

$$\nabla^2 u = \frac{1}{a^2 (\sinh^2 \chi + \cos^2 \theta_2)} \left[\frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial u}{\partial \theta_2} \right) + \frac{1}{\cosh \chi} \frac{\partial}{\partial \chi} \left(\cosh \chi \frac{\partial u}{\partial \chi} \right) \right] + \frac{1}{a^2 \sin^2 \theta_2 \cosh^2 \chi} \frac{\partial^2 u}{\partial \theta_1^2} = 0$$

Define-se

$$u(\theta_1, \theta_2, \chi) = \Theta_1(\theta_1)W(\theta_2, \chi)$$

Aplicando na Equação de Laplace, tem-se:

$$\nabla^2 u = \frac{1}{a^2 (\sinh^2 \chi + \cos^2 \theta_2)} \left[\frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\Theta_1 \sin \theta_2 \frac{\partial W}{\partial \theta_2} \right) + \frac{1}{\cosh \chi} \frac{\partial}{\partial \chi} \left(\Theta_1 \cosh \chi \frac{\partial W}{\partial \chi} \right) \right] + \frac{1}{a^2 \sin^2 \theta_2 \cosh^2 \chi} \frac{\partial^2 \Theta_1}{\partial \theta_1^2} W = 0$$

Fazendo por partes

$$\begin{aligned} \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\Theta_1 \sin \theta_2 \frac{\partial W}{\partial \theta_2} \right) &= \frac{1}{\sin \theta_2} \left[\Theta_1 \cos(\theta_2) \frac{\partial W}{\partial \theta_2} + \Theta_1 \sin \theta_2 \frac{\partial^2 W}{\partial \theta_2^2} \right] = \Theta_1 \cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \Theta_1 \frac{\partial^2 W}{\partial \theta_2^2} \\ \frac{1}{\cosh \chi} \frac{\partial}{\partial \chi} \left(\Theta_1 \cosh \chi \frac{\partial W}{\partial \chi} \right) &= \frac{1}{\cosh \chi} \left[\Theta_1 \sinh(\chi) \frac{\partial W}{\partial \chi} + \Theta_1 \cosh \chi \frac{\partial^2 W}{\partial \chi^2} \right] = \Theta_1 \tanh(\chi) \frac{\partial W}{\partial \chi} + \Theta_1 \frac{\partial^2 W}{\partial \chi^2} \end{aligned}$$

Aplicando na Equação de Laplace

$$\begin{aligned} \frac{1}{a^2 (\sinh^2 \chi + \cos^2 \theta_2)} \left[\Theta_1 \cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \Theta_1 \frac{\partial^2 W}{\partial \theta_2^2} + \Theta_1 \tanh(\chi) \frac{\partial W}{\partial \chi} + \Theta_1 \frac{\partial^2 W}{\partial \chi^2} \right] + \frac{1}{a^2 \sin^2 \theta_2 \cosh^2 \chi} \frac{\partial^2 \Theta_1}{\partial \theta_1^2} W &= 0 \\ \frac{\Theta_1}{a^2 (\sinh^2 \chi + \cos^2 \theta_2)} \left[\cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \frac{\partial^2 W}{\partial \theta_2^2} + \tanh(\chi) \frac{\partial W}{\partial \chi} + \frac{\partial^2 W}{\partial \chi^2} \right] + \frac{1}{a^2 \sin^2 \theta_2 \cosh^2 \chi} \frac{\partial^2 \Theta_1}{\partial \theta_1^2} W &= 0 \end{aligned}$$

Multiplicando ambos os lados por $\frac{a^2 \sin^2 \theta_2 \cosh^2 \chi}{\Theta_1 W}$

$$\begin{aligned} \frac{1}{W} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[\cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \frac{\partial^2 W}{\partial \theta_2^2} + \tanh(\chi) \frac{\partial W}{\partial \chi} + \frac{\partial^2 W}{\partial \chi^2} \right] + \frac{\partial^2 \Theta_1}{\partial \theta_1^2} \frac{1}{\Theta_1} &= 0 \\ -\frac{\partial^2 \Theta_1}{\partial \theta_1^2} \frac{1}{\Theta_1} = \frac{1}{W} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[\cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \frac{\partial^2 W}{\partial \theta_2^2} + \tanh(\chi) \frac{\partial W}{\partial \chi} + \frac{\partial^2 W}{\partial \chi^2} \right] &= \lambda_1 \end{aligned}$$

Daí obtêm-se

$$-\frac{\partial^2 \Theta_1}{\partial \theta_1^2} \frac{1}{\Theta_1} = \lambda_1 \Rightarrow \frac{\partial^2 \Theta_1}{\partial \theta_1^2} = -\Theta_1 \lambda_1 \Rightarrow \frac{\partial^2 \Theta_1}{\partial \theta_1^2} + \Theta_1 \lambda_1 = 0$$

0.1 Resolvendo essa EDO

$$\frac{d^2 \Theta_1}{d\theta_1^2} + \Theta_1 \lambda_1 = 0$$

0.1.1 Caso $\lambda_1 = 0$

$$\frac{d^2 \Theta_1}{d\theta_1^2} = 0$$

$$d10^2 \Theta_1 \frac{10}{d\theta_1^2 = 0 \text{ equation*}}$$

A solução para esse caso é

$$\Theta_1(\theta_1) = A\theta_1 + B$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$A\theta_1 + B = A(\theta_1 + 2\pi) + B$$

Chega-se em

$$\theta_1 = (\theta_1 + 2\pi)$$

Essa conclusão é absurda, então conclui-se que $\lambda_1 \neq 0$.

0.1.2 Caso $\lambda_1 < 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} - \Theta_1\lambda_1 = 0$$

A equação característica é

$$D^2 - \lambda_1 = 0 \Rightarrow D^2 = \lambda_1 \Rightarrow D = \pm\sqrt{\lambda_1}$$

De modo que as raízes são reais e diferentes. Nesse caso a solução geral é

$$\Theta_1(\theta) = Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{\sqrt{\lambda_1}\theta_1} + Be^{-\sqrt{\lambda_1}\theta_1} = Ae^{\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se

$$\begin{cases} e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-\sqrt{\lambda_1}\theta_1} = e^{-\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{\sqrt{\lambda_1}\theta_1} = e^{\sqrt{\lambda_1}\theta_1} e^{\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{2\pi\sqrt{\lambda_1}} = 1$$

Já que $e^{2\pi\sqrt{\lambda_1}} = 1$ também é um absurdo, a única possibilidade restante é $\lambda_1 > 0$. Verifica-se a seguir.

0.1.3 Caso $\lambda_1 > 0$

$$\frac{d^2\Theta_1}{d\theta_1^2} + \Theta_1\lambda_1 = 0$$

A equação característica dessa EDO é

$$D^2 + \lambda_1 = 0 \Rightarrow D^2 = -\lambda_1 \Rightarrow D = \sqrt{-\lambda_1} = i\sqrt{\lambda_1}$$

De modo que, como as raízes são complexas conjugadas $D = \alpha + i\beta$, a solução geral é:

$$\Theta_1(\theta_1) = Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1}$$

Aplicando a condição de contorno $\Theta_1(\theta_1) = \Theta_1(\theta_1 + 2\pi)$

$$Ae^{i\sqrt{\lambda_1}\theta_1} + Be^{-i\sqrt{\lambda_1}\theta_1} = Ae^{i\sqrt{\lambda_1}(\theta_1+2\pi)} + Be^{-i\sqrt{\lambda_1}(\theta_1+2\pi)}$$

Disso extrai-se o seguinte sistema

$$\begin{cases} e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \\ e^{-i\sqrt{\lambda_1}\theta_1} = e^{-i\sqrt{\lambda_1}(\theta_1+2\pi)} \end{cases}$$

$$e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}(\theta_1+2\pi)} \Rightarrow e^{i\sqrt{\lambda_1}\theta_1} = e^{i\sqrt{\lambda_1}\theta_1} e^{i\sqrt{\lambda_1}2\pi} \Rightarrow e^{i\sqrt{\lambda_1}2\pi} = 1 \Rightarrow \cos(\sqrt{\lambda_1}2\pi) + i\sin(\sqrt{\lambda_1}2\pi) = 1 \Rightarrow \sqrt{\lambda_1} = m \Rightarrow \lambda_1 = m^2$$

Então temos

$$\Theta_{1,m}(\theta_1) = Ae^{im\theta_1} + Be^{-im\theta_1}$$

0.2 Continuando na outra equação

$$\frac{1}{W} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[\cot(\theta_2) \frac{\partial W}{\partial \theta_2} + \frac{\partial^2 W}{\partial \theta_2^2} + \tanh(\chi) \frac{\partial W}{\partial \chi} + \frac{\partial^2 W}{\partial \chi^2} \right] = \lambda_1$$

Fazendo

$$W(\theta_2, \chi) = \Theta_2(\theta_2)X(\chi)$$

$$\begin{aligned} \frac{1}{\Theta_2 X} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[X \cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + X \frac{\partial^2 \Theta_2}{\partial \theta_2^2} + \Theta_2 \tanh(\chi) \frac{\partial X}{\partial \chi} + \Theta_2 \frac{\partial^2 X}{\partial \chi^2} \right] &= \lambda_1 \\ \frac{1}{\Theta_2 X} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[X \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \Theta_2 \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) \right] &= \lambda_1 \\ \frac{1}{\Theta_2} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{1}{X} \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) &= \lambda_1 \\ \frac{\sin^2 \theta_2 \cosh^2 \chi}{(\sinh^2 \chi + \cos^2 \theta_2)} \left[\frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) \right] &= \lambda_1 \\ \frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) &= \lambda_1 \frac{(\sinh^2 \chi + \cos^2 \theta_2)}{\sin^2 \theta_2 \cosh^2 \chi} \end{aligned}$$

Utilizando

$$\begin{aligned} \frac{(\sinh^2 \chi + \cos^2 \theta_2)}{\sin^2 \theta_2 \cosh^2 \chi} &= \frac{\sinh^2 \chi}{\sin^2 \theta_2 \cosh^2 \chi} + \frac{\cos^2 \theta_2}{\sin^2 \theta_2 \cosh^2 \chi} = \frac{\cosh^2 \chi}{\sin^2 \theta_2 \cosh^2 \chi} - \frac{1}{\sin^2 \theta_2 \cosh^2 \chi} + \frac{1}{\sin^2 \theta_2 \cosh^2 \chi} - \frac{\sin^2 \theta_2}{\sin^2 \theta_2 \cosh^2 \chi} \\ &= \frac{\cosh^2 \chi}{\sin^2 \theta_2 \cosh^2 \chi} - \frac{\sin^2 \theta_2}{\sin^2 \theta_2 \cosh^2 \chi} = \frac{1}{\sin^2 \theta_2} - \frac{1}{\cosh^2 \chi} \end{aligned}$$

Tem-se

$$\begin{aligned} \frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) &= \lambda_1 \left(\frac{1}{\sin^2 \theta_2} - \frac{1}{\cosh^2 \chi} \right) \\ \frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) &= \frac{\lambda_1}{\sin^2 \theta_2} - \frac{\lambda_1}{\cosh^2 \chi} \\ -\frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{\lambda_1}{\sin^2 \theta_2} - \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) - \frac{\lambda_1}{\cosh^2 \chi} &= 0 \\ -\frac{1}{\Theta_2} \left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) + \frac{\lambda_1}{\sin^2 \theta_2} &= \frac{1}{X} \left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) + \frac{\lambda_1}{\cosh^2 \chi} \end{aligned}$$

Nota-se que

$$\left(\cot(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) = \left(\frac{\cos(\theta_2)}{\sin(\theta_2)} \frac{\partial \Theta_2}{\partial \theta_2} + \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) = \frac{1}{\sin(\theta_2)} \left(\cos(\theta_2) \frac{\partial \Theta_2}{\partial \theta_2} + \sin \theta_2 \frac{\partial^2 \Theta_2}{\partial \theta_2^2} \right) = \frac{1}{\sin(\theta_2)} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial \Theta_2}{\partial \theta_2} \right)$$

$$\left(\tanh(\chi) \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) = \left(\frac{\sinh(\chi)}{\cosh(\chi)} \frac{\partial X}{\partial \chi} + \frac{\partial^2 X}{\partial \chi^2} \right) = \frac{1}{\cosh(\chi)} \left(\sinh(\chi) \frac{\partial X}{\partial \chi} + \cosh(\chi) \frac{\partial^2 X}{\partial \chi^2} \right) = \frac{1}{\cosh \chi} \frac{\partial}{\partial \chi} \left(\cosh \chi \frac{\partial X}{\partial \chi} \right)$$

Então

$$-\frac{1}{\Theta_2 \sin(\theta_2)} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial \Theta_2}{\partial \theta_2} \right) + \frac{\lambda_1}{\sin^2 \theta_2} = \frac{1}{X \cosh \chi} \frac{\partial}{\partial \chi} \left(\cosh \chi \frac{\partial X}{\partial \chi} \right) + \frac{\lambda_1}{\cosh^2 \chi} = \lambda_2$$

$$-\frac{1}{\Theta_2 \sin(\theta_2)} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial \Theta_2}{\partial \theta_2} \right) + \frac{\lambda_1}{\sin^2 \theta_2} = \lambda_2$$

$$\frac{1}{X \cosh \chi} \frac{\partial}{\partial \chi} \left(\cosh \chi \frac{\partial X}{\partial \chi} \right) + \frac{\lambda_1}{\cosh^2 \chi} = \lambda_2$$

Pode-se escrever

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta} \left(\sin \theta_2 \frac{\partial \Theta_2}{\partial \theta_2} \right) + \left(\lambda_2 - \frac{\lambda_1}{\sin^2 \theta_2} \right) \Theta_2 = 0$$

$$\frac{1}{\cosh \chi} \frac{d}{d\chi} \left(\cosh \chi \frac{dX}{d\chi} \right) + \left(-\lambda_2 + \frac{\lambda_1}{\cosh^2 \chi} \right) X = 0$$

0.3 Resolvendo para Θ_2

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta} \left(\sin \theta_2 \frac{\partial \Theta_2}{\partial \theta_2} \right) + \left(\lambda_2 - \frac{m^2}{\sin^2 \theta_2} \right) \Theta_2 = 0$$

0.3.1 Caso em que $m = 0$

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) + \lambda_2 \Theta_2 = 0$$

Fazendo $x = \cos \theta_2$, $x \in [1, -1]$

$$\sin^2(\theta_2) = 1 - \cos^2(\theta_2) = 1 - x^2$$

$$\frac{d\Theta_2}{d\theta_2} = \frac{d\Theta_2}{dx} \frac{dx}{d\theta_2} = \frac{d\Theta_2}{dx} (-\sin(\theta_2))$$

$$\begin{aligned} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) &= \frac{d}{d\theta_2} \left(-\sin^2(\theta_2) \frac{d\Theta_2}{dx} \right) = -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} - \sin^2(\theta_2) \frac{d}{d\theta_2} \frac{d\Theta_2}{dx} \\ &= -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} - \sin^2(\theta_2) \frac{d}{dx} \frac{d\Theta_2}{d\theta_2} = -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} + \sin^3(\theta_2) \frac{d^2 \Theta_2}{dx^2} \\ \frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) &\Rightarrow \frac{1}{\sin(\theta_2)} \left(-2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} + \sin^3(\theta_2) \frac{d^2 \Theta_2}{dx^2} \right) = -2\cos \theta_2 \frac{d\Theta_2}{dx} + \sin^2(\theta_2) \frac{d^2 \Theta_2}{dx^2} \end{aligned}$$

Então ficamos com

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) + \lambda_2 \Theta_2 = 0 \Rightarrow -2\cos \theta_2 \frac{d\Theta_2}{dx} + \sin^2(\theta_2) \frac{d^2 \Theta_2}{dx^2} + \lambda_2 \Theta_2 = 0$$

$$(1 - x^2) \frac{d^2 \Theta_2}{dx^2} - 2x \frac{d\Theta_2}{dx} + \lambda_2 \Theta_2 = 0$$

Essa equação tem soluções não singulares quando $\lambda_2 = n(n+1)$, $n \in \mathbb{Z}_+$.

$$(1 - x^2) \frac{d^2 \Theta_2}{dx^2} - 2x \frac{d\Theta_2}{dx} + n(n+1) \Theta_2 = 0$$

Chagou-se na Equação de Legendre, a solução dessa equação é expressa em termos dos Polinômios de Legendre de modo que

$$\Theta_{2,n}(x) = c_1 P_n(x) + c_2 Q_n(x)$$

Onde $P_n(x)$ são Polinômios de Legendre de primeira espécie e os $Q_n(x)$ são polinômios de legendre de segunda espécie.

OBS: Se a solução tiver que ser limitada em $[-1, 1]$ ela se torna $\Theta_{2,n}(x) = c_1 P_n(x)$ pois

$$\lim_{x \rightarrow 1^-} Q_n(x) = +\infty$$

$$\lim_{x \rightarrow -1^+} Q_n(x) = (-1)^{n+1} \infty$$

Como o intervalo no caso é $[1, -1]$ pode-se manter a solução na forma descrita acima.

0.3.2 Caso em que $m \neq 0$

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) + \left(\lambda_2 - \frac{m^2}{\sin^2 \theta_2} \right) \Theta_2 = 0$$

Aplicando o termo encontrado $\lambda_2 = n(n+1)$, $n \in \mathbb{Z}_+$ de modo que tem-se

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) + \left(n(n+1) - \frac{m^2}{\sin^2 \theta_2} \right) \Theta_2 = 0$$

Fazendo $x = \cos \theta_2$, $x \in [1, -1]$

$$\sin^2(\theta_2) = 1 - \cos^2(\theta_2) = 1 - x^2$$

$$\frac{d\Theta_2}{d\theta_2} = \frac{d\Theta_2}{dx} \frac{dx}{d\theta_2} = -\sin(\theta_2) \frac{d\Theta_2}{dx}$$

$$\begin{aligned} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) &= \frac{d}{d\theta_2} \left(-\sin^2(\theta_2) \frac{d\Theta_2}{dx} \right) = -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} - \sin^2(\theta_2) \frac{d}{d\theta_2} \frac{d\Theta_2}{dx} \\ &= -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} - \sin^2(\theta_2) \frac{d}{dx} \frac{d\Theta_2}{d\theta_2} = -2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} + \sin^3(\theta_2) \frac{d^2 \Theta_2}{dx^2} \end{aligned}$$

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) \Rightarrow \frac{1}{\sin(\theta_2)} \left(-2\cos \theta_2 \sin \theta_2 \frac{d\Theta_2}{dx} + \sin^3(\theta_2) \frac{d^2 \Theta_2}{dx^2} \right) = -2\cos \theta_2 \frac{d\Theta_2}{dx} + \sin^2(\theta_2) \frac{d^2 \Theta_2}{dx^2}$$

Tem-se

$$\frac{1}{\sin(\theta_2)} \frac{d}{d\theta_2} \left(\sin \theta_2 \frac{d\Theta_2}{d\theta_2} \right) + \left(n(n+1) - \frac{m^2}{\sin^2 \theta_2} \right) \Theta_2 = 0 \Rightarrow -2x \frac{d\Theta_2}{dx} + (1-x^2) \frac{d^2 \Theta_2}{dx^2} + \left(n(n+1) - \frac{m^2}{(1-x^2)} \right) \Theta_2 = 0$$

$$(1-x^2) \frac{d^2 \Theta_2}{dx^2} - 2x \frac{d\Theta_2}{dx} + \left(n(n+1) - \frac{m^2}{(1-x^2)} \right) \Theta_2 = 0$$

Essa é a **Equação de Legendre Generalizada** e suas soluções, para $m, n \in \mathbb{Z}$, são dadas em termos de Funções Associadas de Legendre.

$$\Theta_{2,n,m}(x) = c_1 P_n^m(x) + c_2 Q_n^m(x)$$

0.4 Olhando para χ

$$\frac{1}{\cosh \chi} \frac{d}{d\chi} \left(\cosh \chi \frac{dX}{d\chi} \right) + \left(-\lambda_2 + \frac{\lambda_1}{\cosh^2 \chi} \right) X = 0$$

Fazendo $\zeta = \sinh \chi$, $\zeta \in [0, \infty)$ tem-se

$$\cosh^2(\chi) = 1 + \sinh^2(\chi) = 1 + \zeta^2$$

$$\frac{dX}{d\chi} = \frac{dX}{d\zeta} \frac{d\zeta}{d\chi} = \cosh\chi \frac{dX}{d\zeta}$$

$$\begin{aligned} \frac{d}{d\chi} \left(\cosh\chi \frac{dX}{d\chi} \right) &= \frac{d}{d\chi} \left(\cosh^2\chi \frac{dX}{d\zeta} \right) = 2\cosh\chi \sinh\chi \frac{dX}{d\zeta} + \cosh^2\chi \frac{d}{d\chi} \frac{dX}{d\zeta} \\ &= 2\cosh\chi \sinh\chi \frac{dX}{d\zeta} + \cosh^2\chi \frac{d}{d\zeta} \frac{dX}{d\chi} = 2\cosh\chi \sinh\chi \frac{dX}{d\zeta} + \cosh^3\chi \frac{d^2X}{d\zeta^2} \end{aligned}$$

$$\frac{1}{\cosh\chi} \frac{d}{d\chi} \left(\cosh\chi \frac{dX}{d\chi} \right) = \frac{1}{\cosh\chi} \left(2\cosh\chi \sinh\chi \frac{dX}{d\zeta} + \cosh^3\chi \frac{d^2X}{d\zeta^2} \right) = 2\sinh\chi \frac{dX}{d\zeta} + \cosh^2\chi \frac{d^2X}{d\zeta^2}$$

Então, levando em conta $\lambda_1 = m^2$ e $\lambda_2 = n(n+1)$, chega-se em:

$$(1 + \zeta^2) \frac{d^2X}{d\zeta^2} + 2\zeta \frac{dX}{d\zeta} + \left(-n(n+1) + \frac{m^2}{(1 + \zeta^2)} \right) X = 0$$

Aplicando $\alpha = i\zeta$, $i^2 = -1$.
Veja que

$$\alpha = i\zeta \Rightarrow \alpha^2 = i^2\zeta^2 \Rightarrow -\alpha^2 = \zeta^2$$

$$\begin{aligned} \frac{dX}{d\zeta} &= \frac{dX}{d\alpha} \frac{d\alpha}{d\zeta} = \frac{dX}{d\alpha} i \Rightarrow \frac{d^2X}{d\zeta^2} = \frac{d}{d\zeta} \frac{dX}{d\zeta} = i \frac{d}{d\zeta} \frac{dX}{d\alpha} = i \frac{d}{d\alpha} \frac{dX}{d\zeta} = -\frac{d^2X}{d\alpha^2} \\ 2\zeta \frac{dX}{d\zeta} &\Rightarrow 2i\zeta \frac{dX}{d\alpha} = 2\alpha \frac{dX}{d\alpha} \end{aligned}$$

Substituindo esse resultado na Equação

$$-(1 - \alpha^2) \frac{d^2X}{d\alpha^2} + 2\alpha \frac{dX}{d\alpha} + \left(-n(n+1) + \frac{m^2}{(1 - \alpha^2)} \right) X = 0$$

Multiplicando ambos os lados por -1

$$(1 - \alpha^2) \frac{d^2X}{d\alpha^2} - 2\alpha \frac{dX}{d\alpha} + \left(n(n+1) - \frac{m^2}{(1 - \alpha^2)} \right) X = 0$$

Essa é, novamente, a Equação de Legendre. Como visto anteriormente pode-se analisa-la em dois casos.

0.4.1 Caso $m = 0$

Para esse caso, como já determinando antes, a solução é

$$X_n(x) = c_3 P_n(\alpha) + c_4 Q_n(\alpha)$$

0.4.2 Caso $m \neq 0$

Para esse caso, como já determinando antes, a solução é

$$X_{n,m}(x) = c_3 P_n^m(\alpha) + c_4 Q_n^m(\alpha)$$

0.5 Conclusão

Conclui-se que a solução da Equação

$$\nabla^2 u = \frac{1}{a^2 (\sinh^2 \chi + \cos^2 \theta_2)} \left[\frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial u}{\partial \theta_2} \right) + \frac{1}{\cosh \chi} \frac{\partial}{\partial \chi} \left(\cosh \chi \frac{\partial u}{\partial \chi} \right) \right] + \frac{1}{a^2 \sin^2 \theta_2 \cosh^2 \chi} \frac{\partial^2 u}{\partial \theta_1^2} = 0$$

Pode ser dada por

$$u(\theta_1, \theta_2, \chi) = \Theta_1(\theta_1) \Theta_2(x) X(\alpha)$$

$$u(\theta_1, \theta_2, \chi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Theta_{1,m}(\theta_1) \Theta_{2,m,n}(x) X_{m,n}(\alpha)$$