$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial u}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2 u}{\partial\theta_1^2} =$$

Define-se

$$u(\theta_1,\theta_2,\chi) = \Theta_1(\theta_1)W(\theta_2,\chi)$$

Aplicando na Equação de Laplace, tem-se:

$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}$$

Fazendo por partes

$$\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) = \frac{1}{\sin\theta_2}\left[\Theta_1 cos(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\sin\theta_2\frac{\partial^2 W}{\partial\theta_2^2}\right] = \Theta_1 cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2}$$

$$\frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right) = \frac{1}{\cosh\chi}\left[\Theta_1 sinh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\cosh\chi\frac{\partial^2 W}{\partial\chi^2}\right] = \Theta_1 tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}$$

Aplicando na Equação de Laplace

$$\frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\Theta_1 cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2} + \Theta_1 tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W$$

$$\frac{\Theta_1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

Multiplicando ambos os lados por $\frac{a^2\sin^2\theta_2\cosh^2\chi}{\Theta_1 W}$

$$\frac{1}{W}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = 0$$

$$-\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = \frac{1}{W}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] = \lambda_1$$

Daí obtêm-se

$$-\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = \lambda_1 \Rightarrow \frac{\partial^2\Theta_1}{\partial\theta_1^2} = -\Theta_1\lambda_1 \Rightarrow \boldsymbol{\frac{\partial^2\Theta_1}{\partial\theta_1^2} + \Theta_1\lambda_1 = 0}$$

0.1 Resolvendo essa EDO

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_1 = 0$$

0.1.1 Caso $\lambda_1 = 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} = 0$$

A solução para esse caso é

$$\Theta_1(\theta_1) = A\theta_1 + B$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$A\theta_1 + B = A(\theta_1 + 2\pi) + B$$

Chega-se em

$$\theta_1 = (\theta_1 + 2\pi)$$

Essa conclusão é absurda, então conclui-se que $\lambda_1 \neq 0$.

0.1.2 Caso $\lambda_1 < 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} - \Theta_1\lambda_1 = 0$$

A equação característica é

$$D^2 - \lambda_1 = 0 \Rightarrow D^2 = \lambda_1 \Rightarrow D = \pm\sqrt{\lambda_1}$$

De modo que as raízes são reais e diferentes. Nesse caso a solução geral é

$$\Theta_1(\theta) = Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1} = Ae^{\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se

$$\begin{cases} e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-\sqrt{\lambda_1}\theta_1} = e^{-\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}\theta_1}e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{2\pi\sqrt{\lambda_1}} = 1$$

Já que $e^{2\pi\sqrt{\lambda_1}} = 1$ também é um absurdo, a única possibilidade restante é $\lambda_1 > 0$. Verifica-se a seguir.

0.1.3 Caso $\lambda_1 > 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_1 = 0$$

A equação característica dessa EDO é

$$D^2 + \lambda_1 = 0 \Rightarrow D^2 = -\lambda_1 \Rightarrow D = \sqrt{-\lambda_1} = i\sqrt{\lambda_1}$$

De modo que, como as raízes são complexas conjugadas $D = \alpha + i\beta$, a solução geral é:

$$\Theta_1(\theta_1) = Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1} = Ae^{i\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-i\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se o seguinte sistema

$$\begin{cases} e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-i\sqrt{\lambda_1}\theta_1} = e^{-i\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}\theta_1}e^{i\sqrt{\lambda_1}2\pi} \Rightarrow e^{i\sqrt{\lambda_1}2\pi} = 1 \Rightarrow cos\left(\sqrt{\lambda_1}2\pi\right) + i\,sin\left(\sqrt{\lambda_1}2\pi\right) = 1 \Rightarrow$$

Então temos

$$\boldsymbol{\Theta_{1,m}(\theta_1) = Ae^{im\theta_1} + Be^{-im\theta_1}}$$

0.2 Continuando na outra equação

$$\frac{1}{W}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] = \lambda_1$$

Fazendo

$$W(\theta_2, \chi) = \Theta_2(\theta_2)X(\chi)$$

$$\frac{1}{\Theta_2 X}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[Xcot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + X\frac{\partial^2\Theta_2}{\partial\theta_2^2} + \Theta_2 tanh(\chi)\frac{\partial X}{\partial\chi} + \Theta_2\frac{\partial^2 X}{\partial\chi^2}\right] = \lambda_1$$

$$\frac{1}{\Theta_2 X}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[X\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \Theta_2\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right)\right] = \lambda_1$$

$$\frac{1}{\Theta_2}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) = \lambda_1$$

$$\frac{\sin^2\theta_2\cosh^2\chi}{\left(\sinh^2\chi+\cos^2\theta_2\right)}\left[\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)\right]=\lambda_1$$

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=\lambda_1\frac{\left(\sinh^2\chi+\cos^2\theta_2\right)}{\sin^2\theta_2\cosh^2\chi}$$

Utilizando

$$\frac{\left(\sinh^2\chi+\cos^2\theta_2\right)}{\sin^2\theta_2\cosh^2\chi}=\frac{\sinh^2\chi}{\sin^2\theta_2\cosh^2\chi}+\frac{\cos^2\theta_2}{\sin^2\theta_2\cosh^2\chi}=\frac{\cosh^2\chi}{\sin^2\theta_2\cosh^2\chi}-\frac{1}{\sin^2\theta_2\cosh^2\chi}+\frac{1}{\sin^2\theta_2\cosh^2\chi}-\frac{}{\text{si}}$$

$$=\frac{\cosh^2\chi}{\sin^2\theta_2\cosh^2\chi}-\frac{\sin^2\theta_2}{\sin^2\theta_2\cosh^2\chi}=\frac{1}{\sin^2\theta_2}-\frac{1}{\cosh^2\chi}$$

Tem-se

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=\lambda_1\left(\frac{1}{\sin^2\theta_2}-\frac{1}{\cosh^2\chi}\right)$$

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=\frac{\lambda_1}{\sin^2\theta_2}-\frac{\lambda_1}{\cosh^2\chi}$$

$$-\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{\lambda_1}{\sin^2\theta_2}-\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)-\frac{\lambda_1}{\cosh^2\chi}=0$$

$$-\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)+\frac{\lambda_1}{\sin^2\theta_2}=\frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)+\frac{\lambda_1}{\cosh^2\chi}$$

Nota-se que

$$\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)=\left(\frac{cos(\theta_2)}{sin(\theta_2)}\frac{\partial\Theta_2}{\partial\theta_2}+\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)=\frac{1}{sin(\theta_2)}\left(cos(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2}+\sin\theta_2\frac{\partial^2\Theta_2}{\partial\theta_2^2}\right)=\frac{1}{sin(\theta_2)}\frac{\partial}{\partial\theta_2}$$

$$\left(tanh(\chi)\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=\left(\frac{sinh\,(\chi)}{cosh\,(\chi)}\frac{\partial X}{\partial\chi}+\frac{\partial^2 X}{\partial\chi^2}\right)=\frac{1}{cosh\,(\chi)}\left(sinh\,(\chi)\frac{\partial X}{\partial\chi}+cosh\,(\chi)\frac{\partial^2 X}{\partial\chi^2}\right)=\frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}$$

Então

$$-\frac{1}{\Theta_2 sin(\theta_2)}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial\Theta_2}{\partial\theta_2}\right)+\frac{\lambda_1}{\sin^2\theta_2}=\frac{1}{X\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial X}{\partial\chi}\right)+\frac{\lambda_1}{\cosh^2\chi}=\lambda_2$$

$$-\frac{1}{\Theta_2 sin(\theta_2)}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial\Theta_2}{\partial\theta_2}\right)+\frac{\lambda_1}{\sin^2\theta_2}=\lambda_2$$

$$\frac{1}{X\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial X}{\partial\chi}\right)+\frac{\lambda_1}{\cosh^2\chi}=\lambda_2$$

Pode-se escrever

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta}\left(\sin\theta_2\frac{\partial\Theta_2}{\partial\theta_2}\right)+\left(\lambda_2-\frac{\lambda_1}{\sin^2\theta_2}\right)\Theta_2=0$$

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right)+\left(-\lambda_2+\frac{\lambda_1}{\cosh^2\chi}\right)X=0$$

0.3 Resolvendo para Θ_2

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta}\left(\sin\theta_2\frac{\partial\Theta_2}{\partial\theta_2}\right)+\left(\lambda_2-\frac{m^2}{\sin^2\theta_2}\right)\Theta_2=0$$

0.3.1 Caso em que $m = 0$

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)+\lambda_2\Theta_2=0$$

Fazendo $x = cos\theta_2$, $x \in [1, -1]$

$$sin^2(\theta_2)=1-cos^2(\theta_2)=1-x^2$$

$$\frac{d\Theta_2}{d\theta_2}=\frac{d\Theta_2}{dx}\frac{dx}{d\theta_2}=\frac{d\Theta_2}{dx}(-sin(\theta_2))$$

$$\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)=\frac{d}{d\theta_2}\left(-sin^2(\theta_2)\frac{d\Theta_2}{dx}\right)=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}-sin^2(\theta_2)\frac{d}{d\theta_2}\frac{d\Theta_2}{dx}$$

$$=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}-sin^2(\theta_2)\frac{d}{dx}\frac{d\Theta_2}{d\theta_2}=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}+sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)\Rightarrow\frac{1}{sin(\theta_2)}\left(-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}+sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}\right)=-2cos\theta_2\frac{d\Theta_2}{dx}+sin^2(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

Então ficamos com

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)+\lambda_2\Theta_2=0\Rightarrow -2cos\theta_2\frac{d\Theta_2}{dx}+sin^2(\theta_2)\frac{d^2\Theta_2}{dx^2}+\lambda_2\Theta_2=0$$

$$\left(1-x^2\right)\frac{d^2\Theta_2}{dx^2}-2x\frac{d\Theta_2}{dx}+\lambda_2\Theta_2=0$$

Essa equação tem soluções não singulares quando $\lambda_2=n(n+1)$, $n\in\mathbb{Z}_+$.

$$\left(1-x^2\right)\frac{d^2\Theta_2}{dx^2}-2x\frac{d\Theta_2}{dx}+n(n+1)\Theta_2=0$$

Chagou-se na Equação de Legendre, a solução dessa equação é expressa em termos dos Polinômios de Legendre de modo que

$$\boldsymbol{\Theta_{2,n}(x)=c_1P_n(x)+c_2Q_n(x)}$$

Onde $P_n(x)$ são Polinômios de Legendre de primeira espécie e os $Q_n(x)$ são polinômios de legendre de segunda espécie.

OBS: Se a solução tiver que ser limitada em $[-1,1]$ **ela se torna** $\boldsymbol{\Theta_{2,n}(x)=c_1P_n(x)}$ pois

$$\lim_{x\to1^-}Q_n(x)=+\infty$$

$$\lim_{x\to-1^+}Q_n(x)=(-1)^{n+1}\infty$$

Como o intervalo no caso é $[1,\,-1]$ **pode-se manter a solução na forma descrita acima.**

0.3.2 Caso em que $m\neq0$

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)+\left(\lambda_2-\frac{m^2}{\sin^2\theta_2}\right)\Theta_2=0$$

Aplicando o termo encontrado$\lambda_2=n(n+1)$, $n\in\mathbb{Z}_+$ de modo que tem-se

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta_2}\right)\Theta_2=0$$

Fazendo $x=cos\theta_2$, $x\in[1,\,-1]$

$$sin^2(\theta_2)=1-cos^2(\theta_2)=1-x^2$$

$$\frac{d\Theta_2}{d\theta_2}=\frac{d\Theta_2}{dx}\frac{dx}{d\theta_2}=-sin(\theta_2)\frac{d\Theta_2}{dx}$$

$$\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)=\frac{d}{d\theta_2}\left(-sin^2(\theta_2)\frac{d\Theta_2}{dx}\right)=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}-sin^2(\theta_2)\frac{d}{d\theta_2}\frac{d\Theta_2}{dx}$$

$$=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}-sin^2(\theta_2)\frac{d}{dx}\frac{d\Theta_2}{d\theta_2}=-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}+sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)\Rightarrow\frac{1}{sin(\theta_2)}\left(-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx}+sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}\right)=-2cos\theta_2\frac{d\Theta_2}{dx}+sin^2(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

Tem-se

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta_2}\right)\Theta_2=0\Rightarrow-2x\frac{d\Theta_2}{dx}+\left(1-x^2\right)\frac{d^2\Theta_2}{dx^2}+\left(n(n+1)-\frac{m^2}{(1-x^2)}\right.$$

$$\left(1-x^2\right)\frac{d^2\Theta_2}{dx^2}-2x\frac{d\Theta_2}{dx}+\left(n(n+1)-\frac{m^2}{(1-x^2)}\right)\Theta_2=0$$

Essa é a **Equação de Legendre Generalizada** e suas soluções, para $m,n\in\mathbb{Z}$, são dadas em termos de Funções Associadas de Legendre.

$$\boldsymbol{\Theta_{2,n,m}(x)=c_1P_n^m(x)+c_2Q_n^m(x)}$$

0.4 Olhando para χ

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right)+\left(-\lambda_2+\frac{\lambda_1}{\cosh^2\chi}\right)X=0$$

Fazendo $\zeta=\sinh\chi$, $\zeta\in[0,\infty)$ tem-se

$$cosh^2(\chi)=1+sinh^2(\chi)=1+\zeta^2$$

$$\frac{dX}{d\chi} = \frac{dX}{d\zeta}\frac{d\zeta}{d\chi} = cosh\chi\frac{dX}{d\zeta}$$

$$\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) = \frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\zeta}\right) = 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d}{d\chi}\frac{dX}{d\zeta}$$

$$= 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d}{d\zeta}\frac{dX}{d\chi} = 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^3\chi\frac{d^2X}{d\zeta^2}$$

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) = \frac{1}{\cosh\chi}\left(2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^3\chi\frac{d^2X}{d\zeta^2}\right) = 2sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d^2X}{d\zeta^2}$$

Então, levando em conta $\lambda_1 = m^2$ e $\lambda_2 = n(n+1)$, chega-se em:

$$\left(1+\zeta^2\right)\frac{d^2X}{d\zeta^2} + 2\zeta\frac{dX}{d\zeta} + \left(-n(n+1) + \frac{m^2}{(1+\zeta^2)}\right)X = 0$$

Aplicando $\alpha = i\zeta,\ i^2 = -1$.
Veja que

$$\alpha = i\zeta \Rightarrow \alpha^2 = i^2\zeta^2 \Rightarrow -\alpha^2 = \zeta^2$$

$$\frac{dX}{d\zeta} = \frac{dX}{d\alpha}\frac{d\alpha}{d\zeta} = \frac{dX}{d\alpha}i \Rightarrow \frac{d^2X}{d\zeta^2} = \frac{d}{d\zeta}\frac{dX}{d\zeta} = i\frac{d}{d\zeta}\frac{dX}{d\alpha} = i\frac{d}{d\alpha}\frac{dX}{d\zeta} = -\frac{d^2X}{d\alpha^2}$$

$$2\zeta\frac{dX}{d\zeta} \Rightarrow 2i\zeta\frac{dX}{d\alpha} = 2\alpha\frac{dX}{d\alpha}$$

Substituindo esse resultado na Equação

$$-\left(1-\alpha^2\right)\frac{d^2X}{d\alpha^2} + 2\alpha\frac{dX}{d\alpha} + \left(-n(n+1) + \frac{m^2}{(1-\alpha^2)}\right)X = 0$$

Multiplicando ambos os lados por -1

$$\left(1-\alpha^2\right)\frac{d^2X}{d\alpha^2} - 2\alpha\frac{dX}{d\alpha} + \left(n(n+1) - \frac{m^2}{(1-\alpha^2)}\right)X = 0$$

Essa é, novamente, a Equação de Legendre. Como visto anteriormente pode-se analisa-la em dois casos.

0.4.1 Caso $m = 0$

Para esse caso, como já determinando antes, a solução é

$$\boldsymbol{X_n(x) = c_3 P_n(\alpha) + c_4 Q_n(\alpha)}$$

0.4.2 Caso $m \neq 0$

Para esse caso, como já determinando antes, a solução é

$$\boldsymbol{X_{n,m}(x) = c_3 P_n^m(\alpha) + c_4 Q_n^m(\alpha)}$$

0.5 Conclusão

Conclui-se que a solução da Equação

$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial u}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2 u}{\partial\theta_1^2} =$$

Pode ser dada por

$$u(\theta_1, \theta_2, \chi) = \Theta_1(\theta_1)\Theta_2(x)X(\alpha)$$

$$\boldsymbol{u(\theta_1, \theta_2, \chi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Theta_{1,m}(\theta_1)\Theta_{2,m,n}(x)X_{m,n}(\alpha)}$$