$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial u}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2 u}{\partial\theta_1^2} = 0$$

Define-se

$$u(\theta_1,\theta_2,\chi) = \Theta_1(\theta_1)W(\theta_2,\chi)$$

Aplicando na Equação de Laplace, tem-se:

$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

Fazendo por partes

$$\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) = \frac{1}{\sin\theta_2}\left[\Theta_1 cos(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\sin\theta_2\frac{\partial^2 W}{\partial\theta_2^2}\right] = \Theta_1 cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2}$$

$$\frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right) = \frac{1}{\cosh\chi}\left[\Theta_1 sinh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\cosh\chi\frac{\partial^2 W}{\partial\chi^2}\right] = \Theta_1 tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}$$

Aplicando na Equação de Laplace

$$\frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\Theta_1 cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2} + \Theta_1 tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

$$\frac{\Theta_1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

Multiplicando ambos os lados por

$\frac{a^2\sin^2\theta_2\cosh^2\chi}{\Theta_1 W}$

$$\frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = 0$$

$$\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = -\frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] = \lambda_1$$

Daí obtêm-se

$$\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = \lambda_1 \Rightarrow \boldsymbol{\frac{d^2\Theta_1}{d\theta_1^2} = \lambda_1\Theta_1}$$

$$-\frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2}+\frac{\partial^2 W}{\partial\theta_2^2}+tanh(\chi)\frac{\partial W}{\partial\chi}+\frac{\partial^2 W}{\partial\chi^2}\right]=\lambda_1$$

Fazendo

$$W(\theta_2,\chi)=\Theta_2(\theta_2)X(\chi)$$

$$-\frac{1}{\Theta_2 X}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left[X cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+X\frac{\partial^2\Theta_2}{\partial\theta_2^2}+\Theta_2 tanh(\chi)\frac{\partial X}{\partial\chi}+\Theta_2\frac{\partial^2 X}{\partial\chi^2}\right]=\lambda_1$$

$$-\frac{1}{\Theta_2 X}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left[X\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\Theta_2\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)\right]=\lambda_1$$

$$\frac{1}{\Theta_2}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=-\lambda_1$$

$$\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left[\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)\right]=-\lambda_1$$

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=-\lambda_1\frac{\left(\sinh^2\chi+\cos^2\theta_2\right)}{\sin\theta_2\cosh^2\chi}$$