$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial u}{\partial\theta_3}\right) + \frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial u}{\partial\chi}\right)\right) + \frac{1}{a^2\sin^2\theta_3\cosh^2\chi}\nabla^2_{S^2}u = 0$$

Tal que

$$\nabla^2_{S^2}u = \left[\left(\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial u}{\partial\theta_2}\right)\right) + \frac{1}{\sin^2\theta_2}\frac{\partial^2 u}{\partial\theta_1^2}\right]$$

É o Laplaciano na esfera S^2.

Para realizar a separação de variáveis façamos

$$u(\chi,\theta_1,\theta_2,\theta_3) = Y(\theta_1,\theta_2)W(\chi,\theta_3)$$

Aplicando na Equação

$$\nabla^2 u = \frac{Y}{a^2\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial W}{\partial\theta_3}\right) + \frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial W}{\partial\chi}\right)\right) + \frac{W}{a^2\sin^2\theta_3\cosh^2\chi}\left(\nabla^2_{S^2}Y\right) = 0$$

Multiplicando os dois lados por $\frac{a^2\sin^2\theta_3\cosh^2\chi}{WY}$

$$\frac{\sin^2\theta_3\cosh^2\chi}{W\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial W}{\partial\theta_3}\right) + \frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial W}{\partial\chi}\right)\right) + \frac{1}{Y}\left(\nabla^2_{S^2}Y\right) = 0$$

Então, tem-se

$$\frac{\sin^2\theta_3\cosh^2\chi}{W\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial W}{\partial\theta_3}\right) + \frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial W}{\partial\chi}\right)\right) = -\frac{1}{Y}\left(\nabla^2_{S^2}Y\right) = \lambda_1$$

De modo que chega-se em duas EDPs

$$\frac{\sin^2\theta_3\cosh^2\chi}{W\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial W}{\partial\theta_3}\right) + \frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial W}{\partial\chi}\right)\right) = \lambda_1$$

$$\nabla^2_{S^2}Y = -Y\lambda_1$$

Assumindo

$$Y(\theta_1,\theta_2) = \Theta_1(\theta_1)\Theta_2(\theta_2)$$

$$W(\chi,\theta_3) = X(\chi)\Theta_3(\theta_3)$$

Tem-se

1 Parte Esférica

A equação $\nabla^2_{S^2}Y = -Y\lambda_1$ fica da seguinte forma

$$\nabla^2_{S^2}Y = \left[\left(\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial Y}{\partial\theta_2}\right)\right) + \frac{1}{\sin^2\theta_2}\frac{\partial^2 Y}{\partial\theta_1^2}\right] = -Y\lambda_1$$

A solução dessa equação é expressa em termos de harmônicos esféricos.

$$Y(\theta_1,\theta_2) = \Theta_1(\theta_1)\Theta_2(\theta_2) = Y_{n,\,m}(\theta_1,\,\theta_2)$$

Para determinarmos $\lambda_1$vamos desenvolver a expressão

$$\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial Y}{\partial\theta_2}\right) + \frac{1}{\sin^2\theta_2}\frac{\partial^2 Y}{\partial\theta_1^2} + Y\lambda_1 = 0$$

$$Y = \Theta_1\Theta_2$$

Então tem-se

$$\frac{\Theta_1}{\sin\theta_2}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \frac{\Theta_2}{\sin^2\theta_2}\frac{d^2\Theta_1}{d\theta_1^2} + Y\lambda_1 = 0$$

Multiplicando por $\frac{\sin^2\theta_2}{\Theta_1\Theta_2}$

$$\frac{\sin\theta_2}{\Theta_2}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \sin^2\theta_2\lambda_1 + \frac{1}{\Theta_1}\frac{d^2\Theta_1}{d\theta_1^2} = 0$$

Então podemos separar

$$\frac{\sin\theta_2}{\Theta_2}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \sin^2\theta_2\lambda_1 = -\frac{1}{\Theta_1}\frac{d^2\Theta_1}{d\theta_1^2} = \lambda_2$$

Desse modo chega-se em

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_2 = 0 \Rightarrow \Theta_{1,m}(\theta_1) = Ae^{im\theta_1} + Be^{-im\theta_1},\ \boldsymbol{\lambda_2 = m^2}$$

$$\frac{1}{\sin\theta_2}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \left(\lambda_1 - \frac{m^2}{\sin^2\theta_2}\right)\Theta_2 = 0$$

A última expressão é conhecida e dela obtemos $\lambda_1 = n(n+1)$.

2 Analisando a outra EDP

$$\frac{\sin^2\theta_3\cosh^2\chi}{W\left(\sinh^2\chi + \cos^2\theta_3\right)}\left(\frac{X}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) + \frac{\Theta_3}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right)\right) = \lambda_1$$

Multiplica-se os dois lados por $\frac{W\left(\sinh^2\chi+\cos^2\theta_3\right)}{X\Theta_3\sin^2\theta_3\cosh^2\chi}$

$$\frac{1}{X\Theta_3}\left(\frac{X}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) + \frac{\Theta_3}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right)\right) = \lambda_1\frac{\left(\sinh^2\chi + \cos^2\theta_3\right)}{\sin^2\theta_3\cosh^2\chi}$$

$$\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) + \frac{1}{X}\frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right) = \lambda_1\frac{\left(\sinh^2\chi + \cos^2\theta_3\right)}{\sin^2\theta_3\cosh^2\chi}$$

Veja que

$$\frac{\left(\sinh^2\chi + \cos^2\theta_3\right)}{\sin^2\theta_3\cosh^2\chi} = \frac{\sinh^2\chi}{\sin^2\theta_3\cosh^2\chi} + \frac{\cos^2\theta_3}{\sin^2\theta_3\cosh^2\chi} = \frac{\cosh^2\chi - 1}{\sin^2\theta_3\cosh^2\chi} + \frac{1 - \sin^2\theta_3}{\sin^2\theta_3\cosh^2\chi}$$

$$= \frac{1}{\sin^2\theta_3} - \frac{1}{\sin^2\theta_3\cosh^2\chi} + \frac{1}{\sin^2\theta_3\cosh^2\chi} - \frac{1}{\cosh^2\chi} = \frac{1}{\sin^2\theta_3} - \frac{1}{\cosh^2\chi}$$

Então a equação fica

$$\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) + \frac{1}{X}\frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right) = \lambda_1\frac{1}{\sin^2\theta_3} - \lambda_1\frac{1}{\cosh^2\chi}$$

$$\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) - \lambda_1\frac{1}{\sin^2\theta_3} + \frac{1}{X}\frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right) + \lambda_1\frac{1}{\cosh^2\chi} = 0$$

$$\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial\Theta_3}{\partial\theta_3}\right) - \lambda_1\frac{1}{\sin^2\theta_3} = -\frac{1}{X}\frac{1}{\cosh^2\chi}\frac{\partial}{\partial\chi}\left(\cosh^2\chi\frac{\partial X}{\partial\chi}\right) - \lambda_1\frac{1}{\cosh^2\chi} = \lambda_2$$

Fica-se com

$$-\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{d}{d\theta_3}\left(\sin^2\theta_3\frac{d\Theta_3}{d\theta_3}\right)+\lambda_1\frac{1}{\sin^2\theta_3}=-\lambda_3$$
$$-\frac{1}{X}\frac{1}{\cosh^2\chi}\frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\chi}\right)-\lambda_1\frac{1}{\cosh^2\chi}=\lambda_3$$

Ou, de outra forma, tem-se

$$-\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{d}{d\theta_3}\left(\sin^2\theta_3\frac{d\Theta_3}{d\theta_3}\right)+\left(\frac{\lambda_1}{\sin^2\theta_3}+\lambda_3\right)=0$$
$$-\frac{1}{X}\frac{1}{\cosh^2\chi}\frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\chi}\right)-\left(\lambda_3+\frac{\lambda_1}{\cosh^2\chi}\right)=0$$

2.1 Equação para θ_3

$$\frac{1}{\Theta_3}\frac{1}{\sin^2\theta_3}\frac{d}{d\theta_3}\left(\sin^2\theta_3\frac{d\Theta_3}{d\theta_3}\right)-\left(\lambda_1\frac{1}{\sin^2\theta_3}+\lambda_3\right)=0$$

Aplicando troca de variáveis $\xi=\cos\theta_3$

$$sin^2(\theta_3)=1-cos^2(\theta_3)=1-\xi^2$$

$$\sin^2\theta_3\frac{d\Theta_3}{d\theta_3}=\sin^2\theta_3\frac{d\Theta_3}{d\xi}\frac{d\xi}{d\theta_3}=-\sin^3(\theta_3)\frac{d\Theta_3}{d\xi}$$

$$\frac{d}{d\theta_3}\left(\sin^2\theta_3\frac{d\Theta_3}{d\theta_3}\right)=-\frac{d}{d\theta_3}\left(\sin^3(\theta_3)\frac{d\Theta_3}{d\xi}\right)=-\left(3\cos(\theta_3)\sin^2(\theta_3)\frac{d\Theta_3}{d\xi}+\sin^3(\theta_3)\frac{d}{d\theta_3}\frac{d\Theta_3}{d\xi}\right)$$
$$=-\left(3\cos(\theta_3)\sin^2(\theta_3)\frac{d\Theta_3}{d\xi}+\sin^3(\theta_3)\frac{d}{d\xi}\frac{d\Theta_3}{d\theta_3}\right)=-\left(3\cos(\theta_3)\sin^2(\theta_3)\frac{d\Theta_3}{d\xi}-\sin^4(\theta_3)\frac{d^2\Theta_3}{d\xi^2}\right)$$
$$=-\sin^2(\theta_3)\left(3\cos(\theta_3)\frac{d\Theta_3}{d\xi}-\sin^2(\theta_3)\frac{d^2\Theta_3}{d\xi^2}\right)$$

A equação fica

$$-\frac{1}{\Theta_3}\left(3\cos(\theta_3)\frac{d\Theta_3}{d\xi}-\sin^2(\theta_3)\frac{d^2\Theta_3}{d\xi^2}\right)-\left(\lambda_1\frac{1}{\sin^2\theta_3}+\lambda_3\right)=0$$
$$\frac{1}{\Theta_3}\left(3\xi\frac{d\Theta_3}{d\xi}-\left(1-\xi^2\right)\frac{d^2\Theta_3}{d\xi^2}\right)+\left(\frac{n(n+1)}{(1-\xi^2)}+\lambda_3\right)=0$$

2.2 Equação para χ

$$\frac{1}{X}\frac{1}{\cosh^2\chi}\frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\chi}\right)+\left(\lambda_3+\frac{\lambda_1}{\cosh^2\chi}\right)=0$$

Aplicando $\zeta=\sinh\chi$

$$\cosh^2\chi=1+\sinh\chi=1+\zeta^2$$

$$\cosh^2\chi\frac{dX}{d\chi}=\cosh^2\chi\frac{dX}{d\zeta}\frac{d\zeta}{d\chi}=\cosh^3\chi\frac{dX}{d\zeta}$$

$$\frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\chi}\right)=\frac{d}{d\chi}\left(\cosh^3\chi\frac{dX}{d\zeta}\right)=3\cosh^2\chi\sinh\chi\frac{dX}{d\zeta}+\cosh^3\chi\frac{d}{d\chi}\frac{dX}{d\zeta}$$
$$=3\cosh^2\chi\sinh\chi\frac{dX}{d\zeta}+\cosh^3\chi\frac{d}{d\zeta}\frac{dX}{d\chi}=3\cosh^2\chi\sinh\chi\frac{dX}{d\zeta}+\cosh^4\chi\frac{d^2X}{d\zeta^2}$$
$$=\cosh^2\chi\left(3\sinh\chi\frac{dX}{d\zeta}+\cosh^2\chi\frac{d^2X}{d\zeta^2}\right)$$

A equação fica

$$\frac{1}{X}\left(3\zeta\frac{dX}{d\zeta}+\left(1+\zeta^2\right)\frac{d^2X}{d\zeta^2}\right)+\left(\lambda_3+\frac{\lambda_1}{(1+\zeta^2)}\right)=0$$

Agora aplicando $\alpha = i\zeta$

$$-\alpha^2=\zeta^2$$

$$\frac{dX}{d\zeta}=\frac{dX}{d\alpha}\frac{d\alpha}{d\zeta}=i\frac{dX}{d\alpha}\Rightarrow 3\zeta\frac{dX}{d\zeta}=3\zeta i\frac{dX}{d\alpha}=3\alpha\frac{dX}{d\alpha}$$

$$\frac{d^2X}{d\zeta^2}=\frac{d}{d\zeta}\frac{dX}{d\zeta}=i\frac{d}{d\zeta}\frac{dX}{d\alpha}=i\frac{d}{d\alpha}\frac{dX}{d\zeta}=-\frac{d}{d\alpha}\frac{dX}{d\alpha}=-\frac{d^2X}{d\alpha^2}$$

Então, tem-se

$$\frac{1}{X}\left(3\alpha\frac{dX}{d\alpha}-\left(1-\alpha^2\right)\frac{d^2X}{d\alpha^2}\right)+\left(\lambda_3+\frac{n(n+1)}{(1-\alpha^2)}\right)=0$$