;

1 operador de Laplace-Beltrami

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vamos aprender a técnica do operador de Laplace-Beltrami (LB)

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Sec. 2.3 no [relAleff], p. 18 {no arq tem erros de digitação}

;

usando a parametrização como está no site;

so trocados

$$x_1 \leftrightarrow x_2$$

$$\begin{aligned}
x_3 &= a \sinh \chi \sin \theta_2 \\
x_2 &= a \cosh \chi \cos \theta_2 \cos \theta_1 \\
x_1 &= a \cosh \chi \cos \theta_2 \sin \theta_1 \\
0 &\leq \chi < \infty,\ -\pi/2 \leq \theta_2 \leq \pi/2,\ 0 \leq \theta_1 < 2\pi
\end{aligned}$$

O operador de Laplace-Beltrami é definido como (implica soma por índices repetidos):

$$\Delta u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial q_i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial q_j} \right) \ , \tag{1}$$

onde g_{ij} é o tensor métrico, g^{ij} é o tensor métrico inverso e g é o determinante do tensor métrico. O tensor métrico pode ser determinado como [! aqui precisa colocar referência]:

$$g_{ij} = \sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} = G^T G \ . \tag{2}$$

No nosso caso:

$$q_1 = \theta_1,\ q_2 = \theta_2,\ q_3 = \chi$$

Para determinar g_{ij}, g, g^{ij} tenho o programa na Maxima (em anexo em .pdf), no programa $\theta_1 \theta_2 \theta_3$ denotadas por t1,t2,t3 e χ por v. Como coordenadas são ortogonais, o tensor métrico é diagonal e pode ser escrito como

$$g_{ij} = diag\left(h_1, h_2, h_3\right) = diag\left(h_1, h_2, h_2\right)$$

$$h_1 = a^2 \cos^2 \theta_2 \cosh^2 \chi,\ h_2 = h_3 = a^2 \left(\cos^2 \theta_2 \sinh^2 \chi + \sin^2 \theta_2 \cosh^2 \chi\right)$$

$$g^{ij} = g_{ij}^{-1} = diag\left(h_1^{-1}, h_2^{-1}, h_3^{-1}\right) = diag\left(h_1^{-1}, h_2^{-1}, h_2^{-1}\right)$$

$$|g| = h_1 h_2 h_3 = h_1 h_2^2$$

com $\partial_i = \frac{\partial}{\partial q_i}$

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial q_i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial q_j} \right) = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j u \right) = \frac{1}{\sqrt{|g|}} \left[\partial_i \left(\sqrt{|g|} g^{ij} \right) \right] \partial_j u + g^{ij} \partial_i \partial_j u$$

$$\frac{1}{\sqrt{|g|}} \left[\partial_i \left(\sqrt{|g|} g^{ij} \right) \right] \partial_j u = \frac{1}{2\,|g|} \left(\partial_i \,|g| \right) g^{ij} \partial_j u + \left(\partial_i g^{ij} \right) \partial_j u$$

aqui

$$g^{ij} \partial_i \partial_j u = h_1^{-1} \partial_1^2 u + h_2^{-1} \partial_2^2 u + h_3^{-1} \partial_3^2 u = h_1^{-1} \partial_1^2 u + h_2^{-1} \partial_2^2 u + h_2^{-1} \partial_3^2 u = h_1^{-1} \partial_1^2 u + h_2^{-1} \left(\partial_2^2 u + \partial_3^2 u \right)$$

;

$$\left(\partial_i g^{ij} \right) \partial_j u = \left(\partial_1 h_1^{-1} \right) \left(\partial_1 u \right) + \left(\partial_2 h_2^{-1} \right) \left(\partial_2 u \right) + \left(\partial_3 h_2^{-1} \right) \left(\partial_3 u \right) = \left(\partial_2 h_2^{-1} \right) \left(\partial_2 u \right) + \left(\partial_3 h_2^{-1} \right) \left(\partial_3 u \right)$$

pois $\left(\partial_1 h_1^{-1} \right) = 0$

$$\begin{aligned} &= \left(-h_2^{-2} \right) \left(\partial_2 h_2 \right) \left(\partial_2 u \right) + \left(-h_2^{-2} \right) \left(\partial_3 h_2 \right) \left(\partial_3 u \right) \\ &= -\frac{h_1}{h_1 h_2^2} \left[\left(\partial_2 h_2 \right) \left(\partial_2 u \right) + \left(\partial_3 h_2 \right) \left(\partial_3 u \right) \right] \\ &= -\frac{h_1}{|g|} \left[\left(\partial_2 h_2 \right) \left(\partial_2 u \right) + \left(\partial_3 h_2 \right) \left(\partial_3 u \right) \right] \end{aligned}$$

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$$\left(\partial_i g^{ij} \right) \partial_j u = -\frac{h_1}{|g|} \left[\left(\partial_2 h_2 \right) \left(\partial_2 u \right) + \left(\partial_3 h_2 \right) \left(\partial_3 u \right) \right]$$

;

$$\left(\partial_i \,|g| \right) g^{ij} \partial_j u = \left(\partial_1 \,|g| \right) h_1^{-1} \left(\partial_1 u \right) + \left(\partial_2 \,|g| \right) h_2^{-1} \left(\partial_2 u \right) + \left(\partial_3 \,|g| \right) h_2^{-1} \left(\partial_3 u \right)$$

$$= \left(\partial_2 \,|g| \right) h_2^{-1} \left(\partial_2 u \right) + \left(\partial_3 \,|g| \right) h_2^{-1} \left(\partial_3 u \right)$$

pois $\left(\partial_1 \,|g| \right) = 0$ e

$$\partial_i \,|g| = \left(\partial_i h_1 \right) h_2^2 + 2 h_1 h_2 \left(\partial_i h_2 \right) = h_2 \left(h_2 \left(\partial_i h_1 \right) + 2 h_1 \left(\partial_i h_2 \right) \right) = \frac{|g| \left(h_2 \left(\partial_i h_1 \right) + 2 h_1 \left(\partial_i h_2 \right) \right)}{h_1 h_2}, \; i = 2, 3$$

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$$\begin{aligned} \left(\partial_i \,|g| \right) g^{ij} \partial_j u &= \frac{|g| \left(h_2 \left(\partial_2 h_1 \right) + 2 h_1 \left(\partial_2 h_2 \right) \right)}{h_1 h_2} h_2^{-1} \left(\partial_2 u \right) + \frac{|g| \left(h_2 \left(\partial_3 h_1 \right) + 2 h_1 \left(\partial_3 h_2 \right) \right)}{h_1 h_2} h_2^{-1} \left(\partial_3 u \right) \\ &= \frac{|g| \left(h_2 \left(\partial_2 h_1 \right) + 2 h_1 \left(\partial_2 h_2 \right) \right)}{h_1 h_2^2} \left(\partial_2 u \right) + \frac{|g| \left(h_2 \left(\partial_3 h_1 \right) + 2 h_1 \left(\partial_3 h_2 \right) \right)}{h_1 h_2^2} \left(\partial_3 u \right) \\ &= \left(h_2 \left(\partial_2 h_1 \right) + 2 h_1 \left(\partial_2 h_2 \right) \right) \left(\partial_2 u \right) + \left(h_2 \left(\partial_3 h_1 \right) + 2 h_1 \left(\partial_3 h_2 \right) \right) \left(\partial_3 u \right) \end{aligned}$$

;

$$\frac{1}{2\,|g|} \left(\partial_i \,|g| \right) g^{ij} \partial_j u = \frac{1}{2\,|g|} \left[\left(h_2 \left(\partial_2 h_1 \right) + 2 h_1 \left(\partial_2 h_2 \right) \right) \left(\partial_2 u \right) + \left(h_2 \left(\partial_3 h_1 \right) + 2 h_1 \left(\partial_3 h_2 \right) \right) \left(\partial_3 u \right) \right]$$

então

$$\frac{1}{2\left|g\right|}\left(\partial_{i}\left|g\right|\right)g^{ij}\partial_{j}u=\frac{1}{2\left|g\right|}\left[\left(h_{2}\left(\partial_{2}h_{1}\right)+2h_{1}\left(\partial_{2}h_{2}\right)\right)\left(\partial_{2}u\right)+\left(h_{2}\left(\partial_{3}h_{1}\right)+2h_{1}\left(\partial_{3}h_{2}\right)\right)\left(\partial_{3}u\right)\right]$$

$$\left(\partial_{i}g^{ij}\right)\partial_{j}u=-\frac{h_{1}}{\left|g\right|}\left[\left(\partial_{2}h_{2}\right)\left(\partial_{2}u\right)+\left(\partial_{3}h_{2}\right)\left(\partial_{3}u\right)\right]$$

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$$\frac{1}{\sqrt{\left|g\right|}}\left[\partial_{i}\left(\sqrt{\left|g\right|}g^{ij}\right)\right]\partial_{j}u=\frac{1}{2\left|g\right|}\left(\partial_{i}\left|g\right|\right)g^{ij}\partial_{j}u+\left(\partial_{i}g^{ij}\right)\partial_{j}u$$

$$\begin{aligned}=\ & \frac{1}{2\left|g\right|}\left[\left(h_{2}\left(\partial_{2}h_{1}\right)+2h_{1}\left(\partial_{2}h_{2}\right)\right)\left(\partial_{2}u\right)+\left(h_{2}\left(\partial_{3}h_{1}\right)+2h_{1}\left(\partial_{3}h_{2}\right)\right)\left(\partial_{3}u\right)\right]\\ & -\frac{h_{1}}{\left|g\right|}\left[\left(\partial_{2}h_{2}\right)\left(\partial_{2}u\right)+\left(\partial_{3}h_{2}\right)\left(\partial_{3}u\right)\right]\end{aligned}$$

$$\begin{aligned}=\ & \frac{1}{2\left|g\right|}\left[\left(h_{2}\left(\partial_{2}h_{1}\right)+2h_{1}\left(\partial_{2}h_{2}\right)\right)-2h_{1}\left(\partial_{2}h_{2}\right)\right]\left(\partial_{2}u\right)\\ & +\frac{1}{2\left|g\right|}\left[\left(h_{2}\left(\partial_{3}h_{1}\right)+2h_{1}\left(\partial_{3}h_{2}\right)\right)-2h_{1}\left(\partial_{3}h_{2}\right)\right]\left(\partial_{3}u\right)\end{aligned}$$

$$=\frac{h_{2}}{2\left|g\right|}\left[\left(\partial_{2}h_{1}\right)\left(\partial_{2}u\right)+\left(\partial_{3}h_{1}\right)\left(\partial_{3}u\right)\right]$$

então

$$\frac{1}{\sqrt{\left|g\right|}}\left[\partial_{i}\left(\sqrt{\left|g\right|}g^{ij}\right)\right]\partial_{j}u=\frac{h_{2}}{2\left|g\right|}\left[\left(\partial_{2}h_{1}\right)\left(\partial_{2}u\right)+\left(\partial_{3}h_{1}\right)\left(\partial_{3}u\right)\right]$$

e

$$\left(\partial_{2}h_{1}\right)=-2a^{2}\cos\theta_{2}\cosh^{2}\chi\sin\theta_{2}=-2h_{1}\frac{\sin\theta_{2}}{\cos\theta_{2}}=-2h_{1}\tan\theta_{2}$$

$$\left(\partial_{3}h_{1}\right)=2a^{2}\cos^{2}\theta_{2}\cosh\chi\sinh\chi=2h_{1}\frac{\sinh\chi}{\cosh\chi}=2h_{1}\tanh\chi$$

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$$\begin{aligned}\frac{1}{\sqrt{\left|g\right|}}\left[\partial_{i}\left(\sqrt{\left|g\right|}g^{ij}\right)\right]\partial_{j}u & =\frac{h_{2}}{2\left|g\right|}\left[-2h_{1}\tan\theta_{2}\left(\partial_{2}u\right)+2h_{1}\tanh\chi\left(\partial_{3}u\right)\right]\\ & =\frac{2h_{2}h_{1}}{2h_{1}h_{2}^{2}}\left[-\tan\theta_{2}\left(\partial_{2}u\right)+\tanh\chi\left(\partial_{3}u\right)\right]\\ & =\frac{1}{h_{2}}\left[-\tan\theta_{2}\left(\partial_{2}u\right)+\tanh\chi\left(\partial_{3}u\right)\right]\end{aligned}$$

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$$\frac{1}{\sqrt{\left|g\right|}}\partial_{i}\left(\sqrt{\left|g\right|}g^{ij}\partial_{j}u\right)=\frac{1}{\sqrt{\left|g\right|}}\left[\partial_{i}\left(\sqrt{\left|g\right|}g^{ij}\right)\right]\partial_{j}u+g^{ij}\partial_{i}\partial_{j}u$$

$$\begin{aligned}& =\frac{1}{h_{2}}\left[-\tan\theta_{2}\left(\partial_{2}u\right)+\tanh\chi\left(\partial_{3}u\right)\right]+h_{1}^{-1}\partial_{1}^{2}u+h_{2}^{-1}\left(\partial_{2}^{2}u+\partial_{3}^{2}u\right)\\ & =\frac{1}{h_{2}}\left[-\tan\theta_{2}\left(\partial_{2}u\right)+\tanh\chi\left(\partial_{3}u\right)+\partial_{2}^{2}u+\partial_{3}^{2}u\right]+\frac{1}{h_{1}}\partial_{1}^{2}u\\ & =\frac{1}{h_{2}}\left[-\tan\theta_{2}\left(\partial_{2}u\right)+\partial_{2}^{2}u+\tanh\chi\left(\partial_{3}u\right)+\partial_{3}^{2}u\right]+\frac{1}{h_{1}}\partial_{1}^{2}u\end{aligned}$$

aqui

$$\left[-\tan\theta_2\left(\partial_2 u\right)+\partial_2^2 u+\tanh\chi\left(\partial_3 u\right)+\left(\partial_3^2 u\right)\right]=\frac{1}{\cos\theta_2}\partial_2\left(\cos\theta_2\partial_2 u\right)+\frac{1}{\cosh\chi}\partial_3\left(\cosh\chi\partial_3 u\right)$$

pp

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j u\right)=$$

$$=\frac{1}{h_2}\left[\frac{1}{\cos\theta_2}\partial_2\left(\cos\theta_2\partial_2 u\right)+\frac{1}{\cosh\chi}\partial_3\left(\cosh\chi\partial_3 u\right)\right]+\frac{1}{h_1}\partial_1^2 u$$

aplicando

$$h_1=a^2\cos^2\theta_2\cosh^2\chi,\;\; h_2=h_3=a^2\left(\cos^2\theta_2\sinh^2\chi+\sin^2\theta_2\cosh^2\chi\right)$$

obtemos

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j u\right)=$$

$$\begin{aligned} = \;& \frac{1}{a^2\left(\cos^2\theta_2\sinh^2\chi+\sin^2\theta_2\cosh^2\chi\right)}\left[\frac{1}{\cos\theta_2}\partial_2\left(\cos\theta_2\partial_2 u\right)+\frac{1}{\cosh\chi}\partial_3\left(\cosh\chi\partial_3 u\right)\right] \\ & +\frac{1}{a^2\cos^2\theta_2\cosh^2\chi}\partial_1^2 u \end{aligned}$$

observamos também que

$$\left(\cos^2\theta_2\sinh^2\chi+\sin^2\theta_2\cosh^2\chi\right)=\left(1-\sin^2\theta_2\right)\sinh^2\chi+\sin^2\theta_2\cosh^2\chi$$

$$=\sinh^2\chi+\sin^2\theta_2\left(\cosh^2\chi-\sinh^2\chi\right)=\sinh^2\chi+\sin^2\theta_2$$

e podemos escrever

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j u\right)=$$

$$\begin{aligned} = \;& \frac{1}{a^2\left(\sinh^2\chi+\sin^2\theta_2\right)}\left[\frac{1}{\cos\theta_2}\partial_2\left(\cos\theta_2\partial_2 u\right)+\frac{1}{\cosh\chi}\partial_3\left(\cosh\chi\partial_3 u\right)\right] \\ & +\frac{1}{a^2\cos^2\theta_2\cosh^2\chi}\partial_1^2 u \end{aligned}$$

Finalmente obtemos o Laplaciano na coordenadas esferoidais:

$$\begin{aligned} \nabla^2 u \;=\;& \frac{1}{a^2\left(\sinh^2\chi+\sin^2\theta_2\right)}\left[\frac{1}{\cos\theta_2}\partial_2\left(\cos\theta_2\partial_2 u\right)+\frac{1}{\cosh\chi}\partial_3\left(\cosh\chi\partial_3 u\right)\right] \\ & +\frac{1}{a^2\cos^2\theta_2\cosh^2\chi}\partial_1^2 u \end{aligned}$$

ou

$$\begin{aligned} \nabla^2 u \;=\;& \frac{1}{a^2\left(\sinh^2\chi+\sin^2\theta_2\right)}\left[\frac{1}{\cos\theta_2}\frac{\partial}{\partial\theta_2}\left(\cos\theta_2\frac{\partial u}{\partial\theta_2}\right)+\frac{1}{\cosh\chi}\frac{\partial}{\partial\chi}\left(\cosh\chi\frac{\partial u}{\partial\chi}\right)\right] \\ & +\frac{1}{a^2\cos^2\theta_2\cosh^2\chi}\frac{\partial^2 u}{\partial\theta_1^2} \end{aligned}$$

;

2 proxima etapa

;

obter Laplaciano na parametrização

$$\begin{aligned}
x_3 &= a\sinh\chi\cos\theta_2\\
x_2 &= a\cosh\chi\sin\theta_2\cos\theta_1\\
x_1 &= a\cosh\chi\sin\theta_2\sin\theta_1\\
0 &\le \chi<\infty,\ 0\le\theta_2\le\pi,\ 0\le\theta_1<2\pi
\end{aligned}$$

e depois passar para dimensão 4.

Na dimensão 4 usamos as coordenadas:

$$\begin{aligned}
x_4 &= a\sinh\chi\cos\theta_3\\
x_3 &= a\cosh\chi\sin\theta_3\cos\theta_2\\
x_1 &= a\cosh\chi\sin\theta_3\sin\theta_2\cos\theta_1\\
x_2 &= a\cosh\chi\sin\theta_3\sin\theta_2\sin\theta_1\\
0 &\le \chi<\infty,\ 0\le\theta_3,\theta_2\le\pi,\ 0\le\theta_1<2\pi
\end{aligned}$$

por motivo da simetria na parametrização é trocado $x_1 \leftrightarrow x_2$, facilita os calculos (mas nesse caso surge uma questão de orientação de eixos de coordenadas; isso não é drástico, mas guardamos isso na mente).

então, usaremos coordenadas:

$$\begin{aligned}
x_4 &= a\sinh\chi\cos\theta_3\\
x_3 &= a\cosh\chi\sin\theta_3\cos\theta_2\\
x_2 &= a\cosh\chi\sin\theta_3\sin\theta_2\cos\theta_1\\
x_1 &= a\cosh\chi\sin\theta_3\sin\theta_2\sin\theta_1\\
0 &\le \chi<\infty,\ 0\le\theta_3,\theta_2\le\pi,\ 0\le\theta_1<2\pi
\end{aligned}$$

;