De modo geral, para verificar a ortogonalidade do sistema de coordenadas é necessário verificar se os vetores da base do sistema ($\boldsymbol{e}_i$) são ortogonais. Para isso, basta verificar se o produto escalar, entre dois vetores da base, é nulo ou seja:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$$

Porém, antes disso, é necessário encontrar os vetores da base. Para isto, basta tomar $\boldsymbol{r}(q_1, q_2, q_3, ..., q_n)$ e tem-se o seguinte:

$$\boldsymbol{e}_i = \frac{\partial \boldsymbol{r}}{\partial q_i}$$

Verificando Ortogonalidade do Hiperbolóide 3-dim (3-Hiperboloide?) de 1 Folha

Verificar a ortogonalidade do seguinte sistema de coordenadas

$$\begin{array}{rcl} x_3 &=& a\sinh\chi\cos\theta_2 \\ x_2 &=& a\cosh\chi\sin\theta_2\cos\theta_1 \\ x_1 &=& a\cosh\chi\sin\theta_2\sin\theta_1 \\ 0 &\leq& \chi < \infty,\ 0 \leq \theta_2 \leq \pi,\ 0 \leq \theta_1 < 2\pi \end{array}$$

Define-se

$$\boldsymbol{r} = (a\cosh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + (a\cosh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + (a\sinh\chi\cos\theta_2)\widehat{\boldsymbol{k}}$$

Tem-se

$$\begin{aligned} \boldsymbol{e}_{\theta_1} = \frac{\partial \boldsymbol{r}}{\partial \theta_1} &= \left(\frac{\partial}{\partial \theta_1}(a\cosh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial \theta_1}(a\cosh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial \theta_1}(a\sinh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right) \\ &= \left(a\,cosh(\chi)cos(\theta_1)sin(\theta_2)\widehat{\boldsymbol{i}} - a\,cosh(\chi)sin(\theta_1)sin(\theta_2)\widehat{\boldsymbol{j}} + 0\widehat{\boldsymbol{k}}\right) \end{aligned}$$

$$\begin{aligned} \boldsymbol{e}_{\theta_2} = \frac{\partial \boldsymbol{r}}{\partial \theta_2} &= \left(\frac{\partial}{\partial \theta_2}(a\cosh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial \theta_2}(a\cosh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial \theta_2}(a\sinh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right) \\ &= \left(a\,cosh(\chi)sin(\theta_1)cos(\theta_2)\widehat{\boldsymbol{i}} + a\,cosh(\chi)cos(\theta_1)cos(\theta_2)\widehat{\boldsymbol{j}} - a\,sinh(\chi)sin(\theta_2)\widehat{\boldsymbol{k}}\right) \\ \boldsymbol{e}_{\chi} = \frac{\partial \boldsymbol{r}}{\partial \chi} &= \left(\frac{\partial}{\partial \chi}(a\cosh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial \chi}(a\cosh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial \chi}(a\sinh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right) \\ &= \left(a\,sinh(\chi)sin(\theta_1)sin(\theta_2)\widehat{\boldsymbol{i}} + a\,sinh(\chi)cos(\theta_1)sin(\theta_2)\widehat{\boldsymbol{j}} + a\,cosh(\chi)cos(\theta_2)\widehat{\boldsymbol{k}}\right) \end{aligned}$$

Agora basta verificar a ortogonalidade dois a dois entre estes vetores.

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\theta_2} = \left(a^2\,cosh^2(\chi)cos(\theta_1)sin(\theta_2)sin(\theta_1)cos(\theta_2) - a^2\,cosh^2(\chi)sin(\theta_1)sin(\theta_2)cos(\theta_1)cos(\theta_2)\right) = 0$$

$$\boldsymbol{e}_{\theta_1}\cdot\boldsymbol{e}_\chi = \left(a^2\, cosh(\chi)cos(\theta_1)sin^2(\theta_2)sinh(\chi)sin(\theta_1) - a^2\, cosh(\chi)sin(\theta_1)sin^2(\theta_2)sinh(\chi)cos(\theta_1)\right) = 0$$

$$\boldsymbol{e}_{\theta_2}\cdot\boldsymbol{e}_\chi = \big(a^2\, cosh(\chi)sin^2(\theta_1)cos(\theta_2)\, sinh(\chi)sin(\theta_2) + a^2\, cosh(\chi)cos^2(\theta_1)cos(\theta_2)sinh(\chi)sin(\theta_2)$$

$$-a^2\, sinh(\chi)sin(\theta_2)cosh(\chi)cos(\theta_2)\big)$$

$$= a^2\, cosh(\chi)cos(\theta_2)\, sinh(\chi)sin(\theta_2)\left(sin^2(\theta_1)+cos^2(\theta_1)\right) - a^2\, sinh(\chi)sin(\theta_2)cosh(\chi)cos(\theta_2)$$

$$= a^2\, cosh(\chi)cos(\theta_2)\, sinh(\chi)sin(\theta_2) - a^2\, sinh(\chi)sin(\theta_2)cosh(\chi)cos(\theta_2) = 0$$

Verificada a ortogonalidade entre os vetores da base, pode-se dizer que o sistema de coordenadas é ortogonal.

Verificando Ortogonalidade do Hiperbolóide 3-dim (3-Hiperboloide?) de 2 Folhas

Verificar a ortogonalidade do seguinte sistema de coordenadas

$$\begin{aligned}
x_3 &= a\cosh\chi\cos\theta_2\\
x_2 &= a\sinh\chi\sin\theta_2\cos\theta_1\\
x_1 &= a\sinh\chi\sin\theta_2\sin\theta_1\\
0 &\le \chi<\infty,\ 0\le\theta_2\le\pi,\ 0\le\theta_1<2\pi
\end{aligned}$$

Define-se

$$\boldsymbol{r} = (a\sinh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + (a\sinh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + (a\cosh\chi\cos\theta_2)\widehat{\boldsymbol{k}}$$

Tem-se

$$\boldsymbol{e}_{\theta_1} = \frac{\partial\boldsymbol{r}}{\partial\theta_1} = \left(\frac{\partial}{\partial\theta_1}(a\sinh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial\theta_1}(a\sinh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial\theta_1}(a\cosh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right)$$

$$= \left(a\, sinh(\chi)cos(\theta_1)sin(\theta_2)\widehat{\boldsymbol{i}} - a\, sinh(\chi)sin(\theta_1)sin(\theta_2)\widehat{\boldsymbol{j}} + 0\widehat{\boldsymbol{k}}\right)$$

$$\boldsymbol{e}_{\theta_2} = \frac{\partial\boldsymbol{r}}{\partial\theta_2} = \left(\frac{\partial}{\partial\theta_2}(a\sinh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial\theta_2}(a\sinh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial\theta_2}(a\cosh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right)$$

$$= \left(a\, sinh(\chi)sin(\theta_1)cos(\theta_2)\widehat{\boldsymbol{i}} + a\, sinh(\chi)cos(\theta_1)cos(\theta_2)\widehat{\boldsymbol{j}} - a\, cosh(\chi)sin(\theta_2)\widehat{\boldsymbol{k}}\right)$$

$$\boldsymbol{e}_\chi = \frac{\partial\boldsymbol{r}}{\partial\chi} = \left(\frac{\partial}{\partial\chi}(a\sinh\chi\sin\theta_2\sin\theta_1)\widehat{\boldsymbol{i}} + \frac{\partial}{\partial\chi}(a\sinh\chi\sin\theta_2\cos\theta_1)\widehat{\boldsymbol{j}} + \frac{\partial}{\partial\chi}(a\cosh\chi\cos\theta_2)\widehat{\boldsymbol{k}}\right)$$

$$= \left(a\, cosh(\chi)sin(\theta_1)sin(\theta_2)\widehat{\boldsymbol{i}} + a\, cosh(\chi)cos(\theta_1)sin(\theta_2)\widehat{\boldsymbol{j}} + a\, sinh(\chi)cos(\theta_2)\widehat{\boldsymbol{k}}\right)$$

Agora basta verificar a ortogonalidade dois a dois entre estes vetores.

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\theta_2} = \big(a^2\, sinh^2(\chi) cos(\theta_1) sin(\theta_2) sin(\theta_1) cos(\theta_2) - a^2\, sinh^2(\chi) sin(\theta_1) sin(\theta_2) cos(\theta_1) cos(\theta_2)\big) = 0$$

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\chi} = \big(a^2\, cosh(\chi) cos(\theta_1) sin^2(\theta_2) sinh(\chi) sin(\theta_1) - a^2\, cosh(\chi) sin(\theta_1) sin^2(\theta_2) sinh(\chi) cos(\theta_1)\big) = 0$$

$$\boldsymbol{e}_{\theta_2} \cdot \boldsymbol{e}_{\chi} = \big(a^2\, cosh(\chi) sin^2(\theta_1) cos(\theta_2)\, sinh(\chi) sin(\theta_2) + a^2\, cosh(\chi) cos^2(\theta_1) cos(\theta_2) sinh(\chi) sin(\theta_2)$$

$$-a^2\, sinh(\chi) sin(\theta_2) cosh(\chi) cos(\theta_2)\big)$$

$$= a^2\, cosh(\chi) cos(\theta_2)\, sinh(\chi) sin(\theta_2)\, \big(sin^2(\theta_1) + cos^2(\theta_1)\big) - a^2\, sinh(\chi) sin(\theta_2) cosh(\chi) cos(\theta_2)$$

$$= a^2\, cosh(\chi) cos(\theta_2)\, sinh(\chi) sin(\theta_2) - a^2\, sinh(\chi) sin(\theta_2) cosh(\chi) cos(\theta_2) = 0$$

Verificada a ortogonalidade entre os vetores da base, pode-se dizer que o sistema de coordenadas é ortogonal.

Verificando Ortogonalidade do Hiperbolóide 4-dim (4-Hiperboloide?) de 1 Folha

Verificar a ortogonalidade do seguinte sistema de coordenadas

$$\begin{array}{rcl} x_4 &=& a \sinh\chi \cos\theta_3 \\ x_3 &=& a \cosh\chi \sin\theta_3 \cos\theta_2 \\ x_1 &=& a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1 \\ x_2 &=& a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1 \\ 0 &\leq& \chi < \infty,\ 0 \leq \theta_3, \theta_2 \leq \pi,\ 0 \leq \theta_1 < 2\pi \end{array}$$

Define-se

$$\boldsymbol{r} = (a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1,\ a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1,\ a \cosh\chi \sin\theta_3 \cos\theta_2,\ a \sinh\chi \cos\theta_3)$$

Tem-se

$$\boldsymbol{e}_{\theta_1} = \frac{\partial \boldsymbol{r}}{\partial \theta_1} = \left(\frac{\partial}{\partial \theta_1}(a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1),\ \frac{\partial}{\partial \theta_1}(a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1),\ \frac{\partial}{\partial \theta_1}(a \cosh\chi \sin\theta_3 \cos\theta_2)\right.$$

$$\left., \frac{\partial}{\partial \theta_1}(a \sinh\chi \cos\theta_3)\right)$$

$$= (-a \cosh\chi \sin\theta_3 \sin\theta_2 sin\theta_1, a \cosh\chi \sin\theta_3 \sin\theta_2 cos\theta_1,\ 0,\ 0)$$

$$\boldsymbol{e}_{\theta_2} = \frac{\partial \boldsymbol{r}}{\partial \theta_2} = \Bigg(\frac{\partial}{\partial \theta_2}(a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1),\ \frac{\partial}{\partial \theta_2}(a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1),\ \frac{\partial}{\partial \theta_2}(a \cosh\chi \sin\theta_3 \cos\theta_2),$$

$$\frac{\partial}{\partial \theta_2}(a \sinh\chi \cos\theta_3)\Bigg)$$

$$= (a \cosh\chi \sin\theta_3 cos\theta_2 \cos\theta_1,\ a \cosh\chi \sin\theta_3 cos\theta_2 \sin\theta_1,\ -a \cosh\chi \sin\theta_3 sin\theta_2, 0)$$

$$\boldsymbol{e}_{\theta_3} = \frac{\partial \boldsymbol{r}}{\partial \theta_3} = \Bigg(\frac{\partial}{\partial \theta_3}(a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1),\ \frac{\partial}{\partial \theta_3}(a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1),\ \frac{\partial}{\partial \theta_3}(a \cosh\chi \sin\theta_3 \cos\theta_2),$$

$$\frac{\partial}{\partial \theta_3}(a \sinh\chi \cos\theta_3)\Bigg)$$

$$= (a \cosh\chi cos\theta_3 \sin\theta_2 \cos\theta_1,\ a \cosh\chi cos\theta_3 \sin\theta_2 \sin\theta_1, a \cosh\chi cos\theta_3 \cos\theta_2,\ -a \sinh\chi sin\theta_3)$$

$$\boldsymbol{e}_{\chi} = \frac{\partial \boldsymbol{r}}{\partial \chi} = \Bigg(\frac{\partial}{\partial \chi}(a \cosh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1),\ \frac{\partial}{\partial \chi}(a \cosh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1),\ \frac{\partial}{\partial \chi}(a \cosh\chi \sin\theta_3 \cos\theta_2),$$

$$\frac{\partial}{\partial \chi}(a \sinh\chi \cos\theta_3)\Bigg)$$

$$= (a\, sinh\chi \sin\theta_3 \sin\theta_2 \cos\theta_1,\ a\, sinh\chi \sin\theta_3 \sin\theta_2 \sin\theta_1,\ a\, sinh\chi \sin\theta_3 \cos\theta_2,\ a\, cosh\chi \cos\theta_3)$$

Agora basta verificar a ortogonalidade dois a dois entre estes vetores.

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\theta_2} = \big(-a^2 \cosh^2\chi \sin^2\theta_3 \sin\theta_2 sin\theta_1\, cos\theta_2 \cos\theta_1 + a^2 \cosh^2\chi \sin^2\theta_3 \sin\theta_2 cos\theta_1 cos\theta_2 \sin\theta_1\big) = 0$$

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\theta_3} = \big(-a^2 \cosh^2\chi \sin\theta_3 \sin^2\theta_2 sin\theta_1 cos\theta_3 \cos\theta_1 + a^2 \cosh^2\chi \sin\theta_3 \sin^2\theta_2 cos\theta_1 cos\theta_3 \sin\theta_1\big) = 0$$

$$\boldsymbol{e}_{\theta_1} \cdot \boldsymbol{e}_{\chi} = \big(-a^2 \cosh\chi \sin^2\theta_3 \sin^2\theta_2 sin\theta_1 sinh\chi \cos\theta_1 + a^2 \cosh\chi \sin^2\theta_3 \sin^2\theta_2 cos\theta_1\, sinh\chi \sin\theta_1\big) = 0$$

$$\boldsymbol{e}_{\theta_2} \cdot \boldsymbol{e}_{\theta_3} = \big(a^2 \cosh^2\chi \sin\theta_3 cos\theta_2 \cos^2\theta_1 cos\theta_3 \sin\theta_2 + a^2 \cosh^2\chi \sin\theta_3 cos\theta_2 \sin^2\theta_1 cos\theta_3 \sin\theta_2$$

$$-a^2 \cosh^2\chi \sin\theta_3 sin\theta_2 cos\theta_3 \cos\theta_2\big)$$

$$= \big(a^2 \cosh^2\chi \sin\theta_3 cos\theta_2 cos\theta_3 \sin\theta_2 - a^2 \cosh^2\chi \sin\theta_3 sin\theta_2 cos\theta_3 \cos\theta_2\big) = 0$$

$$\boldsymbol{e}_{\theta_2} \cdot \boldsymbol{e}_{\chi} = \big(a^2 \cosh\chi \sin^2\theta_3 cos\theta_2 \cos^2\theta_1 sinh\chi \sin\theta_2 +\ a^2 \cosh\chi \sin^2\theta_3 cos\theta_2 \sin^2\theta_1\, sinh\chi \sin\theta_2$$

$$-a^2 \cosh\chi \sin^2\theta_3 sin\theta_2\, sinh\chi \cos\theta_2\big)$$

$$= \left(a^2 \cosh\chi \sin^2\theta_3 cos\theta_2 sinh\chi \sin\theta_2 - a^2 \cosh\chi \sin^2\theta_3 sin\theta_2\, sinh\chi \cos\theta_2\right) = 0$$

$$\boldsymbol{e}_{\theta_3} \cdot \boldsymbol{e}_\chi = (a^2 \cosh\chi cos\theta_3 \sin^2\theta_2 \cos^2\theta_1 sinh\chi \sin\theta_3 + a^2 \cosh\chi cos\theta_3 \sin^2\theta_2 \sin^2\theta_1\, sinh\chi \sin\theta_3 +$$
$$a^2 \cosh\chi cos\theta_3 \cos^2\theta_2\, sinh\chi \sin\theta_3 - a^2 \sinh\chi sin\theta_3\, cosh\chi \cos\theta_3)$$

$$= a^2 \cosh\chi cos\theta_3 \sin^2\theta_2 sinh\chi \sin\theta_3 + a^2 \cosh\chi cos\theta_3 \cos^2\theta_2\, sinh\chi \sin\theta_3 - a^2 \sinh\chi sin\theta_3\, cosh\chi \cos\theta_3$$

$$= a^2 \cosh\chi cos\theta_3 sinh\chi \sin\theta_3 - a^2 \sinh\chi sin\theta_3\, cosh\chi \cos\theta_3 = 0$$

Verificada a ortogonalidade entre os vetores da base, pode-se dizer que o sistema de coordenadas é ortogonal.

Verificando Ortogonalidade do Hiperbolóide 4-dim (4-Hiperboloide?) de 2 Folhas

Verificar a ortogonalidade do seguinte sistema de coordenadas

$$\begin{aligned}
x_4 &= a\cosh\chi\cos\theta_3 \\
x_3 &= a\sinh\chi\sin\theta_3\cos\theta_2 \\
x_2 &= a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1 \\
x_1 &= a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1 \\
0 &\le \chi < \infty,\ 0 \le \theta_3, \theta_2 \le \pi,\ 0 \le \theta_1 < 2\pi
\end{aligned}$$

Define-se

$$\boldsymbol{r} = (a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1,\ a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1,\ a\sinh\chi\sin\theta_3\cos\theta_2,\ a\cosh\chi\cos\theta_3)$$

Tem-se

$$\boldsymbol{e}_{\theta_1} = \frac{\partial\boldsymbol{r}}{\partial\theta_1} = \left(\frac{\partial}{\partial\theta_1}(a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1),\ \frac{\partial}{\partial\theta_1}(a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1),\ \frac{\partial}{\partial\theta_1}(\,a\sinh\chi\sin\theta_3\cos\theta_2),\right.$$
$$\left.\frac{\partial}{\partial\theta_1}(a\cosh\chi\cos\theta_3)\right)$$

$$= (a\, sinh\chi \sin\theta_3 \sin\theta_2 cos\theta_1, -a\sinh\chi\sin\theta_3\sin\theta_2 sin\theta_1,\ 0,\ 0)$$

$$\boldsymbol{e}_{\theta_2} = \frac{\partial\boldsymbol{r}}{\partial\theta_2} = \left(\frac{\partial}{\partial\theta_2}(a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1),\ \frac{\partial}{\partial\theta_2}(a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1),\ \frac{\partial}{\partial\theta_2}(\,a\sinh\chi\sin\theta_3\cos\theta_2),\right.$$
$$\left.\frac{\partial}{\partial\theta_2}(a\cosh\chi\cos\theta_3)\right)$$

$$= (a\sinh\chi\sin\theta_3 cos\theta_2\sin\theta_1,\ a\sinh\chi\sin\theta_3 cos\theta_2\cos\theta_1,\ -a\sinh\chi\sin\theta_3 sin\theta_2, 0)$$

$$\boldsymbol{e}_{\theta_3} = \frac{\partial \boldsymbol{r}}{\partial \theta_3} = \left(\frac{\partial}{\partial\theta_3}(a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1),\ \frac{\partial}{\partial\theta_3}(a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1),\ \frac{\partial}{\partial\theta_3}(\ a\sinh\chi\sin\theta_3\cos\theta_2),\right.$$

$$\left.\frac{\partial}{\partial\theta_3}(a\cosh\chi\cos\theta_3)\right)$$

$$= (a\sinh\chi cos\theta_3\sin\theta_2\sin\theta_1,\ a\sinh\chi cos\theta_3\sin\theta_2\cos\theta_1, a\sinh\chi cos\theta_3\cos\theta_2,\ -a\cosh\chi sin\theta_3)$$

$$\boldsymbol{e}_{\chi} = \frac{\partial \boldsymbol{r}}{\partial \chi} = \left(\frac{\partial}{\partial\chi}(a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1),\ \frac{\partial}{\partial\chi}(a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1),\ \frac{\partial}{\partial\chi}(\ a\sinh\chi\sin\theta_3\cos\theta_2),\right.$$

$$\left.\frac{\partial}{\partial\chi}(a\cosh\chi\cos\theta_3)\right)$$

$$= (a\, cosh\chi\sin\theta_3\sin\theta_2\sin\theta_1, a\, cosh\chi\sin\theta_3\sin\theta_2\cos\theta_1,\ a\, cosh\chi\sin\theta_3\cos\theta_2,\ a\, sinh\chi\cos\theta_3)$$

Agora basta verificar a ortogonalidade dois a dois entre estes vetores.

$$\boldsymbol{e}_{\theta_1}\cdot\boldsymbol{e}_{\theta_2} = \left(a^2\, sinh^2\chi\sin^2\theta_3\sin\theta_2 cos\theta_1 cos\theta_2\sin\theta_1 - a^2\sinh^2\chi\sin^2\theta_3\sin\theta_2 sin\theta_1\, cos\theta_2\cos\theta_1\right) = 0$$

$$\boldsymbol{e}_{\theta_1}\cdot\boldsymbol{e}_{\theta_3} = \left(a^2\, sinh^2\chi\sin\theta_3\sin^2\theta_2 cos\theta_1 cos\theta_3\sin\theta_1 - a^2\sinh^2\chi\sin\theta_3\sin^2\theta_2 sin\theta_1 cos\theta_3\cos\theta_1\right) = 0$$

$$\boldsymbol{e}_{\theta_1}\cdot\boldsymbol{e}_{\chi} = \left(a^2\, sinh\chi\sin^2\theta_3\sin^2\theta_2 cos\theta_1 cosh\chi\sin\theta_1 - a^2\sinh\chi\sin^2\theta_3\sin^2\theta_2 sin\theta_1 cosh\chi\cos\theta_1\right) = 0$$

$$\boldsymbol{e}_{\theta_2}\cdot\boldsymbol{e}_{\theta_3} = \left(a^2\sinh^2\chi\sin\theta_3 cos\theta_2\sin^2\theta_1 cos\theta_3\sin\theta_2 + a^2\sinh^2\chi\sin\theta_3 cos\theta_2\cos^2\theta_1 cos\theta_3\sin\theta_2\right.$$

$$\left.-a^2\sinh^2\chi\sin\theta_3 sin\theta_2 cos\theta_3\cos\theta_2\right)$$

$$= \left(a^2\sinh^2\chi\sin\theta_3 cos\theta_2 cos\theta_3\sin\theta_2 - a^2\sinh^2\chi\sin\theta_3 sin\theta_2 cos\theta_3\cos\theta_2\right) = 0$$

$$\boldsymbol{e}_{\theta_2}\cdot\boldsymbol{e}_{\chi} = \left(a^2\sinh\chi\sin^2\theta_3 cos\theta_2\sin^2\theta_1 cosh\chi\sin\theta_2 + a^2\sinh\chi\sin^2\theta_3 cos\theta_2\cos^2\theta_1 cosh\chi\sin\theta_2\right.$$

$$\left.-a^2\sinh\chi\sin^2\theta_3 sin\theta_2 cosh\chi\cos\theta_2\right)$$

$$= \left(a^2\sinh\chi\sin^2\theta_3 cos\theta_2 cosh\chi\sin\theta_2 - a^2\sinh\chi\sin^2\theta_3 sin\theta_2 cosh\chi\cos\theta_2\right) = 0$$

$$\boldsymbol{e}_{\theta_3}\cdot\boldsymbol{e}_{\chi} = (a^2\sinh\chi cos\theta_3\sin^2\theta_2\sin^2\theta_1 cosh\chi\sin\theta_3 + a^2\sinh\chi cos\theta_3\sin^2\theta_2\cos^2\theta_1 cosh\chi\sin\theta_3$$

$$+a^2\sinh\chi cos\theta_3\cos^2\theta_2 cosh\chi\sin\theta_3 - a^2\cosh\chi sin\theta_3 sinh\chi\cos\theta_3)$$

$$= a^2\sinh\chi cos\theta_3\sin^2\theta_2 cosh\chi\sin\theta_3 + a^2\sinh\chi cos\theta_3\cos^2\theta_2 cosh\chi\sin\theta_3 - a^2\cosh\chi sin\theta_3 sinh\chi\cos\theta_3$$

$$= a^2\sinh\chi cos\theta_3 cosh\chi\sin\theta_3 - a^2\cosh\chi sin\theta_3 sinh\chi\cos\theta_3 = 0$$

Verificada a ortogonalidade entre os vetores da base, pode-se dizer que o sistema de coordenadas é ortogonal.

1 Conclusão

Mostrou-se a ortogonalidade das 4 parametrizações estudadas. Dada essa característica dos sistemas de coordenadas vale, de acordo com Riley(2006), a seguinte propriedade:

$$g_{ij} = \begin{cases} h_i^2 & ,i=j \\ 0 & ,i\neq j \end{cases}$$