$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial u}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2 u}{\partial\theta_1^2} = 0$$

Define-se

$$u(\theta_1,\theta_2,\chi) = \Theta_1(\theta_1)W(\theta_2,\chi)$$

Aplicando na Equação de Laplace, tem-se:

$$\nabla^2 u = \frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) + \frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right)\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

Fazendo por partes

$$\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\Theta_1\sin\theta_2\frac{\partial W}{\partial\theta_2}\right) = \frac{1}{\sin\theta_2}\left[\Theta_1\cos(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\sin\theta_2\frac{\partial^2 W}{\partial\theta_2^2}\right] = \Theta_1\cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2}$$

$$\frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\Theta_1\cosh\chi\frac{\partial W}{\partial\chi}\right) = \frac{1}{\cosh\chi}\left[\Theta_1\sinh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\cosh\chi\frac{\partial^2 W}{\partial\chi^2}\right] = \Theta_1\tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}$$

Aplicando na Equação de Laplace

$$\frac{1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\Theta_1\cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \Theta_1\frac{\partial^2 W}{\partial\theta_2^2} + \Theta_1\tanh(\chi)\frac{\partial W}{\partial\chi} + \Theta_1\frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

$$\frac{\Theta_1}{a^2\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + \tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{1}{a^2\sin^2\theta_2\cosh^2\chi}\frac{\partial^2\Theta_1}{\partial\theta_1^2}W = 0$$

Multiplicando ambos os lados por $\frac{a^2\sin^2\theta_2\cosh^2\chi}{\Theta_1 W}$

$$\frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + \tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] + \frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = 0$$

$$-\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = \frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{\left(\sinh^2\chi + \cos^2\theta_2\right)}\left[\cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + \tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] = \lambda_1$$

Daí obtêm-se

$$-\frac{\partial^2\Theta_1}{\partial\theta_1^2}\frac{1}{\Theta_1} = \lambda_1 \Rightarrow \frac{\partial^2\Theta_1}{\partial\theta_1^2} = -\Theta_1\lambda_1 \Rightarrow \boldsymbol{\frac{\partial^2\Theta_1}{\partial\theta_1^2} + \Theta_1\lambda_1 = 0}$$

1 Resolvendo a EDO para Θ_1

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_1 = 0$$

1.1 Caso $\lambda_1 = 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} = 0$$

A solução para esse caso é

$$\Theta_1(\theta_1) = A\theta_1 + B$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$A\theta_1 + B = A(\theta_1 + 2\pi) + B$$

Chega-se em

$$\theta_1 = (\theta_1 + 2\pi)$$

Essa conclusão é absurda, então conclui-se que $\lambda_1 \neq 0$.

1.2 Caso $\lambda_1 < 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} - \Theta_1\lambda_1 = 0$$

A equação característica é

$$D^2 - \lambda_1 = 0 \Rightarrow D^2 = \lambda_1 \Rightarrow D = \pm\sqrt{\lambda_1}$$

De modo que as raízes são reais e diferentes. Nesse caso a solução geral é

$$\Theta_1(\theta) = Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1} = Ae^{\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se

$$\begin{cases} e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-\sqrt{\lambda_1}\theta_1} = e^{-\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}\theta_1}e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{2\pi\sqrt{\lambda_1}} = 1$$

Já que $e^{2\pi\sqrt{\lambda_1}} = 1$ também é um absurdo, a única possibilidade restante é $\lambda_1 > 0$. Verifica-se a seguir.

1.3 Caso $\lambda_1 > 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_1 = 0$$

A equação característica dessa EDO é

$$D^2 + \lambda_1 = 0 \Rightarrow D^2 = -\lambda_1 \Rightarrow D = \sqrt{-\lambda_1} = i\sqrt{\lambda_1}$$

De modo que, como as raízes são complexas conjugadas $D = \alpha + i\beta$, a solução geral é:

$$\Theta_1(\theta_1) = Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1} = Ae^{i\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-i\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se o seguinte sistema

$$\begin{cases} e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-i\sqrt{\lambda_1}\theta_1} = e^{-i\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}\theta_1}e^{i\sqrt{\lambda_1}2\pi} \Rightarrow e^{i\sqrt{\lambda_1}2\pi} = 1$$

$$e^{i\sqrt{\lambda_1}2\pi} = 1 \Rightarrow cos\left(\sqrt{\lambda_1}2\pi\right) + i\,sin\left(\sqrt{\lambda_1}2\pi\right) = 1 \Rightarrow \sqrt{\lambda_1} = m \Rightarrow \boldsymbol{\lambda_1 = m^2}$$

Então temos

$$\boldsymbol{\Theta_1(\theta_1) = Ae^{im\theta_1} + Be^{-im\theta_1}}$$

2 Continuando na outra equação

$$\frac{1}{W}\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left[cot(\theta_2)\frac{\partial W}{\partial\theta_2} + \frac{\partial^2 W}{\partial\theta_2^2} + tanh(\chi)\frac{\partial W}{\partial\chi} + \frac{\partial^2 W}{\partial\chi^2}\right] = \lambda_1$$

Fazendo

$$W(\theta_2,\chi) = \Theta_2(\theta_2)X(\chi)$$

$$\frac{1}{\Theta_2 X}\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left[Xcot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + X\frac{\partial^2\Theta_2}{\partial\theta_2^2} + \Theta_2 tanh(\chi)\frac{\partial X}{\partial\chi} + \Theta_2\frac{\partial^2 X}{\partial\chi^2}\right] = \lambda_1$$

$$\frac{1}{\Theta_2 X}\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left[X\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \Theta_2\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right)\right] = \lambda_1$$

$$\frac{1}{\Theta_2}\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) = \lambda_1$$

$$\frac{\sin\theta_2\cosh^2\chi}{(\sinh^2\chi + \cos^2\theta_2)}\left[\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right)\right] = \lambda_1$$

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) = \lambda_1\frac{(\sinh^2\chi + \cos^2\theta_2)}{\sin\theta_2\cosh^2\chi}$$

Utilizando

$$\frac{(\sinh^2\chi + \cos^2\theta_2)}{\sin^2\theta_2\cosh^2\chi} = \frac{1}{\sin^2\theta_2} + \frac{1}{\cosh^2\chi}$$

Tem-se

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) = \lambda_1\left(\frac{1}{\sin^2\theta_2} + \frac{1}{\cosh^2\chi}\right)$$

$$\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) = \frac{\lambda_1}{\sin^2\theta_2} + \frac{\lambda_1}{\cosh^2\chi}$$

$$-\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{\lambda_1}{\sin^2\theta_2} - \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) + \frac{\lambda_1}{\cosh^2\chi} = 0$$

$$-\frac{1}{\Theta_2}\left(cot(\theta_2)\frac{\partial\Theta_2}{\partial\theta_2} + \frac{\partial^2\Theta_2}{\partial\theta_2^2}\right) + \frac{\lambda_1}{\sin^2\theta_2} = \frac{1}{X}\left(tanh(\chi)\frac{\partial X}{\partial\chi} + \frac{\partial^2 X}{\partial\chi^2}\right) - \frac{\lambda_1}{\cosh^2\chi}$$

Nota-se que

$$\left(cot(\theta_2)\frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2}\right) = \left(\frac{cos(\theta_2)}{sin(\theta_2)}\frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2}\right) = \frac{1}{sin(\theta_2)}\left(cos(\theta_2)\frac{\partial \Theta_2}{\partial \theta_2} + \sin\theta_2\frac{\partial^2 \Theta_2}{\partial \theta_2^2}\right) = \frac{1}{sin(\theta_2)}\frac{\partial}{\partial \theta_2}\left(\sin\theta_2\frac{\partial \Theta_2}{\partial \theta_2}\right)$$

$$\left(tanh(\chi)\frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2}\right) = \left(\frac{sinh(\chi)}{cosh(\chi)}\frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2}\right) = \frac{1}{cosh(\chi)}\left(sinh(\chi)\frac{\partial X}{\partial \chi} + cosh(\chi)\frac{\partial^2 X}{\partial \chi^2}\right) = \frac{1}{\cosh\chi}\frac{\partial}{\partial \chi}\left(\cosh\chi\frac{\partial X}{\partial \chi}\right)$$

Então

$$-\frac{1}{\Theta_2 sin(\theta_2)}\frac{\partial}{\partial \theta_2}\left(\sin\theta_2\frac{\partial \Theta_2}{\partial \theta_2}\right) + \frac{\lambda_1}{\sin^2\theta_2} = \frac{1}{X\cosh\chi}\frac{\partial}{\partial \chi}\left(\cosh\chi\frac{\partial X}{\partial \chi}\right) - \frac{\lambda_1}{\cosh^2\chi} = \lambda_2$$

$$-\frac{1}{\Theta_2 sin(\theta_2)}\frac{\partial}{\partial \theta_2}\left(\sin\theta_2\frac{\partial \Theta_2}{\partial \theta_2}\right) + \frac{\lambda_1}{\sin^2\theta_2} = \lambda_2$$

$$\frac{1}{X\cosh\chi}\frac{\partial}{\partial \chi}\left(\cosh\chi\frac{\partial X}{\partial \chi}\right) - \frac{\lambda_1}{\cosh^2\chi} = \lambda_2$$

Pode-se escrever

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta}\left(\sin\theta_2\frac{\partial \Theta_2}{\partial \theta_2}\right) + \left(\lambda_2 - \frac{\lambda_1}{\sin^2\theta_2}\right)\Theta_2 = 0$$

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) - \left(\lambda_2 + \frac{\lambda_1}{\cosh^2\chi}\right)X = 0$$

2.1 Resolvendo para Θ_2

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \left(\lambda_2 - \frac{m^2}{\sin^2\theta_2}\right)\Theta_2 = 0$$

Determina-se $\lambda_2 = n(n+1)$, $n \in \mathbb{Z}_+$ de modo que tem-se

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \left(n(n+1) - \frac{m^2}{\sin^2\theta_2}\right)\Theta_2 = 0$$

Fazendo $x = cos\theta_2$

$$sin^2(\theta_2) = 1 - cos^2(\theta_2) = 1 - x^2$$

$$\frac{d\Theta_2}{d\theta_2} = \frac{d\Theta_2}{dx}\frac{dx}{d\theta_2} = -sin(\theta_2)\frac{d\Theta_2}{dx}$$

$$\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) = \frac{d}{d\theta_2}\left(-sin^2(\theta_2)\frac{d\Theta_2}{dx}\right) = -2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx} - sin^2(\theta_2)\frac{d}{d\theta_2}\frac{d\Theta_2}{dx}$$

$$= -2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx} - sin^2(\theta_2)\frac{d}{dx}\frac{d\Theta_2}{d\theta_2} = -2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx} + sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) \Rightarrow \frac{1}{sin(\theta_2)}\left(-2cos\theta_2 sin\theta_2\frac{d\Theta_2}{dx} + sin^3(\theta_2)\frac{d^2\Theta_2}{dx^2}\right) = -2cos\theta_2\frac{d\Theta_2}{dx} + sin^2(\theta_2)\frac{d^2\Theta_2}{dx^2}$$

Tem-se

$$\frac{1}{sin(\theta_2)}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{d\Theta_2}{d\theta_2}\right) + \left(n(n+1) - \frac{m^2}{\sin^2\theta_2}\right)\Theta_2 = 0 \Rightarrow -2x\frac{d\Theta_2}{dx} + \left(1-x^2\right)\frac{d^2\Theta_2}{dx^2} + \left(n(n+1) - \frac{m^2}{(1-x^2)}\right)\Theta_2 = 0$$

$$\left(1-x^2\right)\frac{d^2\Theta_2}{dx^2} - 2x\frac{d\Theta_2}{dx} + \left(n(n+1) - \frac{m^2}{(1-x^2)}\right)\Theta_2 = 0$$

Essa é a **Equação de Legendre Generalizada** e suas soluções, para $m, n \in \mathbb{Z}$, são dadas em termos de Polinômios Associados de Legendre.

$$\Theta_{2,n,m}(x) = P_n^m(x)$$

Quando $m = 0$ tem-se o caso especial em que a solução reduz-se a um Polinômio de Legendre

$$\Theta_{2,n,0}(x) = P_n^0(x) = P_n(x)$$

2.2 Olhando para χ

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) - \left(\lambda_2 + \frac{\lambda_1}{\cosh^2\chi}\right)X = 0$$

Fazendo $\zeta = \sinh\chi$ tem-se

$$cosh^2(\chi) = 1 - sinh^2(\chi) = 1 - \zeta^2$$

$$\frac{dX}{d\chi} = \frac{dX}{d\zeta}\frac{d\zeta}{d\chi} = cosh\chi\frac{dX}{d\zeta}$$

$$\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) = \frac{d}{d\chi}\left(\cosh^2\chi\frac{dX}{d\zeta}\right) = 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d}{d\chi}\frac{dX}{d\zeta}$$

$$= 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d}{d\zeta}\frac{dX}{d\chi} = 2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^3\chi\frac{d^2X}{d\zeta^2}$$

$$\frac{1}{\cosh\chi}\frac{d}{d\chi}\left(\cosh\chi\frac{dX}{d\chi}\right) = \frac{1}{\cosh\chi}\left(2cosh\chi sinh\chi\frac{dX}{d\zeta} + \cosh^3\chi\frac{d^2X}{d\zeta^2}\right) = 2sinh\chi\frac{dX}{d\zeta} + \cosh^2\chi\frac{d^2X}{d\zeta^2}$$

Então, considerando $\lambda_1 = m^2$ e $\lambda_2 = n(n+1)$, chega-se em:

$$\left(1 - \zeta^2\right) \frac{d^2 X}{d\zeta^2} + 2\zeta \frac{dX}{d\zeta} - \left(n(n+1) + \frac{m^2}{(1 - \zeta^2)}\right) X = 0$$