1 Laplaciano hiperboloide de 2 folhas (3-dim)

Dada a seguinte parametrização:

$$\begin{aligned}
x_3 &= a\cosh\chi\cos\theta_2 \\
x_2 &= a\sinh\chi\sin\theta_2\cos\theta_1 \\
x_1 &= a\sinh\chi\sin\theta_2\sin\theta_1 \\
0 &\leq \chi < \infty,\ 0 \leq \theta_2 \leq \pi,\ 0 \leq \theta_1 < 2\pi
\end{aligned}$$

com

$$q_1 = \theta_1,\ q_2 = \theta_2,\ q_3 = \chi$$

Utiliza-se o Operador de Laplace-Beltrami para encontrar o Laplaciano:

$$\Delta u = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q_i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial q_j}\right)$$

De acordo com Arfken(2017), pode-se determinar o tensor métrico g_{ij}

$$G = (g_{ij}) = \sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$$

Na dada parametrização, forma-se um **sistema de coordenadas ortogonal** de modo que, segundo Riley(2006), os elementos do tensor métrico podem ser dados por:

$$g_{ij} = \begin{cases} h_i^2 & ,i = j \\ 0 & ,i \neq j \end{cases}$$

Onde $h_i^2 = \sum_j \left[\left(\frac{\partial x_j}{\partial q_i}\right)^2\right]$. Ou seja, h_i é associado a q_i. Como nesse caso $q_1 = \theta_1$, $q_2 = \theta_2$, $q_3 = \chi$ então tem-se $h_1 = h_{\theta_1}$, $h_2 = h_{\theta_2}$ e $h_3 = \chi$.

Ou seja, a matriz $G = (g_{ij})$ é diagonal com os elementos h_1^2, h_2^2 e h_3^2. Em outras palavras $G = diag(h_1^2,\, h_2^2,\, h_3^2)$.

$$G = (g_{ij}) = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

Ainda de acordo com Riley(2006), o tensor métrico inverso, para um sistema ortogonal de coordenadas, pode ser dado por:

$$G^{-1} = (g^{ij}) = \begin{cases} 1/h_i^2 & ,i = j \\ 0 & ,i \neq j \end{cases}$$

$$G^{-1} = (g^{ij}) = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix}$$

E, por fim, $g = det(g_{ij}) = h_1^2 h_2^2 h_3^2$. Consequentemente, $\sqrt{|g|} = h_1 h_2 h_3$.

Os coeficientes métricos são encontrados da seguinte forma:

$$h_1^2 = h_{\theta_1}^2 = \left(\frac{\partial x_1}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_1}\right)^2$$

$$h_2^2 = h_{\theta_2}^2 = \left(\frac{\partial x_1}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_2}\right)^2$$

$$h_3^2 = h_{\chi}^2 = \left(\frac{\partial x_1}{\partial \chi}\right)^2 + \left(\frac{\partial x_2}{\partial \chi}\right)^2 + \left(\frac{\partial x_3}{\partial \chi}\right)^2$$

Calcula-se tais coeficientes:

$$h_1^2 = h_{\theta_1}^2 = \left(\frac{\partial x_1}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_1}\right)^2 = a^2 sinh^2(\chi) cos^2(\theta_1) sin^2(\theta_2) + a^2 sinh^2(\chi) sin^2(\theta_1) sin^2(\theta_2)$$

$$= a^2 sinh^2(\chi) sin^2(\theta_2) \left[cos^2(\theta_1) + sin^2(\theta_1)\right]$$

$$\boldsymbol{h_1^2 = h_{\theta_1}^2 = a^2 sinh^2(\chi) sin^2(\theta_2)}$$

$$h_2^2 = h_{\theta_2}^2 = \left(\frac{\partial x_1}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_2}\right)^2 = a^2 sinh^2(\chi) sin^2(\theta_1) cos^2(\theta_2)$$

$$+ a^2 sinh^2(\chi) cos^2(\theta_1) cos^2(\theta_2) + a^2 cosh^2(\chi) sin^2(\theta_2)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_2) \left[sin^2(\theta_1) + cos^2(\theta_1)\right] + a^2 cosh^2(\chi) sin^2(\theta_2)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_2) + a^2 cosh^2(\chi) sin^2(\theta_2)$$

$$\boldsymbol{h_2^2 = h_{\theta_2}^2 = a^2 \left[sinh^2(\chi) cos^2(\theta_2) + cosh^2(\chi) sin^2(\theta_2)\right]}$$

$$h_3^2 = h_{\chi}^2 = \left(\frac{\partial x_1}{\partial \chi}\right)^2 + \left(\frac{\partial x_2}{\partial \chi}\right)^2 + \left(\frac{\partial x_3}{\partial \chi}\right)^2 = a^2 cosh^2(\chi) sin^2(\theta_1) sin^2(\theta_2) + a^2 cosh^2(\chi) cos^2(\theta_1) sin^2(\theta_2)$$

$$+ a^2 sinh^2(\chi) cos^2(\theta_2)$$

$$= a^2 cosh^2(\chi) sin^2(\theta_2) \left[sin^2(\theta_1) + cos^2(\theta_1)\right] + a^2 sinh^2(\chi) cos^2(\theta_2)$$

$$= a^2 cosh^2(\chi) sin^2(\theta_2) + a^2 sinh^2(\chi) cos^2(\theta_2)$$

$$\boldsymbol{h_3^2 = h_{\chi}^2 = a^2 \left[cosh^2(\chi) sin^2(\theta_2) + sinh^2(\chi) cos^2(\theta_2)\right]}$$

Observa-se que $h_2^2 = h_{\theta_2}^2 = h_3^2 = h_{\chi}^2$ então $\sqrt{|g|} = h_1 h_2^2$. Agora é possível construir a matriz $\sqrt{|g|} g^{ij}$:

$$\sqrt{|g|} g^{ij} = h_1 h_2^2 \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix} = \begin{pmatrix} \frac{h_2^2}{h_1} & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \end{pmatrix}$$

Aplica-se os resultados no Operador de Laplace-Beltrami e chega-se em:

$$\Delta u = \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial q_1}\left(\sqrt{|g|}g^{11}\frac{\partial u}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\sqrt{|g|}g^{22}\frac{\partial u}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\sqrt{|g|}g^{33}\frac{\partial u}{\partial q_3}\right)\right]$$

$$\Delta u = \frac{1}{h_1 h_2^2}\left[\frac{\partial}{\partial \theta_1}\left(\frac{h_2^2}{h_1}\frac{\partial u}{\partial \theta_1}\right) + \frac{\partial}{\partial \theta_2}\left(h_1\frac{\partial u}{\partial \theta_2}\right) + \frac{\partial}{\partial \chi}\left(h_1\frac{\partial u}{\partial \chi}\right)\right]$$

$$\Delta u = \frac{1}{h_1 h_2^2}\left[\frac{\partial}{\partial \theta_1}\left(\frac{a\left[sinh^2(\chi)cos^2(\theta_2) + cosh^2(\chi)sin^2(\theta_2)\right]}{sinh(\chi)sin(\theta_2)}\frac{\partial u}{\partial \theta_1}\right)\right.$$
$$\left. + \frac{\partial}{\partial \theta_2}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \theta_2}\right) + \frac{\partial}{\partial \chi}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \chi}\right)\right]$$

$$\Delta u = \frac{1}{h_1 h_2^2}\left[\frac{a\left[sinh^2(\chi)cos^2(\theta_2) + cosh^2(\chi)sin^2(\theta_2)\right]}{sinh(\chi)sin(\theta_2)}\frac{\partial^2 u}{\partial \theta^2{}_1} + \frac{\partial}{\partial \theta_2}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \theta_2}\right)\right.$$
$$\left. + \frac{\partial}{\partial \chi}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \chi}\right)\right]$$

Sabendo que $\frac{1}{h_1 h_2^2} = \frac{1}{a^3 sinh(\chi)sin(\theta_2)[sinh^2(\chi)cos^2(\theta_2)+cosh^2(\chi)sin^2(\theta_2)]}$, o Laplaciano fica:

$$\Delta u = \frac{1}{h_1 h_2^2}\left[\frac{\partial}{\partial \theta_2}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \theta_2}\right) + \frac{\partial}{\partial \chi}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \chi}\right)\right] + \frac{1}{a^2 sinh^2(\chi)sin^2(\theta_2)}\frac{\partial^2 u}{\partial \theta^2{}_1}$$

$$\boldsymbol{\Delta u = \frac{1}{a^3 sinh(\chi)sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_2) + cosh^2(\chi)sin^2(\theta_2)\right]}\left[\frac{\partial}{\partial \theta_2}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \theta_2}\right)\right.}$$
$$\boldsymbol{\left. + \frac{\partial}{\partial \chi}\left(asinh(\chi)sin(\theta_2)\frac{\partial u}{\partial \chi}\right)\right] + \frac{1}{a^2 sinh^2(\chi)sin^2(\theta_2)}\frac{\partial^2 u}{\partial \theta_1^2}}$$

OBS: Não consegui simplificar $a^3 sinh(\chi)sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_2) + cosh^2(\chi)sin^2(\theta_2)\right]$**.**

2 Laplaciano hiperboloide de 2 folhas (4-dim)

Dada a seguinte parametrização:

$$\begin{aligned}
x_4 &= a\cosh\chi\cos\theta_3 \\
x_3 &= a\sinh\chi\sin\theta_3\cos\theta_2 \\
x_2 &= a\sinh\chi\sin\theta_3\sin\theta_2\cos\theta_1 \\
x_1 &= a\sinh\chi\sin\theta_3\sin\theta_2\sin\theta_1 \\
0 &\leq \chi < \infty,\ 0 \leq \theta_3, \theta_2 \leq \pi,\ 0 \leq \theta_1 < 2\pi
\end{aligned}$$

com

$$q_1 = \theta_1,\ q_2 = \theta_2,\ q_3 = \theta_3,\ q_4 = \chi$$

Utiliza-se o Operador de Laplace-Beltrami para encontrar o Laplaciano:

$$\Delta u = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q_i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial q_j}\right)$$

De acordo com Arfken(2017), pode-se determinar o tensor métrico g_{ij}

$$G = (g_{ij}) = \sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$$

Na dada parametrização, forma-se um **sistema de coordenadas ortogonal** de modo que, segundo Riley(2006), os elementos do tensor métrico podem ser dados por:

$$g_{ij} = \begin{cases} h_i^2 & ,i = j \\ 0 & ,i \neq j \end{cases}$$

Onde $h_i^2 = \sum_j \left[\left(\frac{\partial x_j}{\partial q_i}\right)^2\right]$. Ou seja, h_i é associado a q_i. Como nesse caso $q_1 = \theta_1$, $q_2 = \theta_2$, $q_3 = \theta_3$, $q_4 = \chi$ então tem-se $h_1 = h_{\theta_!}$, $h_2 = h_{\theta_2}$, $h_3 = h_{\theta_3}$ e $h_4 = \chi$.

Ou seja, a matriz $G = (g_{ij})$ é diagonal com os elementos h_1^2, h_2^2 , h_3^2 e h_4^2. Em outras palavras $G = diag(h_1^2, h_2^2, h_3^2, h_4^2)$.

$$G = (g_{ij}) = \begin{pmatrix} h_1^2 & 0 & 0 & 0 \\ 0 & h_2^2 & 0 & 0 \\ 0 & 0 & h_3^2 & 0 \\ 0 & 0 & 0 & h_4^2 \end{pmatrix}$$

Ainda de acordo com Riley(2006), o tensor métrico inverso, para um sistema ortogonal de coordenadas, pode ser dado por:

$$G^{-1} = (g^{ij}) = \begin{cases} 1/h_i^2 & ,i = j \\ 0 & ,i \neq j \end{cases}$$

$$G^{-1} = (g^{ij}) = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 & 0 \\ 0 & 0 & \frac{1}{h_3^2} & 0 \\ 0 & 0 & 0 & \frac{1}{h_4^2} \end{pmatrix}$$

E, por fim, $g = det(g_{ij}) = h_1^2 h_2^2 h_3^2 h_4^2$. Consequentemente, $\sqrt{|g|} = h_1 h_2 h_3 h_4$.

Os coeficientes métricos são encontrados da seguinte forma:

$$h_1^2 = h_{\theta_1}^2 = \left(\frac{\partial x_1}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_4}{\partial \theta_1}\right)^2$$

$$h_2^2 = h_{\theta_2}^2 = \left(\frac{\partial x_1}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_4}{\partial \theta_2}\right)^2$$

$$h_3^2 = h_{\theta_3}^2 = \left(\frac{\partial x_1}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_4}{\partial \theta_3}\right)^2$$

$$h_4^2 = h_{\chi}^2 = \left(\frac{\partial x_1}{\partial \chi}\right)^2 + \left(\frac{\partial x_2}{\partial \chi}\right)^2 + \left(\frac{\partial x_3}{\partial \chi}\right)^2 + \left(\frac{\partial x_4}{\partial \chi}\right)^2$$

Calcula-se tais coeficientes:

$$h_1^2 = h_{\theta_1}^2 = \left(\frac{\partial x_1}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_1}\right)^2 = a^2 sinh^2(\chi) cos^2(\theta_1) sin^2(\theta_2) sin^2(\theta_3) + a^2 sinh^2(\chi) sin^2(\theta_1) sin^2(\theta_2) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3) \left[cos^2(\theta_1) + sin^2(\theta_1)\right]$$

$$= a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)$$

$$\boldsymbol{h_1^2 = h_{\theta_1}^2 = a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)}$$

$$h_2^2 = h_{\theta_2}^2 = \left(\frac{\partial x_1}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_2}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_2}\right)^2 = a^2 sinh^2(\chi) sin^2(\theta_1) cos^2(\theta_2) sin^2(\theta_3) + a^2 sinh^2(\chi) cos^2(\theta_1) cos^2(\theta_2) sin^2(\theta_3)$$

$$+ a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_2) sin^2(\theta_3) \left[sin^2(\theta_1) + cos^2(\theta_1)\right] + a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_2) sin^2(\theta_3) + a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) sin^2(\theta_3) \left[cos^2(\theta_2) + sin^2(\theta_2)\right]$$

$$\boldsymbol{h_2^2 = h_{\theta_2}^2 = a^2 sinh^2(\chi) sin^2(\theta_3)}$$

$$h_3^2 = h_{\theta_3}^2 = \left(\frac{\partial x_1}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_2}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_3}{\partial \theta_3}\right)^2 + \left(\frac{\partial x_4}{\partial \theta_3}\right)^2 = a^2 sinh^2(\chi) sin^2(\theta_1) sin^2(\theta_2) cos^2(\theta_3)$$

$$+ a^2 sinh^2(\chi) cos^2(\theta_1) sin^2(\theta_2) cos^2(\theta_3)$$

$$+ a^2 sinh^2(\chi) cos^2(\theta_2) cos^2(\theta_3) + a^2 cosh^2(\chi) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) sin^2(\theta_2) cos^2(\theta_3) \left[sin^2(\theta_1) + cos^2(\theta_1)\right] + a^2 sinh^2(\chi) cos^2(\theta_2) cos^2(\theta_3) + a^2 cosh^2(\chi) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) sin^2(\theta_2) cos^2(\theta_3) + a^2 sinh^2(\chi) cos^2(\theta_2) cos^2(\theta_3) + a^2 cosh^2(\chi) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_3) \left[sin^2(\theta_2) + cos^2(\theta_2)\right] + a^2 cosh^2(\chi) sin^2(\theta_3)$$

$$= a^2 sinh^2(\chi) cos^2(\theta_3) + a^2 cosh^2(\chi) sin^2(\theta_3)$$

$$\boldsymbol{h_3^2 = h_{\theta_3}^2 = a^2 \left[sinh^2(\chi) cos^2(\theta_3) + cosh^2(\chi) sin^2(\theta_3)\right]}$$

$$h_4^2 = h_\chi^2 = \left(\frac{\partial x_1}{\partial \chi}\right)^2 + \left(\frac{\partial x_2}{\partial \chi}\right)^2 + \left(\frac{\partial x_3}{\partial \chi}\right)^2 + \left(\frac{\partial x_4}{\partial \chi}\right)^2 = a^2 cosh^2(\chi) sin^2(\theta_1) sin^2(\theta_2) sin^2(\theta_3)$$

$$+ a^2 cosh^2(\chi) cos^2(\theta_1) sin^2(\theta_2) sin^2(\theta_3)$$

$$+a^2cosh^2(\chi)cos^2(\theta_2)sin^2(\theta_3)+a^2sinh^2(\chi)cos^2(\theta_3)$$

$$=a^2cosh^2(\chi)sin^2(\theta_2)sin^2(\theta_3)\left[sin^2(\theta_1)+cos^2(\theta_1)\right]+a^2cosh^2(\chi)cos^2(\theta_2)sin^2(\theta_3)+a^2sinh^2(\chi)cos^2(\theta_3)$$

$$=a^2cosh^2(\chi)sin^2(\theta_2)sin^2(\theta_3)+a^2cosh^2(\chi)cos^2(\theta_2)sin^2(\theta_3)+a^2sinh^2(\chi)cos^2(\theta_3)$$

$$=a^2cosh^2(\chi)sin^2(\theta_3)\left[sin^2(\theta_2)+cos^2(\theta_2)\right]+a^2sinh^2(\chi)cos^2(\theta_3)$$

$$=a^2cosh^2(\chi)sin^2(\theta_3)+a^2sinh^2(\chi)cos^2(\theta_3)$$

$$\boldsymbol{h_4^2=h_\chi^2=a^2\left[cosh^2(\chi)sin^2(\theta_3)+sinh^2(\chi)cos^2(\theta_3)\right]} \tag{1}$$

Observa-se que $h_3^2=h_{\theta_3}^2=h_4^2=h_\chi^2$ então $\sqrt{|g|}=h_1h_2h_3^2$. Agora é possível construir a matriz $\sqrt{|g|}g^{ij}$:

$$\sqrt{|g|}g^{ij}=h_1h_2h_3^2\begin{pmatrix}\frac{1}{h_1^2}&0&0&0\\0&\frac{1}{h_2^2}&0&0\\0&0&\frac{1}{h_3^2}&0\\0&0&0&\frac{1}{h_4^2}\end{pmatrix}=\begin{pmatrix}\frac{h_2h_3^2}{h_1}&0&0&0\\0&\frac{h_1h_3^2}{h_2}&0&0\\0&0&h_1h_2&0\\0&0&0&h_1h_2\end{pmatrix}$$

Aplica-se os resultados no Operador de Laplace-Beltrami e chega-se em:

$$\Delta u=\frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial q_1}\left(\sqrt{|g|}g^{11}\frac{\partial u}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\sqrt{|g|}g^{22}\frac{\partial u}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\sqrt{|g|}g^{33}\frac{\partial u}{\partial q_3}\right)++\frac{\partial}{\partial q_4}\left(\sqrt{|g|}g^{44}\frac{\partial u}{\partial q_4}\right)\right]$$

$$\Delta u=\frac{1}{h_1h_2h_3^2}\left[\frac{\partial}{\partial\theta_1}\left(\frac{h_2h_3^2}{h_1}\frac{\partial u}{\partial\theta_1}\right)+\frac{\partial}{\partial\theta_2}\left(\frac{h_1h_3^2}{h_2}\frac{\partial u}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_3}\left(h_1h_2\frac{\partial u}{\partial\theta_3}\right)+\frac{\partial}{\partial\chi}\left(h_1h_2\frac{\partial u}{\partial\chi}\right)\right]$$

Dado que

$$\frac{1}{h_1h_2h_3^2}=\frac{1}{a^4sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_3)+cosh^2(\chi)sin^2(\theta_3)\right]}$$

$$\frac{h_2h_3^2}{h_1}=\frac{a^2\left[sinh^2(\chi)cos^2(\theta_3)+cosh^2(\chi)sin^2(\theta_3)\right]}{sin(\theta_2)}$$

$$\frac{h_1h_3^2}{h_2}=sin(\theta_2)a^2\left[sinh^2(\chi)cos^2(\theta_3)+cosh^2(\chi)sin^2(\theta_3)\right]$$

$$h_1h_2=a^2sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)$$

Vê-se que $\frac{h_2h_3^2}{h_1}$ não depende de θ_1, logo:

$$\frac{\partial}{\partial\theta_1}\left(\frac{h_2h_3^2}{h_1}\frac{\partial u}{\partial\theta_1}\right)=\frac{h_2h_3^2}{h_1}\frac{\partial^2u}{\partial\theta_1^2}=\frac{a^2\left[sinh^2(\chi)cos^2(\theta_3)+cosh^2(\chi)sin^2(\theta_3)\right]}{sin(\theta_2)}\frac{\partial^2u}{\partial\theta_1^2}$$

O Laplaciano fica então:

$$\Delta u=\frac{1}{h_1h_2h_3^2}\left[\frac{h_2h_3^2}{h_1}\frac{\partial^2u}{\partial\theta_1^2}+\frac{\partial}{\partial\theta_2}\left(\frac{h_1h_3^2}{h_2}\frac{\partial u}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_3}\left(h_1h_2\frac{\partial u}{\partial\theta_3}\right)+\frac{\partial}{\partial\chi}\left(h_1h_2\frac{\partial u}{\partial\chi}\right)\right]$$

$$\Delta u = \frac{1}{h_1 h_2 h_3^2}\left[\frac{\partial}{\partial\theta_2}\left(\frac{h_1 h_3^2}{h_2}\frac{\partial u}{\partial\theta_2}\right) + \frac{\partial}{\partial\theta_3}\left(h_1 h_2 \frac{\partial u}{\partial\theta_3}\right) + \frac{\partial}{\partial\chi}\left(h_1 h_2 \frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{h_1^2}\frac{\partial^2 u}{\partial\theta_1^2}$$

Para finalmente encontrar a expressão do Laplaciano, aplica-se as expressões para dos coeficientes métricos:

$$\boldsymbol{\Delta u = \frac{1}{a^4 sinh^2(\chi) sin^2(\theta_3) sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_3) + cosh^2(\chi)sin^2(\theta_3)\right]}}$$

$$\boldsymbol{\left[\frac{\partial}{\partial\theta_2}\left(sin(\theta_2)a^2\left[sinh^2(\chi)cos^2(\theta_3) + cosh^2(\chi)sin^2(\theta_3)\right]\frac{\partial u}{\partial\theta_2}\right)\right.}$$

$$\boldsymbol{\left.+\frac{\partial}{\partial\theta_3}\left(a^2 sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\frac{\partial u}{\partial\theta_3}\right) + \frac{\partial}{\partial\chi}\left(a^2 sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\frac{\partial u}{\partial\chi}\right)\right]}$$

$$\boldsymbol{+\frac{1}{a^2 sinh^2(\chi) sin^2(\theta_2) sin^2(\theta_3)}\frac{\partial^2 u}{\partial\theta_1^2}}$$

OBS1: Não consegui simplificar $a^4 sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_3) + cosh^2(\chi)sin^2(\theta_3)\right]$**.**

OBS2: Utilizando outro Software de Matemática Simbólica (Python utilizando o pacote SymPy) eu encontro a expressão (1) simplificada na forma $h_3^2 = h_{\theta_3}^2 = h_4^2 = h_\chi^2 = a^2\left[sin^2(\theta_3) + sinh^2(\chi)\right]$**. Porém, fazendo à mão, eu não consegui entender como ele chegou nisso e, portanto, não usei essa simplificação.**

3 Comparando Resultados

Ao comparar o **Laplaciano Hiperboloide de 1 Folha** (3-dim):

$$\Delta u = \frac{1}{a^3 cosh(\chi)\, sin(\theta_2)\left[sinh^2(\chi)sin^2(\theta_2) + cosh^2(\chi)cos^2(\theta_2)\right]}\left[\frac{\partial}{\partial\theta_2}\left(a\, cosh(\chi)\, sin(\theta_2)\frac{\partial u}{\partial\theta_2}\right)\right.$$
$$\left.+\frac{\partial}{\partial\chi}\left(a\, cosh(\chi)\, sin(\theta_2)\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\, cosh^2(\chi)\, sin^2(\theta_2)}\frac{\partial^2 u}{\partial\theta_1^2}$$

Com o **Laplaciano Hiperboloide de 2 Folhas** (3-dim):

$$\Delta u = \frac{1}{a^3 sinh(\chi) sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_2) + cosh^2(\chi)sin^2(\theta_2)\right]}\left[\frac{\partial}{\partial\theta_2}\left(a\, sinh(\chi)\, sin(\theta_2)\frac{\partial u}{\partial\theta_2}\right)\right.$$
$$\left.+\frac{\partial}{\partial\chi}\left(a\, sinh(\chi)\, sin(\theta_2)\frac{\partial u}{\partial\chi}\right)\right] + \frac{1}{a^2\, sinh^2(\chi)\, sin^2(\theta_2)}\frac{\partial^2 u}{\partial\theta_1^2}$$

Percebe-se que a principal diferença são algumas trocas de $sin(\theta_2)$ por $cos(\theta_2)$ e, principalmente, de $cosh(\chi)$ por $sinh(\chi)$.

Quando compara-se o **Laplaciano Hiperboloide de 1 Folha** (4-dim)

$$\Delta u = \frac{1}{a^4 sin(\theta_2) sin^2(\theta_3) cosh^2(\chi)[\cosh^2(\chi) - \sin^2(\theta_3)]}\left[\frac{\partial}{\partial\theta_2}\left(-a^2 sin(\theta_2)[\sin^2(\theta_3) - \cosh^2(\chi)]\frac{\partial u}{\partial\theta_2}\right)\right.$$
$$\left.+\frac{\partial}{\partial\theta_3}\left(a^2\sin(\theta_2)\sin^2(\theta_3)\cosh^2(\chi)\frac{\partial u}{\partial\theta_3}\right) + \frac{\partial}{\partial\chi}\left(a^2\sin(\theta_2)\sin^2(\theta_3)\cosh^2(\chi)\frac{\partial u}{\partial\chi}\right)\right]$$
$$-\left(\frac{1}{a^2 sin^2(\theta_2) sin^3(\theta_3) cosh^2(\chi)}\right)\frac{\partial^2 u}{\partial\theta_1^2}$$

Com o **Laplaciano Hiperboloide de 2 Folha** (4-dim).

$$\Delta u = \frac{1}{a^4 sinh^2(\chi) sin^2(\theta_3) sin(\theta_2)\left[sinh^2(\chi)cos^2(\theta_3) + cosh^2(\chi)sin^2(\theta_3)\right]}$$

$$\left[\frac{\partial}{\partial\theta_2}\left(sin(\theta_2)a^2\left[sinh^2(\chi)cos^2(\theta_3) + cosh^2(\chi)sin^2(\theta_3)\right]\frac{\partial u}{\partial\theta_2}\right)\right.$$

$$\left.+\frac{\partial}{\partial\theta_3}\left(a^2 sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\frac{\partial u}{\partial\theta_3}\right) + \frac{\partial}{\partial\chi}\left(a^2 sinh^2(\chi)sin^2(\theta_3)sin(\theta_2)\frac{\partial u}{\partial\chi}\right)\right]$$

$$+\frac{1}{a^2 sinh^2(\chi)sin^2(\theta_2)sin^2(\theta_3)}\frac{\partial^2 u}{\partial\theta_1^2}$$

Percebe-se que a principal diferença são algumas trocas de $cosh(\chi)$ por $sinh(\chi)$.