

$$\nabla^2 u = \frac{1}{a^2 (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial u}{\partial \theta_3} \right) + \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial u}{\partial \chi} \right) \right) + \frac{1}{a^2 \sin^2 \theta_3 \cosh^2 \chi} \nabla_{S^2}^2 u = 0$$

Tal que

$$\nabla_{S^2}^2 u = \left[\left(\frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial u}{\partial \theta_2} \right) \right) + \frac{1}{\sin^2 \theta_2} \frac{\partial^2 u}{\partial \theta_1^2} \right]$$

É o Laplaciano na esfera S^2 .

Para realizar a separação de variáveis façamos

$$u(\chi, \theta_1, \theta_2, \theta_3) = Y(\theta_1, \theta_2) W(\chi, \theta_3)$$

Aplicando na Equação

$$\nabla^2 u = \frac{Y}{a^2 (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial W}{\partial \theta_3} \right) + \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial W}{\partial \chi} \right) \right) + \frac{W}{a^2 \sin^2 \theta_3 \cosh^2 \chi} (\nabla_{S^2}^2 Y) = 0$$

Multiplicando os dois lados por $\frac{a^2 \sin^2 \theta_3 \cosh^2 \chi}{W Y}$

$$\frac{\sin^2 \theta_3 \cosh^2 \chi}{W (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial W}{\partial \theta_3} \right) + \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial W}{\partial \chi} \right) \right) + \frac{1}{Y} (\nabla_{S^2}^2 Y) = 0$$

Então, tem-se

$$\frac{\sin^2 \theta_3 \cosh^2 \chi}{W (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial W}{\partial \theta_3} \right) + \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial W}{\partial \chi} \right) \right) = -\frac{1}{Y} (\nabla_{S^2}^2 Y) = \lambda_1$$

De modo que chega-se em duas EDPs

$$\frac{\sin^2 \theta_3 \cosh^2 \chi}{W (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial W}{\partial \theta_3} \right) + \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial W}{\partial \chi} \right) \right) = \lambda_1$$

$$\nabla_{S^2}^2 Y = -Y \lambda_1$$

Assumindo

$$Y(\theta_1, \theta_2) = \Theta_1(\theta_1) \Theta_2(\theta_2)$$

$$W(\chi, \theta_3) = X(\chi) \Theta_3(\theta_3)$$

Tem-se

0.1 Parte Esférica

A equação $\nabla_{S^2}^2 Y = -Y \lambda_1$ fica da seguinte forma

$$\nabla_{S^2}^2 Y = \left[\left(\frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial Y}{\partial \theta_2} \right) \right) + \frac{1}{\sin^2 \theta_2} \frac{\partial^2 Y}{\partial \theta_1^2} \right] = -Y \lambda_1$$

A solução dessa equação é expressa em termos de harmônicos esféricos.

$$Y(\theta_1, \theta_2) = \Theta_1(\theta_1) \Theta_2(\theta_2) = Y_{n,m}(\theta_1, \theta_2)$$

0.2 Analisando a outra EDP

$$\frac{\sin^2 \theta_3 \cosh^2 \chi}{W (\sinh^2 \chi + \cos^2 \theta_3)} \left(\frac{X}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) + \frac{\Theta_3}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) \right) = \lambda_1$$

Multiplica-se os dois lados por $\frac{W(\sinh^2 \chi + \cos^2 \theta_3)}{X \Theta_3 \sin^2 \theta_3 \cosh^2 \chi}$

$$\begin{aligned} \frac{1}{X \Theta_3} \left(\frac{X}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) + \frac{\Theta_3}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) \right) &= \lambda_1 \frac{(\sinh^2 \chi + \cos^2 \theta_3)}{\sin^2 \theta_3 \cosh^2 \chi} \\ \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) + \frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) &= \lambda_1 \frac{(\sinh^2 \chi + \cos^2 \theta_3)}{\sin^2 \theta_3 \cosh^2 \chi} \end{aligned}$$

Veja que

$$\begin{aligned} \frac{(\sinh^2 \chi + \cos^2 \theta_3)}{\sin^2 \theta_3 \cosh^2 \chi} &= \frac{\sinh^2 \chi}{\sin^2 \theta_3 \cosh^2 \chi} + \frac{\cos^2 \theta_3}{\sin^2 \theta_3 \cosh^2 \chi} = \frac{\cosh^2 \chi - 1}{\sin^2 \theta_3 \cosh^2 \chi} + \frac{1 - \sin^2 \theta_3}{\sin^2 \theta_3 \cosh^2 \chi} \\ &= \frac{1}{\sin^2 \theta_3} - \frac{1}{\sin^2 \theta_3 \cosh^2 \chi} + \frac{1}{\sin^2 \theta_3 \cosh^2 \chi} - \frac{1}{\cosh^2 \chi} = \frac{1}{\sin^2 \theta_3} - \frac{1}{\cosh^2 \chi} \end{aligned}$$

Então a equação fica

$$\begin{aligned} \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) + \frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) &= \lambda_1 \frac{1}{\sin^2 \theta_3} - \lambda_1 \frac{1}{\cosh^2 \chi} \\ \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) - \lambda_1 \frac{1}{\sin^2 \theta_3} + \frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) + \lambda_1 \frac{1}{\cosh^2 \chi} &= 0 \\ \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial \Theta_3}{\partial \theta_3} \right) - \lambda_1 \frac{1}{\sin^2 \theta_3} &= -\frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left(\cosh^2 \chi \frac{\partial X}{\partial \chi} \right) - \lambda_1 \frac{1}{\cosh^2 \chi} = \lambda_2 \end{aligned}$$

Fica-se com

$$\begin{aligned} \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{d}{d\theta_3} \left(\sin^2 \theta_3 \frac{d\Theta_3}{d\theta_3} \right) - \lambda_1 \frac{1}{\sin^2 \theta_3} &= \lambda_2 \\ \frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{d}{d\chi} \left(\cosh^2 \chi \frac{dX}{d\chi} \right) + \lambda_1 \frac{1}{\cosh^2 \chi} &= -\lambda_2 \end{aligned}$$

Ou, de outra forma, tem-se

$$\begin{aligned} \frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{d}{d\theta_3} \left(\sin^2 \theta_3 \frac{d\Theta_3}{d\theta_3} \right) - \left(\frac{\lambda_1}{\sin^2 \theta_3} + \lambda_2 \right) &= 0 \\ \frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{d}{d\chi} \left(\cosh^2 \chi \frac{dX}{d\chi} \right) + \left(\lambda_2 + \frac{\lambda_1}{\cosh^2 \chi} \right) &= 0 \end{aligned}$$

0.2.1 Equação para θ_3

$$\frac{1}{\Theta_3} \frac{1}{\sin^2 \theta_3} \frac{d}{d\theta_3} \left(\sin^2 \theta_3 \frac{d\Theta_3}{d\theta_3} \right) - \left(\lambda_1 \frac{1}{\sin^2 \theta_3} + \lambda_2 \right) = 0$$

Aplicando troca de variáveis $\xi = \cos \theta_3$

$$\sin^2(\theta_3) = 1 - \cos^2(\theta_3) = 1 - \xi^2$$

$$\sin^2 \theta_3 \frac{d\Theta_3}{d\theta_3} = \sin^2 \theta_3 \frac{d\Theta_3}{d\xi} \frac{d\xi}{d\theta_3} = -\sin^3(\theta_3) \frac{d\Theta_3}{d\xi}$$

$$\frac{d}{d\theta_3} \left(\sin^2 \theta_3 \frac{d\Theta_3}{d\theta_3} \right) = -\frac{d}{d\theta_3} \left(\sin^3(\theta_3) \frac{d\Theta_3}{d\xi} \right) = -\left(3 \cos(\theta_3) \sin^2(\theta_3) \frac{d\Theta_3}{d\xi} + \sin^3(\theta_3) \frac{d}{d\theta_3} \frac{d\Theta_3}{d\xi} \right)$$

$$\begin{aligned}
&= - \left(3 \cos(\theta_3) \sin^2(\theta_3) \frac{d\Theta_3}{d\xi} + \sin^3(\theta_3) \frac{d}{d\xi} \frac{d\Theta_3}{d\theta_3} \right) = - \left(3 \cos(\theta_3) \sin^2(\theta_3) \frac{d\Theta_3}{d\xi} - \sin^4(\theta_3) \frac{d^2\Theta_3}{d\xi^2} \right) \\
&= - \sin^2(\theta_3) \left(3 \cos(\theta_3) \frac{d\Theta_3}{d\xi} - \sin^2(\theta_3) \frac{d^2\Theta_3}{d\xi^2} \right)
\end{aligned}$$

A equação fica

$$\begin{aligned}
&-\frac{1}{\Theta_3} \left(3 \cos(\theta_3) \frac{d\Theta_3}{d\xi} - \sin^2(\theta_3) \frac{d^2\Theta_3}{d\xi^2} \right) - \left(\lambda_1 \frac{1}{\sin^2 \theta_3} + \lambda_2 \right) = 0 \\
&\frac{1}{\Theta_3} \left(3\xi \frac{d\Theta_3}{d\xi} - (1 - \xi^2) \frac{d^2\Theta_3}{d\xi^2} \right) + \left(\frac{\lambda_1}{(1 - \xi^2)} + \lambda_2 \right) = 0
\end{aligned}$$

0.2.2 Equação para χ

$$\frac{1}{X} \frac{1}{\cosh^2 \chi} \frac{d}{d\chi} \left(\cosh^2 \chi \frac{dX}{d\chi} \right) + \left(\lambda_2 + \frac{\lambda_1}{\cosh^2 \chi} \right) = 0$$

Aplicando $\zeta = \sinh \chi$

$$\cosh^2 \chi = 1 + \sinh \chi = 1 + \zeta^2$$

$$\cosh^2 \chi \frac{dX}{d\chi} = \cosh^2 \chi \frac{dX}{d\zeta} \frac{d\zeta}{d\chi} = \cosh^3 \chi \frac{dX}{d\zeta}$$

$$\begin{aligned}
\frac{d}{d\chi} \left(\cosh^2 \chi \frac{dX}{d\chi} \right) &= \frac{d}{d\chi} \left(\cosh^3 \chi \frac{dX}{d\zeta} \right) = 3 \cosh^2 \chi \sinh \chi \frac{dX}{d\zeta} + \cosh^3 \chi \frac{d}{d\chi} \frac{dX}{d\zeta} \\
&= 3 \cosh^2 \chi \sinh \chi \frac{dX}{d\zeta} + \cosh^3 \chi \frac{d}{d\zeta} \frac{dX}{d\chi} = 3 \cosh^2 \chi \sinh \chi \frac{dX}{d\zeta} + \cosh^4 \chi \frac{d^2 X}{d\zeta^2} \\
&= \cosh^2 \chi \left(3 \sinh \chi \frac{dX}{d\zeta} + \cosh^2 \chi \frac{d^2 X}{d\zeta^2} \right)
\end{aligned}$$

A equação fica

$$\frac{1}{X} \left(3\zeta \frac{dX}{d\zeta} + (1 + \zeta^2) \frac{d^2 X}{d\zeta^2} \right) + \left(\lambda_2 + \frac{\lambda_1}{(1 + \zeta^2)} \right) = 0$$