} \mid \theta_{A,B}, H_1)$, respectively.
- Let $\pi_0(\mathrm{d}\theta_{A,B})$ and $\pi_1(\mathrm{d}\theta_{A,B})$ be the prior measures on $\theta_{A,B}$ under H_0 and H_1.
- The Bayes factor of H_1 to H_0 is defined as

$$B_{10} = \frac{\int p(\mathcal{D} \mid \theta_{A,B}, H_1)\pi_1(\mathrm{d}\theta_{A,B})}{\int p(\mathcal{D} \mid \theta_{A,B}, H_0)\pi_0(\mathrm{d}\theta_{A,B})} = \frac{p(\mathcal{D} \mid H_1)}{p(\mathcal{D} \mid H_0)}.$$

- If B_{10} is large, the data support H_1. If B_{10} is near 0, the data support H_0.

Limit consistency

Definition

Let the data from group A be fixed and assume that the sample size of group B goes to infinity.

Then, B_{10} is **limit consistent** if $\lim B_{10} = \ell$ with $0 < \ell < \infty$.

Problems

There are 2 things I don't like about the definition.

1. **It only applies to Bayes factors.** What about Bayes decisions in general, or non-Bayesian approaches?
2. **It isn't "ambitious enough."** I would like to converge to a one-sample problem where θ_B is known.

For that reason, I propose an alternative criterion: **limit compatibility**.

Limit compatibility

Definition

Consider the hypothesis tests:

1. One-sample test: $H_{01} : P_{\theta_A} = P_{\theta_B^*}$ against $H_{11} : P_{\theta_A} \neq P_{\theta_B^*}$ with P_{θ_A} unknown and $P_{\theta_B^*}$ known.
2. Two-sample test: $H_{02} : P_{\theta_A} = P_{\theta_B}$ against $H_{12} : P_{\theta_A} \neq P_{\theta_B}$ with θ_A and θ_B unknown.

Let δ_1 and δ_2 be the decision rules for the one- and two-sample test, respectively.

If the data from group A are fixed and the sample size of group B goes to infinity, δ_1 and δ_2 are **limit-compatible** if $\delta_2 \rightarrow \delta_*$ such that $\delta_* = H_{h2} \Leftrightarrow \delta_1 = H_{h1}$ for $h \in \{0, 1\}$.

Application:
Independent normal means

Two normal means

- **Data:** Independent normal samples of sizes n_A and n_B from groups A and B, respectively:

$$y_{iA} \overset{\text{iid}}{\sim} N(\mu_A, \sigma^2), \; y_{jB} \overset{\text{iid}}{\sim} N(\mu_B, \sigma^2),$$

  where σ^2 is known. Group means are denoted $\overline{y}_A$ and $\overline{y}_B$ and the overall mean is $\overline{y}$.
- **Goal:** Compare limits of decision rules for two-sample test $H_{02} : \mu_A = \mu_B$ against $H_{12} : \mu_A \neq \mu_B$ to decisions for analogous one-sample test where $\mu_B = \mu_B^*$ is known.

z-tests

- One- and two-sample z–tests are limit-compatible.
- Indeed, let $Z_n = \sqrt{\kappa}(\overline{y}_A - \overline{y}_B)/\sigma$, $Z_* = \sqrt{n_A}(\overline{y}_A - \mu_B^*)/\sigma$, $z_\alpha/2$ be the $(1 - \alpha/2)\%$ quantile of a $N(0, 1)$.
- Then, the decision rules

$$\delta_2 = H_{02}\,\mathbb{1}\{|Z_n| \leq |z_{\alpha/2}|\} + H_{12}\,\mathbb{1}\{|Z_n| > |z_{\alpha/2}|\}$$
$$\delta_1 = H_{01}\,\mathbb{1}\{|Z_*| \leq |z_{\alpha/2}|\} + H_{11}\,\mathbb{1}\{|Z_*| > |z_{\alpha/2}|\}$$

are limit-compatible as $n_B \to \infty$ because $|Z_n| \to_d |Z_*|$ and the indicators converge.

Bayesian framework

- Losses L_1 and L_2 for one- and two-sample tests of the type:

$$\begin{aligned} L_j(\mu_A, \mu_B, H_{0j}) &= \mathbb{1}(\mu_A \neq \mu_B)\gamma_1 f(|\mu_A - \mu_B|) \\ L_j(\mu_A, \mu_B, H_{1j}) &= \mathbb{1}(\mu_A = \mu_B)\gamma_0, \end{aligned}$$

  where $\gamma_0, \gamma_1, f(|\mu_A - \mu_B|) > 0$.
- Priors of hypotheses are $\mathbb{P}(H_{01}) = \mathbb{P}(H_{02}) = \pi_0$.
- Includes decisions under $\{0, 1\}$-loss by letting $\gamma_0 = \gamma_1$ and $f(|\mu_A - \mu_B|) = 1$.
- Other loss functions are possible, such as adaptations of the ones in Robert and Casella (1994) or Robert (1996).

Bayes decisions

- Let the data be $\mathcal{D}$ and $\bar{f}_j = \mathbb{E}_j[f(|\mu_A - \mu_B|) \mid H_{1j}, \mathcal{D}]$.
- Given this framework, the Bayes decisions are

$$\delta_j^{\pi} = \begin{cases} H_{0j} & \text{if } \mathbb{P}_j(H_{0j} \mid \mathcal{D}) \geq \frac{\gamma_1 \bar{f}_j}{\gamma_0 + \gamma_1 \bar{f}_j} \\ H_{1j} & \text{otherwise,} \end{cases}$$

where

$$\mathbb{P}_j(H_{0j} \mid \mathcal{D}) = \left[1 + \frac{1 - \pi_0}{\pi_0} B_{10,j}\right]^{-1}.$$

- Bayes decisions depend on $\mathcal{D}$ only through the Bayes factor $B_{10,j}$ and $\bar{f}_j$.

Parametrizations

- It turns out that, in this problem, parametrizations matter... a lot.
- We'll consider 3 parametrizations:
  1. **Independent means:** μ_A and μ_B separately.
  2. **Effect-size:** $\mu_A = \mu - \delta/2$ and $\mu_B = \mu + \delta/2$.
  3. **Baseline:** $\mu_A = \mu_A$ and $\mu_B = \mu_A + \alpha$.

Independent priors

Test $H_{02} : \mu_A = \mu_B = \mu$ against $H_{12} : \mu_A \neq \mu_B$ with priors

$$\mu \mid H_{02} \sim N(0, \sigma^2/\omega)$$
$$\mu_A, \mu_B \mid H_{12} \sim N(\mu_A \mid 0, \sigma^2/\omega_A) N(\mu_B \mid 0, \sigma^2/\omega_B).$$

The Bayes factor of H_{12} to H_{02} is

$$B_{10,2} = \left[\frac{\omega_A \omega_B (\omega + n)}{(\omega_A + n_A)(\omega_B + n_B)\omega} \right]^{1/2} \exp\left\{ \frac{Q}{2} \right\}.$$
$$\sigma^2 Q = \frac{n_A^2 \overline{y}_A^2}{n_A + \omega_A} + \frac{n_B^2 \overline{y}_B^2}{n_B + \omega_B} - \frac{n^2 \overline{y}^2}{n + \omega}$$

Independent priors

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As $n_B \to \infty$,

$$B_{10,2} \to_d B_l = \left(\frac{\omega}{\omega_B} + \frac{\omega n_A}{\omega_A \omega_B} \right)^{-1/2} \exp\left\{ \frac{Q_*}{2} \right\}$$

$$Q_* = Z_*^2 + \frac{(\omega - \omega_B)(\mu_B^*)^2}{\sigma^2} - \frac{n_A \omega_A \bar{y}_A^2}{\sigma^2(n_A + \omega_A)}.$$

Non-vanishing dependence on prior for μ_B unless $\omega_B = \omega$!

Question

- Is there **any** prior specification for the one-sample test where μ_B^* is known that yields B_l as a Bayes factor?
- That is, is there any limit-compatible prior?

Let's find out

- In the one-sample test $H_{11} : \mu_A \neq \mu_B^*$ against $H_{01} : \mu_A = \mu_B^*$, the Bayes factor of H_{11} to H_{01} can be written as

  $$B_{10,1} = \frac{\int N(\overline{y}_A \mid \mu_A, \sigma^2/n_A)\pi_1(\mu_A \mid H_{11})\mathrm{d}\mu_A}{N(\overline{y}_A \mid \mu_B^*, \sigma^2/n_A)}.$$

  The integral of the numerator with respect to $\overline{y}_A$ must be equal to 1

Independent priors

- If B_l were to be $B_{10,1}$, the numerator should be

$$\left(\frac{\omega}{\omega_B}\right)^{-1/2} \exp\left\{\frac{(\omega - \omega_B)(\mu_B^*)^2}{2\sigma^2}\right\} N\left(\bar{y}_A \,|\, 0, \frac{\sigma^2(n_A + \omega_A)}{n_A \omega_A}\right).$$

- There are only 2 choices of $\omega, \omega_B > 0$ so that the integral wrt $\bar{y}_A$ is equal to 1: $\omega = \omega_B$, and another one where the values of ω depend on μ_B^* (which is cheating).

Independent priors

- The choice $\omega = \omega_B$ converges to a Bayes factor the one-sample problem $H_{01} : \mu_A = \mu_B^*$ against $H_{11} : \mu_A \neq \mu_B^*$ with μ_B^* known.
- The prior for the one-sample problem is $\mu_A \mid H_{11} \sim N(0, \sigma^2/\omega_A)$.
- Hard to justify as a default choice unless $\mu_B^* = 0$, since it assumes a preference of values of μ_A near 0 whenever $\mu_A \neq \mu_B^*$.
- For example, the value of $\bar{y}_A$ that minimizes $B_{10,1}$ is $\bar{y}_A = (1 + \omega/n_A)\mu_B^*$ instead of $\bar{y}_A = \mu_B^*$.

Independent priors

- **Recall:** Limit compatibility requires convergence of Bayes factors and **posterior expectations.**
- For any $\omega_B > 0$, the posterior of μ_B collapses to a point mass at μ_B^* and the posterior of μ_A doesn't change.
- The posterior is compatible with the prior $\mu_A \mid H_{11} \sim N(0, \sigma^2/\omega_A)$ for the one-sample test.
- **NB:** Unlike with the Bayes factor, compatibility doesn't depend on the value of ω_B.

Summary

- With independent priors, limit compatibility does not hold unless $\omega_B = \omega$.
- Even then, the compatible prior isn't entirely satisfactory: it is centered at 0 instead of μ_B^*.

Effect-size

- Let $\mu_A = \mu - \delta/2$ and $\mu_B = \mu + \delta/2$.
- δ is the effect size $\delta = \mu_B - \mu_A$.
- Two-sample test becomes $H_{02} : \delta = 0$ against $H_{12} : \delta \neq 0$.
- Prior specification: $\mu \sim N(0, \sigma^2/\lambda)$ independent of $\delta \mid H_{12} \sim N(0, \sigma^2/\omega)$.

Effect-size

As $n_B \to \infty$,

$$B^{\delta}_{10,2} \to_d B_l = (1 + \kappa_l/\omega)^{-1/2} \exp\left\{ \frac{\kappa_l}{2(\kappa_l + \omega)} Z_l^2 \right\}$$

$$\kappa_l = \lambda/4 + n_A$$

$$Z_l = \frac{n_A(\mu^*_B - \overline{y}_A)}{\sigma\sqrt{\kappa_l}} + \frac{\lambda \mu^*_B}{2\sigma\sqrt{\kappa_l}}.$$

Bayes factor

- Only two values of $\eta = 1 + \lambda/(4\omega)$ have compatible Bayes factors wrt the one-sample problem.
  1. $\eta = 1$, which comprises the uninteresting case $\omega \to \infty$ (the prior for δ is point mass at 0) and $\lambda \to 0$, which corresponds to a flat prior on the common parameter.
  2. Root of $4\omega(\mu_B^*)^2(\eta - 1)^2/\sigma^2 = \eta \log \eta$ greater than 1, which depends on μ_B^*.
- If $\lambda = 0$, we converge to the Bayes factor under the **desirable** prior $\mu_A \mid H_{11} \sim N(\mu_B^*, \sigma^2/\omega_A)$, centered at hypothesized value under H_{01}.

Posterior

- The posterior of $\mu_A = \mu - \delta/2$ converges weakly to $N(n_A \bar{y}_A + (\omega - \lambda/4)\mu_B^*)/(n_A + \lambda/4 + \omega), \sigma^2/(n_A + \lambda/4 + \omega))$.
- There are compatible posteriors if $\lambda = 0$ or $\lambda = 4\omega$.
- If $\lambda = 4\omega$, the compatible prior is $\mu_A \sim N(0, \sigma^2/(2\omega))$, which is centered at 0 (not μ_B^*). For this choice of λ there is no compatible Bayes factor.

Summary

- Putting a **flat prior** on common parameter is the **only** possibility that yields limit compatibility.
- The compatible prior is **reasonable**: a normal prior on μ_A under the alternative, centered at μ_B^*.
- Flat priors on common parameters are recommended in the related literature (see e.g. Bayarri et al. (2012)). This can be seen as another justification.

Baseline

- Suppose we parametrize the problem so that the mean of group A is μ_A and the mean of group B is $\mu_A + \alpha$.
- The test becomes $H_{02} : \alpha = 0$ against $H_{12} : \alpha \neq 0$.
- Priors: $\mu_A \sim N_1(0, \sigma^2/\lambda)$ independent of $\alpha \mid H_{12} \sim N_1(0, \sigma^2/\omega)$.

Baseline

Without taking any limits, the Bayes factor of H_{12} to H_{02} is

$$
\begin{aligned}
B^{\alpha}_{10,2} &= (1 + \kappa_\alpha/\omega)^{-1/2} \exp\left\{ \frac{\kappa_\alpha Z_\alpha^2}{2(\kappa_\alpha + \omega)} \right\} \\
\kappa_\alpha &= \frac{\lambda n_B + n_A n_B}{n + \lambda} \\
Z_\alpha &= \frac{n_B n_A (\bar{y}_B - \bar{y}_A) + \lambda n_B \bar{y}_B}{\sigma \sqrt{k_\alpha}(n + \lambda)}.
\end{aligned}
$$

It depends the choice of baseline unless $\lambda = 0$!

Summary

- As with the effect-size parametrization, there is no realistic Bayes factor that is compatible for the one-sample test unless $\lambda = 0$.
- In the case $\lambda = 0$, the baseline and effect-size parametrization yield identical Bayes factors and posteriors on μ_A, μ_B.

Conditional compatibility

Another criterion

- In a Bayesian two-sample test, conditioning on $\mu_B = \mu_B^*$ should yield a Bayes decision that arises under a prior specification the one-sample problem where μ_B^* is known.
- Ideally, conditioning on $\mu_B = \mu_B^*$ should also be the same as taking the limit as $n_B \to \infty$.
- Surprisingly, conditioning and taking limits aren't always the same.

Cond'l compatibility

Definition

Consider the Bayesian hypothesis tests:

1. $H_{01} : P_{\theta_A} = P_{\theta_B^*}$ against $H_{11} : P_{\theta_A} \neq P_{\theta_B^*}$ with P_{θ_A} unknown and $P_{\theta_B^*}$ known, with prior specification π_1.
2. $H_{02} : P_{\theta_A} = P_{\theta_B}$ against $H_{12} : P_{\theta_A} \neq P_{\theta_B}$ with θ_A and θ_B unknown, with prior specification π_2.

Let δ_1 be the Bayes decision for the one-sample test and δ_* be the conditional Bayes decision for the two-sample test upon conditioning on $\theta_B = \theta_B^*$.

The decision rules δ_1 and δ_* are **conditionally compatible** if $\delta_* = H_{h2} \Leftrightarrow \delta_1 = H_{h1}$ for $h \in \{0, 1\}$.

Summary

- With independent priors or effect-size and baseline parametrizations with proper priors, **everything breaks down.**
- With baseline or effect-size parametrization and **flat** priors on “common parameter”, **everything is fine.**

Baseline

- Parametrize $\mu_B = \mu_A + \alpha$ and work with priors $\mu_A \sim N(0, \sigma^2/\lambda)$ and $\alpha \mid H_{12} \sim N(0, \sigma^2/\omega)$.
- After conditioning on $\mu_B = \mu_B^*$, the prior is

$$\mu_A \mid \mu_B = \mu_B^*, H_{21} \sim N\left[\frac{\omega}{\lambda+\omega}\mu_B^*, \frac{\sigma^2}{\lambda+\omega}\right]$$

Not centered at μ_B^* unless $\lambda = 0$.

Baseline

The ratio of the limiting Bayes factor as $n_B \to \infty$ to the conditional Bayes factor is

$$\frac{B_l}{B_{\text{cond}}} = (1 + \lambda/\omega)^{-1/2} \exp\left\{\frac{\lambda(\mu_B^*)^2}{2\sigma^2(\lambda + \omega)}\right\}$$

- In general, not equal to 1 unless $\lambda = 0$.
- If $\mu_B^* = 0$, we have $B_l < B_{\text{cond}}$. If $\mu_B^* \neq 0$, generally $B_l > B_{\text{cond}}$ unless μ^*/σ is very small.

Effect-size

- Similar story as with baseline parametrization.
- Let $\mu_A = \mu - \delta/2$ and $\mu_B = \mu + \delta/2$.
- Priors $\mu \sim N(0, \sigma^2/\lambda)$ and $\delta \mid H_{12} \sim N(0, \sigma^2/\omega)$.
- Ratio "limit /conditional:"

$$\frac{B_l}{B_{\text{cond}}} = [1 + \lambda/(4\omega)]^{-1/2} \exp\left\{ \frac{\lambda(\mu_B^*)^2}{2\sigma^2(\lambda + 4\omega)} \right\}$$

Beyond normal priors

Non-local priors

- **What are they?** Consider the hypothesis test $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$. A prior for $\theta \mid H_1$ is said to be non-local if it vanishes as θ approaches Θ_0 (Johnson and Rossell, 2010).
- Bayes factors under non-local priors have been seen to attain faster rates of convergence to the "truth" under the null hypothesis (Johnson and Rossell, 2010, 2012)

Local vs non-local

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- As noted in Consonni et al. (2013), Bayes decisions under non-local priors and $\{0, 1\}$-loss are equivalent to Bayes decisions under local priors with respect to loss functions that involve parameters.
- The results I showed you (sort of) include this case.
- **Result:** Non-local priors proposed in the literature have the same behavior as local priors.

Local vs non-local

- **Framework:** Losses L_1 and L_2 for one- and two-sample tests of the type:

$$L_j(\mu_A, \mu_B, H_{0j}) = \mathbb{1}(\mu_A \neq \mu_B)\gamma f(|\mu_A - \mu_B|)$$
$$L_j(\mu_A, \mu_B, H_{1j}) = \mathbb{1}(\mu_A = \mu_B)\gamma,$$

  where $\gamma, f(|\mu_A - \mu_B|) > 0$.
- Decisions with non-local priors and $f(|\mu_A - \mu_B|) = 1$ are equivalent to decisions with local priors and choices of f which depend on $|\mu_A - \mu_B|$.

Local vs non-local

- Let $\delta = |\mu_A - \mu_B|$ and consider the "penalty" functions

$$f_M(\delta) = \omega\delta^2/\sigma^2$$
$$f_E(\delta) = \exp\left\{\sqrt{2} - \frac{\sigma^2}{\omega\delta^2}\right\}$$
$$f_I(\delta) = \frac{\sqrt{2}\sigma^2}{\omega\delta^2}\exp\left\{-\frac{\sigma^2}{\omega\delta^2} + \frac{\omega\delta^2}{2\sigma^2}\right\}$$

- Decisions with normal priors and f_M, f_E, and f_I are equivalent to decisions with non-local moment, exponential, and inverse-moment priors and $f(\delta) = 1$, respectively.

g-priors

- Let's work with the baseline parametrization where $\mu_B = \mu_A + \alpha$ and $\pi(\mu_A) \propto 1$.
- In this context, Zellner's g-prior (after centering the design matrix) is $\alpha \mid H_{12} \sim N(0, gn\sigma^2/(n_A n_B))$ which yields

$$B_{10,2} = (g+1)^{-1/2} \exp\left\{ \frac{g Z_n^2}{2(g+1)} \right\}.$$

- A common choice is $g = n$, but that implies $B_{10,2} \to 0$ as $n_B \to \infty$ (it isn't conditionally compatible, either...)

g-priors

- An alternative to $g = n$ is $g = n_A n_B / n$.
- As $n_B \to \infty$, g behaves like n_A and yields compatible Bayes factors with unit information priors for the one-sample problem.
- As both $n_A, n_B \to \infty$, $n_A n_B / n$ acts as the usual polynomial model complexity penalty in local Bayes factors.
- Note that g constant isn't desirable because $B_{10,2}$ wouldn't be consistent under H_{02}.

Thicker-tailed priors

- Preliminary results with mixtures of normals show **identical behavior** as with normal priors.

HPD regions

- Some people like avoiding Bayes factors and like making decisions by checking whether high posterior density regions include the hypothesized value under the null.

Results

- **Indep. priors:** Compatibility with prior on μ_A centered at 0.
- **Effect-size:** Flat prior on common parameter is compatible with prior on μ_A centered at μ_B^*. Non-flat priors on common parameter aren't compatible with any prior unless $\lambda = 4\omega$,which is compatible with a prior on μ_A centered at 0.
- **Baseline:** Flat prior on grand mean is OK; any other choice isn't compatible.

Conclusions and future work

Conclusions

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- Proposed new criteria to defend or propose prior choice in two-sample problems.
- In testing normal means, they **strongly recommend parametrizations with common parameters, and flat priors on them**.
- Have to be careful in how we specify the prior scale of the prior.

Future work

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- Normal means: More than 2 groups.
- Beyond normal means: testing 2 proportions, nonparametric tests of equality of distributions, etc.
- Suggestions?

Thanks!

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