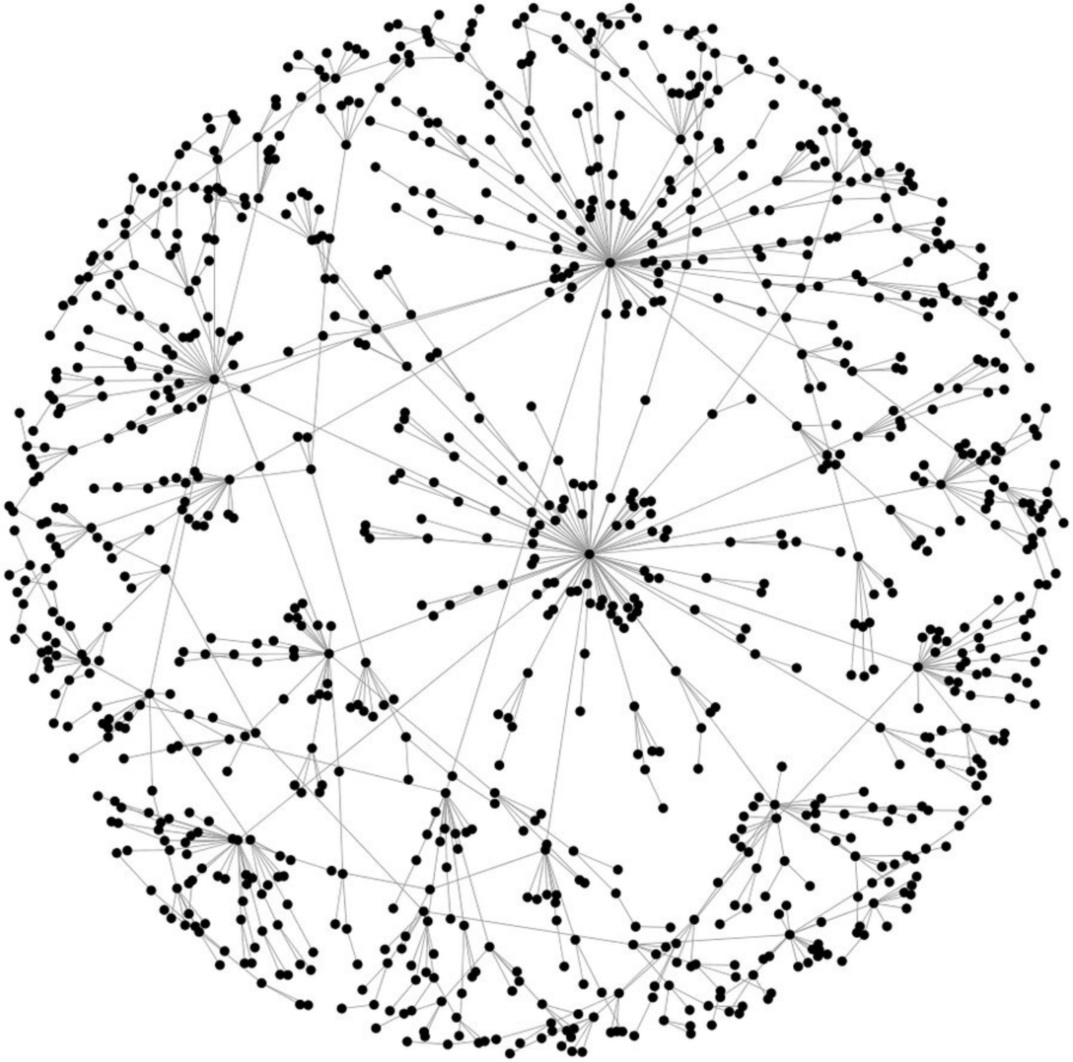
DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

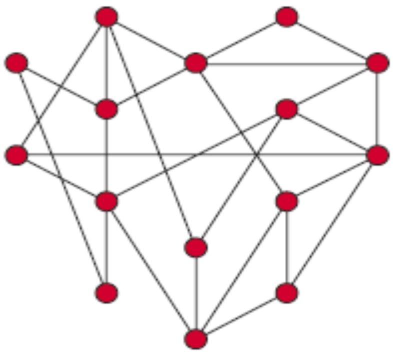
Network revolution(s)



Network Science, 2023/24

DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

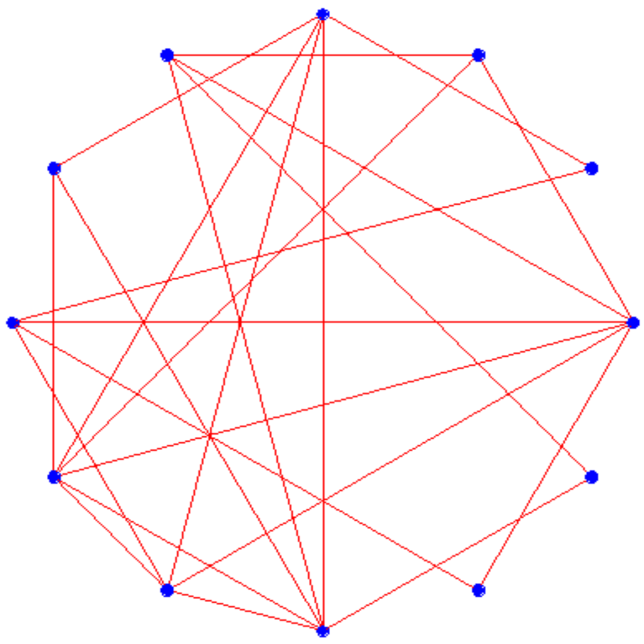
Last class: Random graphs



Network Science, 2023/24

Random network model

- A random network consists of *N* nodes where each pair of nodes is connected with probability ***p***.

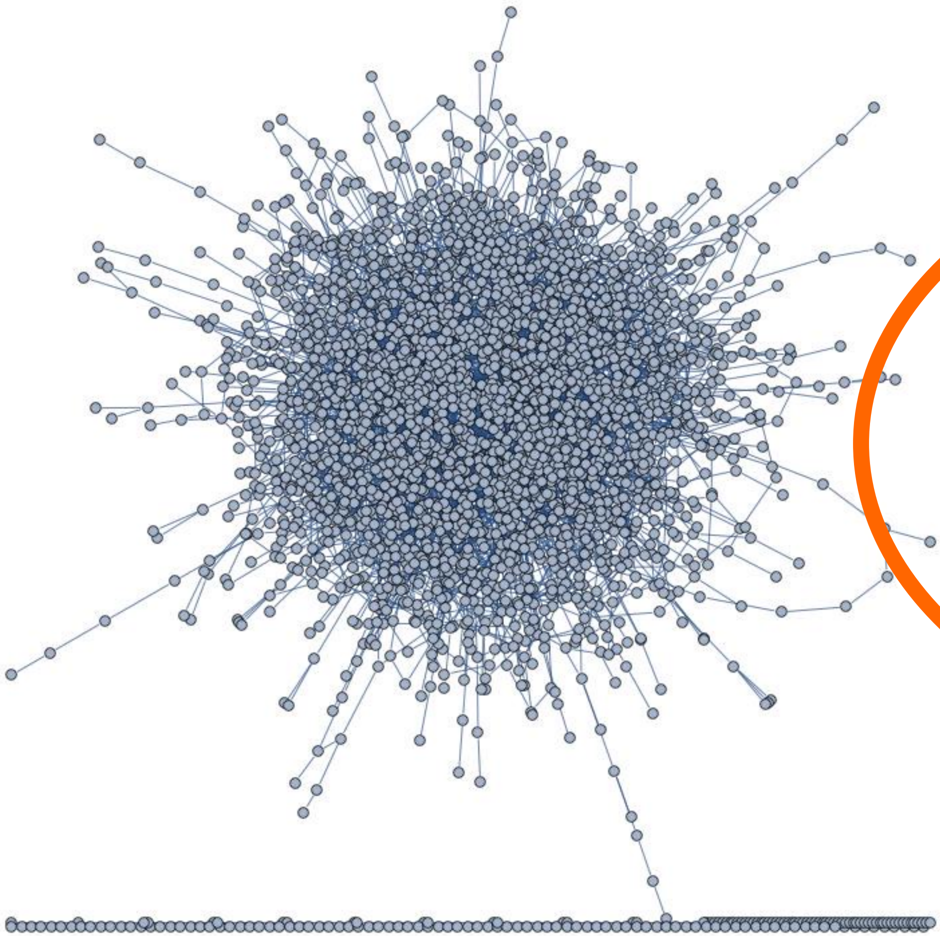


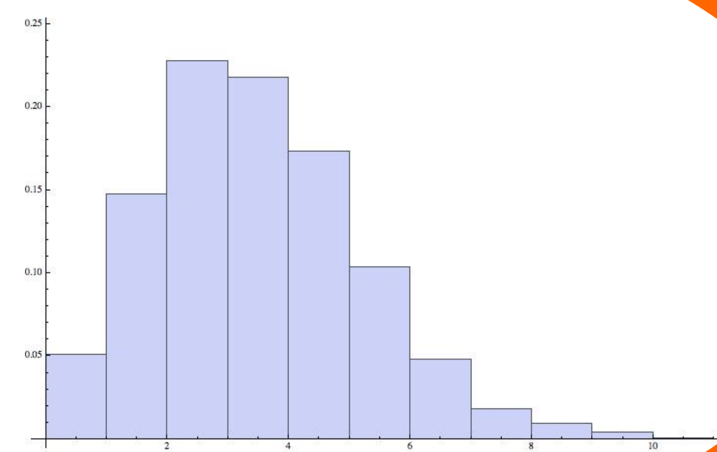
Random network model or the *Erdös and Renyi (ER) Model*

The null model of network science

Random network model

N = 3x10^3 nodes, p = 10^{-3}





Degree distribution of a random graph

- The probability that a node i has exactly k partners is given by ***a binomial distribution***

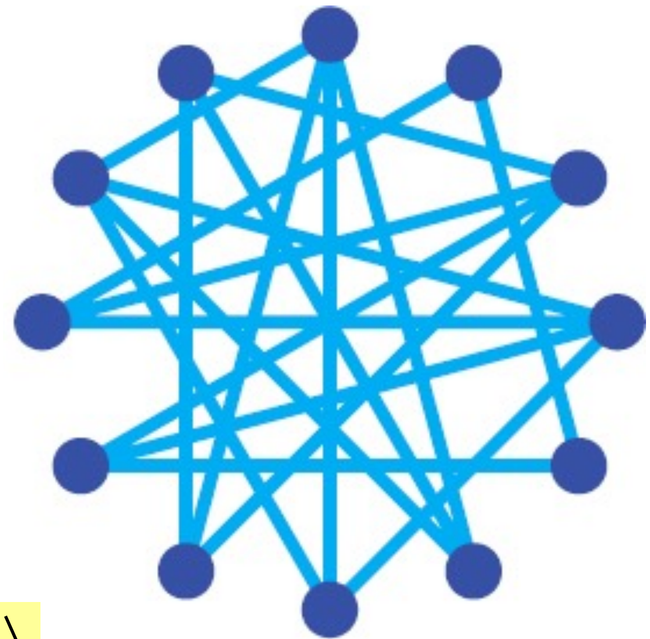
$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

- Real networks are sparse *(i.e., $\langle k \rangle << N$)*
- For large/sparse networks the binomial dist. is well approximated by ***a Poisson distribution***:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

Random graphs portray a very low clustering coeff.

Average number of links e_i in the neighborhood of a node i with $\boldsymbol{k_i}$ partners?



$$p \times \frac{k_i(k_i - 1)}{2}$$

Thus, the clustering coefficient reads

$$C_i = \frac{e_i}{k_i(k_i - 1)/2} = p\frac{k_i(k_i - 1)/2}{k_i(k_i - 1)/2} = p = \frac{\langle k \rangle}{N}$$

which, for sparse & large graphs, C→0 !!

Random graphs portray a very low clustering coeff.

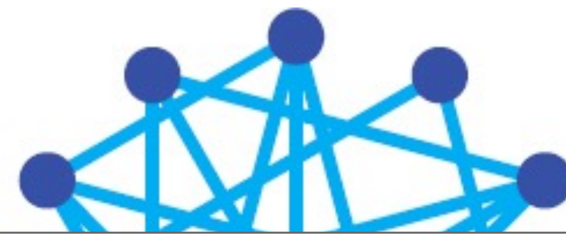
Real world networks often show large Clustering Coeff.

Film actors	$\rightarrow$	C=0.2
Company directors	$\rightarrow$	C=0.59
Math co-authorship	$\rightarrow$	C=0.15
Physics co-authorship	$\rightarrow$	C=0.45
Physics co-authorship	$\rightarrow$	C=0.45
WWW	$\rightarrow$	C=0.11
Neural networks	$\rightarrow$	C=0.18

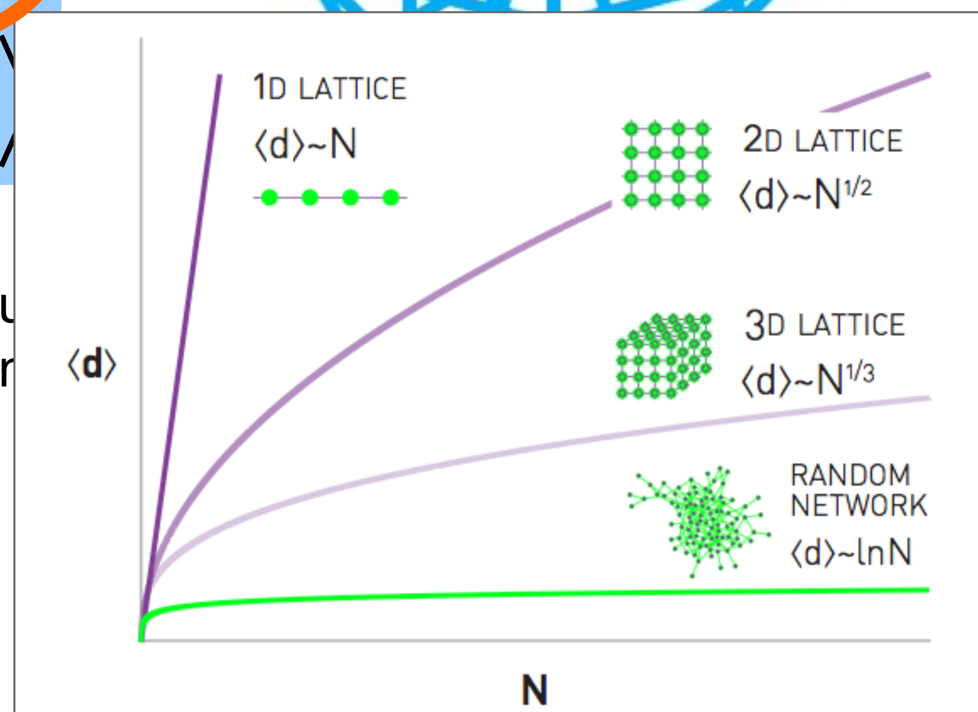
$$C = \frac{\langle k \rangle}{N} \quad ???$$

Average path length of a random graph

$$APL = \langle L \rangle \approx \frac{\ln N}{\ln\langle k \rangle}$$

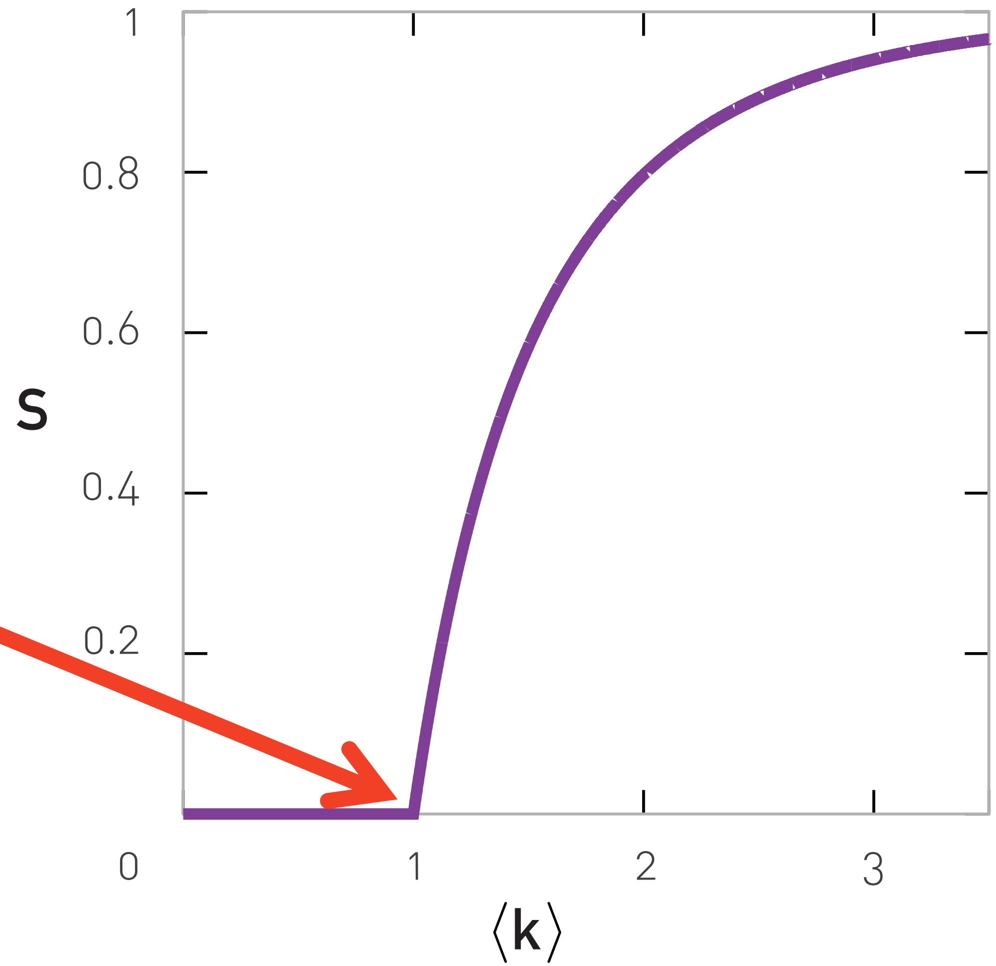


random network model can accou
emergence of small world phenor



Critical transitions

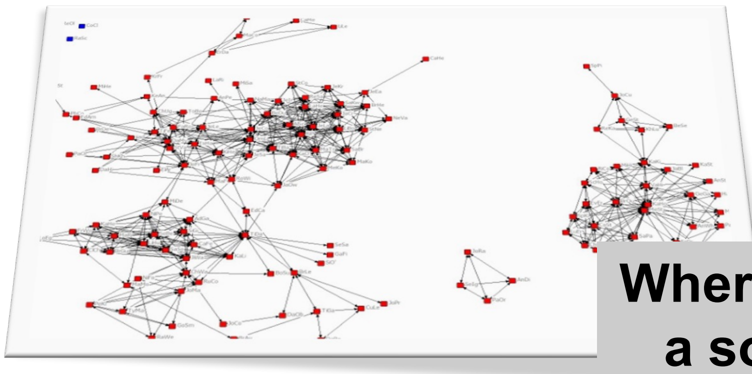
What happens here?



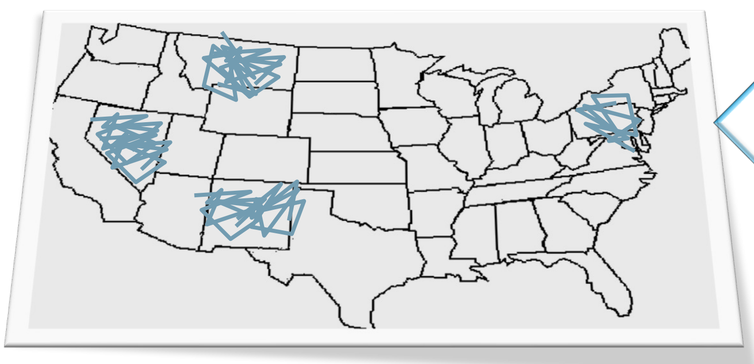
Clustering *vs* Randomness

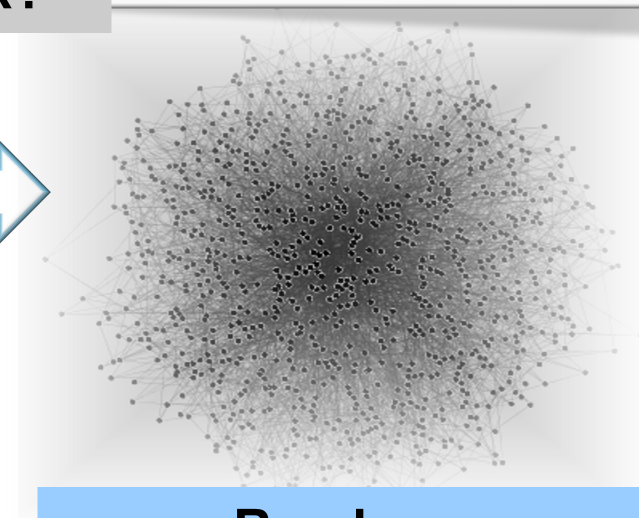
Clustering implies locality

Randomness enables shortcuts



Where should we put a social network?





Locally Structured

Random

Could a network which is so strongly locally structured be at the same time a small world?

Merging structure and randomness

The Watts-Strogatz Small World model (Nature ‘98)

Watts & Strogatz invented a very simple model (***1 parameter!***) which interpolates between regular and random graphs.

Duncan J. Watts
Microsoft Research

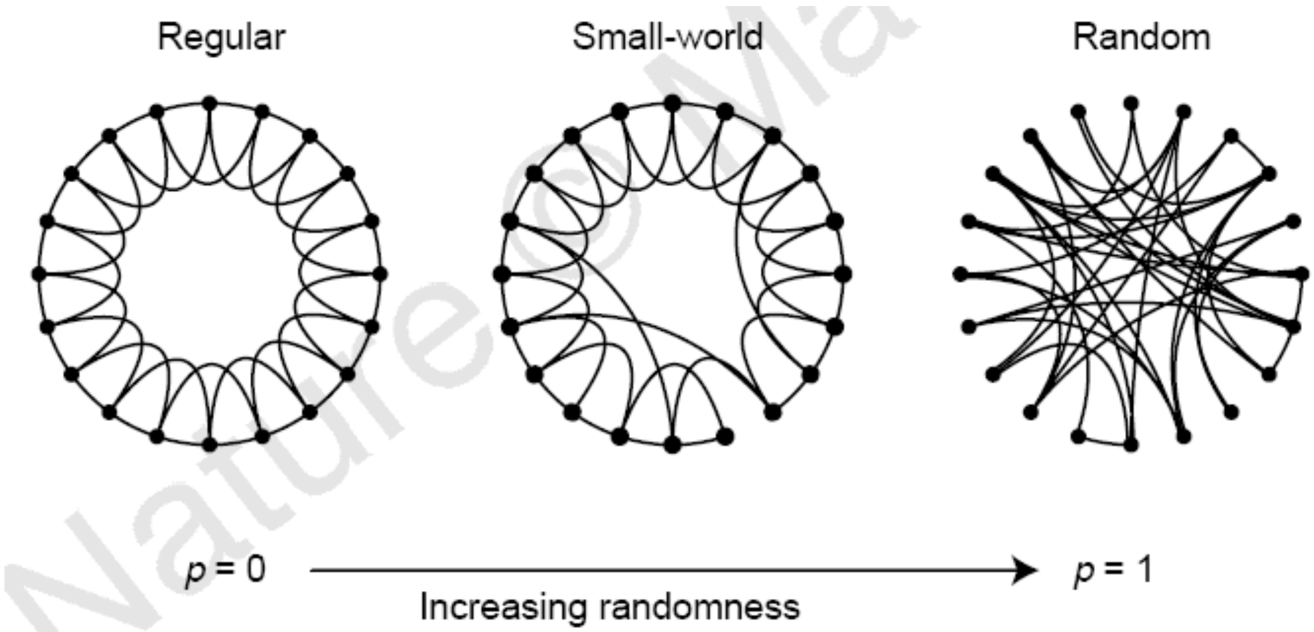
Steven H. Strogatz
Cornell Univ.

Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of ‘small-world’networks.” Nature 393, no. 6684 (1998): 440.

Merging structure and randomness

The Watts-Strogatz Small World model (Nature '98)

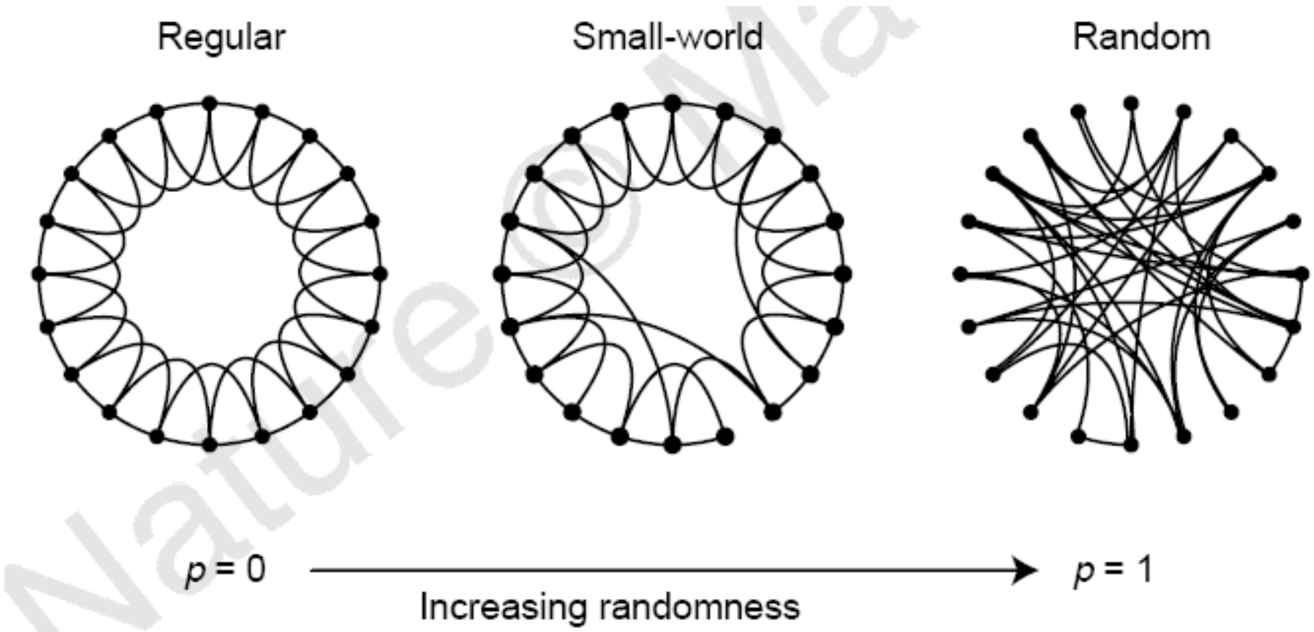
Watts & Strogatz invented a very simple model (***1 parameter!***) which interpolates between regular and random graphs.



Merging structure and randomness

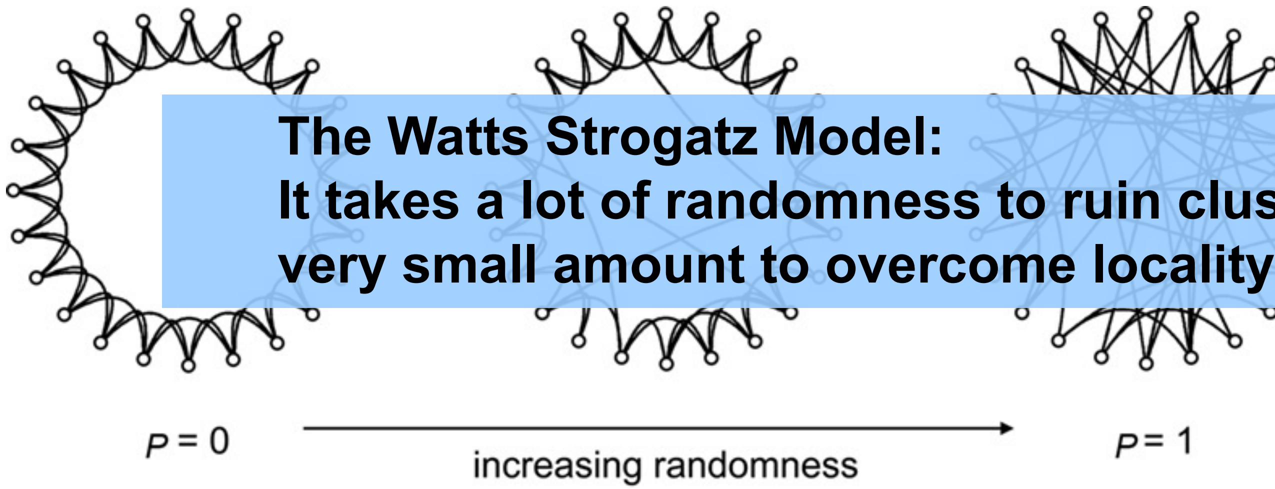
The Watts-Strogatz Small World model (Nature '98)

recipe : start from a regular graph (left); choose a circulating direction (say, clockwise); each edge one encounters is randomly re-directed with a probability ***p***; no repeated edges are allowed; stop when reaching the starting point;

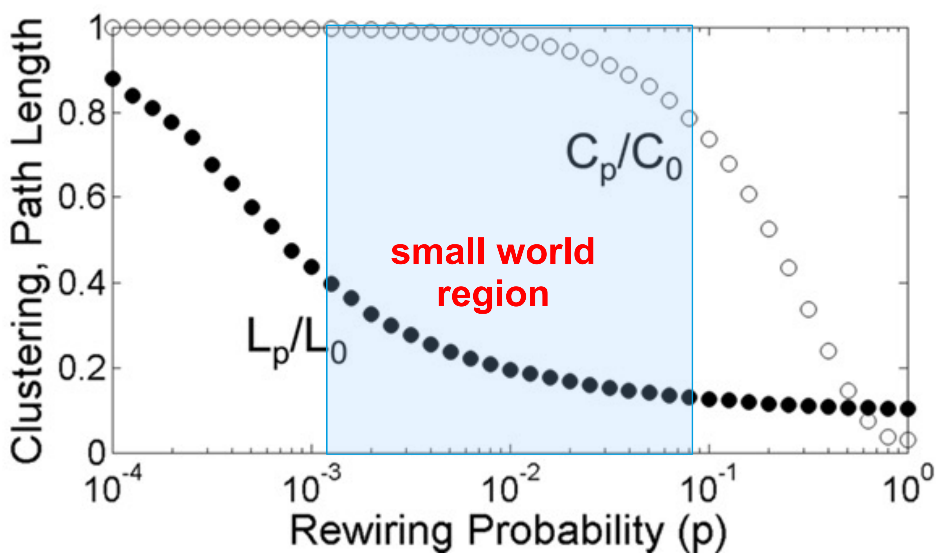


Merging structure and randomness

The Watts-Strogatz Small World model (Nature ‘98)

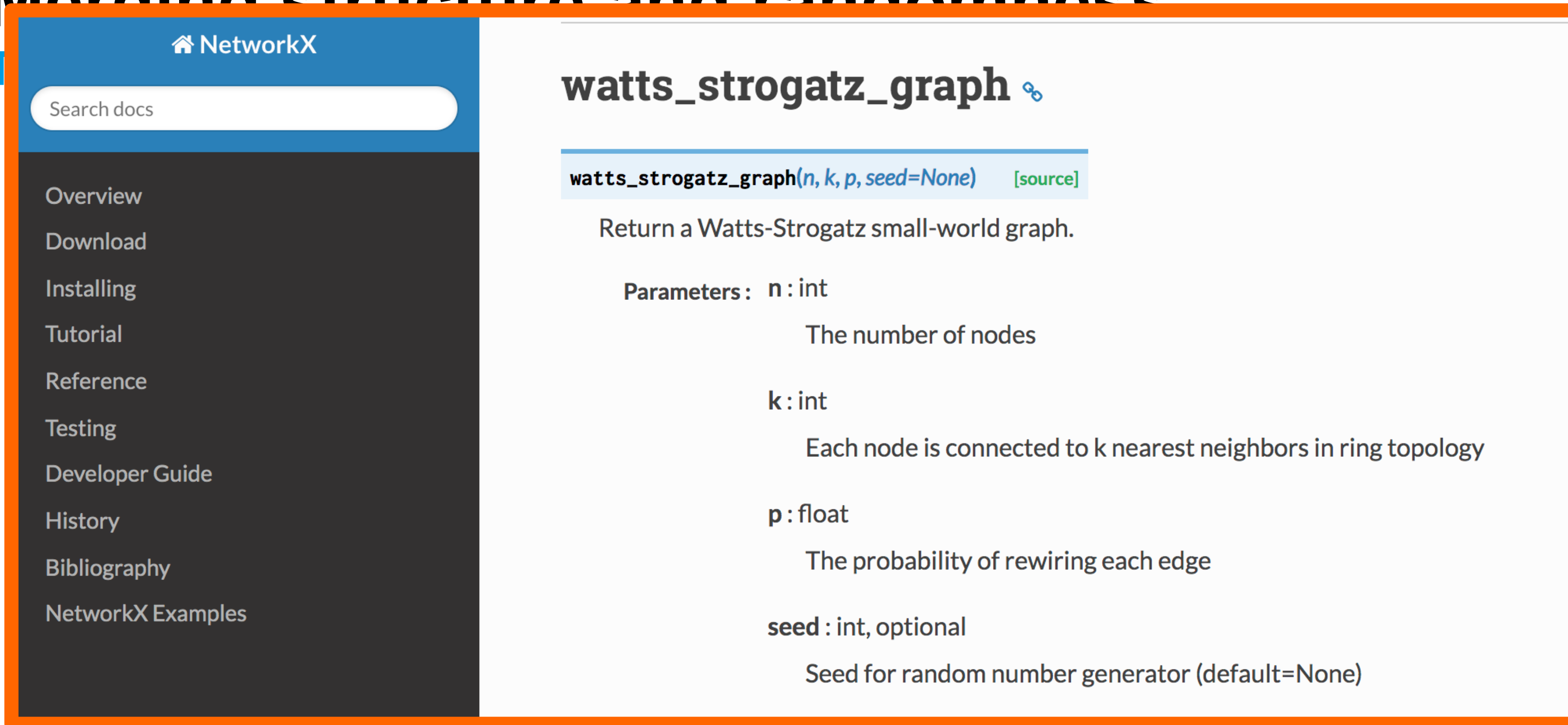


The Watts Strogatz Model:
It takes a lot of randomness to ruin clustering, but a very small amount to overcome locality



Challenge:
Can you reproduce this result?

Merging structure and randomness



NetworkX

Search docs

- Overview
- Download
- Installing
- Tutorial
- Reference
- Testing
- Developer Guide
- History
- Bibliography
- NetworkX Examples

watts_strogatz_graph

`watts_strogatz_graph(n, k, p, seed=None)` [source]

Return a Watts-Strogatz small-world graph.

Parameters :

n : int

The number of nodes

k : int

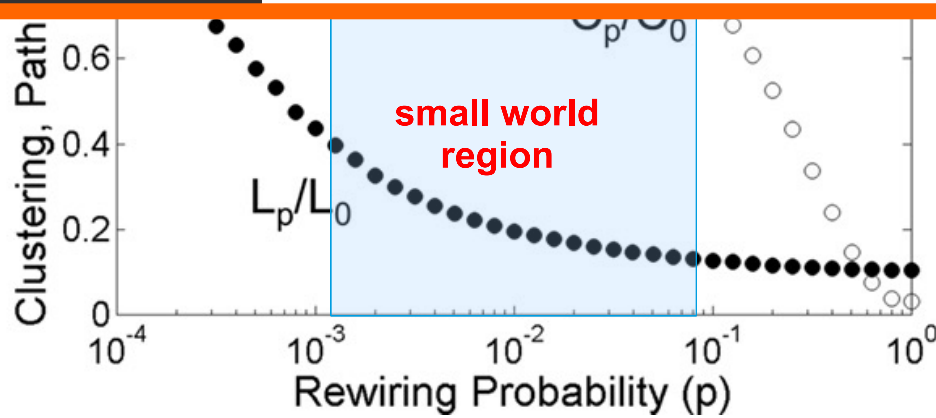
Each node is connected to k nearest neighbors in ring topology

p : float

The probability of rewiring each edge

seed : int, optional

Seed for random number generator (default=None)



Challenge: Can you reproduce this result?

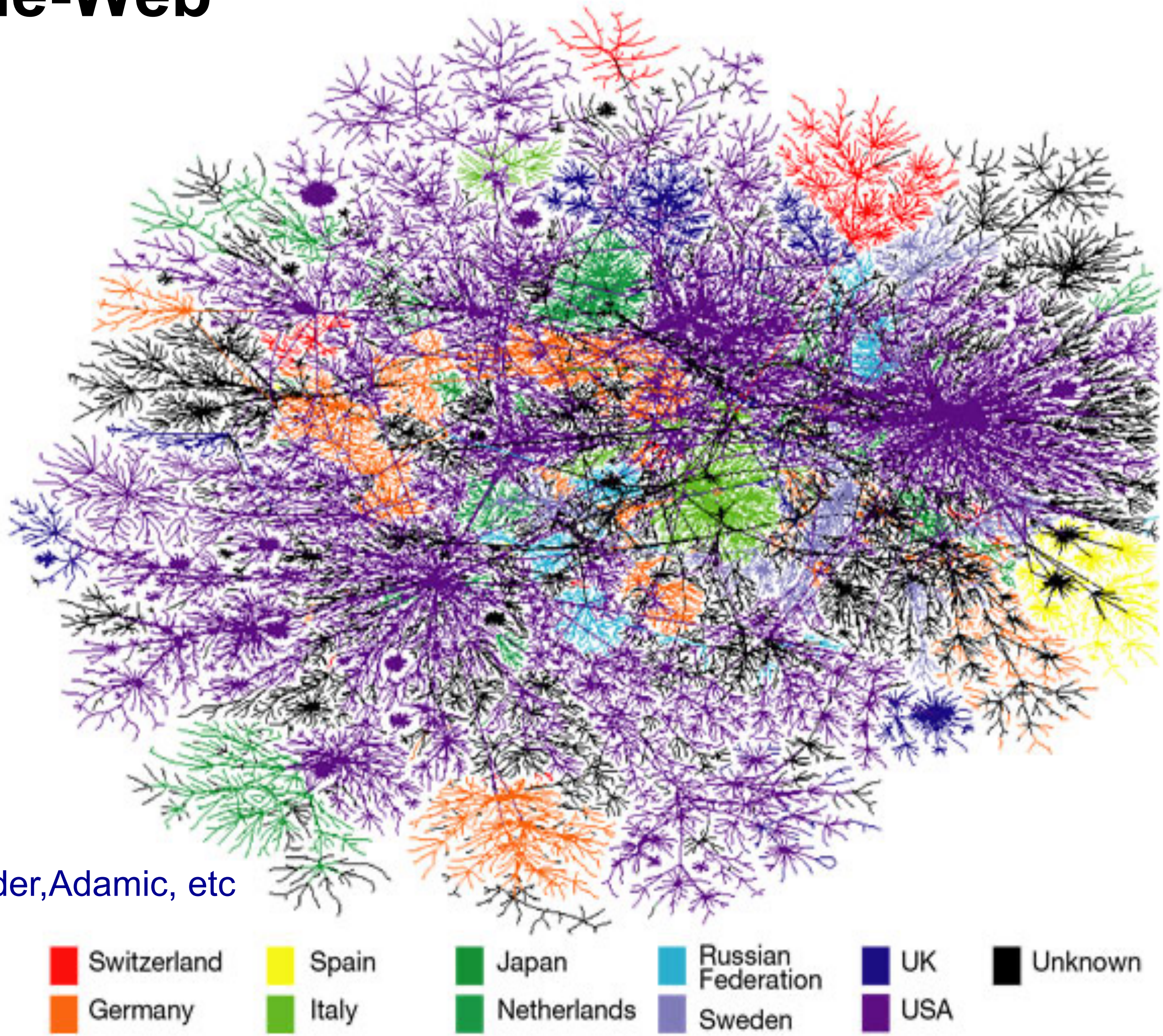
DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

2nd network revolution

Barabási, Albert, Jeong, Dorogovtsev, Mendes, Havlin, Cohen…
(>1999)

Network Science, 2023/24

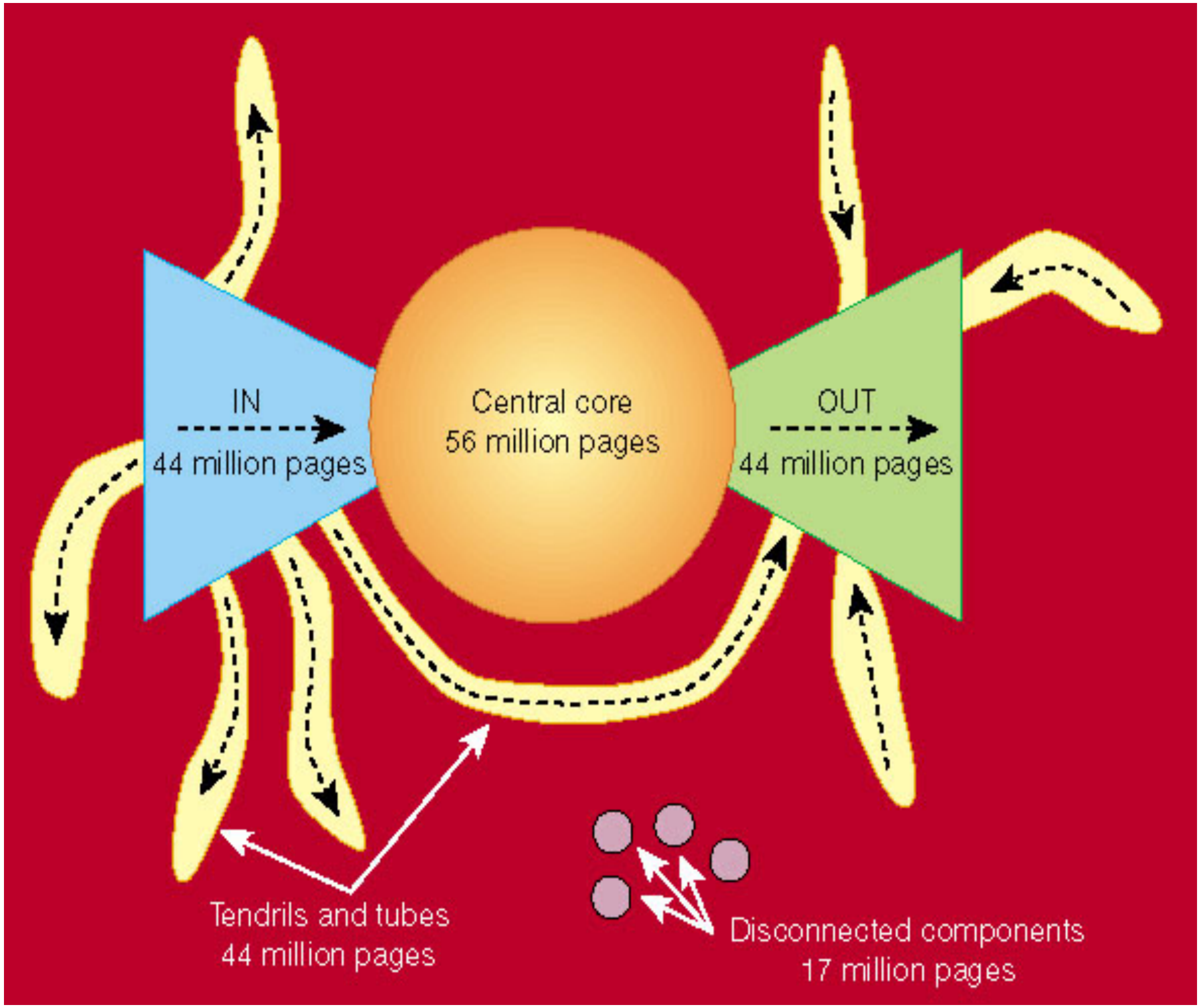
World-Wide-Web

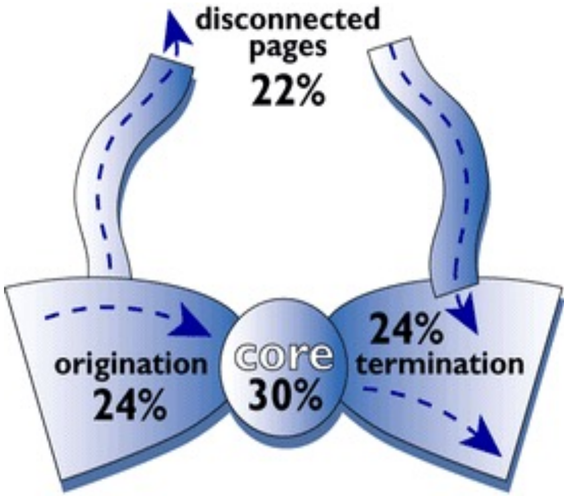


Kleinberg, Barabasi, Broder,Adamic, etc

World-Wide-Web is a “bow-tie”

200 million pages / 1500 million hyperlinks



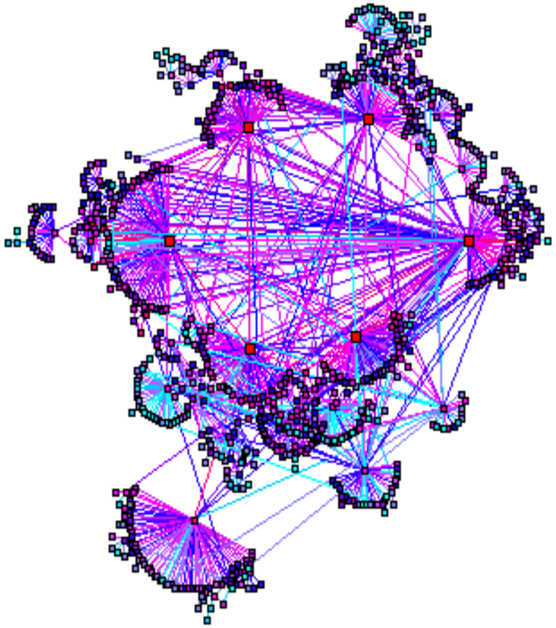


World-Wide-Web

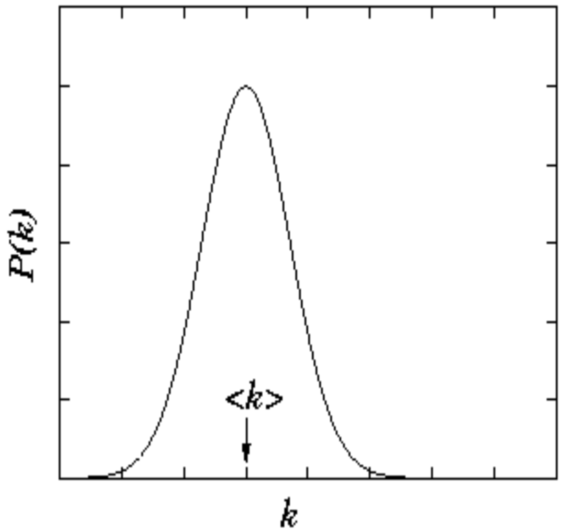
Nodes: **WWW documents**
Links: **URL links**

Over 3 billion documents

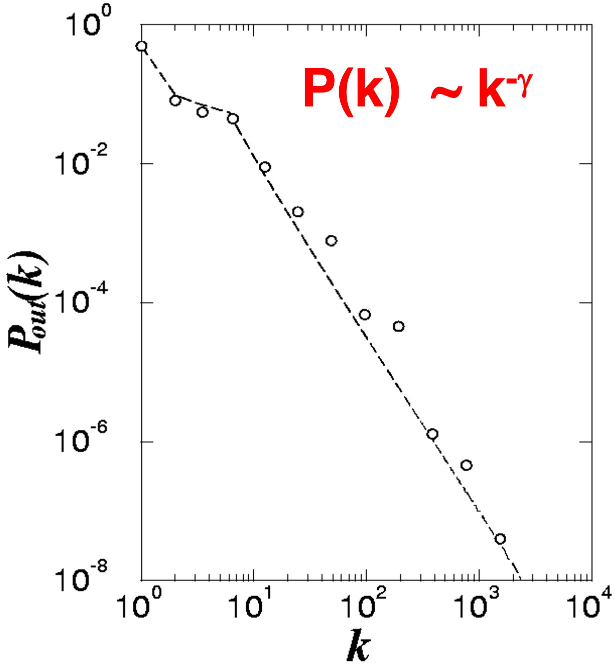
ROBOT: collects all URL's found in a document and follows them recursively



R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).



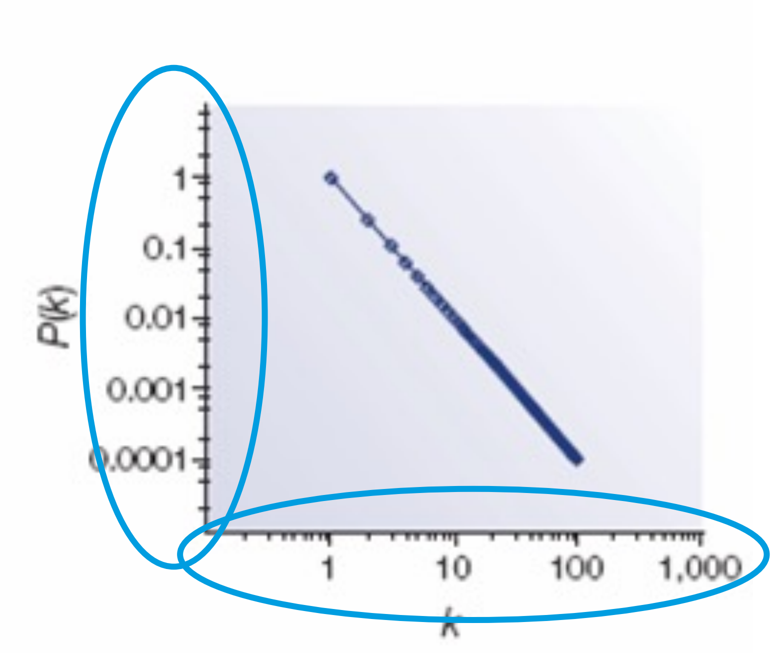
Expected



Found

Power-law degree distributions

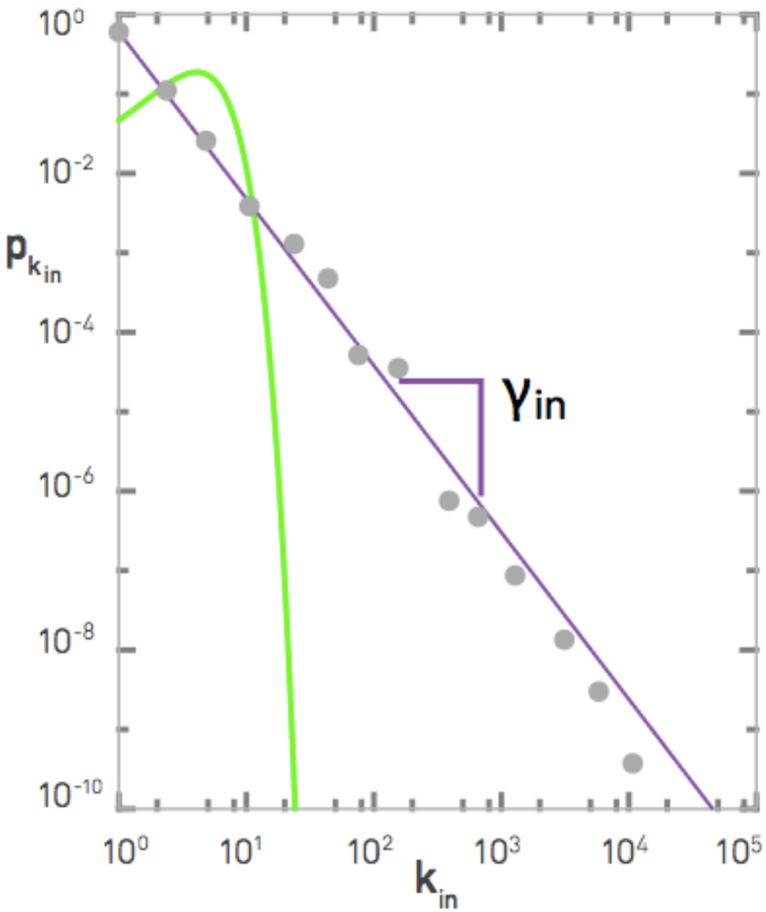
$$P_k \sim k^{-\gamma}$$

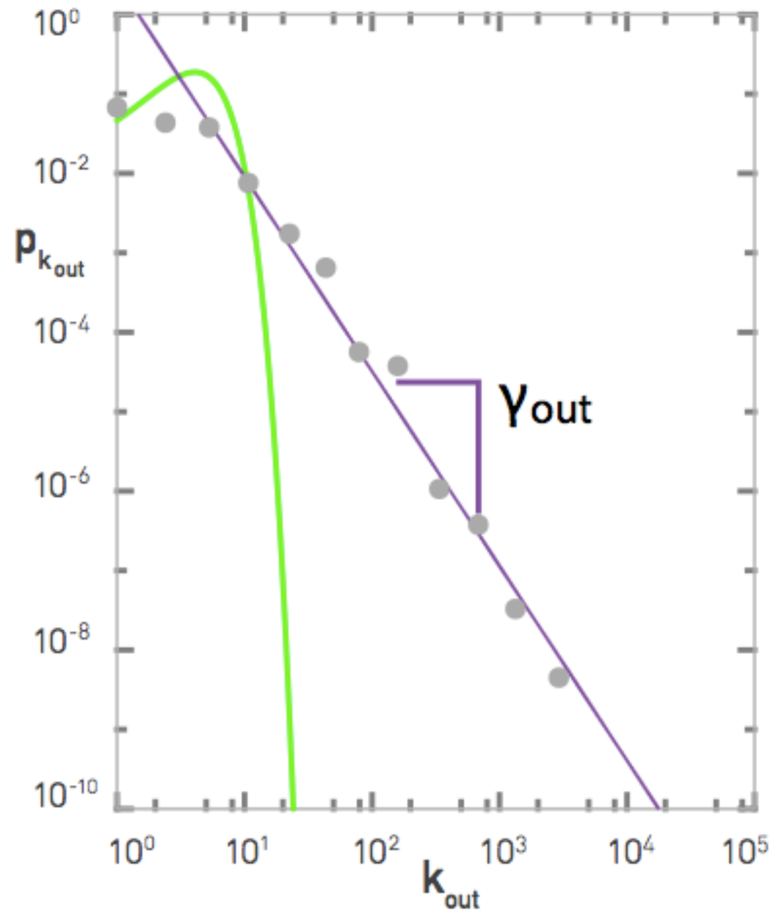


$$\ln P_k \sim \ln k^{-\gamma}$$

$$\ln P_k \sim -\gamma \ln k$$

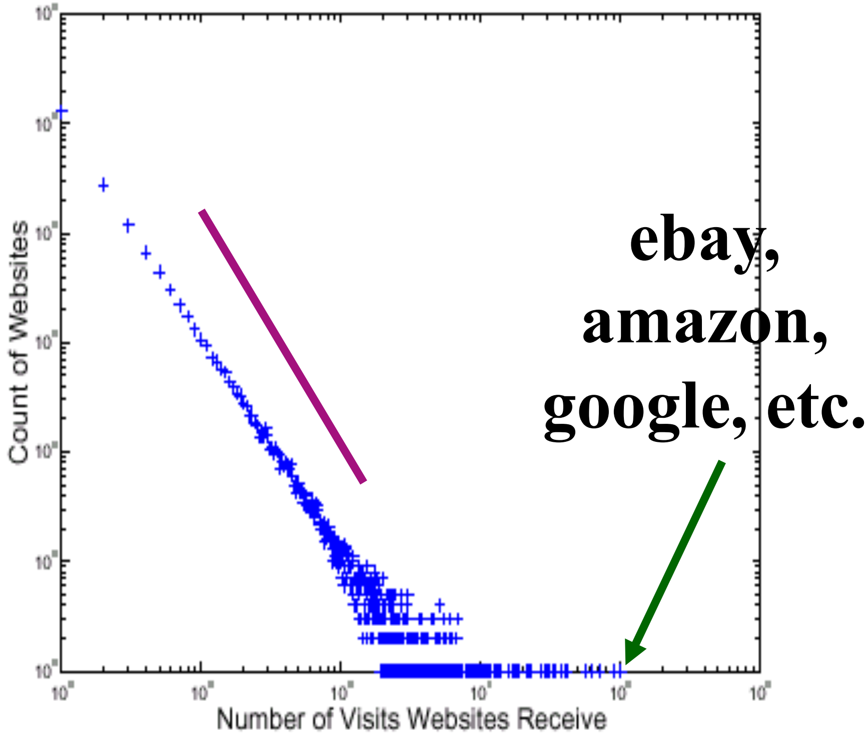
World-Wide-Web as a directed graph:
in & out degree distributions

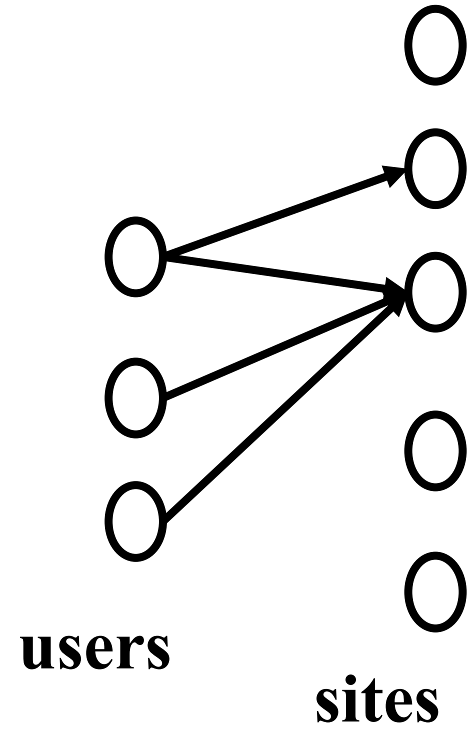




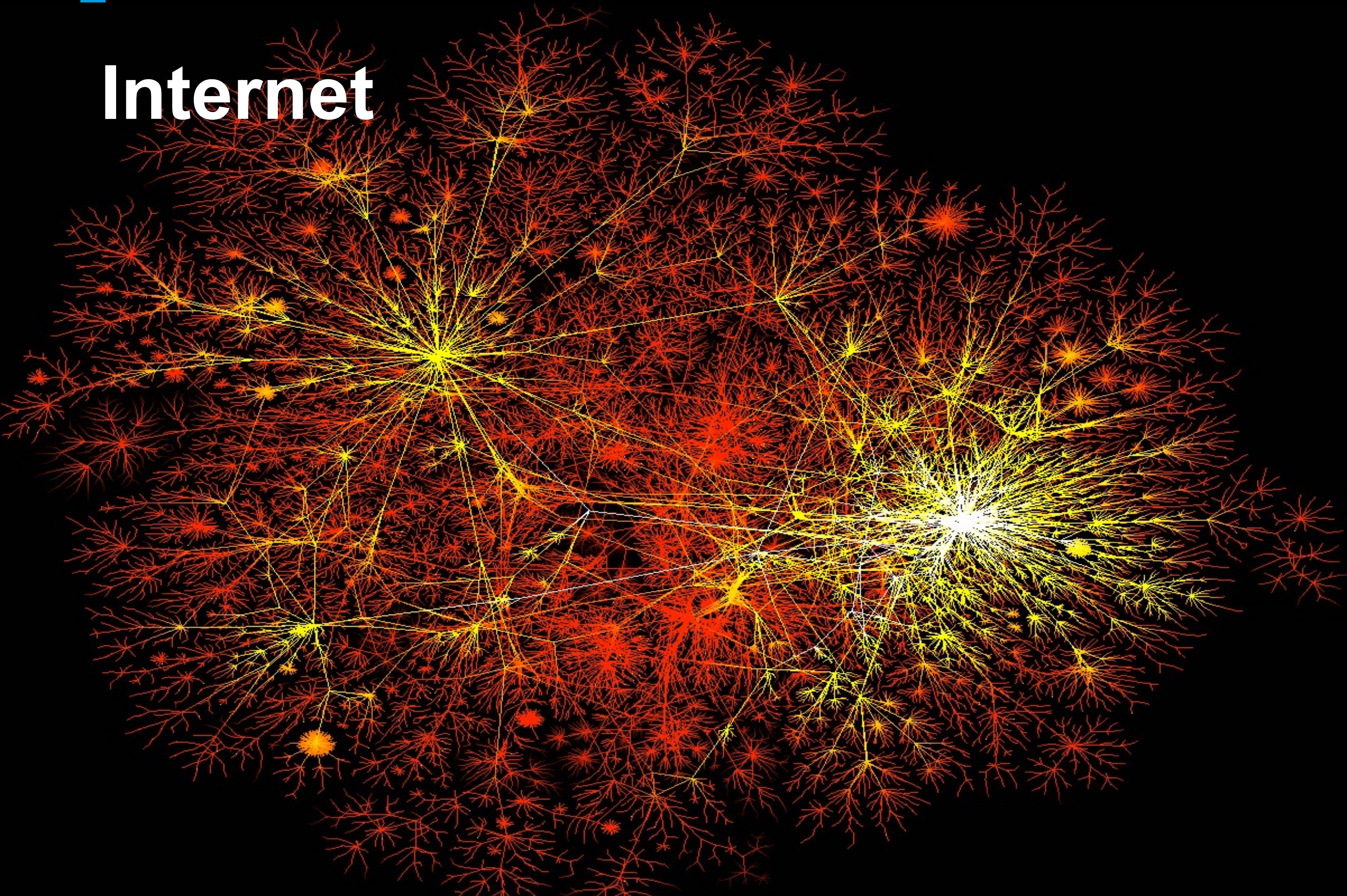
Power-laws are everywhere in the WWW

Web Site Traffic





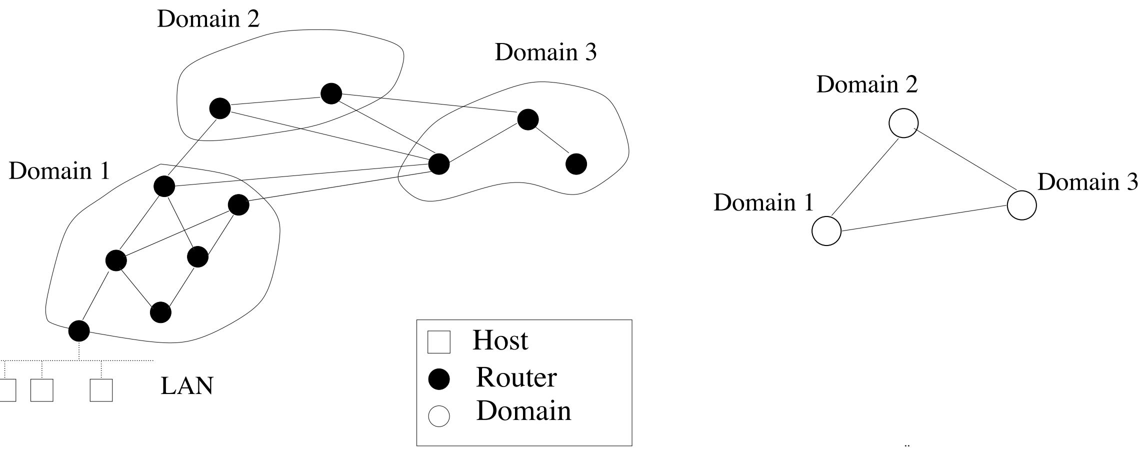
Internet



Internet

Internet (Faloutsos et al, 1999)
Nodes = computers, and edges physical connections among them
… quite difficult

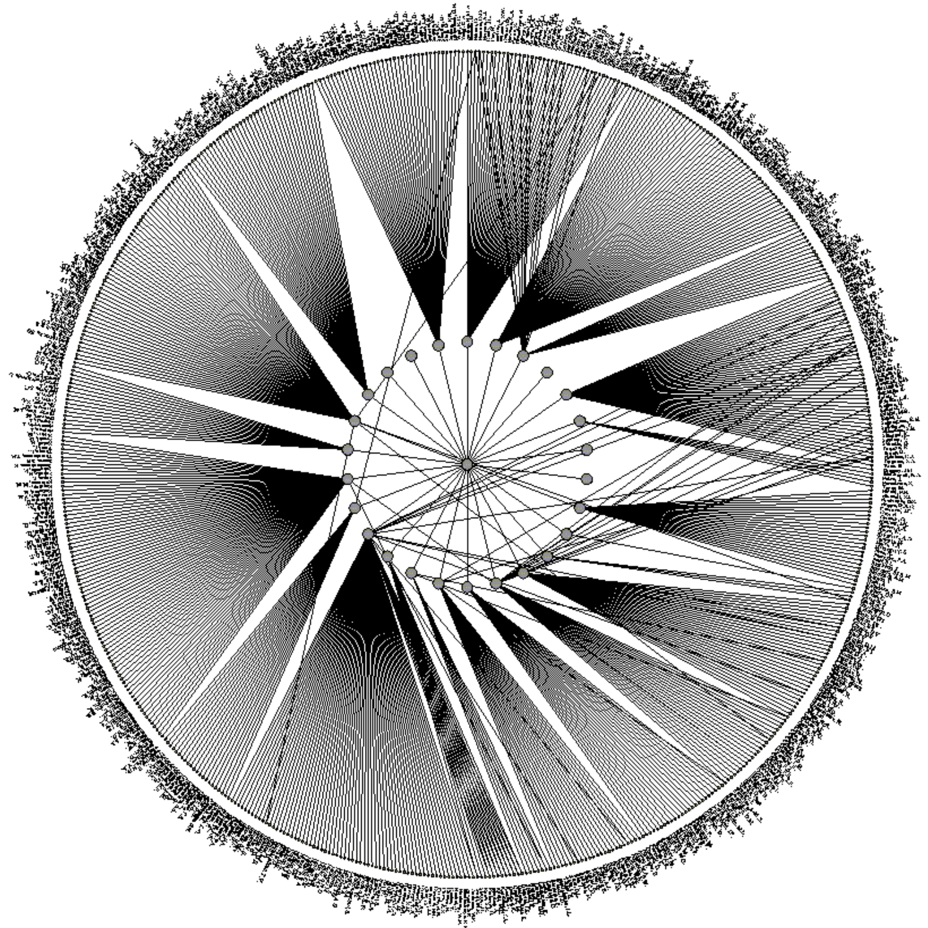
2 levels
Nodes = routers or
Nodes = domains



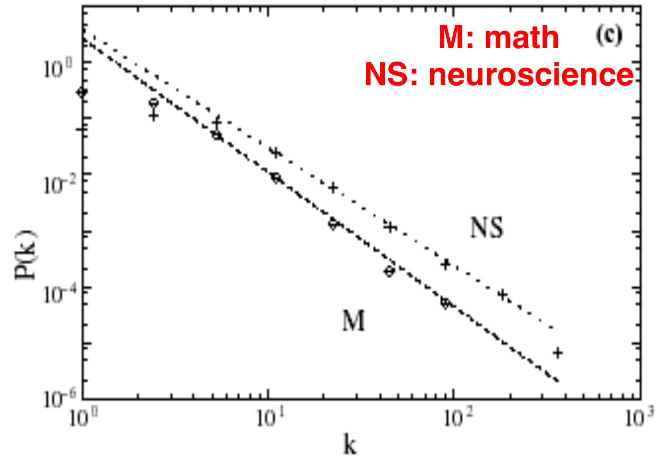
Both showed a scale-free degree dist.

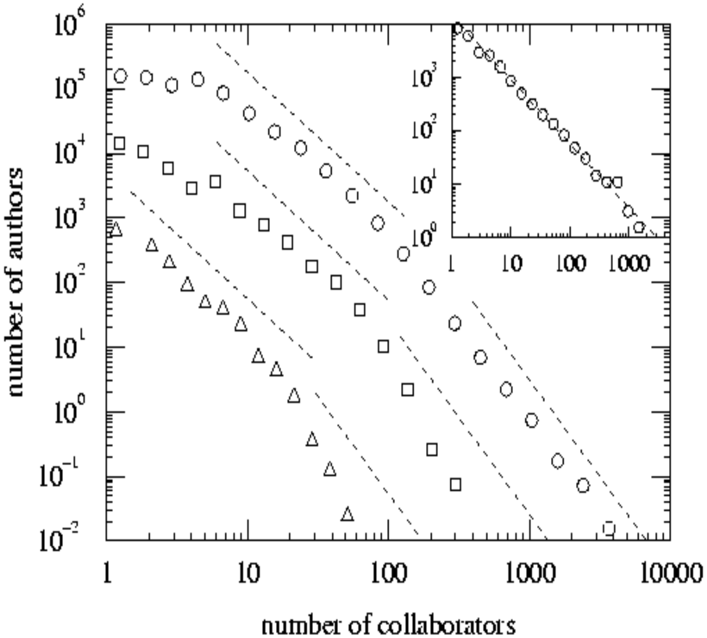
Science collaborations

Nodes: scientist (authors)
Links: joint publication



(Newman, 2000, Barabasi et al 2001)

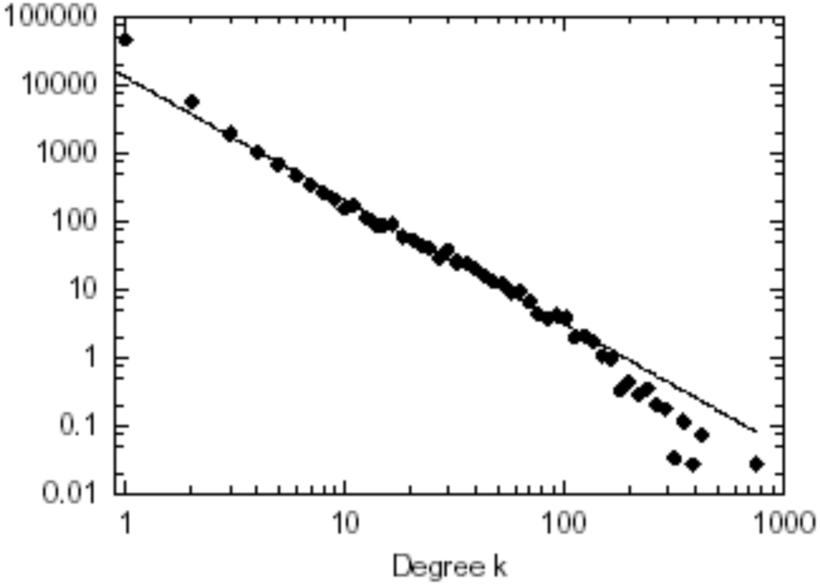




Online communities

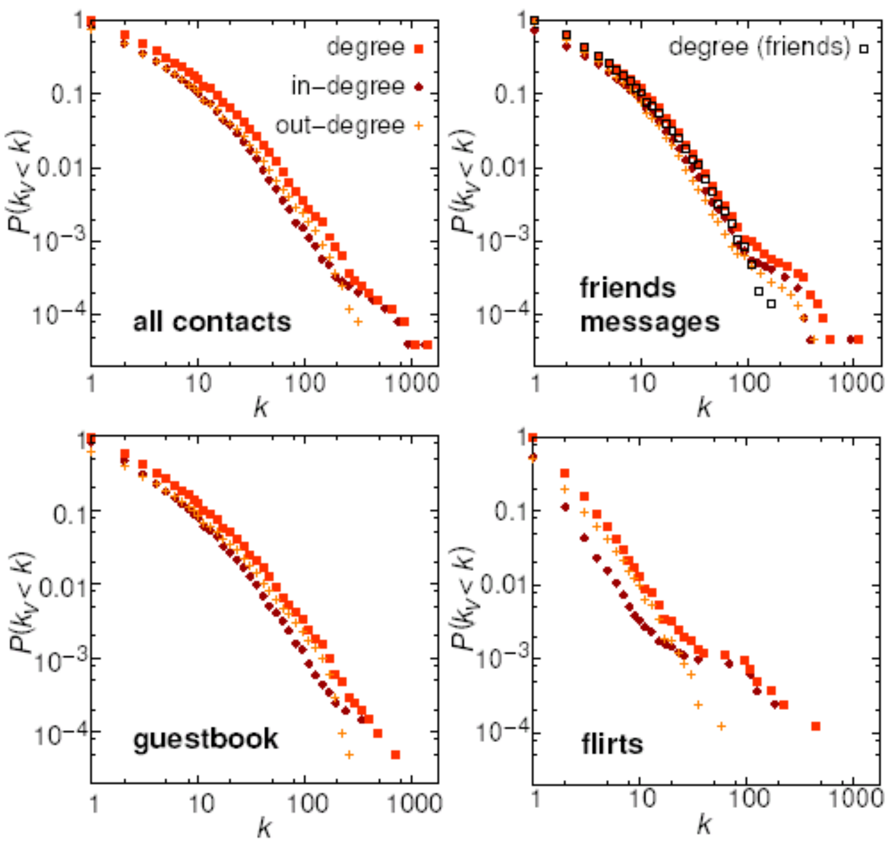
Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



Ebel, Mielsch, Bornholdtz, PRE 2002.

Pussokram.com online community;
512 days, 25,000 users.

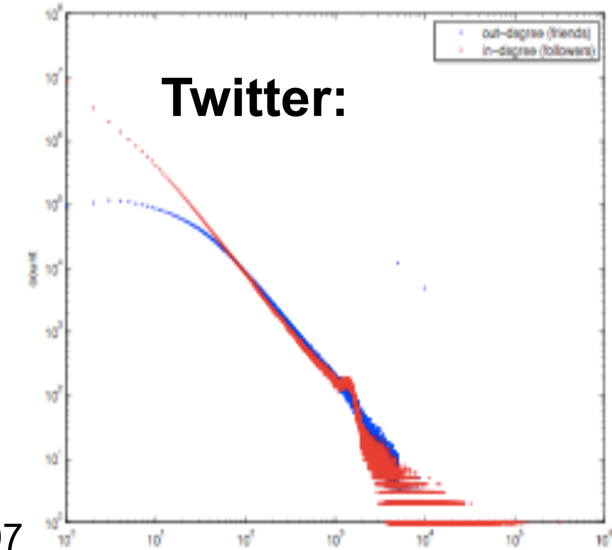


Holme, Edling, Liljeros, 2002.

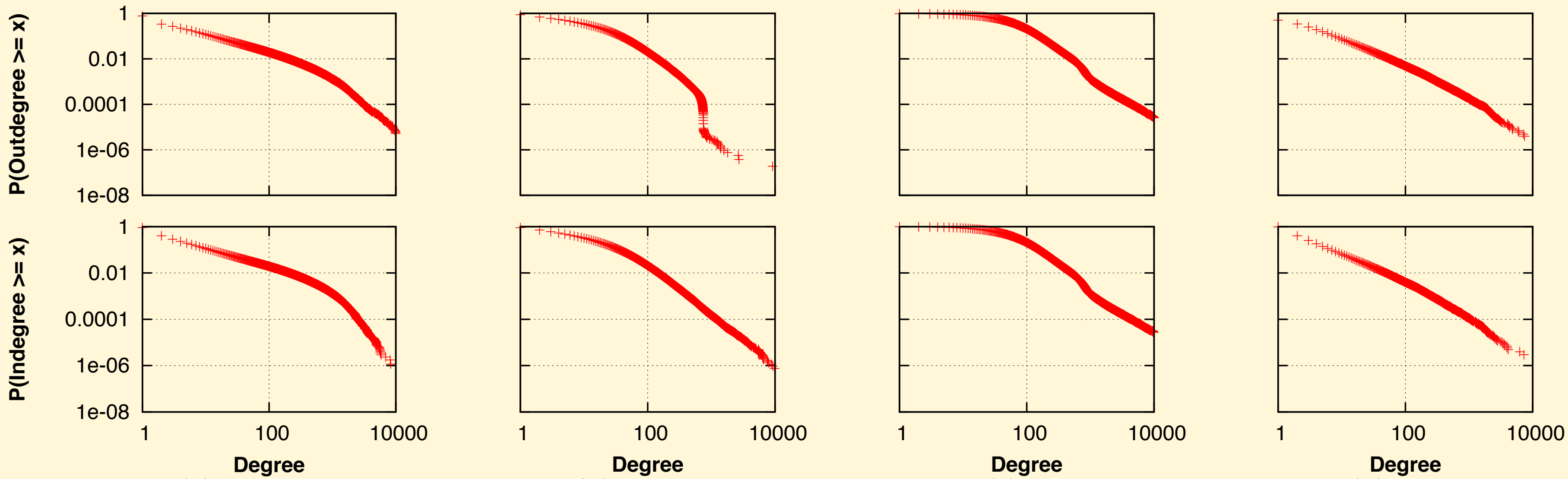
Online communities

All distribtions show a fat-tail behavior:
you have degrees spreading several orders of magnitude.

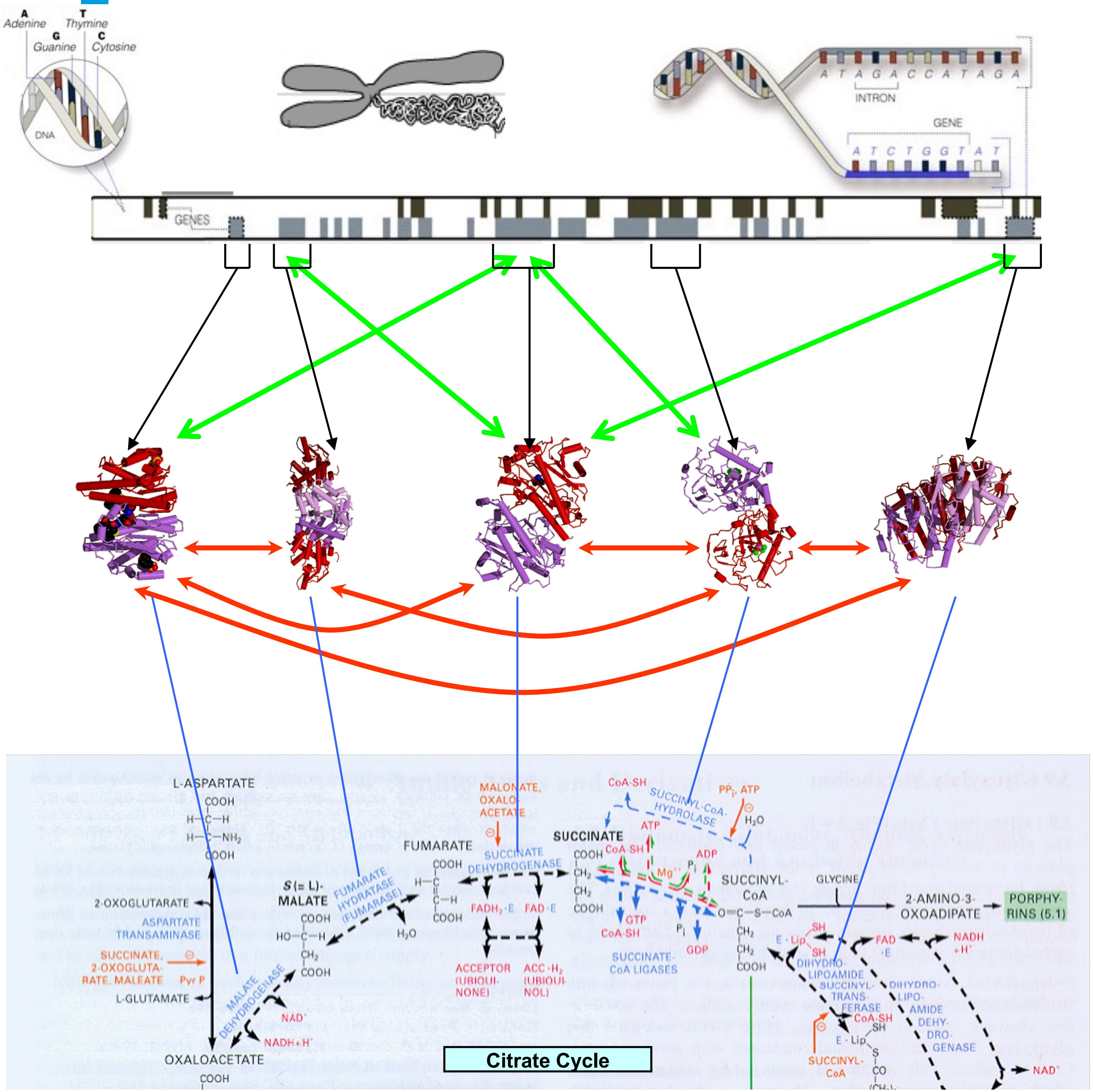
Mislove et al., Measurement and Analysis of Online Social Networks, IMC'2007



Jake Hoffman, Yahoo,



(a) Flickr (b) LiveJournal (c) Orkut (d) YouTube



GENOME

protein-gene interactions

PROTEOME

protein-protein interactions

METABOLISM

Bio-chemical reactions

Network Science, 2023/24

@ our website…

NATURE REVIEWS | GENETICS

NETWORK BIOLOGY: UNDERSTANDING THE CELL'S FUNCTIONAL ORGANIZATION

Albert-László Barabási & Zoltán N. Oltvai‡*

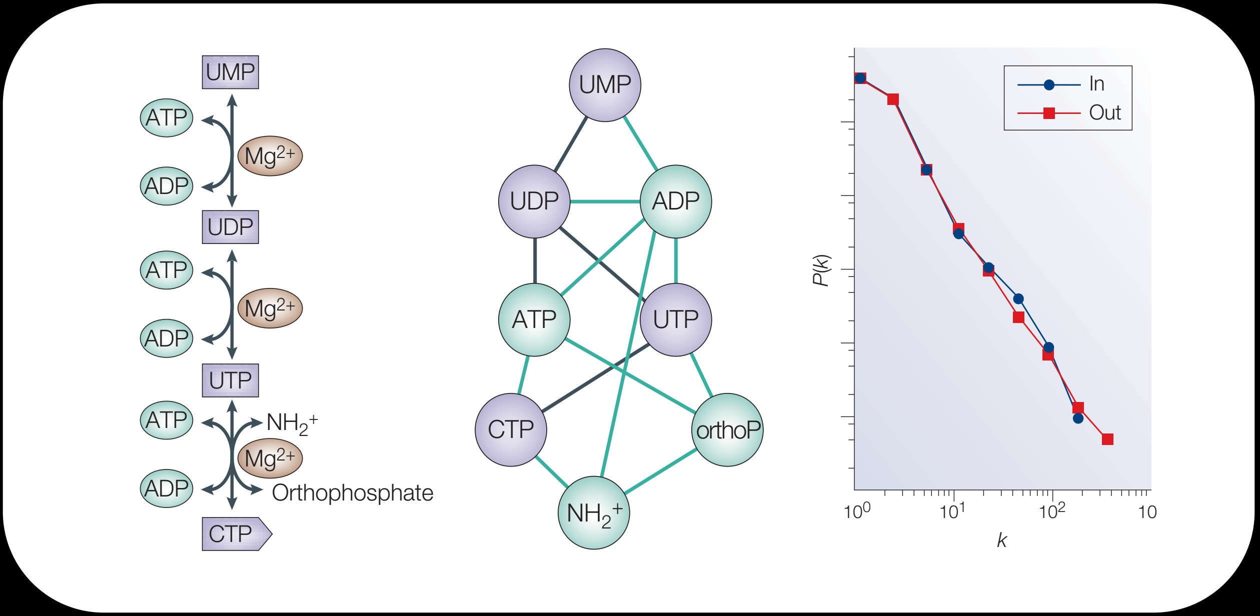
NATURE REVIEWS | GENETICS

Network medicine: a network-based approach to human disease

Albert-László Barabási‡§, Natali Gulbahce*‡|| and Joseph Loscalzo§*

Biological networks: Metabolic networks

Jeong, Albert, Oltvai, Barabási, Fell, Wagner, etc (after 2000).



Nodes : metabolites
Edges : enzyme-catalyzed biochemical reactions

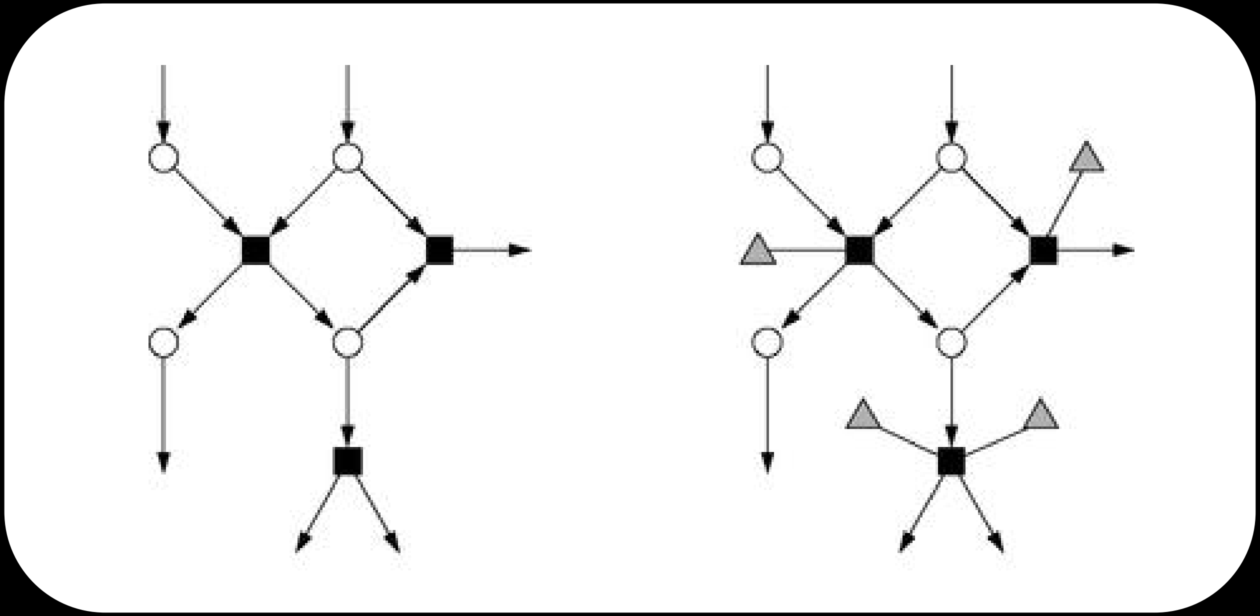
Again, this is a directed graph, as many reactions are irreversible.

The *in* (*out*) degree denotes the number of reactions that produce (consume) a given metabolite.

The analysis of these nets for 43 organisms indicates that "cellular biochemical nets" are *scale-free* (hubs = water, ATP, etc).

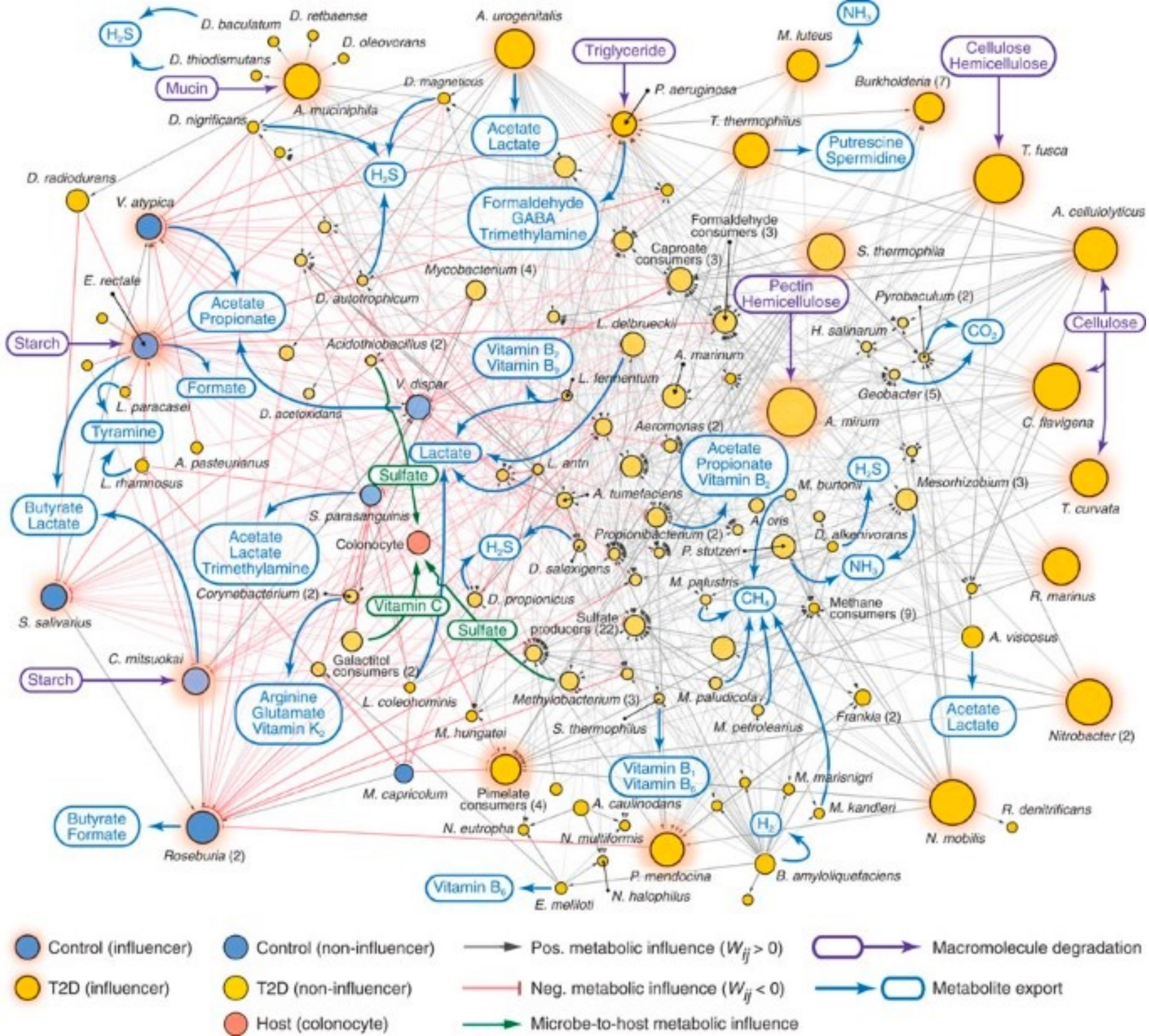
Biological networks: Metabolic networks

Jeong, Albert, Oltvai, Barabási, Fell, Wagner, etc (after 2000).



Bipartide representation (LEFT): metabolites (circles) and reactions (squares) and directed edges indicating which metabolites are substrates (inputs) and products (outputs) of which reactions.

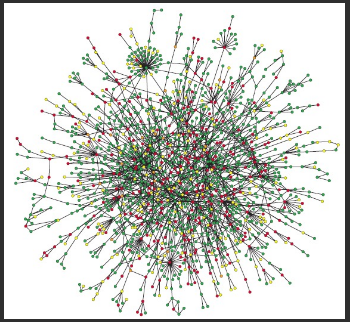
Tripartide representation (RIGHT): A third type of vertex (triangles) can be introduced to represent enzymes, with undirected edges linking them to the reactions they catalyze. The resulting network is a mixed directed/undirected tripartite network.


H2S
D. baculatum
D. retbaense
A. urogenitalis
NH3
M. luteus
D. oleovorans
D. thiodismutans
Triglyceride
Cellulose
Hemicellulose
Mucin
A. muciniphila
D. magneticus
P. aeruginosa
Burkholderia (7)
D. nigrificans
Acetate
Lactate
T. thermophilus
Putrescine
Spermidine
T. fusca
H2S
D. radiodurans
V. atypica
Formaldehyde
GABA
Trimethylamine
Formaldehyde consumers (3)
Caproate consumers (3)
A. cellulolyticus
S. thermophila
E. rectale
Mycobacterium (4)
Pectin
Hemicellulose
Acetate
Propionate
D. autotrophicum
Pyrobaculum (2)
Starch
L. delbrueckii
H. salinarum
CO2
Cellulose
Acidothiobacillus (2)
Vitamin B7
Vitamin B9
Formate
A. marinum
L. fermentum
V. dispar
Geobacter (5)
L. paracasei
D. acetoxidans
A. mirum
C. flavigena
Tyramine
Aeromonas (2)
Lactate
Acetate
Propionate
Vitamin B2
A. pasteurianus
L. antri
H2S
L. rhamnosus
Sulfate
M. burtonii
Mesorhizobium (3)
T. curvata
A. tumefaciens
Butyrate
Lactate
S. parasanguinis
A. oris
Acetate
Lactate
Trimethylamine
Propionibacterium (2)
D. alkenivorans
Colonocyte
H2S
P. stutzeri
D. salexigens
NH3
M. palustris
Corynebacterium (2)
CH4
R. marinus
D. propionicus
Methane consumers (9)
S. salivarius
Vitamin C
Sulfate producers (22)
Sulfate
A. viscosus
C. mitsuokai
Starch
Galactitol consumers (2)
Arginine
Glutamate
Vitamin K2
L. coleohominis
Methylobacterium (3)
M. paludicola
Frankia (2)
Acetate
Lactate
Nitrobacter (2)
M. hungatei
S. thermophilus
M. petrolearius
Vitamin B1
Vitamin B6
M. marisnigri
M. capricolum
Pimelate consumers (4)
A. caulinodans
M. kandleri
R. denitrificans
Butyrate
Formate
N. eutropha
H2
N. multiformis
N. mobilis
Roseburia (2)
P. mendocina
B. amyloliquefaciens
Vitamin B6
N. halophilus
E. meliloti
Control (influencer)
Control (non-influencer)
Pos. metabolic influence (Wij > 0)
Macromolecule degradation
T2D (influencer)
T2D (non-influencer)
Neg. metabolic influence (Wij < 0)
Metabolite export
Host (colonocyte)
Microbe-to-host metabolic influence

Biological networks: Protein-interaction networks

Uetz et al, Ito et al, Giot et al, Li et al, Jeong et al, Sneppen, Wagner, etc (after 2000).

Nodes : proteins
Edges : experimental evidence that they bind creating protein complexes



Biological networks: genetic regulatory nets

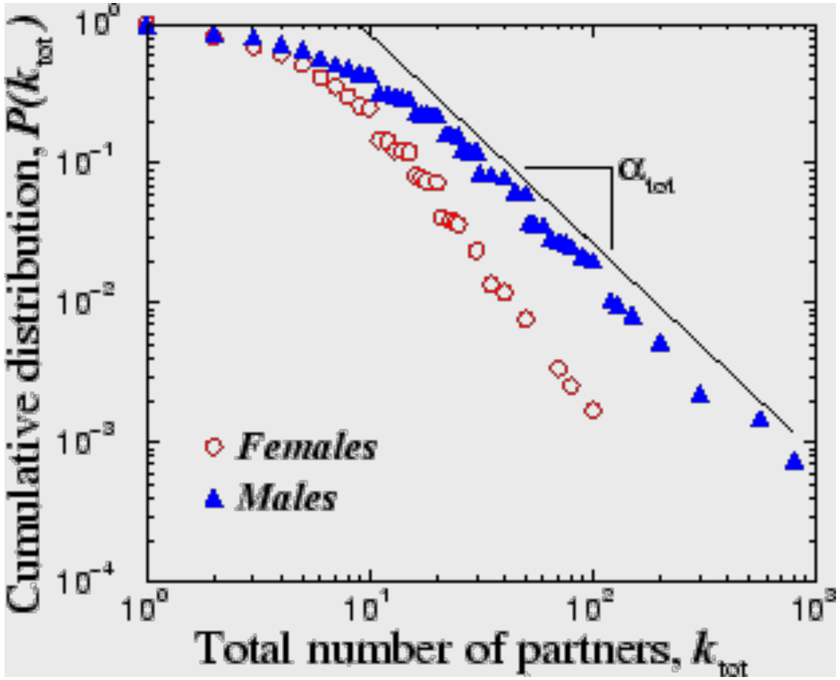
Agrawal 2002. Shaw 2003. Provero 2002, Farkas 2003.

Ex:
Nodes : DNA segments
Edges : indirect interaction through RNA and protein expression products

➡ Plethora of Modelling challenges (ex: Boolean nets, evolutionary networks) and application of "bioinformatics" techniques, DB, etc.

Swedish sexual interaction network

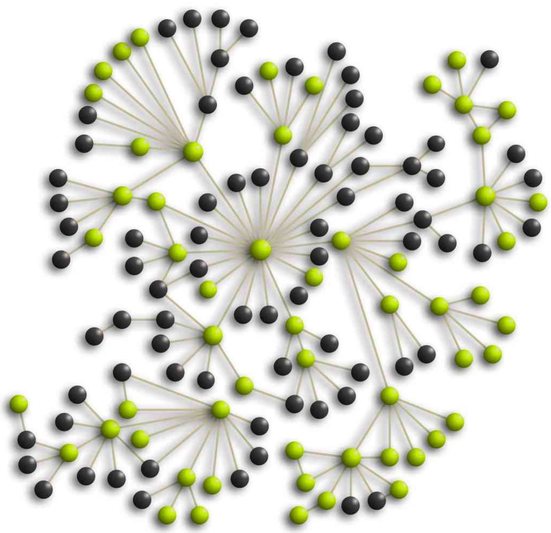
Nodes: people (Females; Males)
Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.

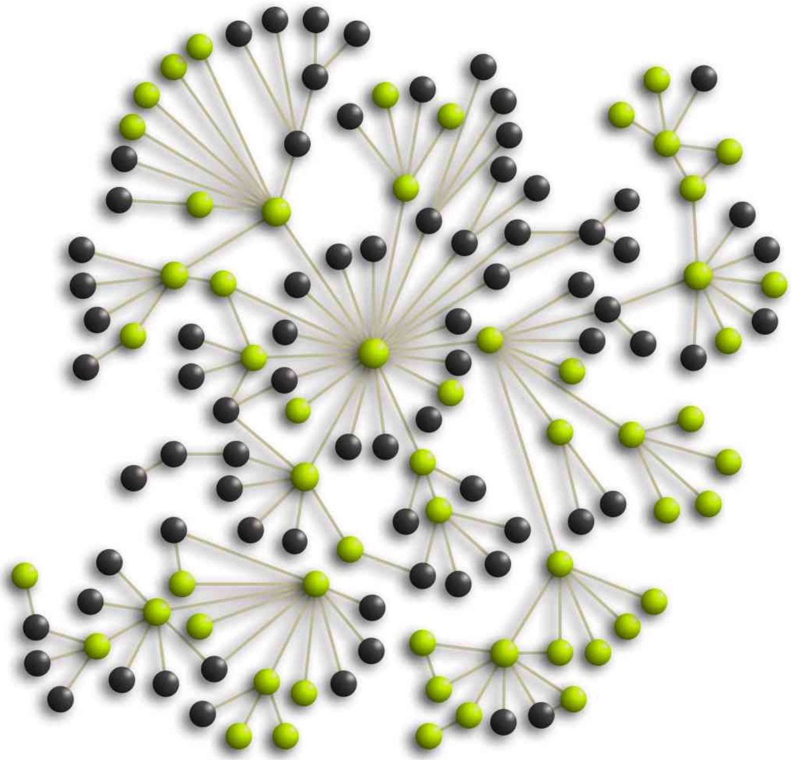
Liljeros et al. Nature 2001

Many real-world nets are scale-free

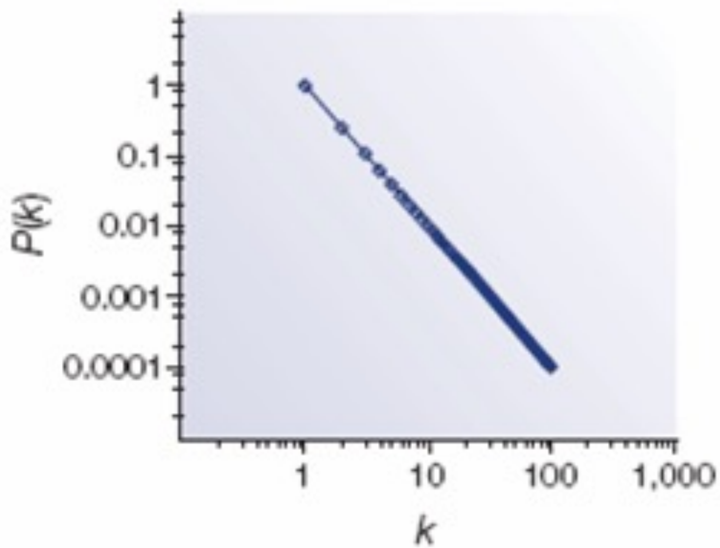


WWW, Internet (routers and domains), electronic circuits, computer software, movie actors, coauthorship networks, sexual web, instant messaging, email web, citations, phone calls, metabolic, protein interaction, protein domains, brain function web, linguistic networks, comic book characters, international trade, bank system, encryption trust net, energy landscapes, astrophysical network…

Universality? I will get back to this point...



$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$

A closer look at power-law distributions

discrete representation: $P_k = Ck^{-\gamma}$ with C given by

$$\sum_{k_{\min}}^{\infty} P_k = 1 \Rightarrow C = \frac{1}{\sum_{k_{\min}}^{\infty} k^{-\gamma}} = \frac{1}{\varsigma(\gamma)}$$

Riemann-zeta function

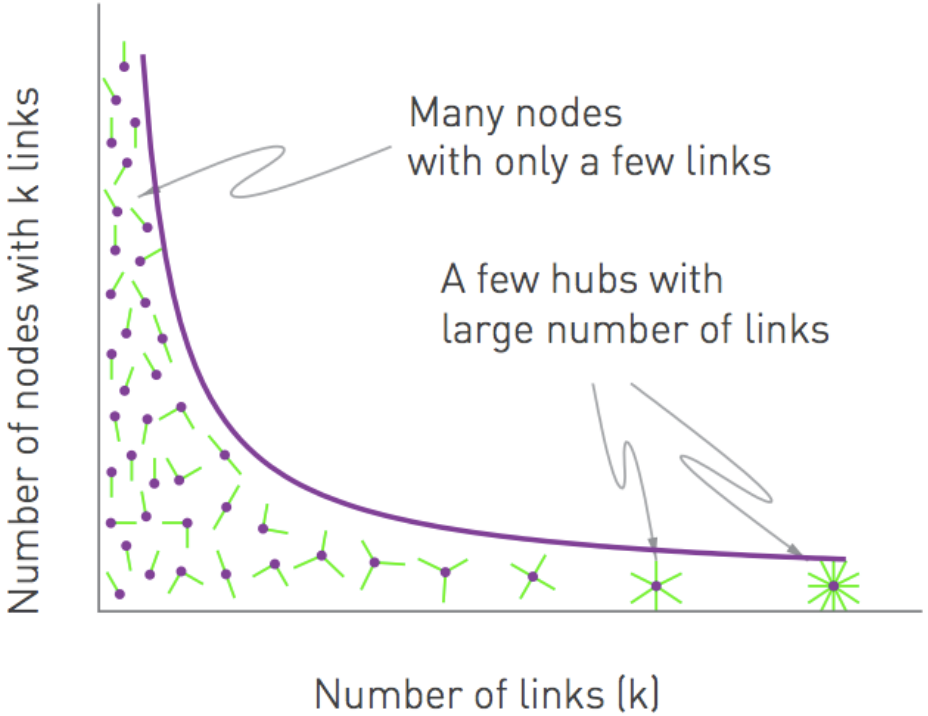
continuum description: $P_k = Ck^{-\gamma}$ with C given by

$$C = \frac{1}{\int_{K_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1)k_{\min}^{\gamma-1}$$

Largest hub

- Just for the fun of it: Imagine that we have a **scale-free network** with the size and average degree of the WWW ($N=10^{12}$, $\langle k \rangle=4.6$ and $\gamma=2.1$). What's the probability of having a node with a degree $k \geq 100$?

$$P_k = \left[(\gamma - 1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$

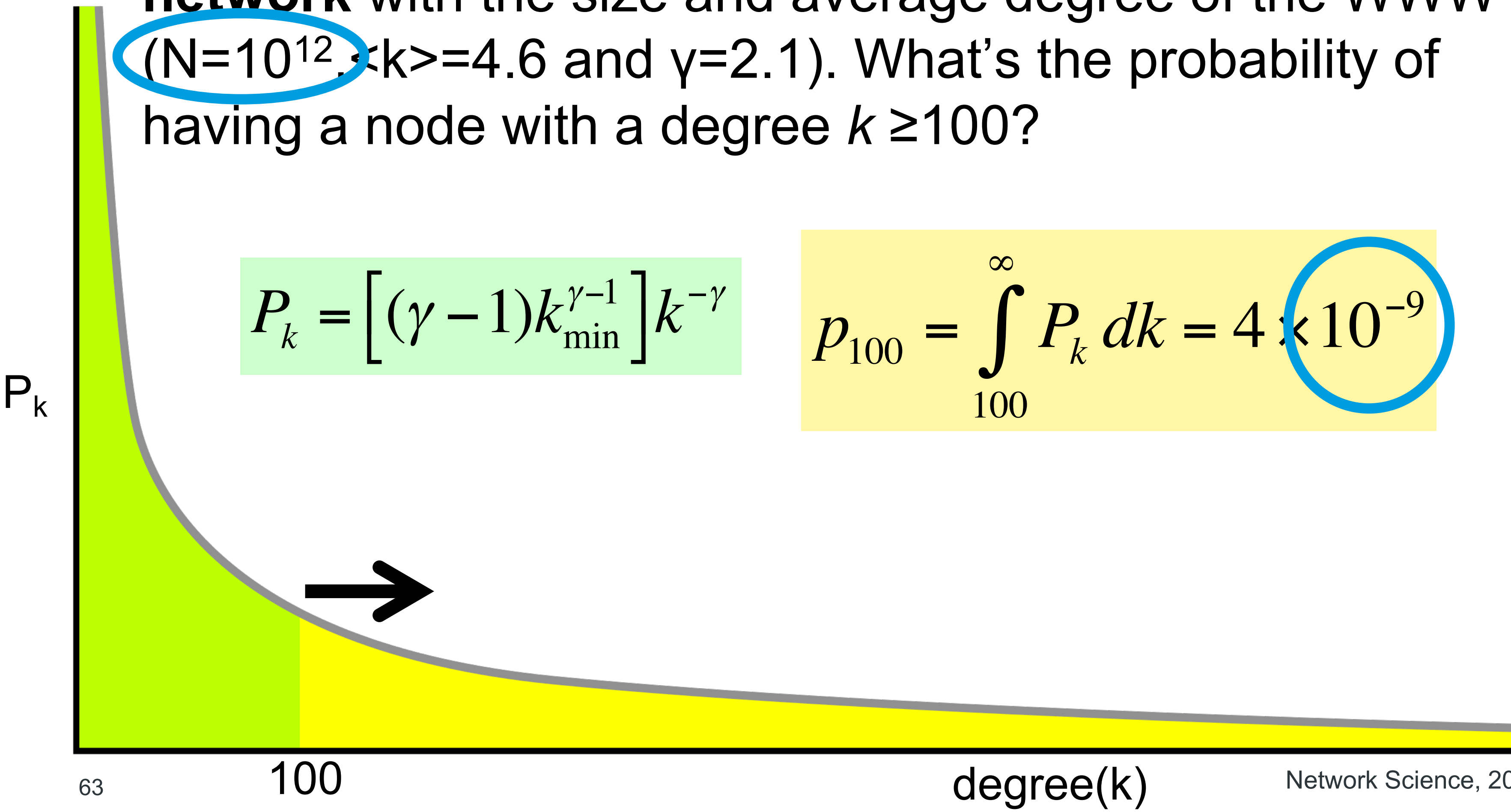


Largest hub

- Just for the fun of it: Imagine that we have a **scale-free network** with the size and average degree of the WWW ($N=10^{12}$, $\langle k \rangle=4.6$ and $\gamma=2.1$). What's the probability of having a node with a degree $k \geq 100$?

$$P_k = \left[(\gamma-1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$

$$p_{100} = \int_{100}^{\infty} P_k \, dk = 4 \times 10^{-9}$$



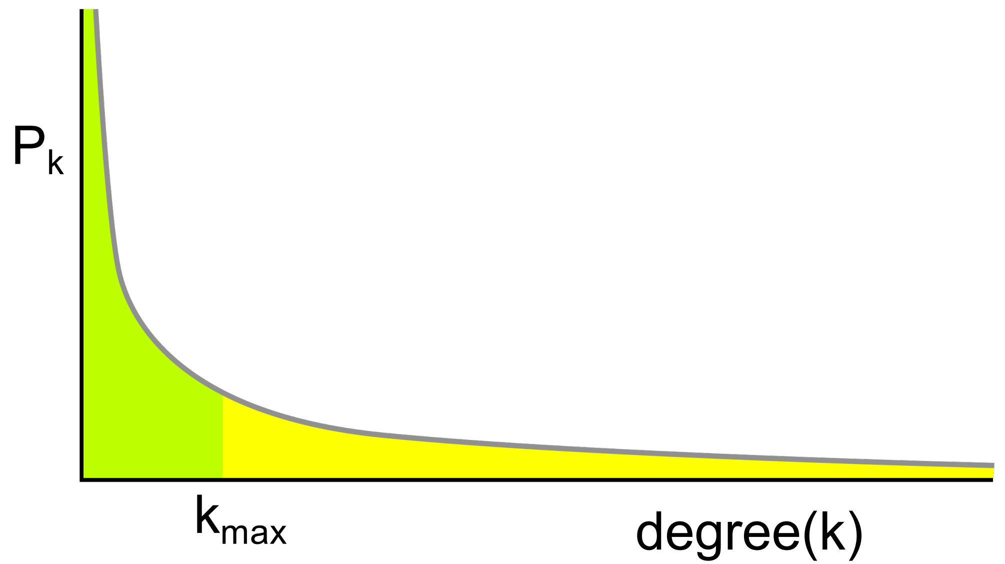
Can we estimate the largest hub?

- We have the degree dist:

$$P_k = \left[(\gamma - 1)k_{\min}^{\gamma-1}\right]k^{-\gamma}$$

- k_{max} will be given by

$$\int_{k_{\max}}^{\infty} P_k \, dk < \frac{1}{N}$$

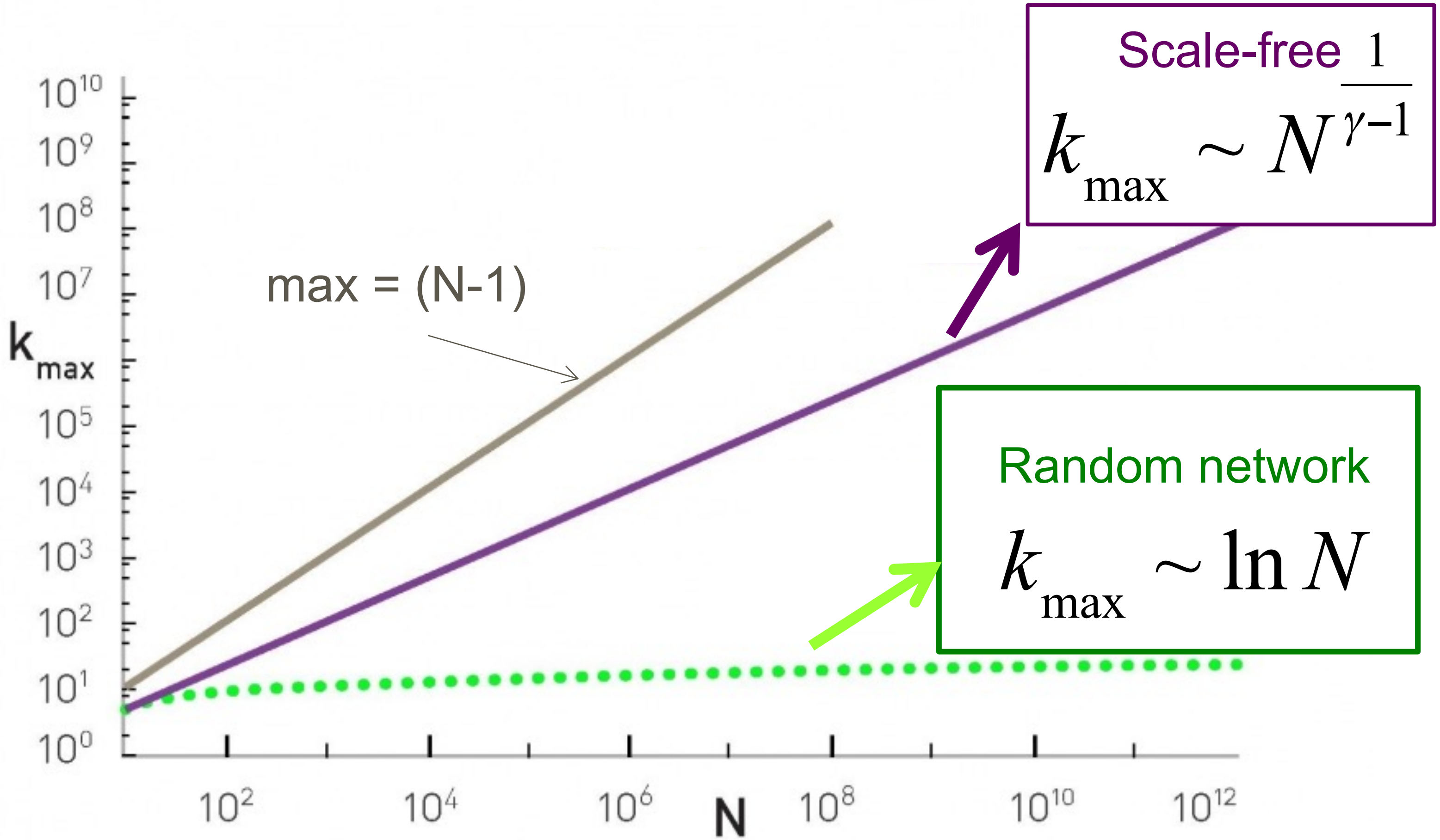


- Which yields

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

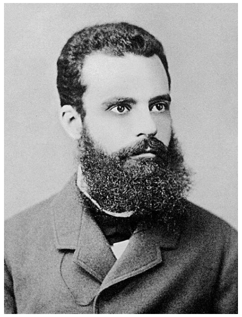
- Example: WWW gives k_{max}=95000

Can we estimate the largest hub?



Revisiting Pareto

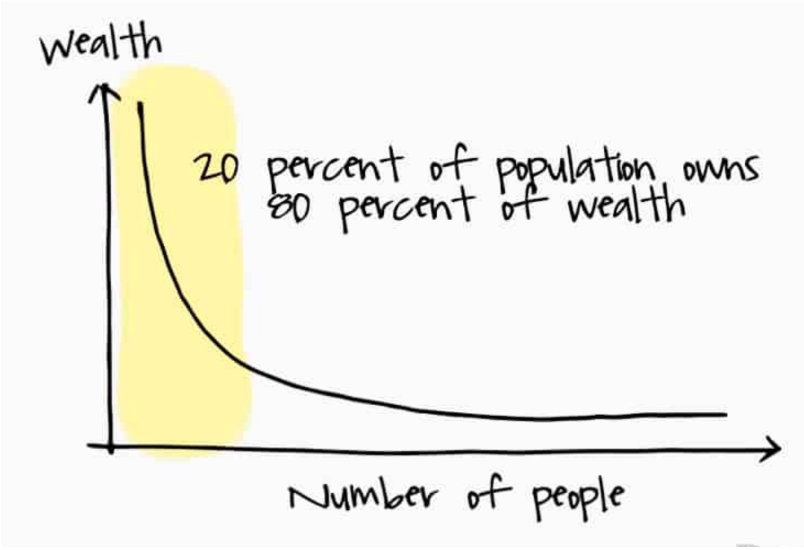
Vilfred Pareto
(1848-1923)



- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule:
  Roughly 80 percent of money is earned by only 20 percent of the population.

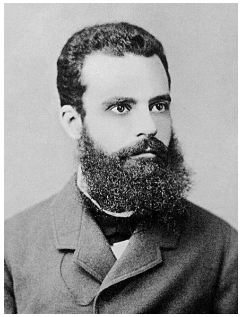
Distribution of world GDP

Quintile of population	Income
Richest 20%	82.70%
Second 20%	11.75%
Third 20%	2.30%
Fourth 20%	1.85%
Poorest 20%	1.40%



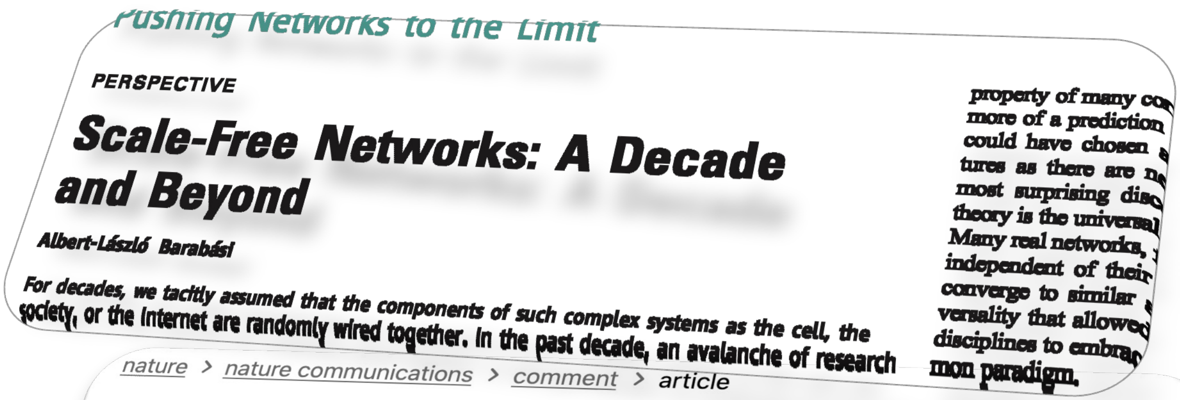
Revisiting Pareto

Vilfred Pareto
(1848-1923)



- Power-law distributions are also called Pareto distributions.
- Power-laws is the principle behind the 80/20 rule: *Roughly 80 percent of money is earned by only 20 percent of the population*.
  - WWW: 80% of the links point to 15% of the pages.
  - Citations: 80% of all citations point to 38% of scientists
  - Hollywood: 80% of all links connect 30% of actors.
  - USA: 1% of the population earns 15% of the total US income.

Scale-free networks? What does it mean?



Pushing Networks to the Limit

PERSPECTIVE

Scale-Free Networks: A Decade and Beyond

Albert-László Barabási

For decades, we tacitly assumed that the components of such complex systems as the cell, the society, or the Internet are randomly wired together. In the past decade, an avalanche of research

property of many co… more of a prediction could have chosen … tures as there are n… most surprising disc… theory is the universal… Many real networks, … independent of their converge to similar … versality that allowed disciplines to embrac… mon paradigm.

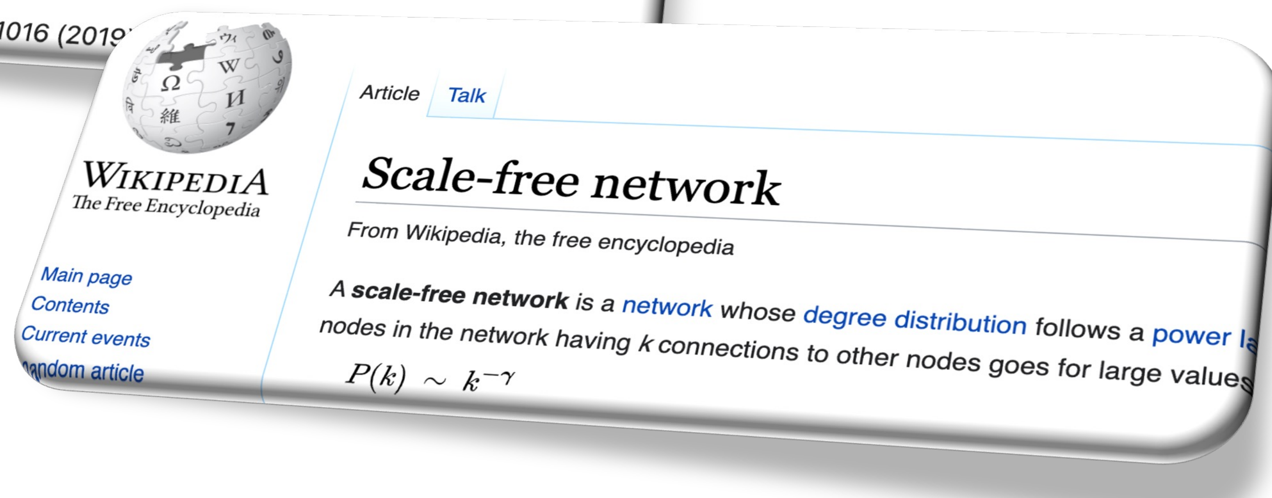
nature > nature communications > comment > article

Comment | Open Access | Published: 04 March 2019

Rare and everywhere: Perspectives on scale-free networks

Petter Holme

Nature Communications 10, Article number: 1016 (2019)



WIKIPEDIA
The Free Encyclopedia

Main page
Contents
Current events
…andom article

Article | Talk

Scale-free network

From Wikipedia, the free encyclopedia

A **scale-free network** is a network whose degree distribution follows a power la… nodes in the network having k connections to other nodes goes for large values

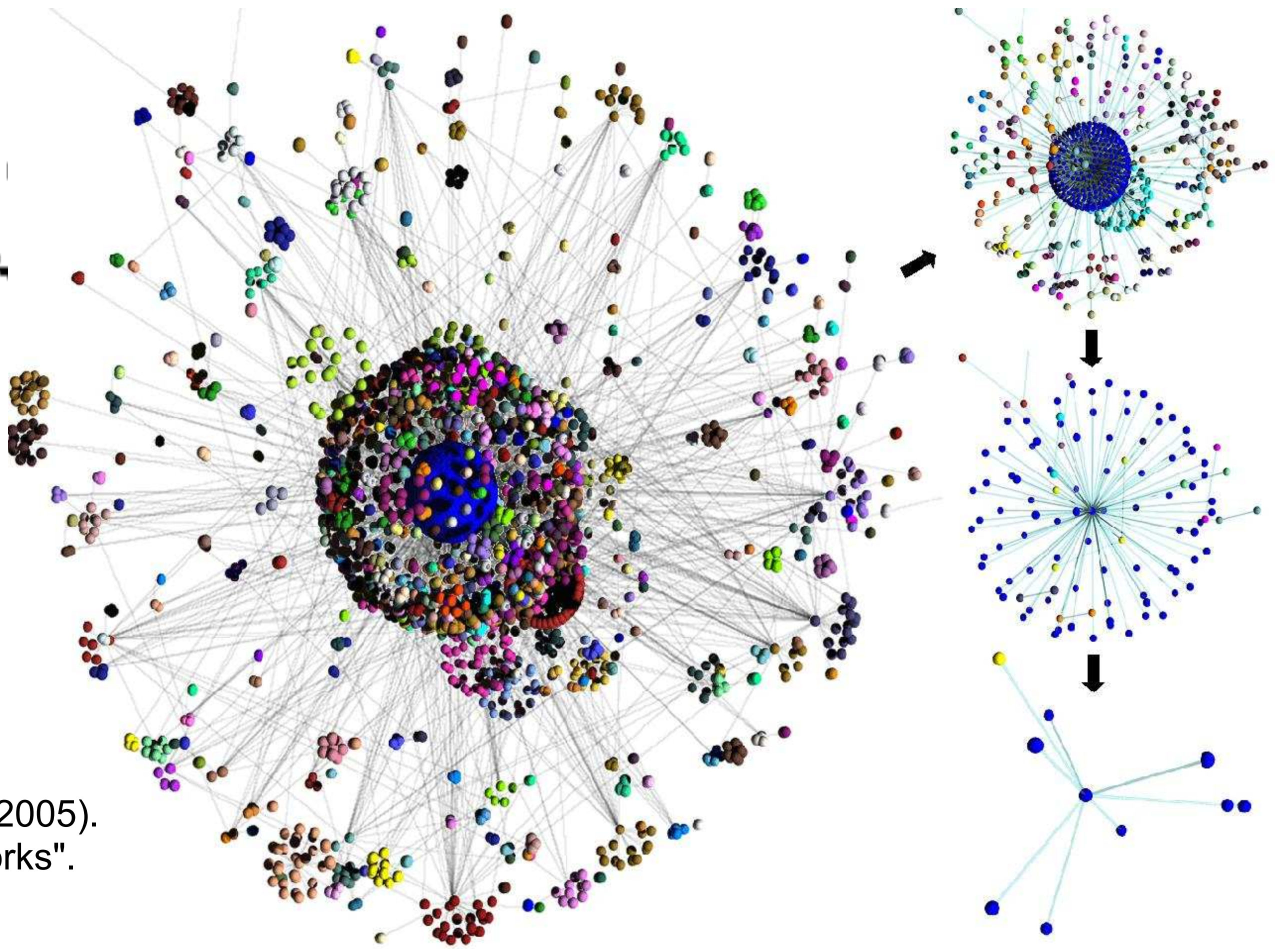
$$P(k) \sim k^{-\gamma}$$

Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance, self-similarity, etc.

a



Song, C.; Havlin, S.; Makse, H.(2005). "Self-similarity of complex networks". Nature 433 (7024): 392–395.

Scale-free? What does it mean?

1st explanation: Scale invariance

Scale-invariance?

$$P(k) = k^{-\gamma}$$

$$P(ck) = (ck)^{-\gamma} = Const.P(k) \propto P(k)$$

- Scaling the argument ***k*** by a constant factor ***c*** causes only a proportionate scaling of the function itself.
- In other words, all power laws (with a given exponent) are equivalent up to constant factors, since each is simply a scaled version of the others.

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- ***n^{th} moments of a distribution***

$$\langle k^n \rangle = \sum_{k_{\min}}^{\infty} k^n P_k \approx \int_{k_{\min}}^{\infty} k^n P_k \, dk$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***…

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$$

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \approx \int_{k_{min}}^{k_{max}} k^n P_k \, dk = C \frac{k_{max}^{n-\gamma+1} - k_{min}^{n-\gamma+1}}{n-\gamma+1}$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

Scales with N

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

- For Scale-free networks we have

$$\langle k^n \rangle \approx \int_{k_{min}}^{k_{max}} k^n P_k \, dk = C \frac{k_{max}^{n-\gamma+1} - k_{min}^{n-\gamma+1}}{n-\gamma+1}$$

Do not scale with N

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

Scales with N

- n=0 sums to one.
- n=1 gives the **average** degree
- n=2 helps us to calculate the **variance**
- n=3 determines the **skewness**

Scale-free? What does it mean?

2nd explanation: Lack of well-defined average value

For large N, it diverges for γ<2

- For Scale-free networks we have (for large N)

$$\langle k^n \rangle \xrightarrow{N\to\infty} N^{\frac{n-\gamma+1}{\gamma-1}}$$

$$\langle k \rangle \to N^{\frac{2-\gamma}{\gamma-1}}$$

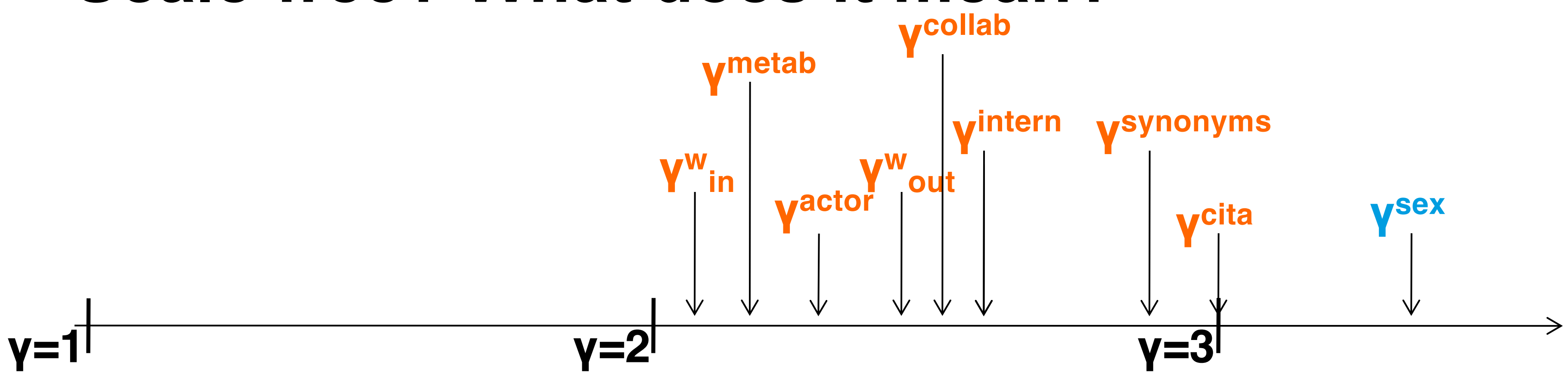
$$\langle k^2 \rangle \to N^{\frac{3-\gamma}{\gamma-1}}$$

- n=0 sums to one.
- n=1 gives the ***average*** degree
- n=2 helps us to calculate the ***variance***
- n=3 determines the ***skewness***

Diverges for γ<3

$$k = \langle k \rangle \pm \infty$$

Scale-free? What does it mean?



$\langle k^2 \rangle$ diverges | $\langle k^2 \rangle$ finite

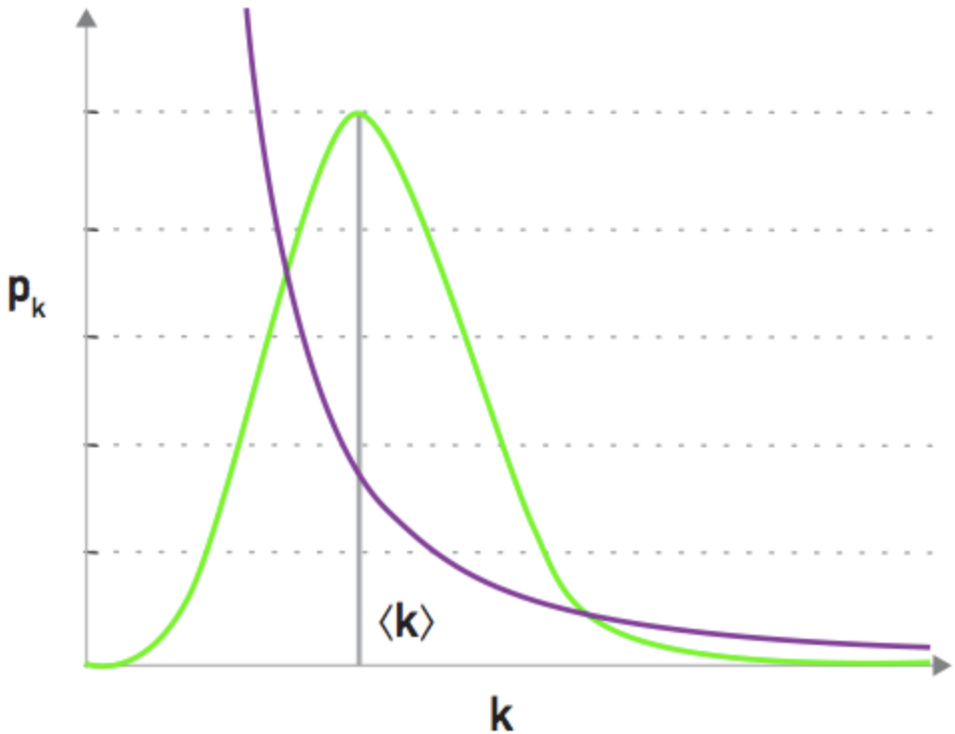
$\langle k \rangle$ diverges | $\langle k \rangle$ finite

Most networks are in a regime in which the variance diverges for large N ☺

$$k = \langle k \rangle \pm \infty$$

For large N, average values are not meaningful, as fluctuations are too large!

Scale-free? What does it mean?



Random Network

Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$

Scale: $\langle k \rangle$

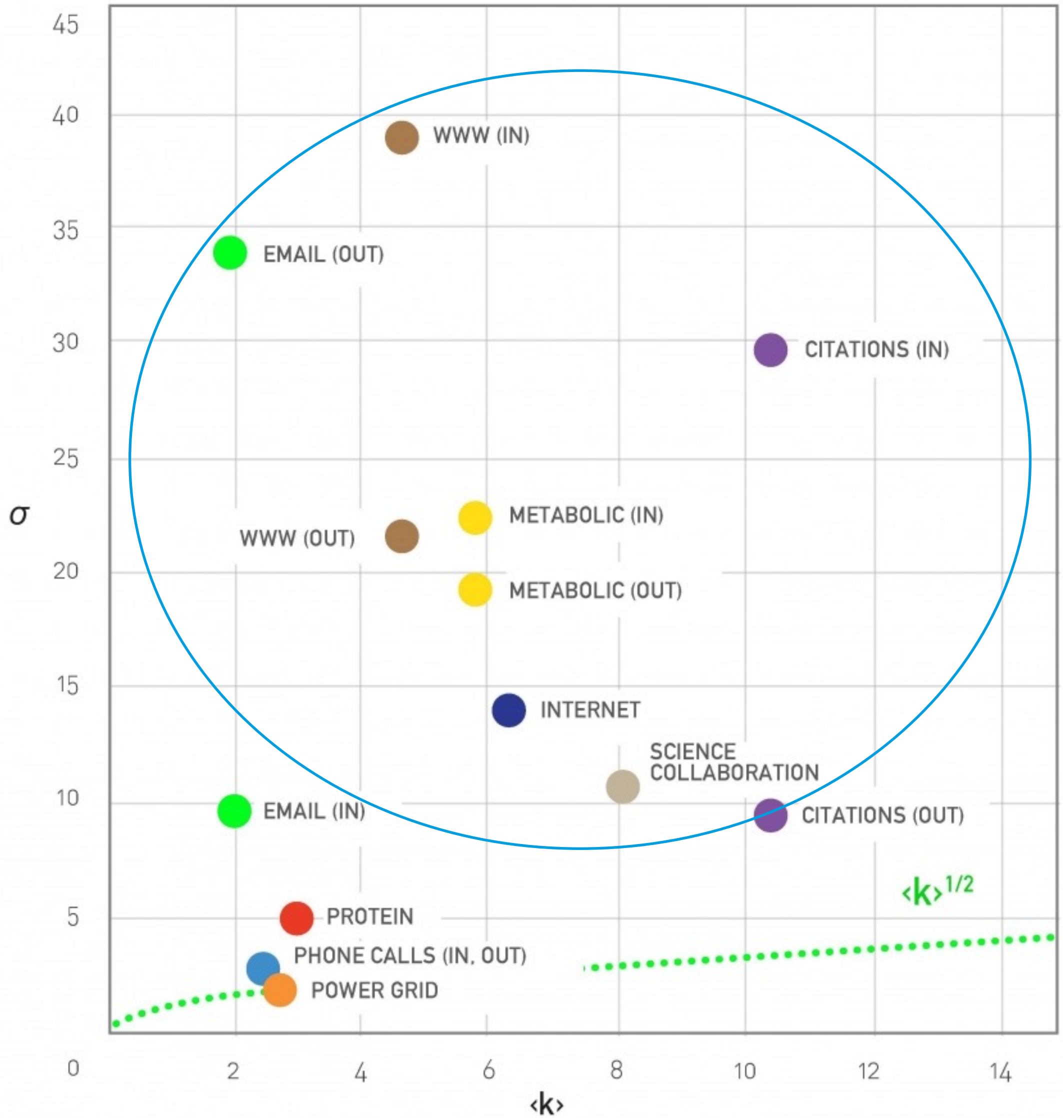
Scale-Free Network

Randomly chosen node: $k = \langle k \rangle \pm \infty$

Scale: none

Scale-free? Networks are finite, yet…

Standard deviation is very large in real networks



Do hubs affect the small world property?

Let's compute the average path length (APL) for a scale-free network...

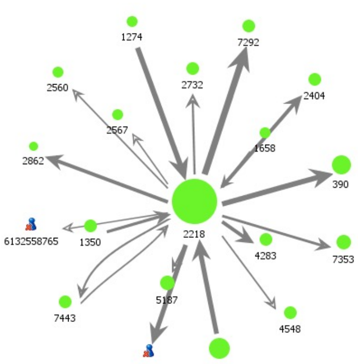
Do we live in a ultra-small-world?

$$APL = \langle L \rangle \approx \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Anomalous regime (γ≤2)

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

K_{max} grows faster than *N*



const.	$\gamma=2$
$\ln \ln N$	$2<\gamma<3$
$\dfrac{\ln N}{\ln \ln N}$	$\gamma=3$
$\ln N$	$\gamma>3$

Ultra-small world regime

$$APL = \langle L \rangle \approx \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

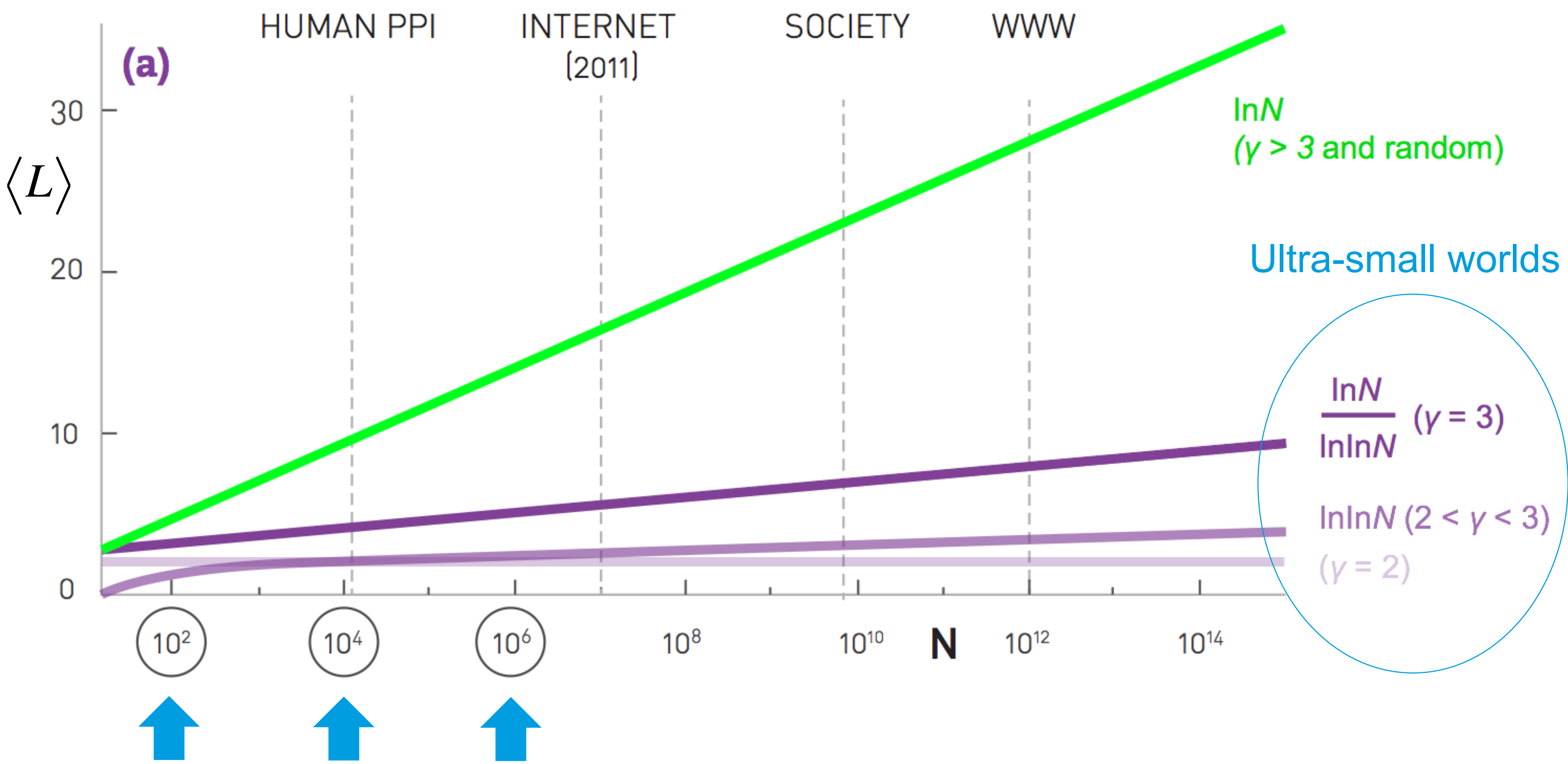
Critical regime

$$APL = \langle L \rangle \approx \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

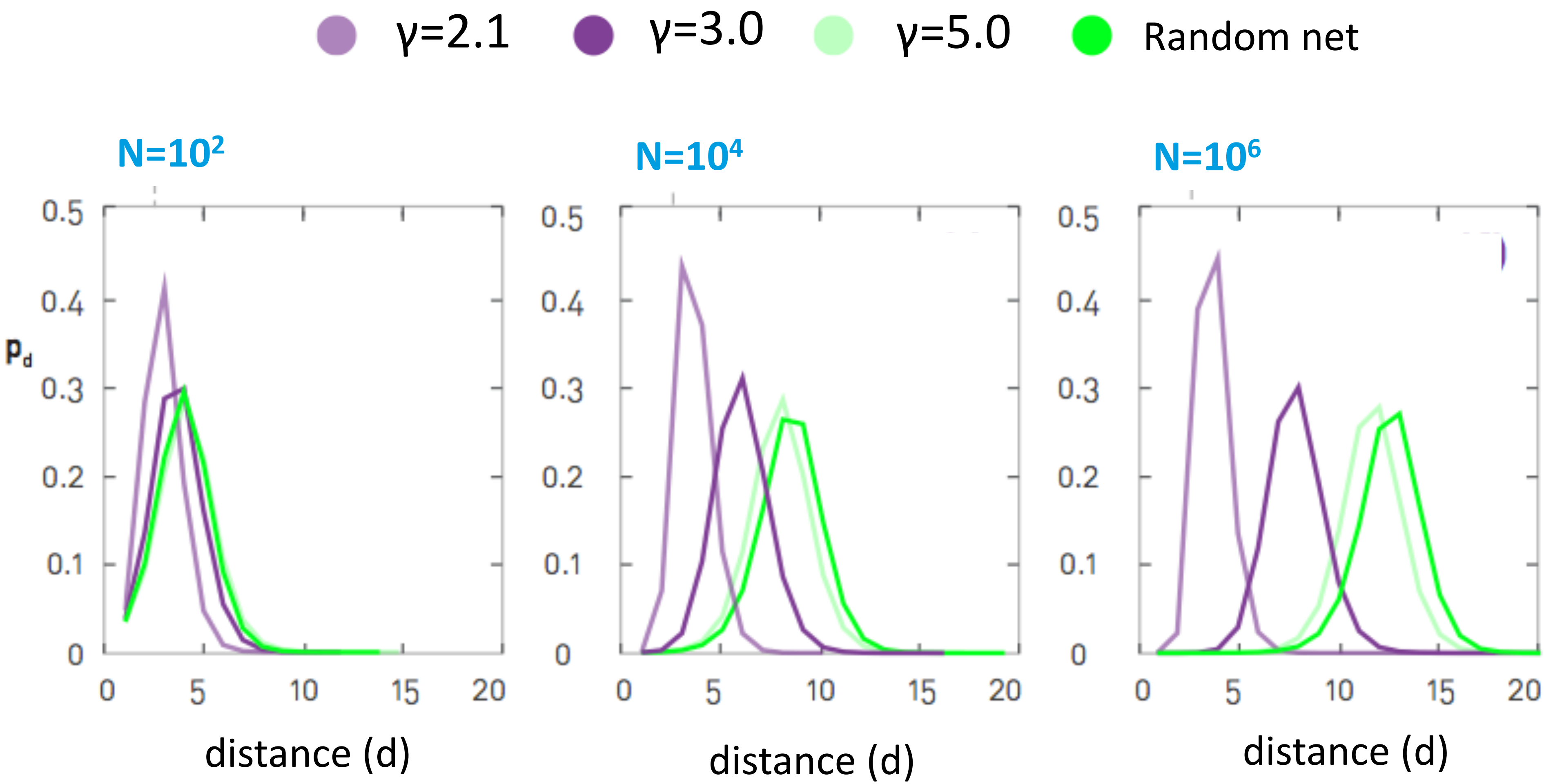
Small-world regime (same as random graphs)

$$APL = \langle L \rangle \approx \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

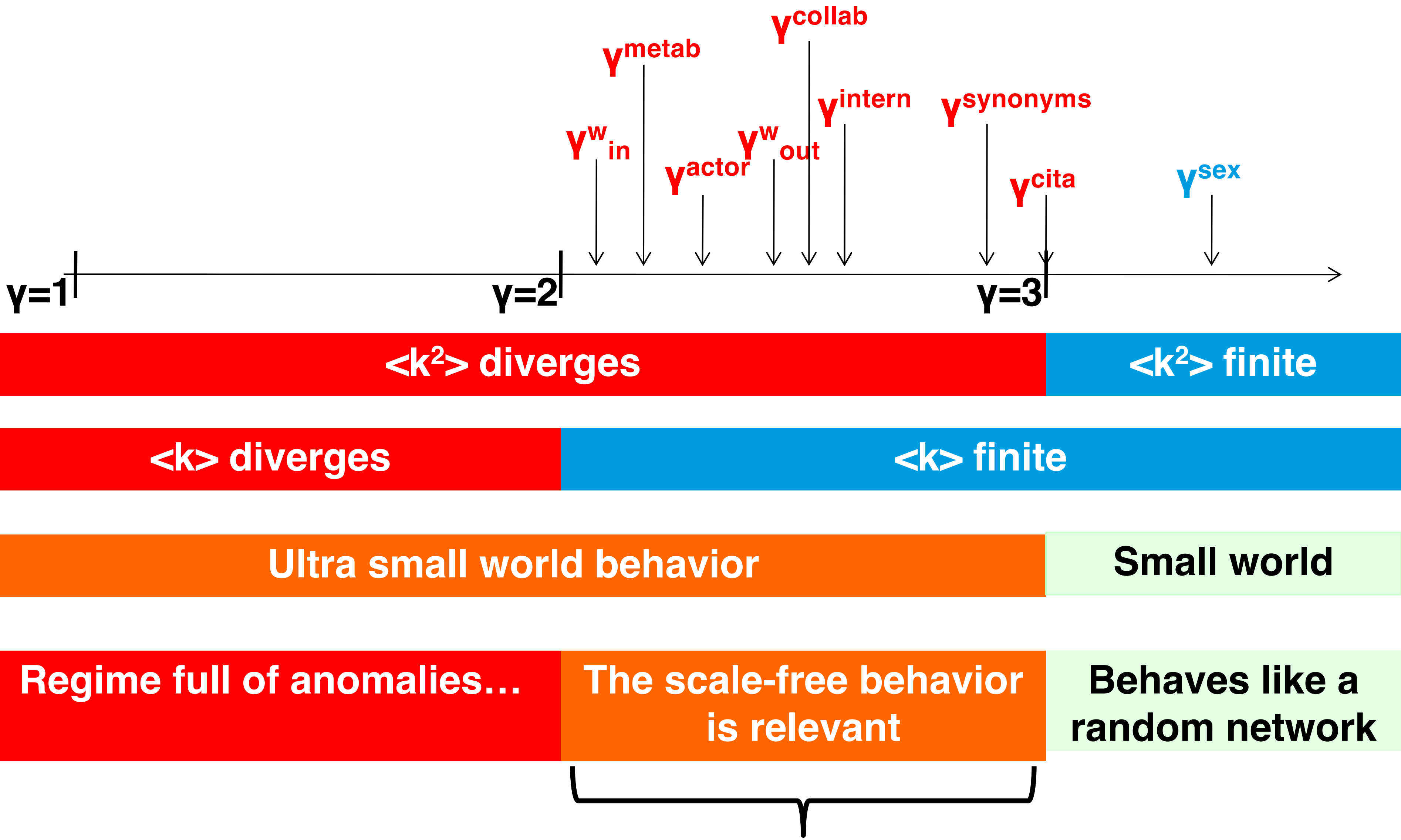
Do we live in a ultra-small-world?



Do we live in a ultra-small-world?



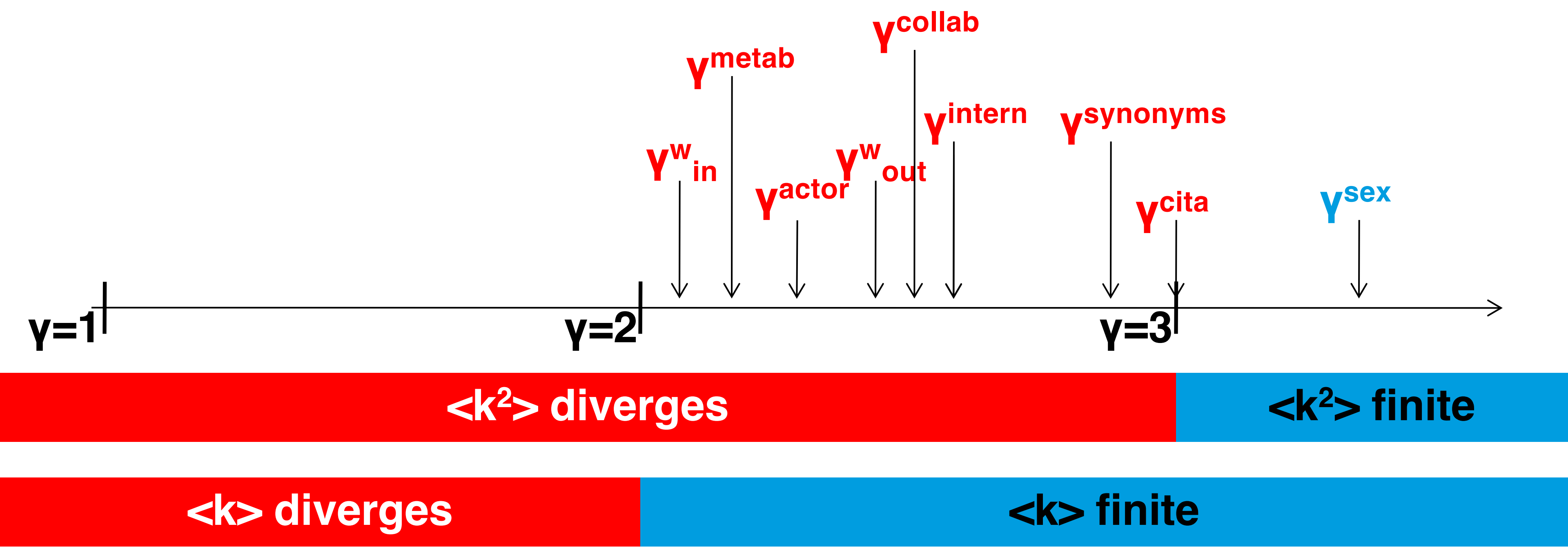
The universe of scale-free networks



Why are most exponents in this regime?

A world of magic exponents

Can you find a good argument which justifies this picture?

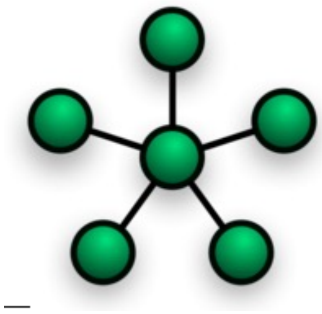


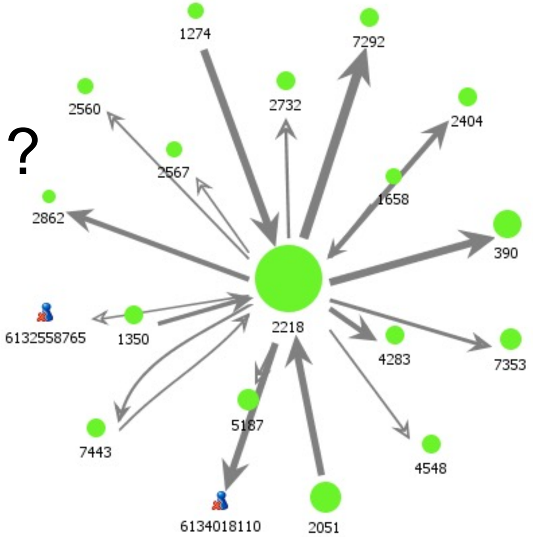
Magic exponents?

- Why is it hard to find networks with γ≤2 ?

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

K_{max} grows faster than *N*





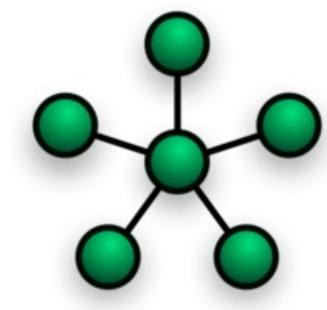
- Why is it hard to find networks with γ>>3 ?

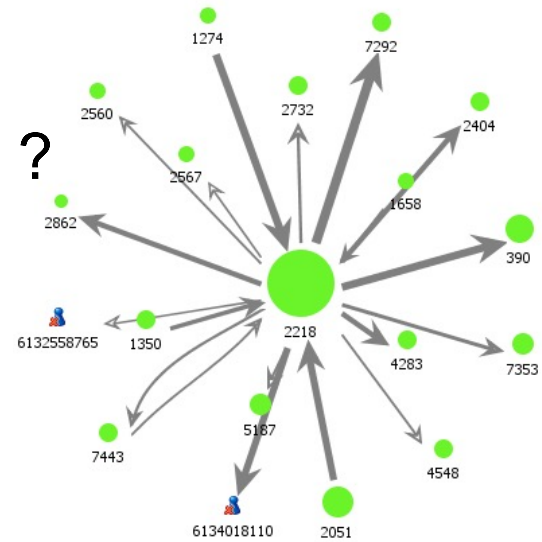
Magic exponents?

- Why is it hard to find networks with γ≤2 ?

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

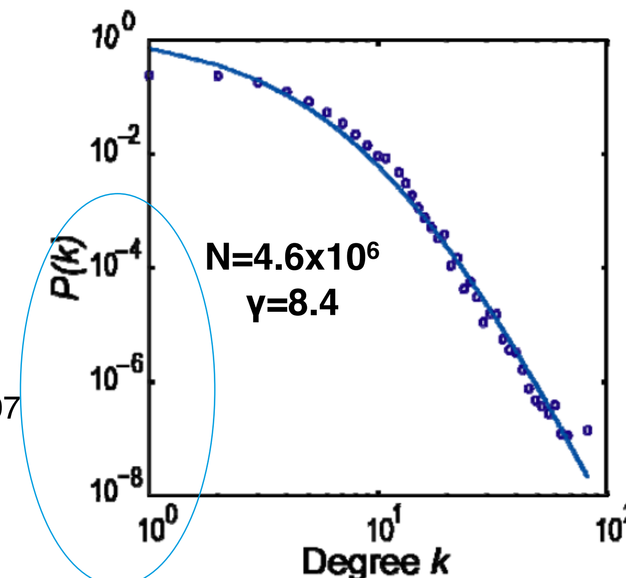
K_{max} grows faster than *N*





- **Why is it hard to find networks with γ>>3 ?**

Mobile Call Network



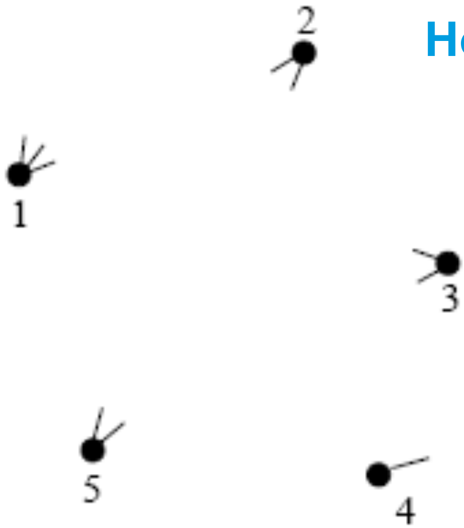
Onella et al. PNAS 2007

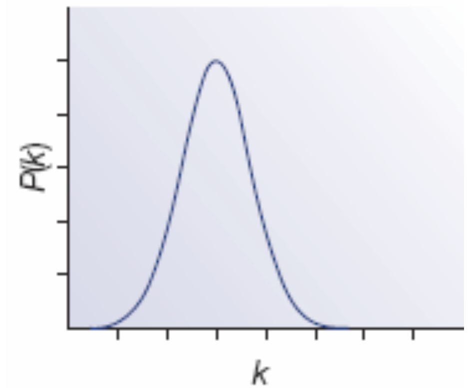
Configuration model

Challenge:
Can you imagine an algorithm capable of creating a random graph with an arbitrary degree distribution?

Configuration model

How to generate a graph compatible with a given $P(k)$



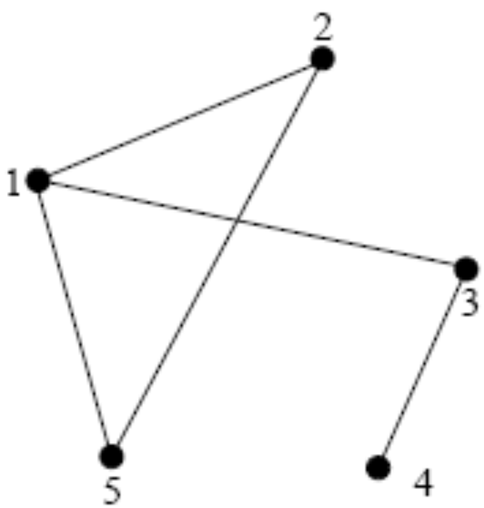


1. **Create a histogram of $P(k)$ (discretization) the sum of which gives N<k>**

2. **add stubs to n's according to histogram**

3. **connect the stubs at random.**

4. **this leads to a random graph with a pre-defined $P(k)$.**

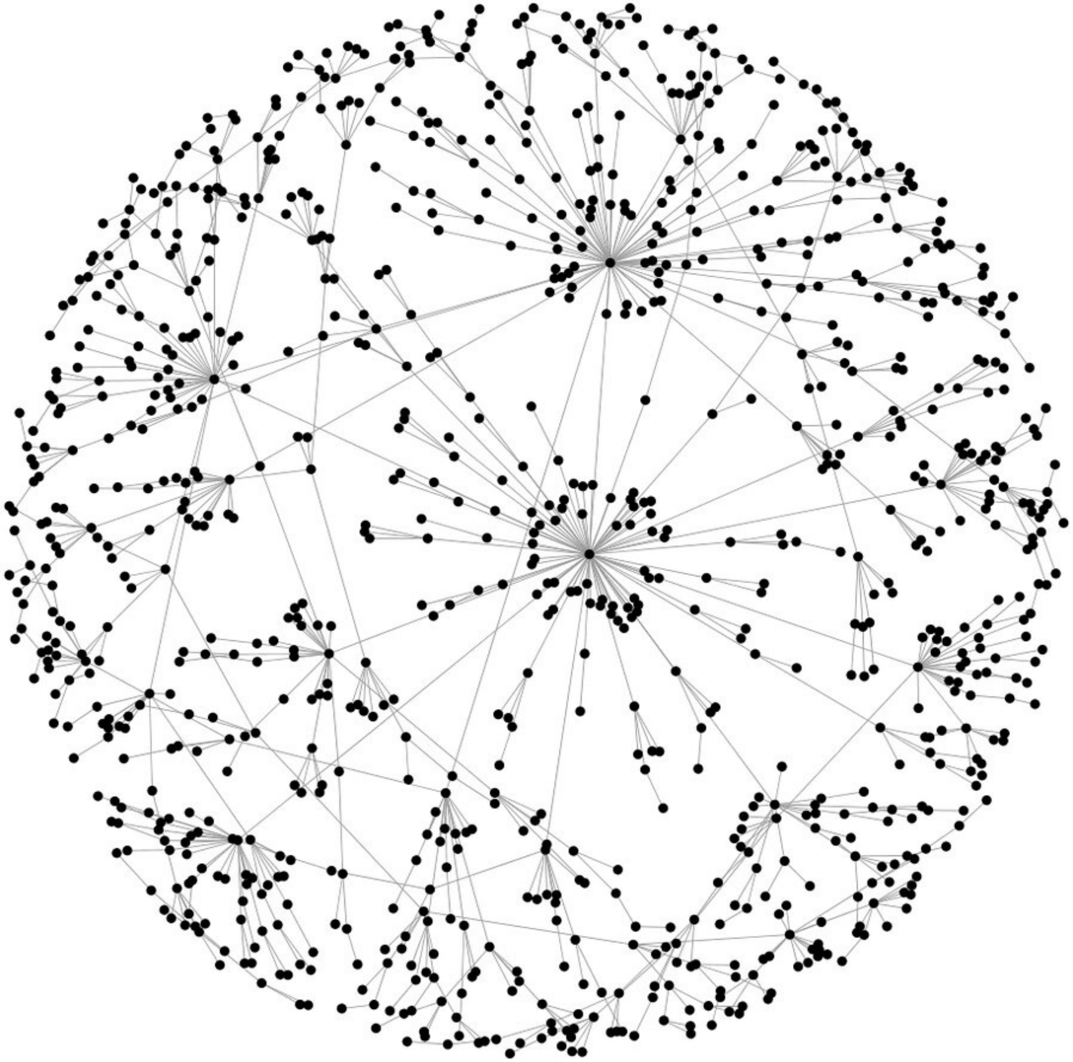


2^{nd} challenge of the day

Challenge:
Can you create an algorithm capable of increasing its clustering coefficient of a network without changing its degree distribution?

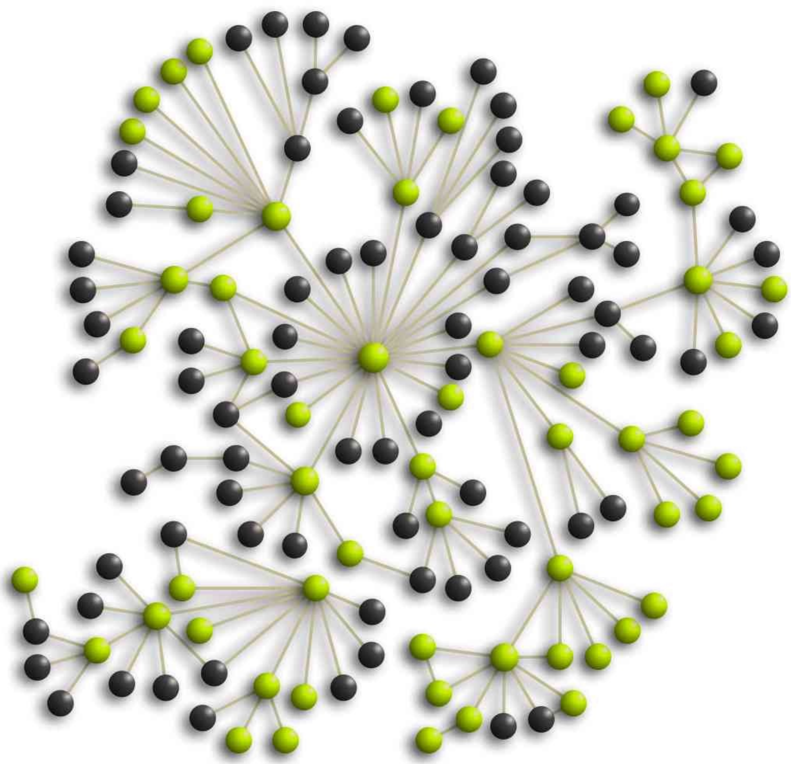
DEI
DEPARTAMENTO
DE ENGENHARIA INFORMÁTICA
TÉCNICO LISBOA

Models of Evolving Networks

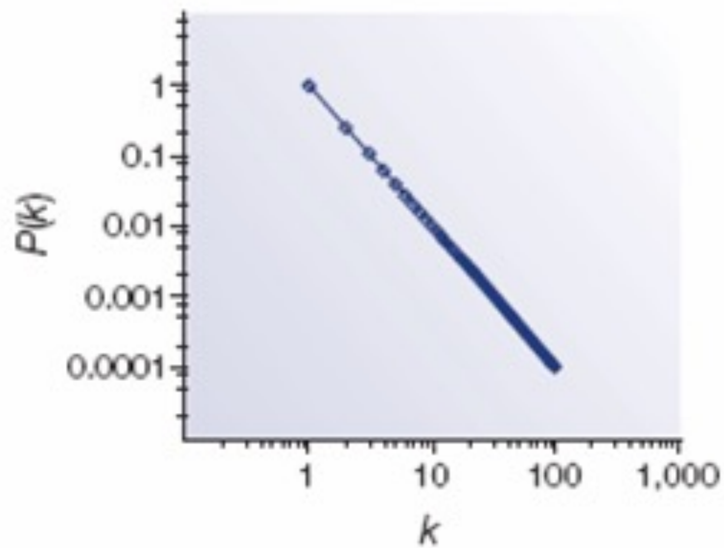


Network Science, 2023/24

Next challenge: Universality?



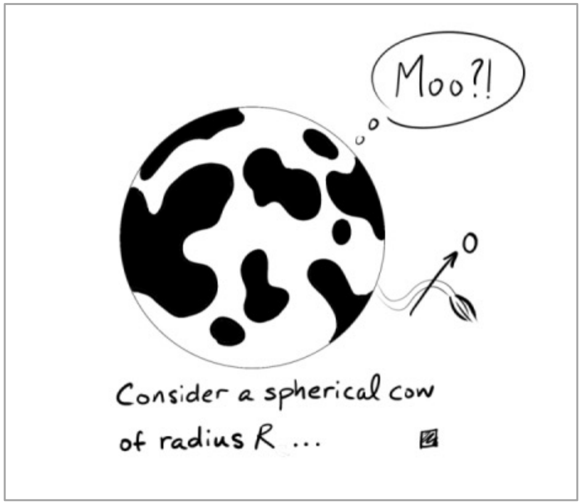
$$P_k \sim k^{-\gamma}$$



WWW	actors	citations	sex	cellular	phones	linguistics
$\gamma = 2.1$	$\gamma = 2.3$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 2.1$	$\gamma = 2.1$	$\gamma = 2.8$

Next challenge: Universality?

Challenge:
Can we identify the main principles leading to the emergence of scale-free networks?

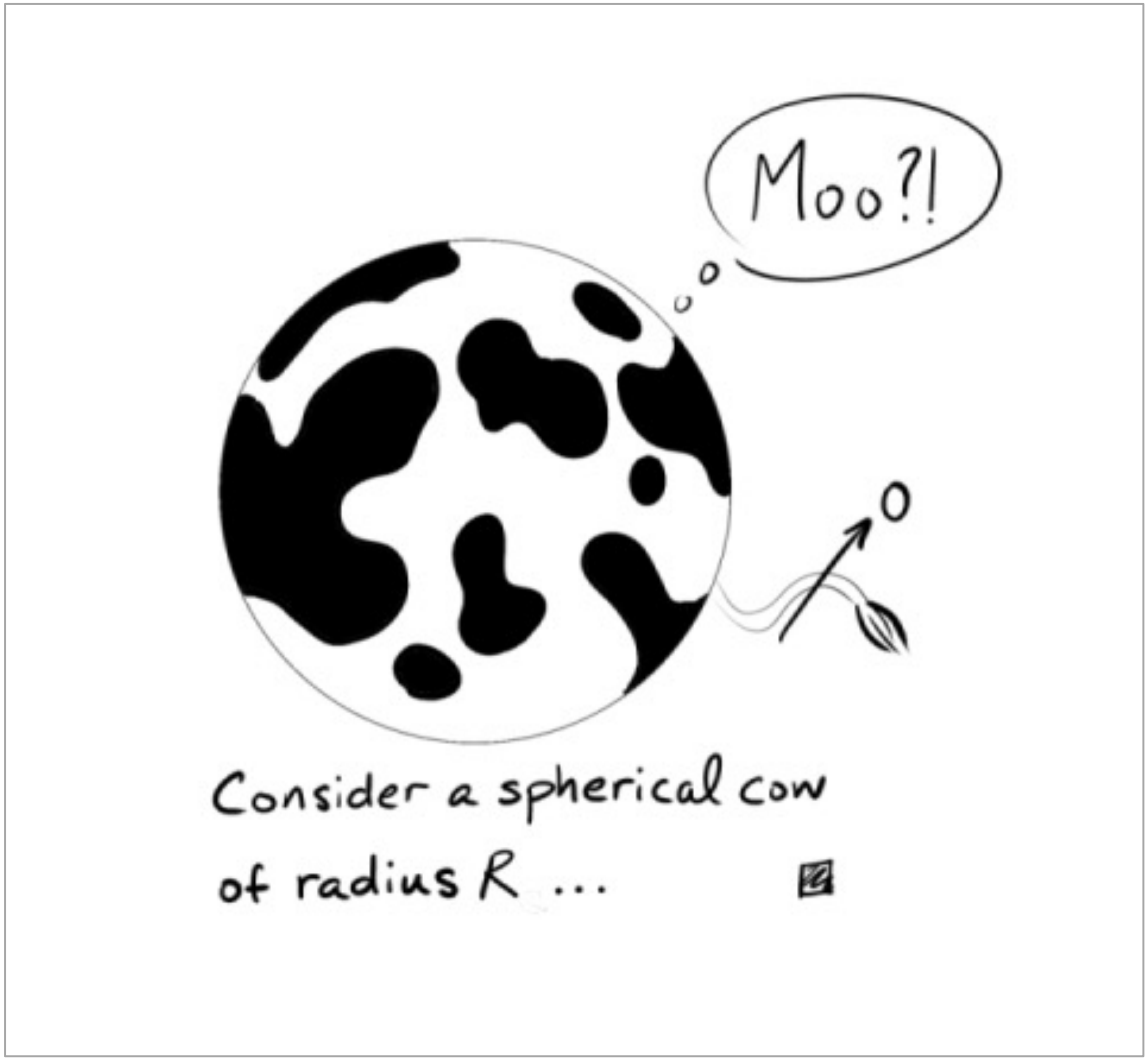


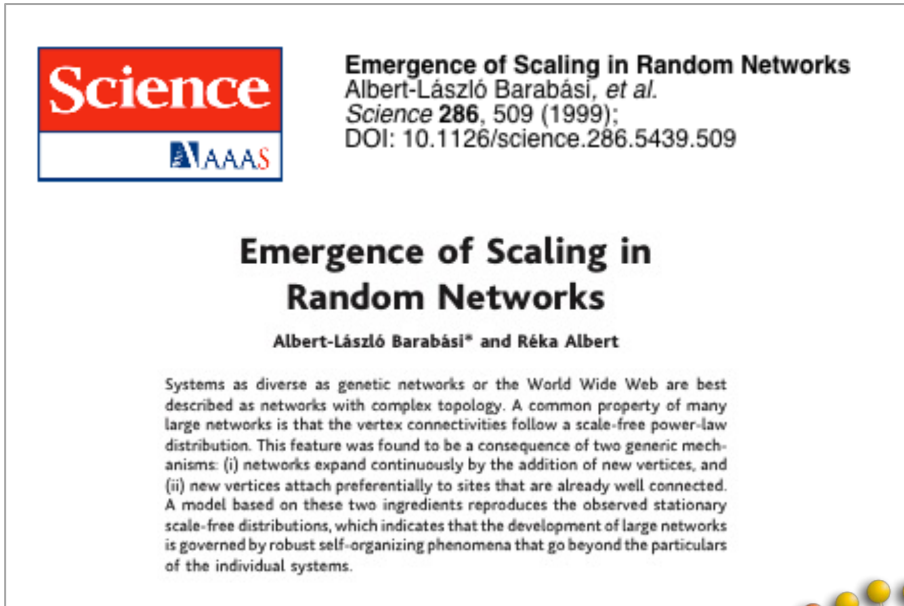
Moving forward: modeling complex systems

Self-organization: process where collective patterns arise out from local interactions between smaller component parts.

The whole is greater than the sum of its parts

The beauty of modeling core principles





Emergence of Scaling in Random Networks
Albert-László Barabási, *et al.*
Science **286**, 509 (1999);
DOI: 10.1126/science.286.5439.509

Emergence of Scaling in Random Networks

Albert-László Barabási* and Réka Albert

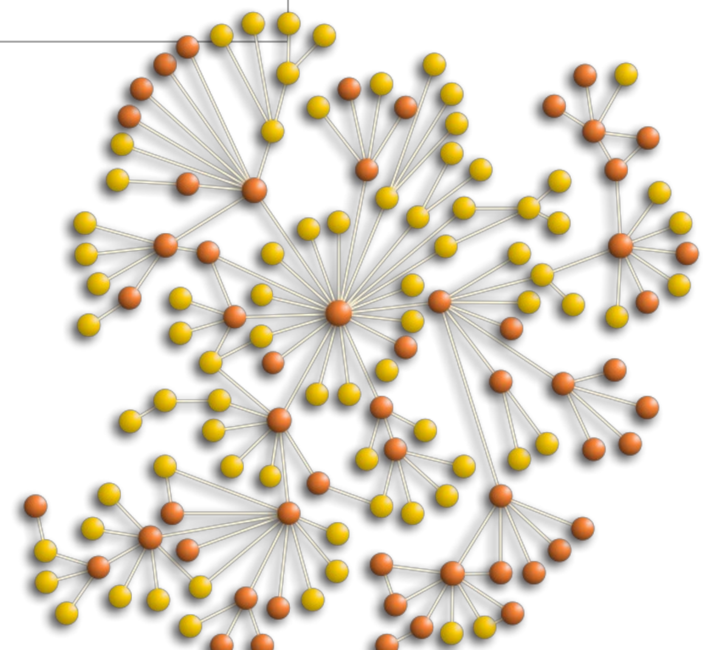
Systems as diverse as genetic networks or the World Wide Web are best described as networks with complex topology. A common property of many large networks is that the vertex connectivities follow a scale-free power-law distribution. This feature was found to be a consequence of two generic mechanisms: (i) networks expand continuously by the addition of new vertices, and (ii) new vertices attach preferentially to sites that are already well connected. A model based on these two ingredients reproduces the observed stationary scale-free distributions, which indicates that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of the individual systems.

Reka Albert

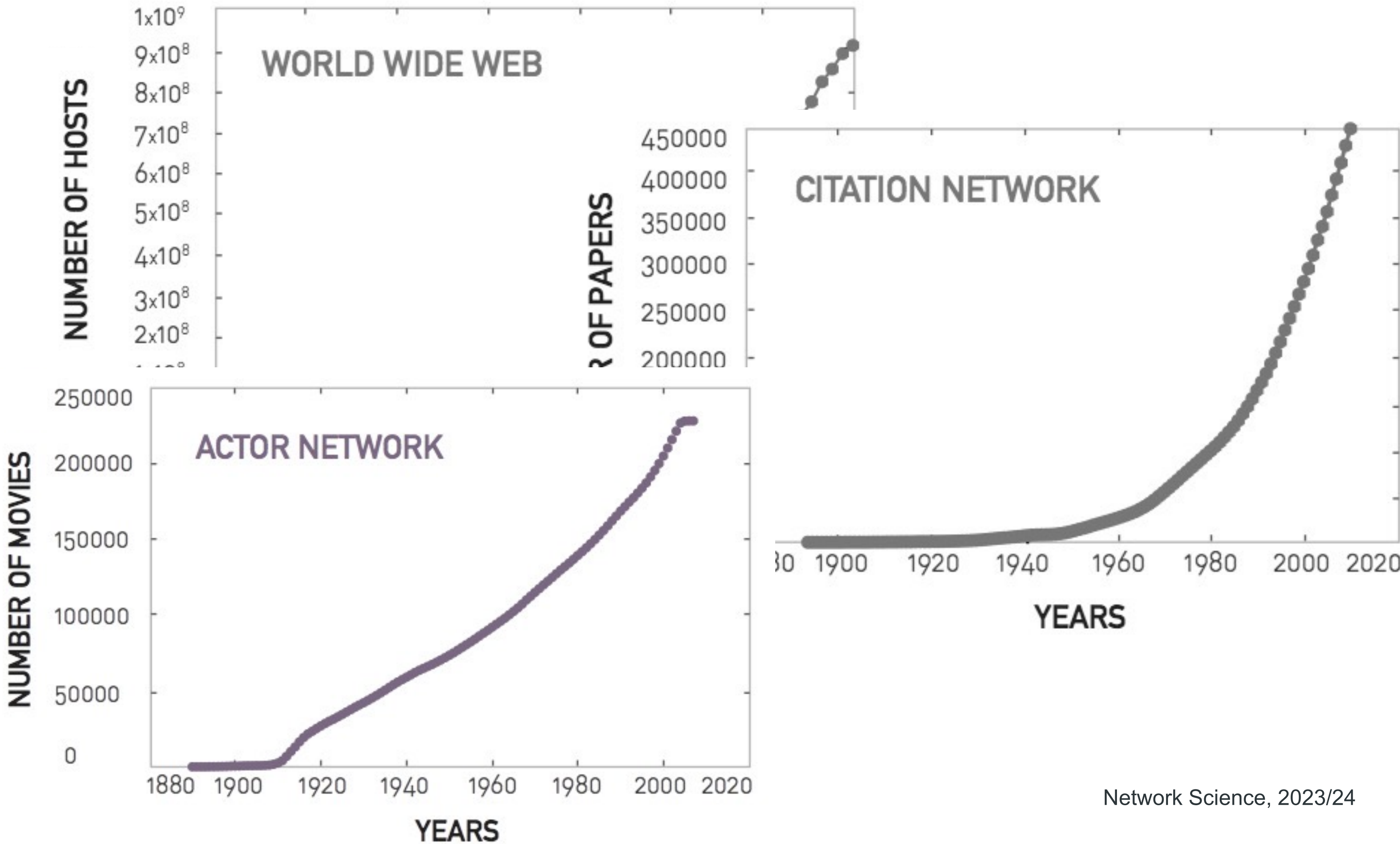
Penn State University

Albert-László Barabási

Northeastern University



First idea: Networks grow!



Second idea: popularity is attractive

- **WWW**: the nodes we know are not entirely random. Our knowledge is biased towards the most popular web documents. Thus, we are more likely to link (e.g., our webpage) to high degree nodes.

- **Citation network**: Our time is limited. We cannot read all papers published in the world! The more cited a paper is, the more likely that we hear about it, and read it. As we cite what we read, we tend to cite paper with high number of citations.

Second idea: popularity is attractive

- **Actor network**: the more movies an actor has played, the higher the chances that she/he will be considered for a new role…

- **This idea is often referred in the literature as Preferential Attachment.**

Related concepts

- ***Herbert Simon*** (Turing award ‘75) showed how ***preferential attachment*** can give rise to fat-tailed distributions describing ***city sizes and wealth distributions***.

- ***Proportional growth*** in business. Larger firms grow faster than smaller firms (R. Gibrat).

- ***Rich get richer effect*** in wealth distributions (Zipf law).

- Derek Prize’s “***Cumulative advantage***” principle, applied to citation statistics.

Related concepts

- ***Matthew effect*** (R. Merton) in sociology:

"For everyone who has will be given more, and he will have an abundance."

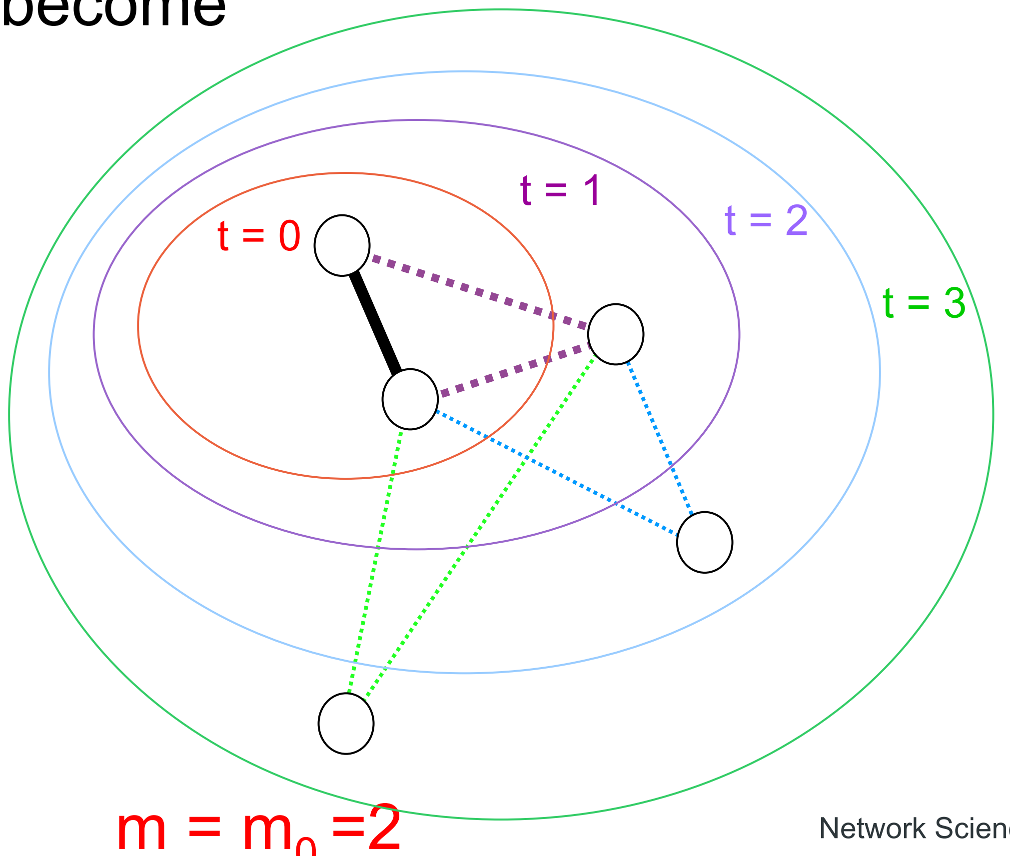
Matthew 25:29, King James Version.

Pois a quem tem mais se lhe dará, e terá
em abundância; mas ao que não tem, até
aquilo que tem ser-lhe-á tirado.
Mateus 25:29, Novo testamento

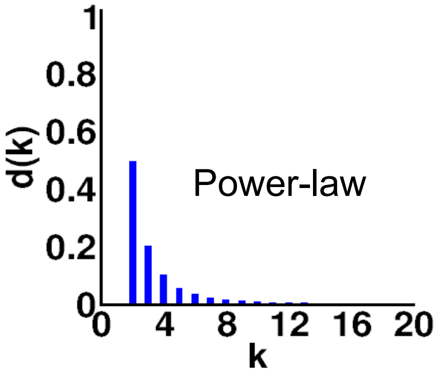
The Barabási-Albert model

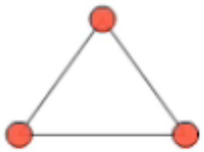
- ***Growth***: add nodes sequentially.
- ***Preferential attachment***: the more popular you are, the more popular you become

n	k
1	2
2	4
3	4
4	2
5	2



Growth + Preferential Attachment = Scale-free networks!





The Barabási-Albert model

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

The Barabási-Albert model

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

- ***Preferential attachment***:

when establishing the connections, each new node is connected to older nodes with a probability proportional to the degree of the older nodes.

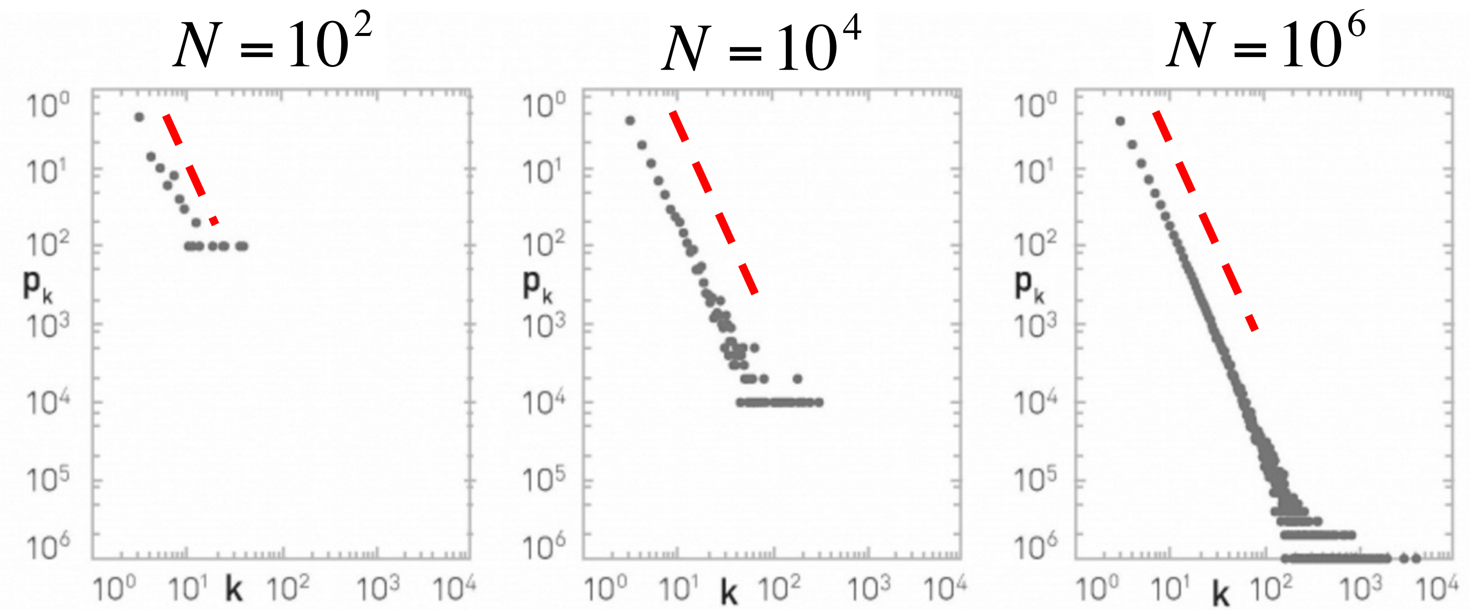
$$\Pi_i = \frac{k_i}{\sum_j k_j}$$

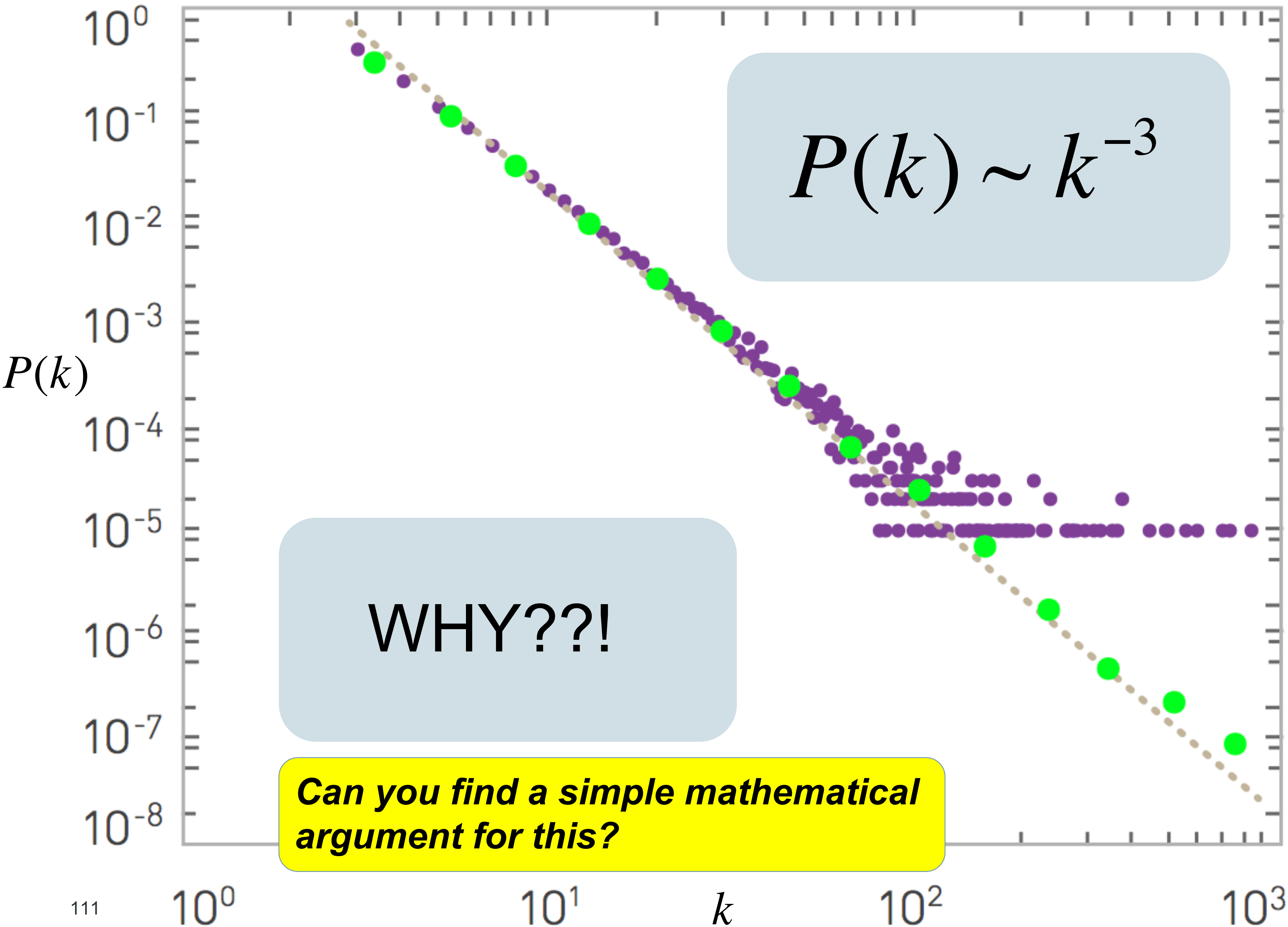
the more popular you are, the more popular you become

The Barabási-Albert model – Average degree?

- After t time-steps (t>>1) the graph has m_0+t nodes and *mt* edges; hence z =<k>≈ ??

<k>~2m





$$P(k) \sim k^{-3}$$

WHY??!

Can you find a simple mathematical argument for this?

BA-model: Degree distribution

- For an exact analysis of the structure of growing networks please check the works by

Sergey Dorogovtsev
Dep. Physics, Univ. Aveiro

José Fernando Mendes
Dep. Physics, Univ. Aveiro

Ex: Dorogovtsev et al. (2000). Structure of growing networks with preferential linking. Phys. Rev. Lett. , 85(21), 4633. (*note: this would be a neat analytical project*)

BA-model: Time evolution of degrees

Back-of-the-envelope argument (Barabasi & Albert, Science 1999)

- The rate at which an existing node *i* acquires links as a result of new nodes (with *m* links) connecting to it is

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

k_i = degree of node ***i***

BA-model: Time evolution of degrees

Back-of-the-envelope argument (Barabasi & Albert, Science 1999)

- The rate at which an existing node *i* acquires links as a result of new nodes (with *m* links) connecting to it is

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

<k> × N = 2m × *t*

m = number of links of each new node.

In other words, each existing node *i* has *m* chances to be chosen

BA-model: Time evolution of degrees

- Since $$\sum_{j=1}^{N-1} k_j \approx 2mt$$

i joins the network at time t_i

- We have $$\frac{dk_i}{dt} = m\frac{k_i}{2mt} = \frac{k_i}{2t}$$

$$\left(k_i(t_i) = m\right)$$

- We get $$\boxed{k_i(t) = Const.\left(t\right)^{1/2}}$$

$$\boxed{Const = \frac{m}{t_i^{1/2}}}$$

BA-model: Time evolution of degrees

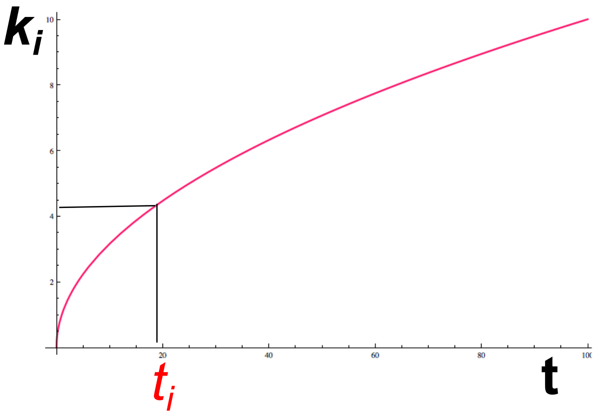
- Since $$\sum_{j=1}^{N-1} k_j \approx 2mt$$

i joins the network at time t_i

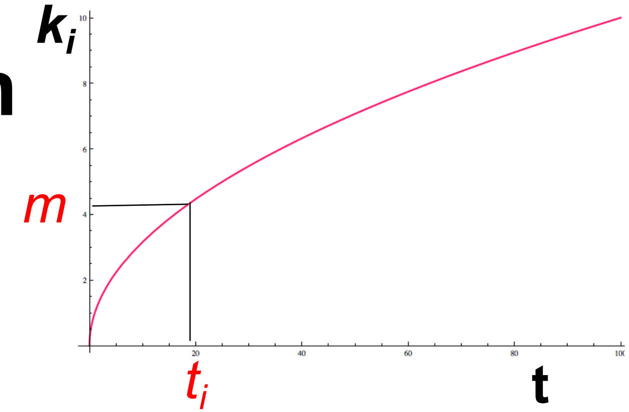
- We have $$\frac{dk_i}{dt} = m\frac{k_i}{2mt} = \frac{k_i}{2t}$$

$$\left(k_i(t_i) = m\right)$$

- We get $$\boxed{k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}}$$



BA-model: Degree distribution



$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}$$

I can also invert this eq. to say that, <u>if the degree of *i* is smaller than *k*, then *i* appeared at the network after time t_i</u>

$$k_i < k \Rightarrow t_i > t\left(\frac{m}{k}\right)^2$$

Since we add a node at each time-step ($N = t$),
the ***number of nodes appearing after t_i*** is given by $N - t_i$

$\Updownarrow$

number of nodes with degree < k = $N - t\left(\frac{m}{k}\right)^2$

BA-model: Degree distribution

- (normalizing by N) the ***probability*** *$P(k_i<k)$* of having a node with degree lower than k is

$$P(k_i < k) = \frac{N - t\left(\frac{m}{k}\right)^2}{N} \approx 1 - \left(\frac{m}{k}\right)^2$$

For large N

$$\left(N = m_0 + t \approx t\right)$$

i.e., we get the ***C***umulative ***D***istribution ***F***unction (CDF):

$$CDF(k) = P(k_i < k) = \int_0^k P(k')\,dk' = 1 - \left(\frac{m}{k}\right)^2$$

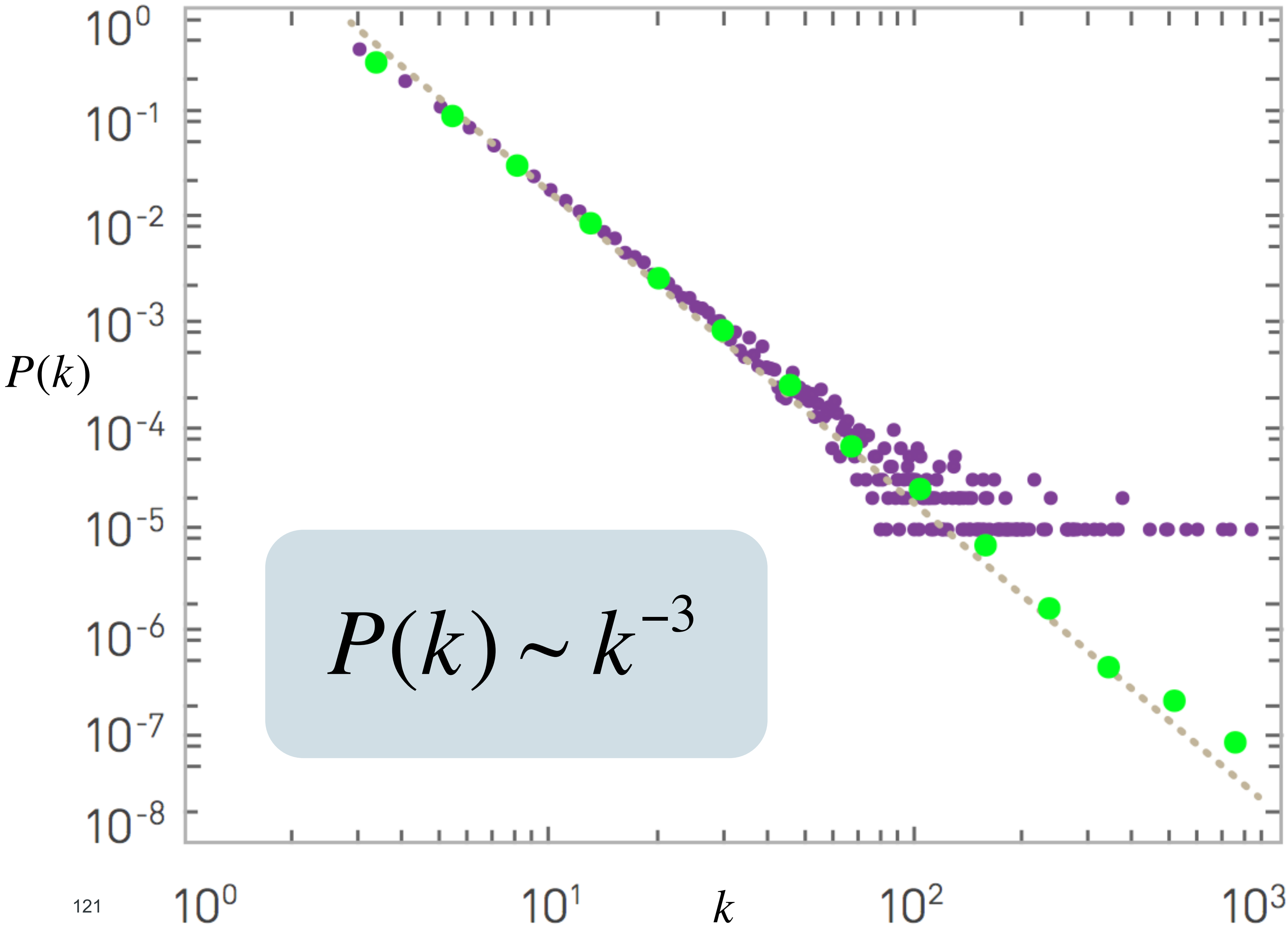
BA-model: Degree distribution

- …if we take its derivative, we get the degree distribution

$$P(k) = \frac{\partial CDF(k)}{\partial k} = 2m^2 \left(\frac{1}{k}\right)^3$$

Et voilà ☺!

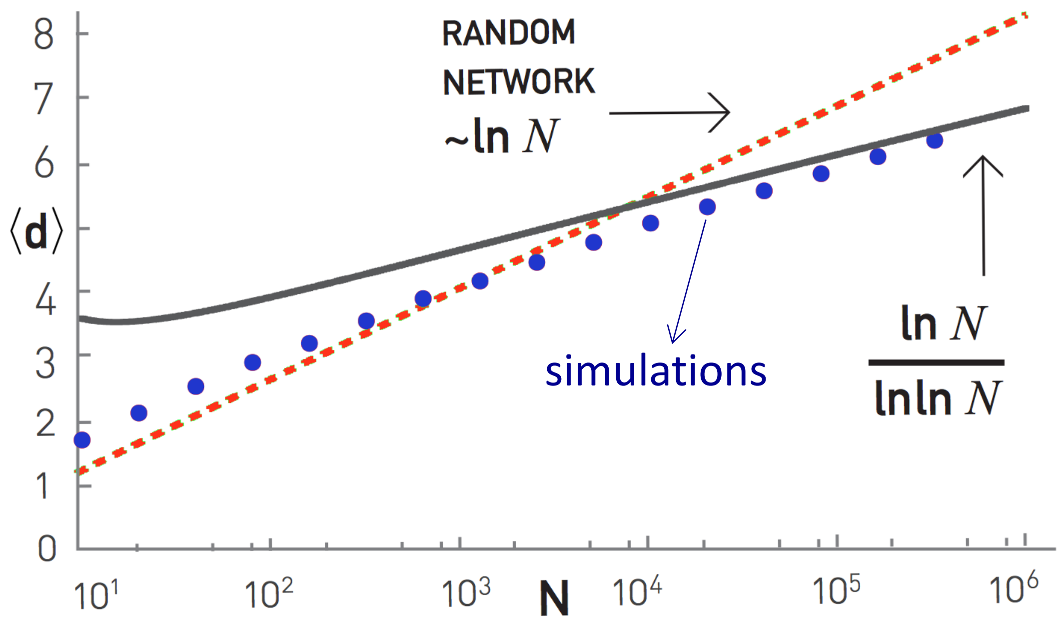
$$P(k) \sim k^{-3}$$


$P(k)$
k
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^0
10^1
10^2
10^3
$P(k) \sim k^{-3}$

APL of the Barabási-Albert model?

- Average-Path-Length (γ=3):

$$\langle L \rangle \sim \frac{\ln N}{\ln\ln N}$$

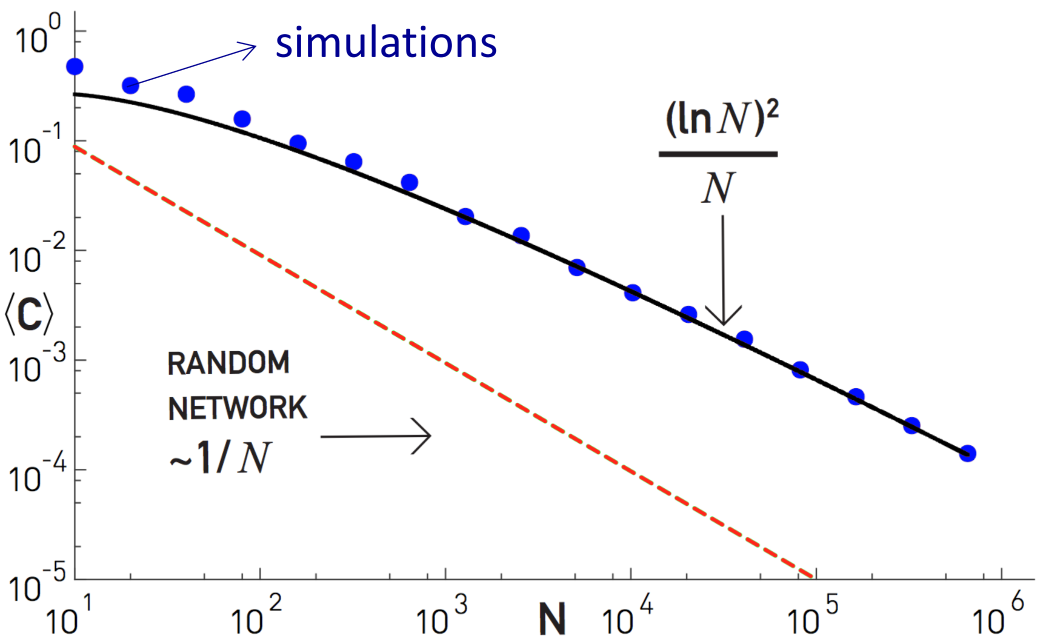


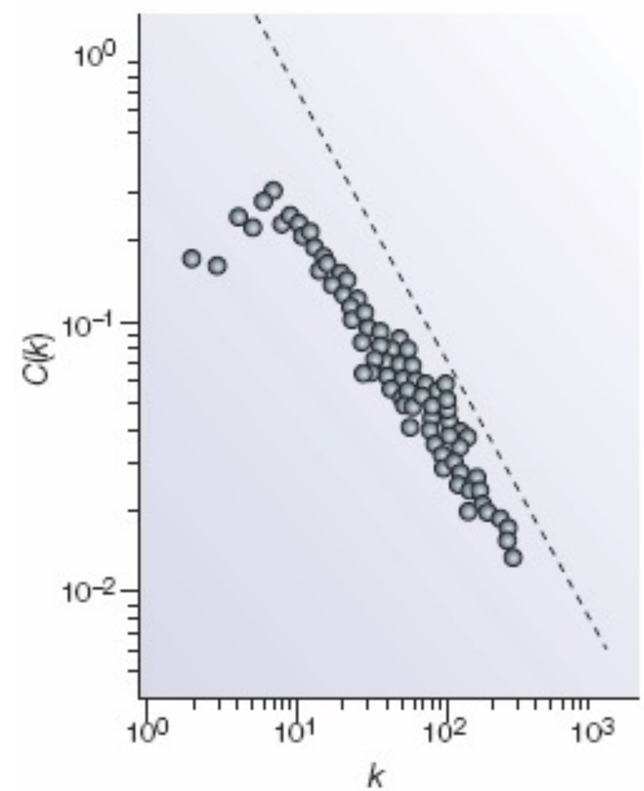
Clustering of the Barabási-Albert model

Real nets

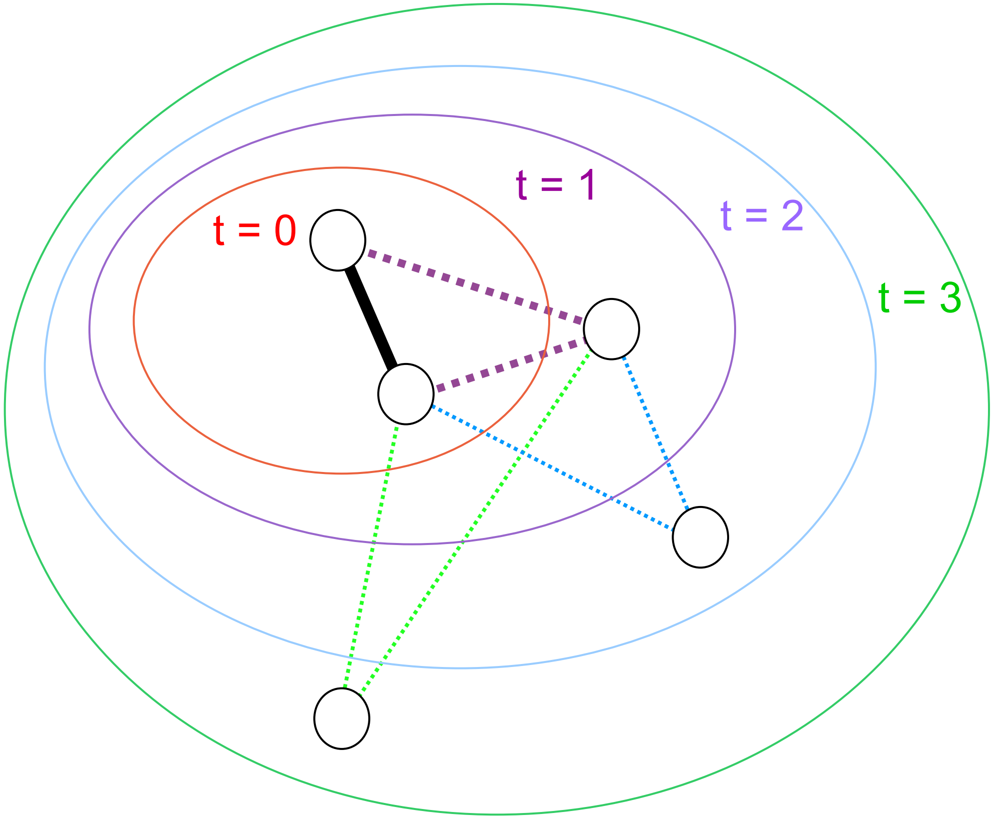
$$C_i \sim \frac{(ln\ N)^2}{N}$$

1. For a fixed <k>, larger the network, smaller the clustering coefficient. ✗
2. Clustering coefficient is independent of the node's degree. ✗





Other features: Age correlations



We are creating a league of gentleman !

Try it!!! How would you implement a preferential attachment model?

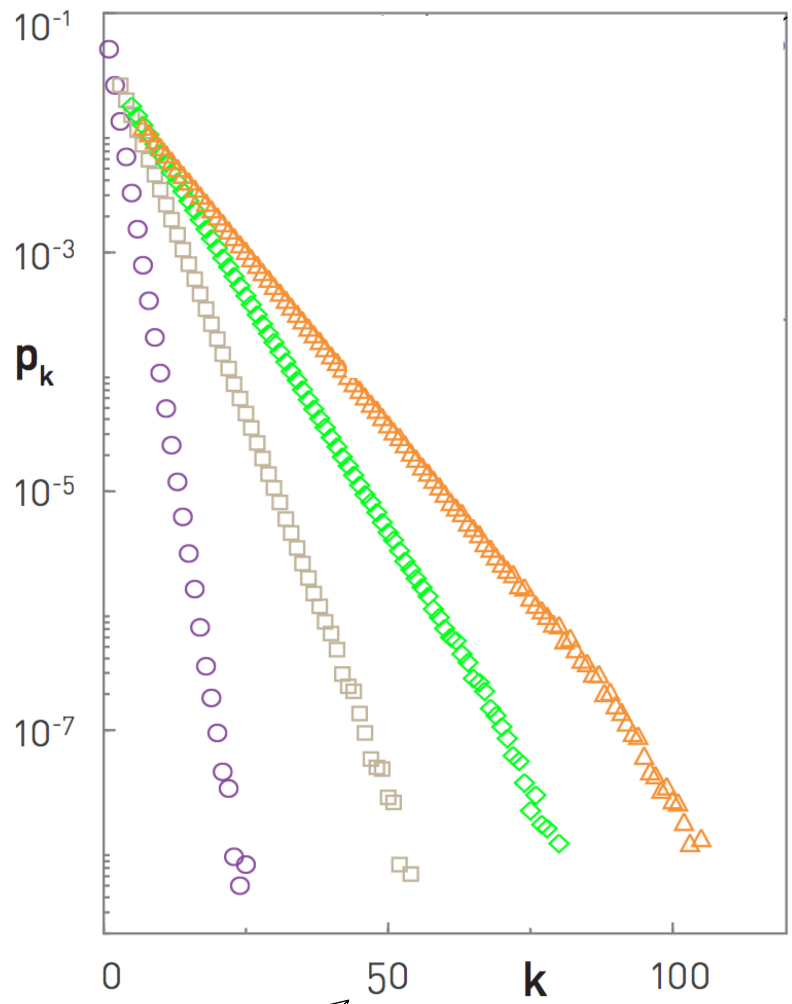
Do we need both growth and Preferential attachment?

- Model A (growth only)

$$P(k) \sim e^{-k/m}$$

Can you explain why ? ☺

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{1}{N(t)-1} \sim m\frac{1}{t}$$

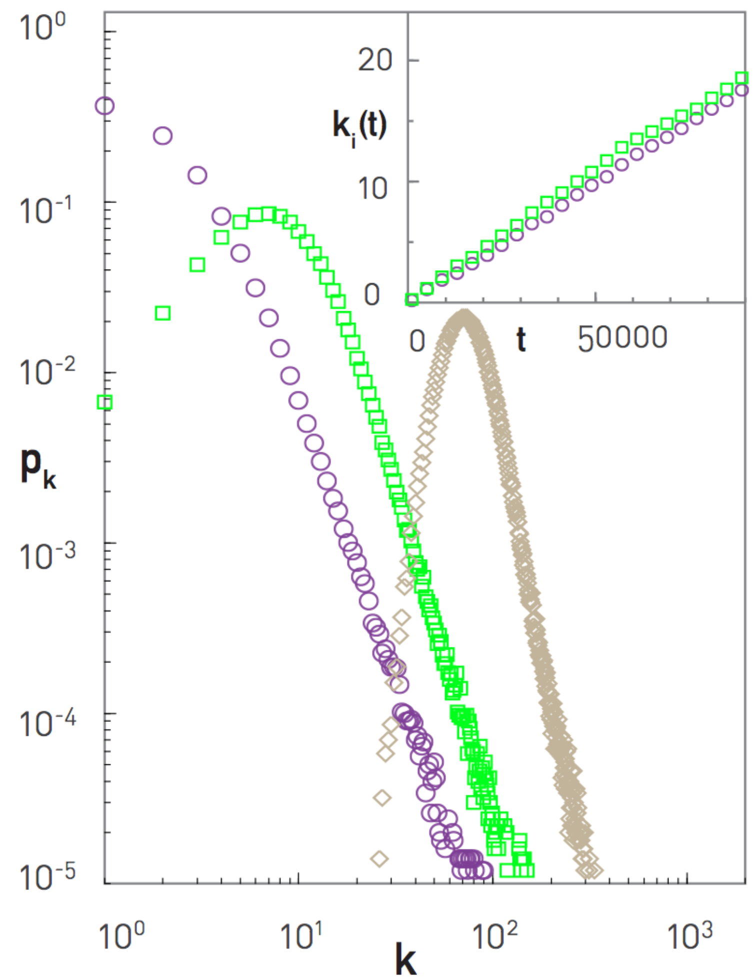


linear scale

Do we need both growth and Preferential attachment?

- Model B
  (only preferential attachment)

- Begin with a fixed number of disconnected nodes and add links, preferentially choosing high degree nodes as link destinations.
- Though the degree distribution early in the simulation looks scale-free, the distribution is not stable, and it eventually becomes nearly Gaussian as the network nears saturation.
- Thus, preferential attachment alone is not sufficient to produce a scale-free structure.



Can we be sure of having a preferential attachment mechanism?

- Changes in degree should follow

$$\Delta k_i = k_i(t+\Delta t) - k_i(t)$$

i.e.,

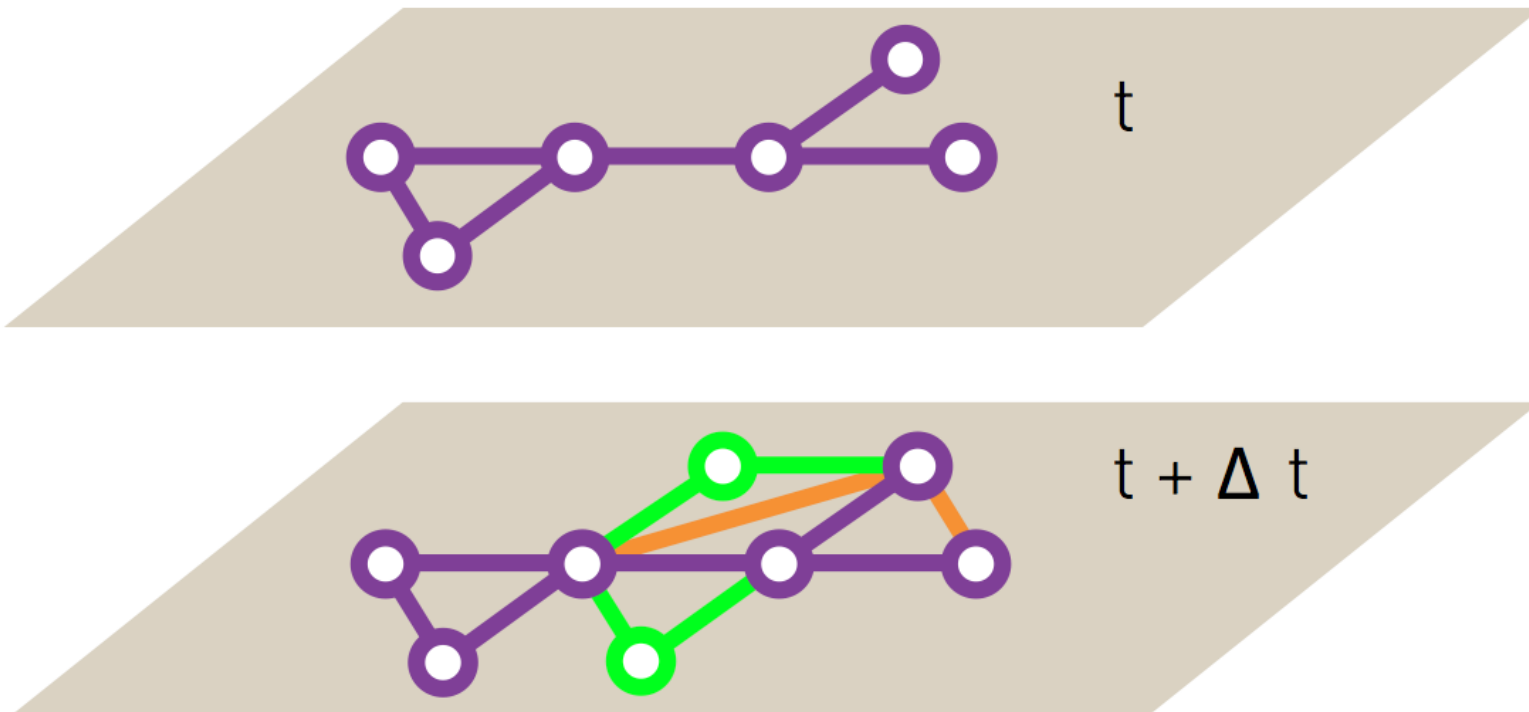
$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i}{\sum_j k_j}$$

The BA model assumes a linear form of preferential attachment

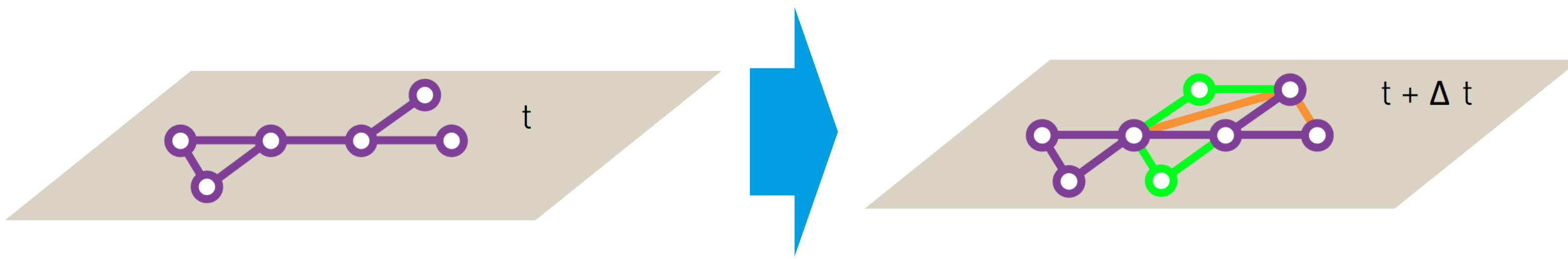
We can average this variation for empirical data and compare it with different scenarios

$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

Can we be sure of having a preferential attachment mechanism?

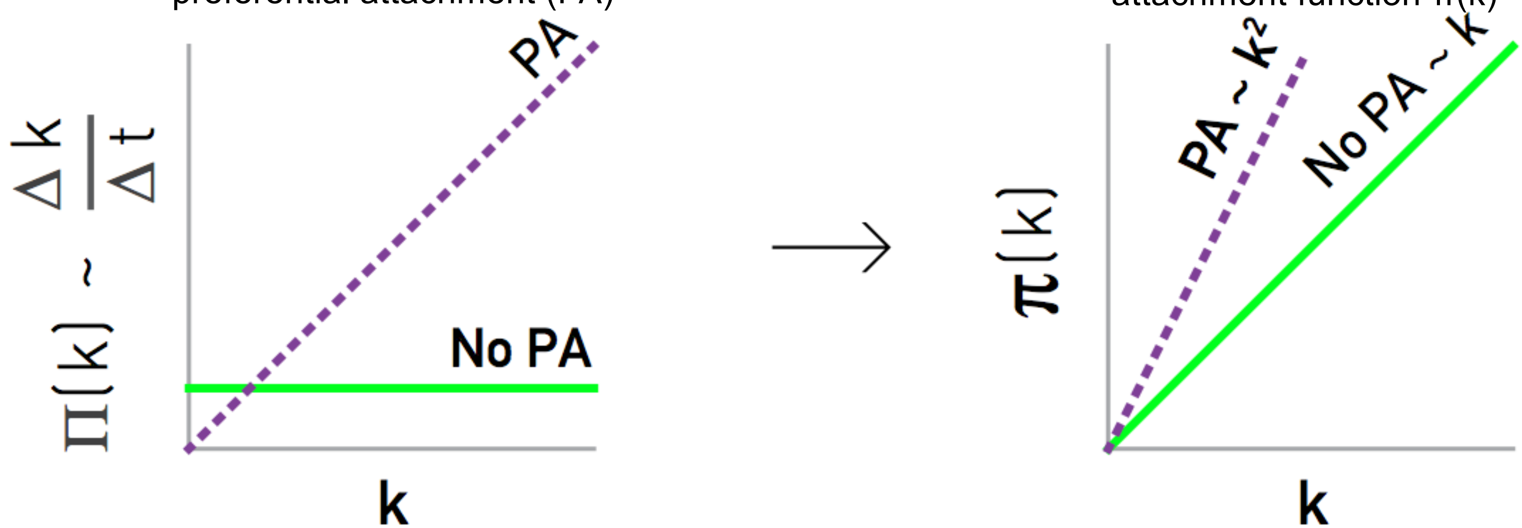


Can we be sure of having a preferential attachment mechanism?



preferential attachment (PA)

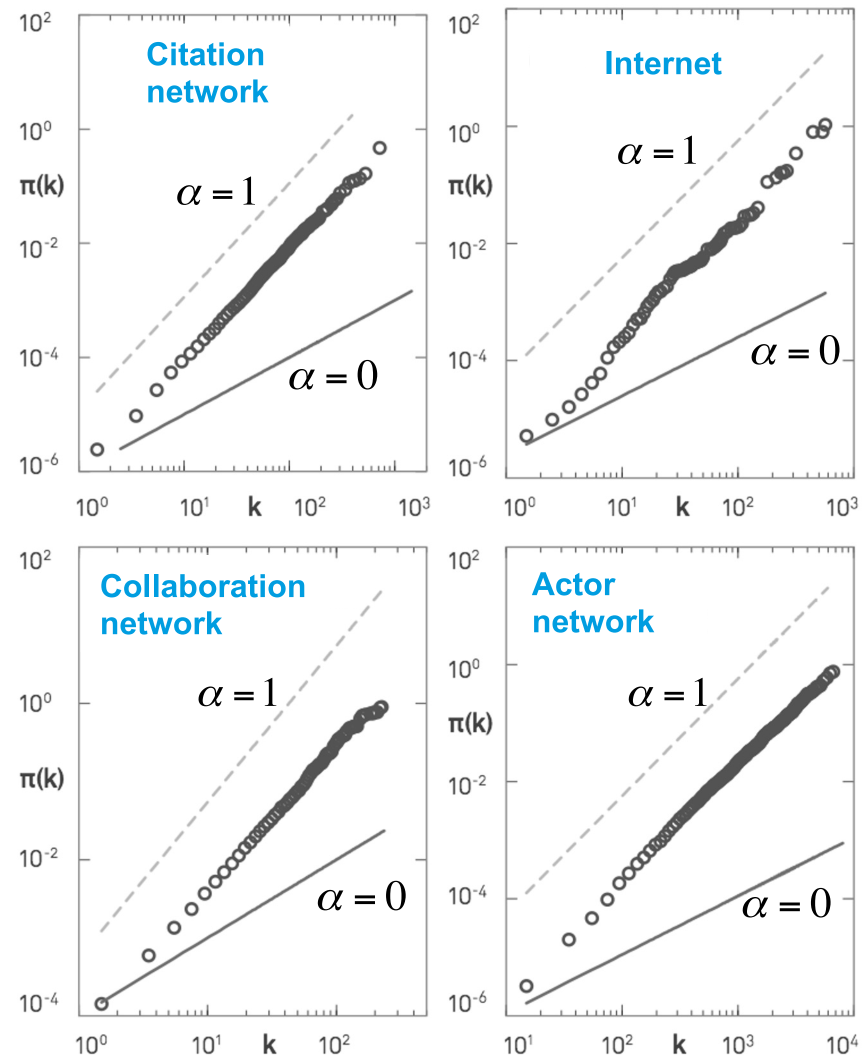
cumulative preferential attachment function π(k)



Log-log scales

Can we be sure of having a preferential attachment mechanism?

cumulative preferential attachment function π(k)



- Preferential attachment is present
- Yet, we may have a non-linear preferential attachment!!

$$\frac{\Delta k_i}{\Delta t} \sim \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

Non-linear preferential attachment:
Does it change anything?

- ***Growth***: add nodes sequentially.

At t=0 consider (e.g.) a ring with m_0 nodes; for each time-step, add 1 node and connect it via m new edges with existing nodes;

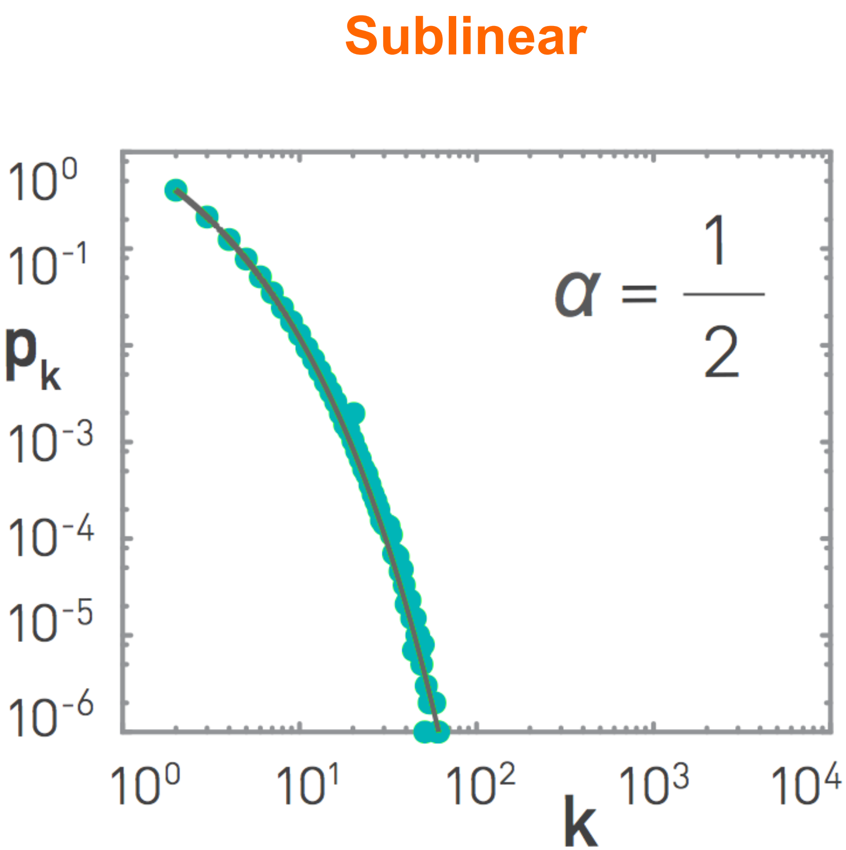
- ***Preferential attachment***:

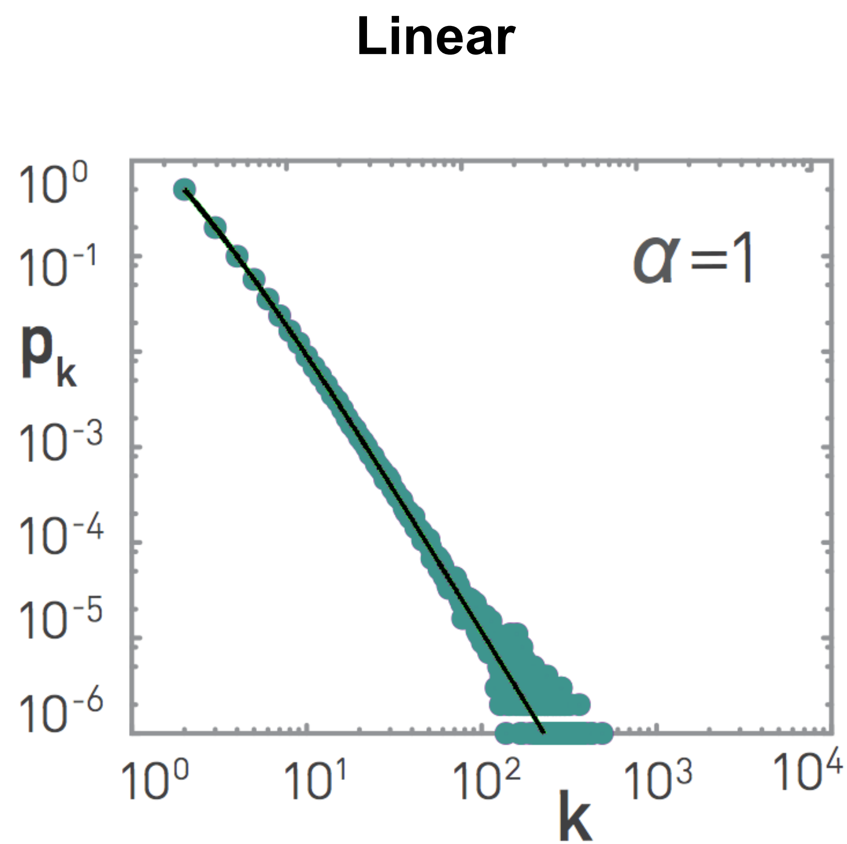
when establishing the connections, each new node is connected to older nodes with a probability proportional to ***degree***$^{\alpha}$ of the older nodes.

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

the more popular you are, the more popular you become, yet with different strengths

Degree distributions (sublinear regime)





Stretched Exponential

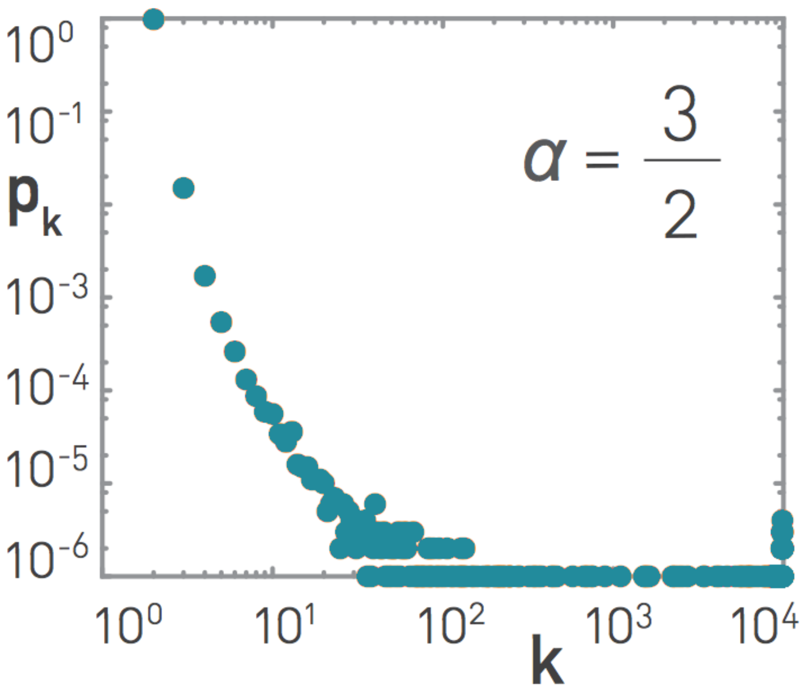
$$k_{\max} \sim (\ln t)^{1/(1-\alpha)}$$

Power law

$$k_{\max} \sim t^{1/2}$$

Degree distributions (superlinear regime)

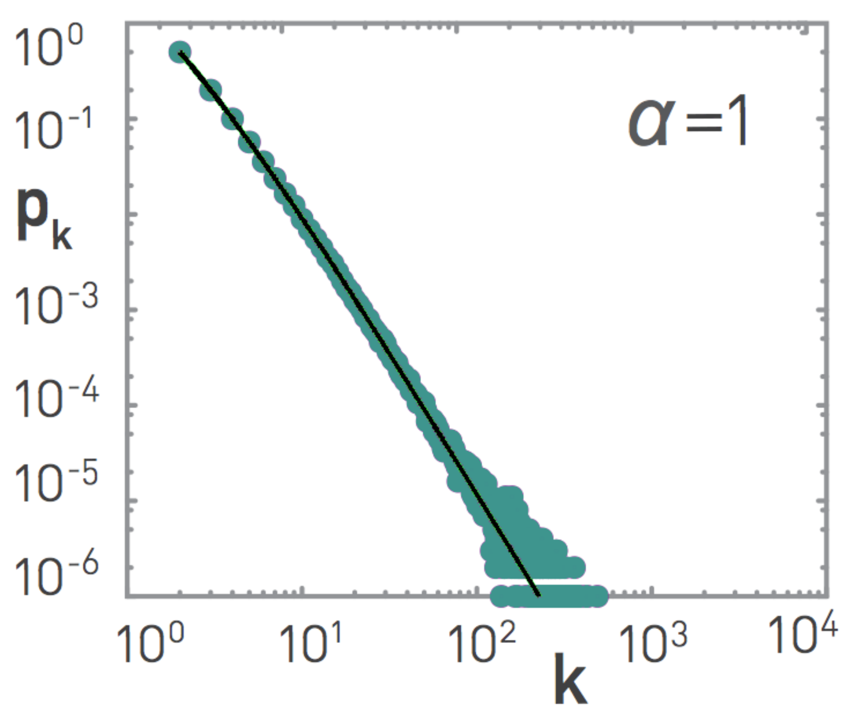
Superlinear



Winners take it all!

$$k_{\max} \sim t$$

Linear



Power law

$$k_{\max} \sim t^{1/2}$$

Non-linear preferential attachment

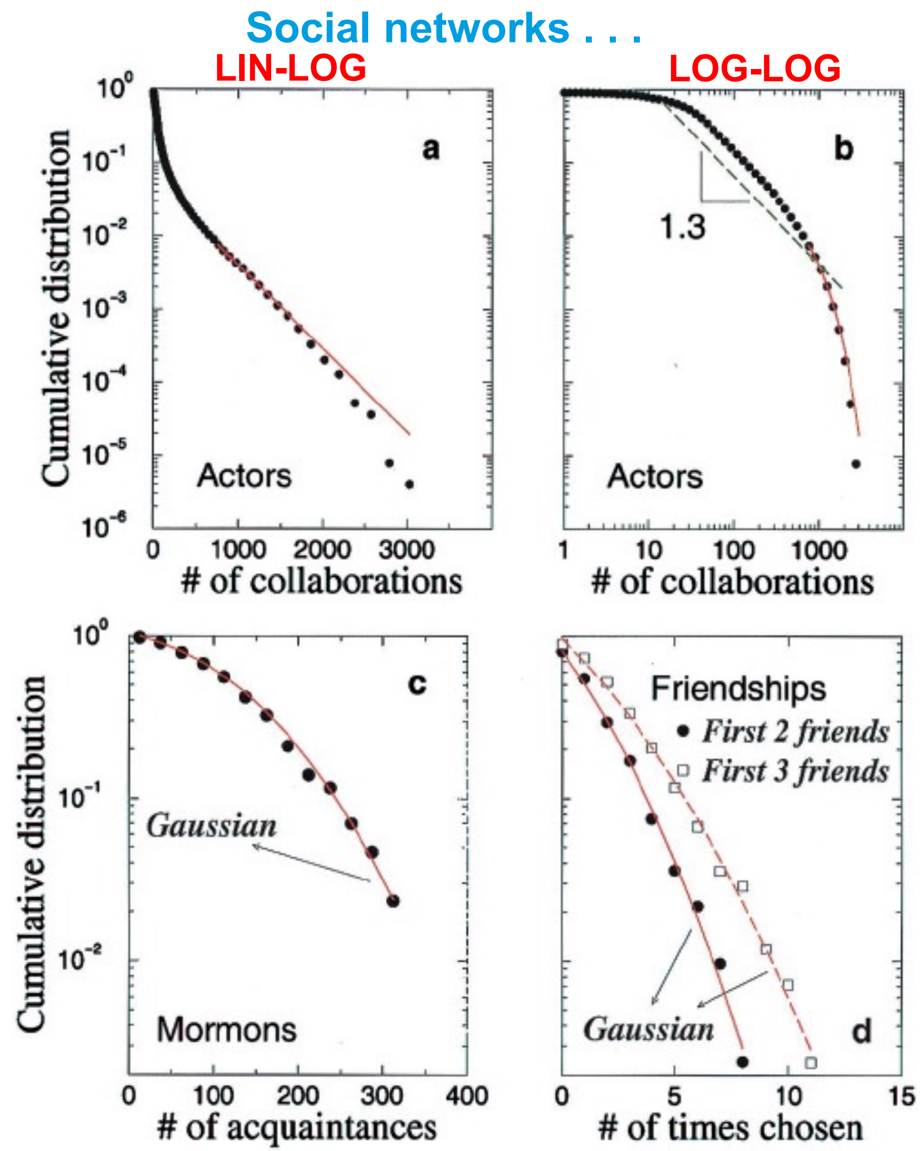
Conclusion:

Only linear preferential attachment produces scale-free growing networks. On the other hand, non-linear preferential attachment is more than sufficient to provide a wide spectrum of fat-tailed degree distributions.

The idea of “Scale-free networks” is a subtle and fragile concept

Scale-free nets is a useful concept…
but be careful with fits

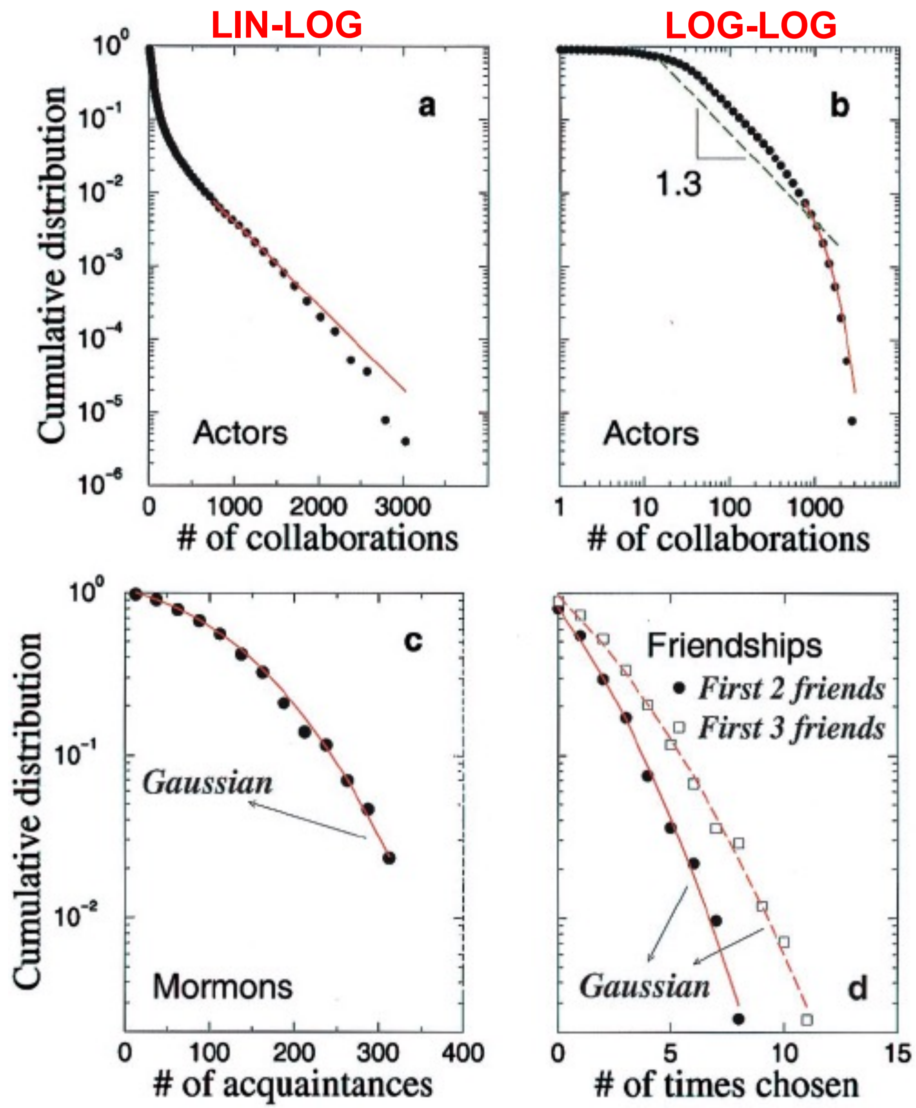
Luís Nunes Amaral
Northwestern University



Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

Scale-free nets is a useful concept…
but be careful with fits

Social networks . . .



…exponential decays
are also common

Networks are finite!!

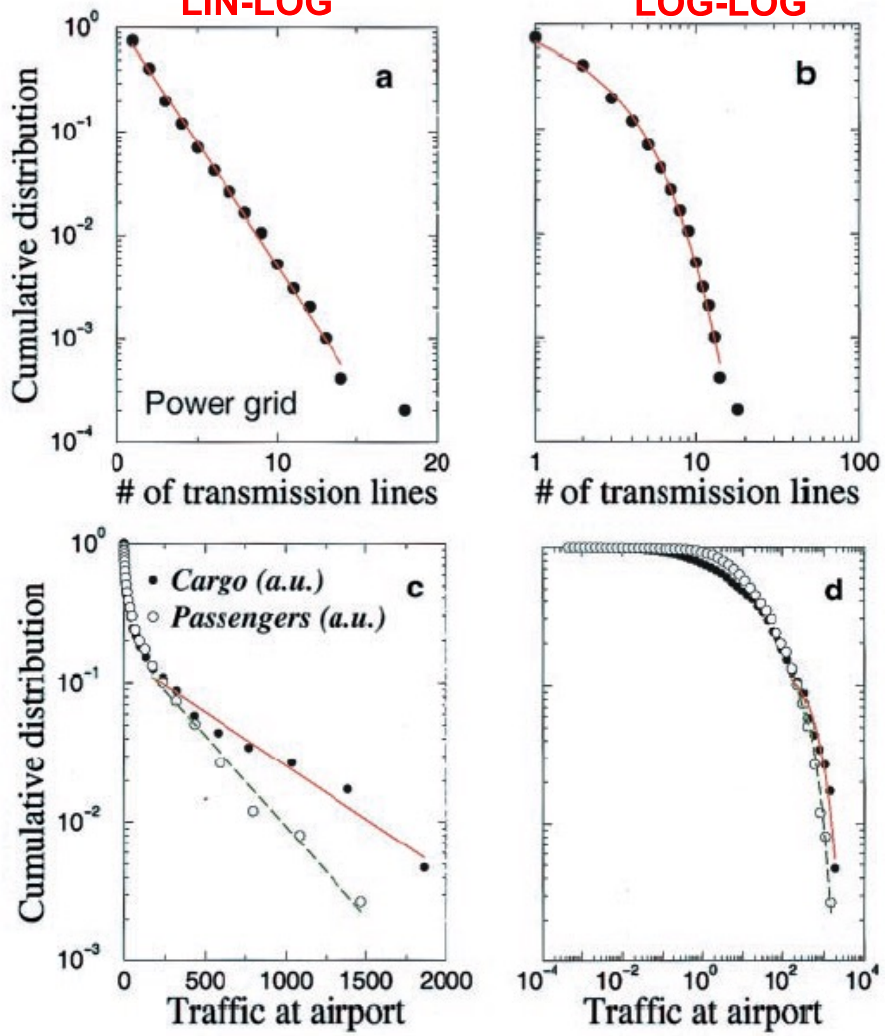
Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

Scale-free nets is a useful concept…
but be careful with fits

…exponential decays
are also common

Networks are finite!!

Technological networks . . .



Amaral, Scala, Barthélémy, and Stanley,
Classes of small-world networks,
PNAS 97 (21) 11149 (2000).

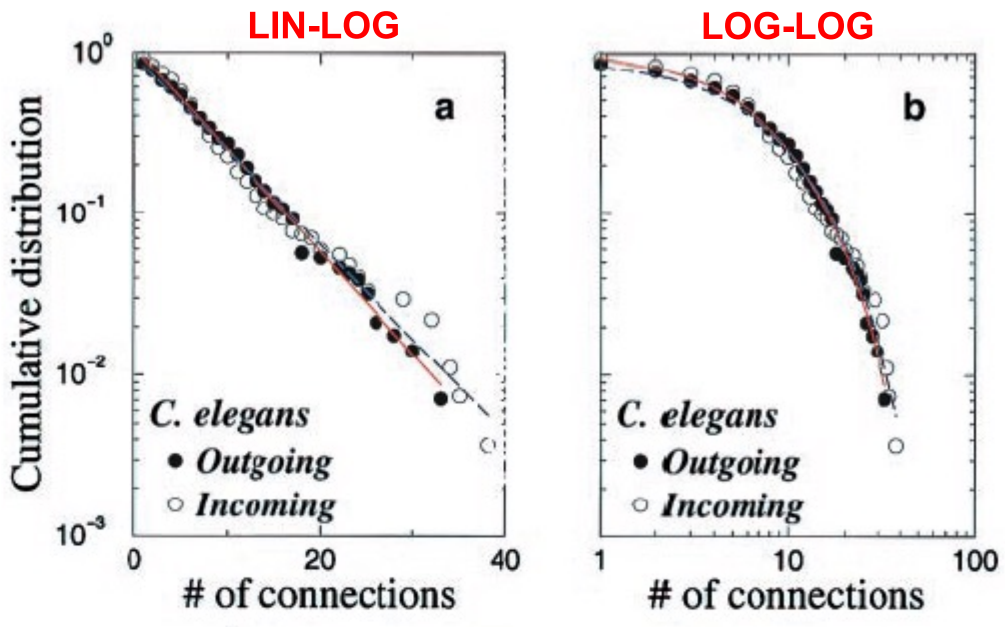
Scale-free nets is a useful concept…

but be careful with fits

biological networks . . .

…exponential decays
are also common

Networks are finite!!



Amaral, Scala, Barthélémy, and Stanley,
Classes of small-world networks,
PNAS 97 (21) 11149 (2000).

Classes of small-world networks

We can state (based on existing empirical analysis of real nets) that there are 3 main classes of graphs :

single-scale : individual degrees do not deviate appreciably from the average degree of the graph (type most compatible with WS-model);

broad-scale : degrees span a wider interval, with degree distribution falling off exponentially for large k ;

scale-free : those graphs in which the degree distributions decays with a power law, exhibiting the same behavior at all scales.

Amaral, Scala, Barthélémy, and Stanley,
Classes of small-world networks,
PNAS 97 (21) 11149 (2000).

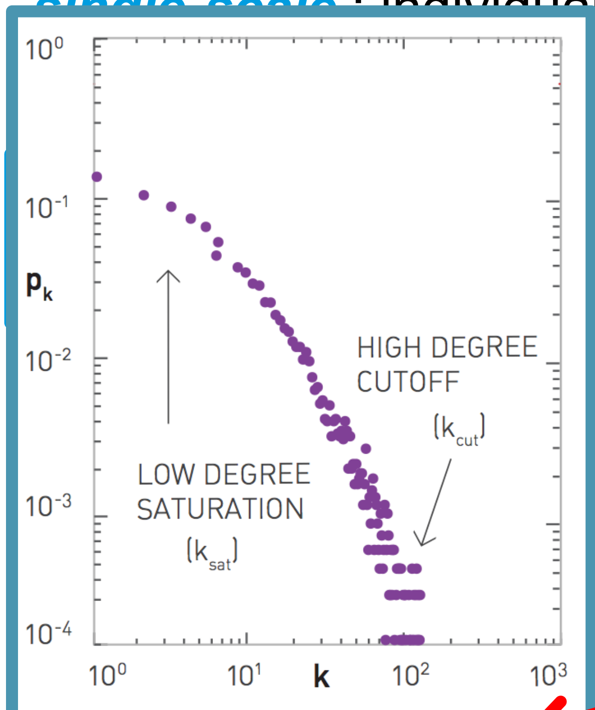
Classes of small-world networks

We can state (based on existing empirical analysis of real nets) that there are 3 main classes of graphs :

single-scale : individual degrees do not deviate appreciably from the
raph (type most compatible with WS-model);

span a wider interval, with degree distribution
for large *k* ;

ns in
he s



Dorogovtsev, Mendes, & Samukhin (2001). Size-dependent degree distribution of a scale-free growing network. Phys Rev E, 63(6), 062101.

Exponential cutoffs & finite size effects:

- A net is SF only in the limit of very large N (try it!).
- There's a maximum degree a node can have. This introduces natural (exponential) cutoffs in the degree dist.

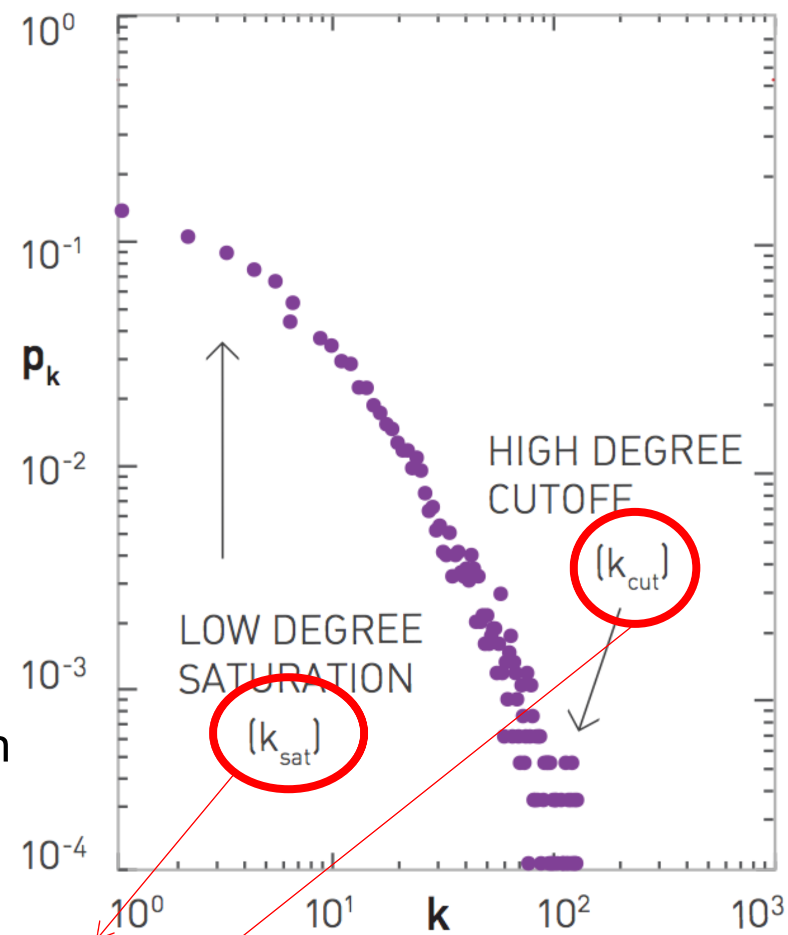
NAME	$p_x/p(x)$	$\langle x \rangle$	$\langle x^2 \rangle$
Poisson (discrete)	$e^{-\mu}\mu^x/x!$	μ	$\mu(1+\mu)$
Exponential (discrete)	$(1-e^{-\lambda})e^{-\lambda x}$	$1/(e^\lambda - 1)$	$(e^\lambda + 1)/(e^\lambda - 1)^2$
Exponential (continuous)	$\lambda e^{-\lambda x}$	$1/\lambda$	$2/\lambda^2$
Power law (discrete)	$x^{-\alpha}/\zeta(\alpha)$	$\begin{cases}\zeta(\alpha-2)/\zeta(\alpha), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1\end{cases}$	$\begin{cases}\zeta(\alpha-1)/\zeta(\alpha), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2\end{cases}$
Power law (continuous)	$\alpha x^{-\alpha}$	$\begin{cases}\alpha/(\alpha-1), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1\end{cases}$	$\begin{cases}\alpha/(\alpha-2), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2\end{cases}$
Power law with cutoff (continuous)	$\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)} x^{-\alpha} e^{-\lambda x}$	$\lambda^{-1}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$	$\lambda^{-2}\frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$
Stretched exponential (continuous)	$\beta\lambda^\beta x^{\beta-1} e^{-(\lambda x)^\beta}$	$\lambda^{-1}\Gamma(1+\beta^{-1})$	$\lambda^{-2}\Gamma(1+2\beta^{-1})$
Log-normal (continuous)	$\frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$e^{\mu+\sigma^2/2}$	$e^{2(\mu+\sigma^2)}$

take-home message

In real systems we rarely find perfect power-laws

Low-degree saturation: is a common deviation from the power-law behavior. Its signature is a flattened P(k) for $k < k_{sat}$. This indicates that we have fewer small degree nodes than expected for a pure power law.

High-degree saturation: High-degree cutoff appears as a rapid drop in P(k) for $k > k_{cut}$. These cutoffs can emerge from different sources, some of which will be analyzed in th next class.



Amaral, Scala, Barthélémy, and Stanley, Classes of small-world networks, PNAS 97 (21) 11149 (2000).

Note however that these cutoffs are hard to determine…

Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Goal: estimate γ from a discrete set of data points

1. Choose a k_{sat} and a k_{cut} between an interval of possible k_{min} and k_{max}. Optimize the best value of the degree exponent corresponding to this pair, using a statistical test to evaluate the quality of the fit.

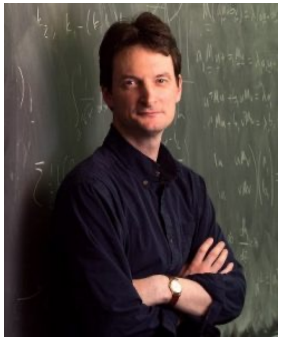
1. Repeat 1, scanning the entire interval of possible values of k_{sat}, k_{cut} , keeping the best combination {γ, $k_{sat,}$ k_{cut}} in what concerns the “goodness” of the fit.

Aaron Clauset

Santa Fe Inst. & Univ. of Colorado

Mark E. J. Newman

Michigan Univ.

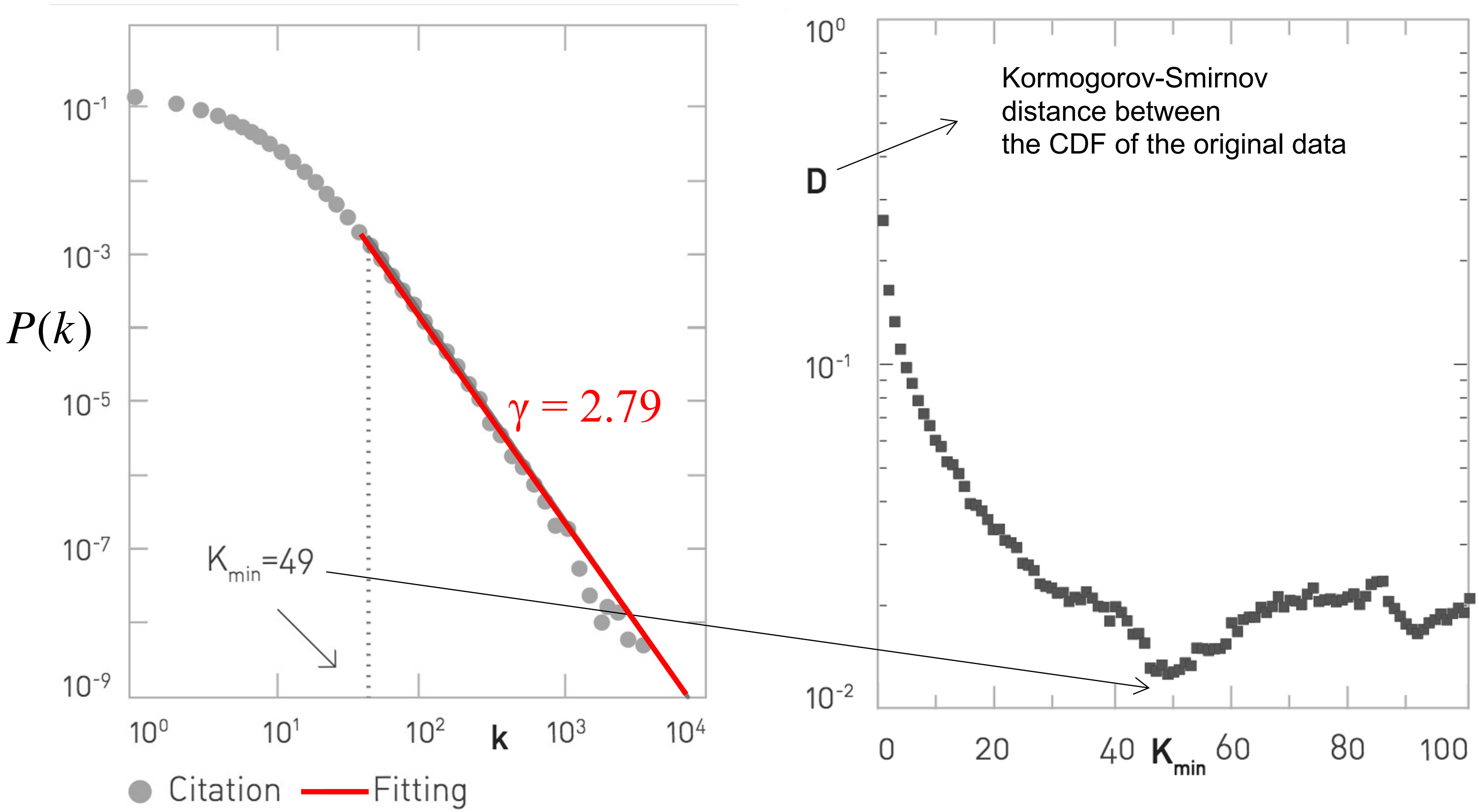


A. Clauset, et al. *Power-law distributions in empirical data*. SIAM Review S1: 661-703, 2009.

Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Example: citation networks



Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Example: email networks

Let's resort to the input file enron.outdegrees
(available here: https://dl.dropboxusercontent.com/s/e1kr9ri2btfwg9v/enron.outdegrees?dl=0) and enron.outdegree
(available here: https://dl.dropboxusercontent.com/s/79n6w3p3cyat5lx/enron.outdegree?dl=0)
created with webgraph in exercise 6 of our problem set 1.

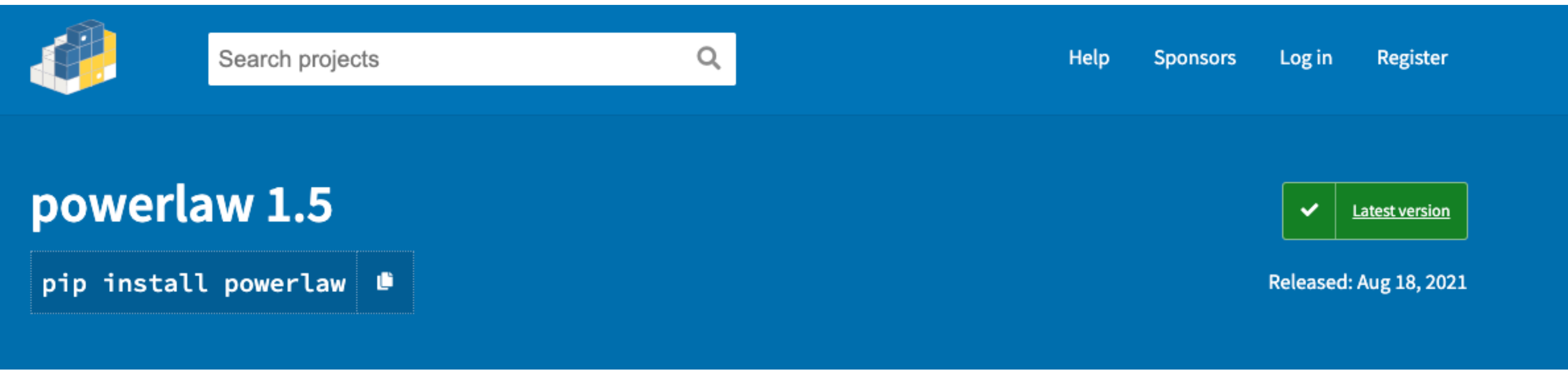
You may try the implementation in R of Clauset's algorithm (last problem in our 2nd set of exercises

```
[madonna@tecnico lab02]$ R
> install.packages("igraph")
> library('igraph')
> degs <- read.table("enron.outdegrees")
> deg_pmf <- read.table("enron.outdegree")
> degs_pl_fit <- power.law.fit(degs$V1)
> degs_pl_fit > plot(deg_pmf$V1, log="xy", xlab="degree", ylab="#vertices")
> plot(rev(cumsum(rev(deg_pmf$V1))), log="xy", xlab="degree", ylab="#vertices")
> plot(sort(degs$V1, decreasing = TRUE), 1:length(degs$V1), log="xy", xlab="degree",
ylab="rank")
```

Clauset et al. method: General idea

For details (and code) on this procedure, see: http://tuvalu.santafe.edu/~aaronc/powerlaws/

Same idea, but in Python:
https://pypi.org/project/powerlaw/



Conclusion:
Be careful with bold statements...

- The scale-free property is a rather fragile concept for small N.

- Moreover, it is unlikely to have a perfect ***linear*** preferential attachment, we may have constraints on the number of links and vertices, etc.

- **Take-home message:**

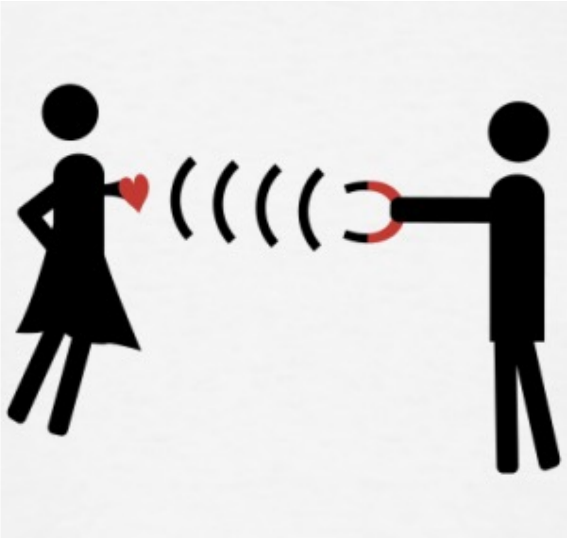
Perfect power-laws are useful concepts to have in mind as a reference point. Reality is often more complex, portraying different classes of complex networks.

Amaral, Scala, Barthélémy, and Stanley, PNAS 97 (21) 11149 (2000).

Broido, A. D., & Clauset, A.. Nature communications, 10(1), 1-10. (2019)

What are the mechanisms behind these variations?

Example:
Is the degree the only thing that matters?



Next class

Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

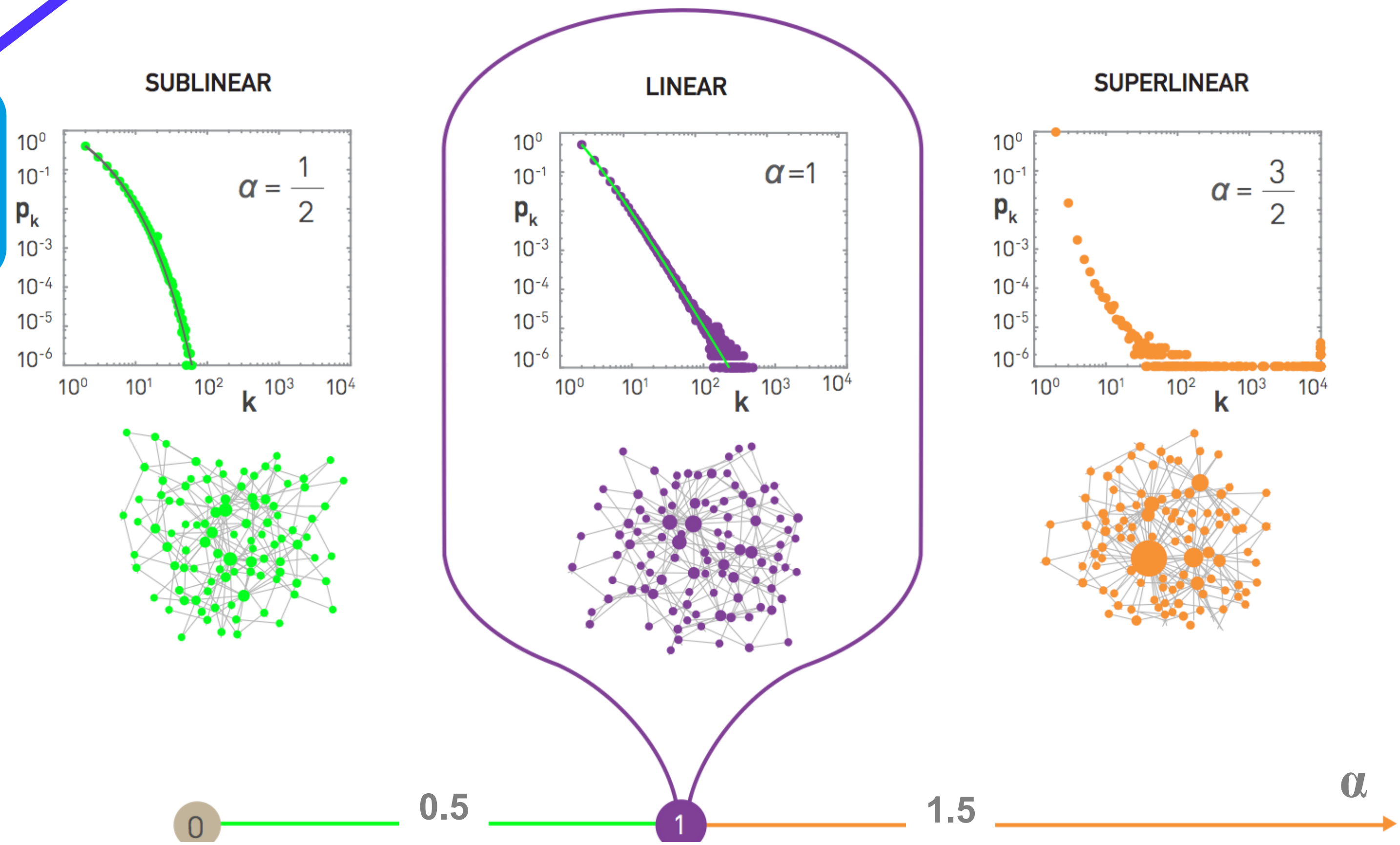
Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^\alpha}{\sum_j k_j^\alpha}$$

- $\alpha<1$ ➔ exponential dist
- $\alpha=1$ ➔ power-law
- $\alpha>1$ ➔ winners-take all

Barabási-Albert Model

$\Pi \sim k$ ➔ $\gamma=3$



Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

Barabási-Albert Model
$\Pi \sim k \rightarrow \gamma = 3$

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

$\alpha < 1 \rightarrow$ exponential dist
$\alpha = 1 \rightarrow$ power-law
$\alpha > 1 \rightarrow$ winners-take all

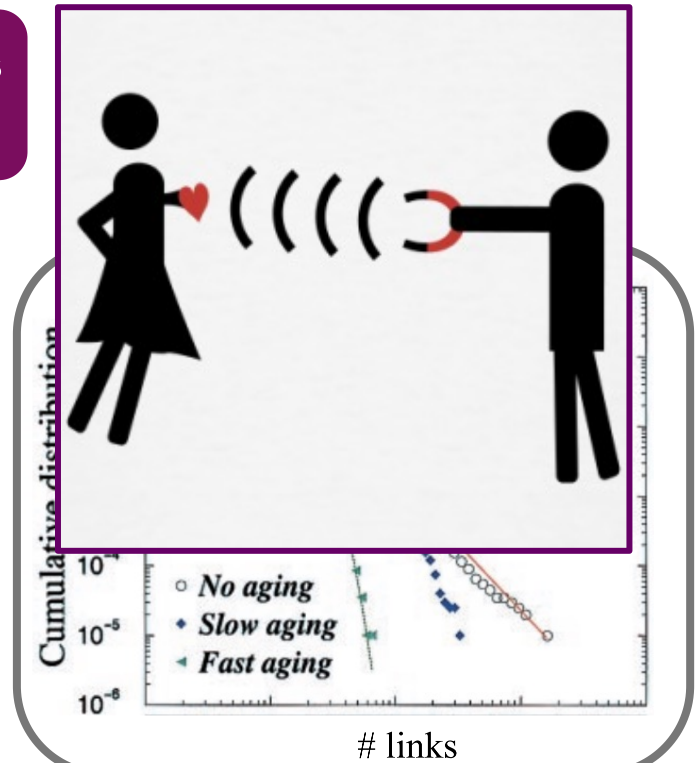
Initial attractiveness
$\Pi \sim A + k \rightarrow \gamma = 3 + A/m$

Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$



Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

Barabási-Albert Model
Π~k ➔ γ=3

Non-linear Pref. Attachment

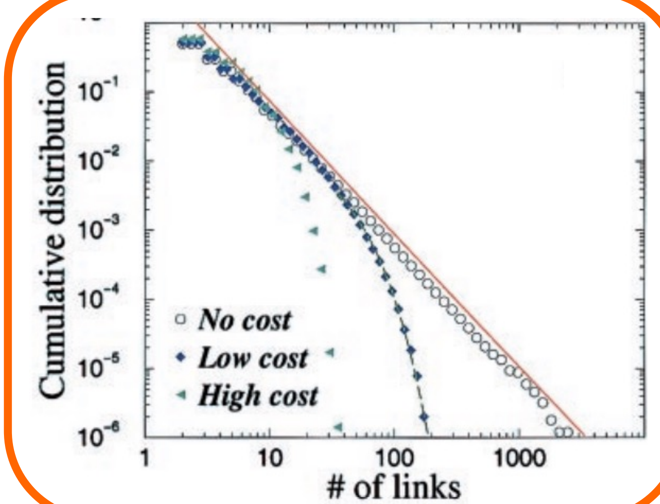
$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

α<1 ➔ exponential dist
α=1 ➔ power-law
α>1 ➔ winners-take all

Initial attractiveness
Π~A+k ➔ γ=3+A/m

Growth w/costs

$$k > k_{\max}$$



Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$

Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

Barabási-Albert Model
Π~k ➔ γ=3

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

α<1 ➔ exponential dist
α=1 ➔ power-law
α>1 ➔ winners-take all

Initial attractiveness
Π~A+k ➔ γ=3+A/m

Growth w/costs

$$k > k_{max}$$

n internal links

i) double pref. attachment

$$\gamma = 2 + \frac{m}{m+2n} < 3$$

ii) random attachment

$$\gamma = 3 + \frac{2n}{2m} > 3$$

Fitness models

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$

Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

Barabási-Albert Model
$\Pi \sim k \rightarrow \gamma = 3$

Non-linear Pref. Attachment

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_j k_j^{\alpha}}$$

$\alpha < 1 \rightarrow$ exponential dist
$\alpha = 1 \rightarrow$ power-law
$\alpha > 1 \rightarrow$ winners-take all

Growth with removal of nodes and/or links

Initial attractiveness
$\Pi \sim A + k \rightarrow \gamma = 3 + A/m$

n internal links

i) double pref. attachment

$$\gamma = 2 + \frac{m}{m + 2n} < 3$$

ii) random attachment

$$\gamma = 3 + \frac{2n}{2m} > 3$$

Growth w/costs

$$k > k_{\max}$$

Fitness models

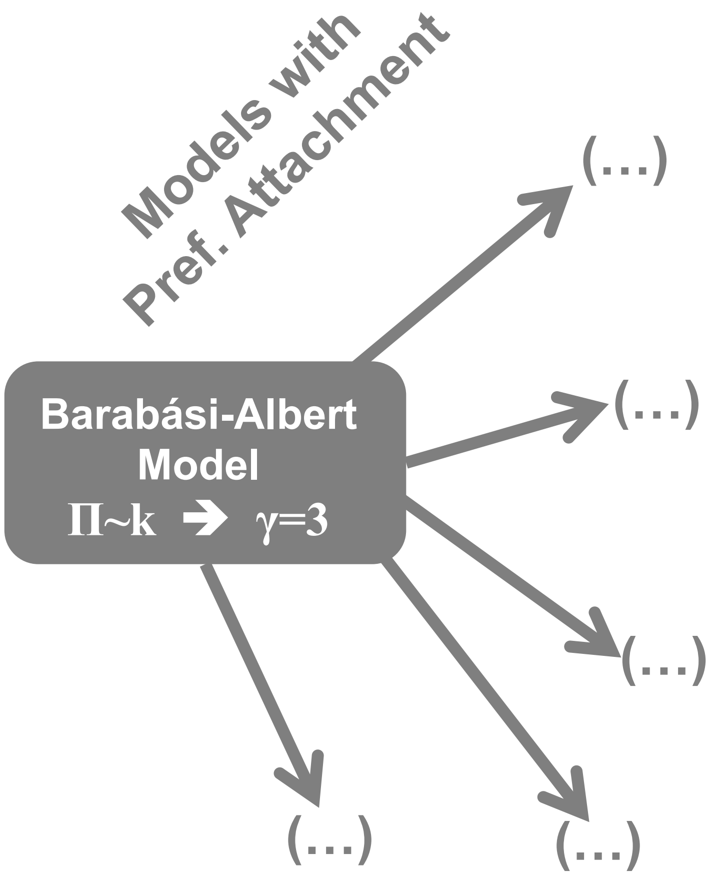
$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Ageing models

$$(t_i - t)^{-\nu}$$

Next class ☺!

Let a million variants bloom!



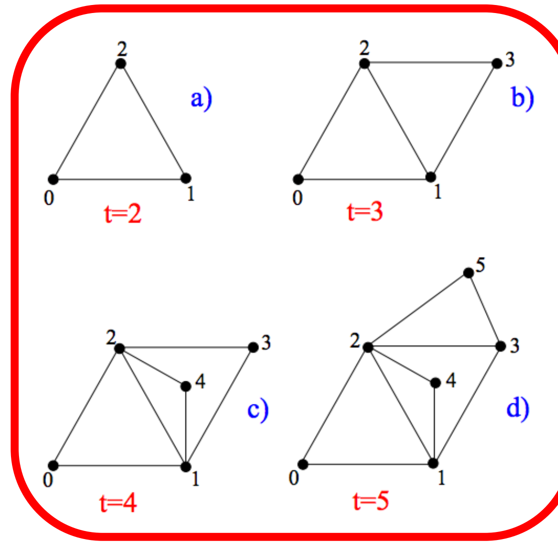
Models without Pref. Attachment?

Next class ☺!

Let a million variants bloom!

Models without Pref. Attachment

Minimal or link-selection model
$\Pi \sim k \rightarrow \gamma=3$



Let a million variants bloom!

Next class ☺!

Models with Pref. Attachment

Barabási-Albert Model
$\Pi \sim k \rightarrow \gamma = 3$

(…)

(…)

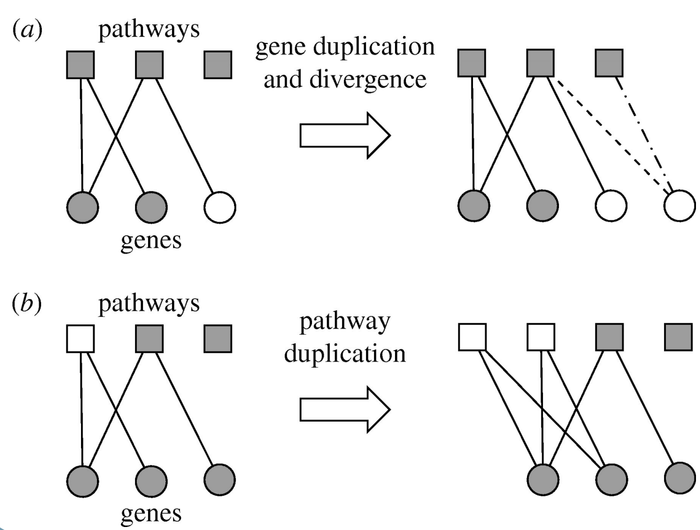
(…)

Models without Pref. Attachment

Ranking models

$$\Pi_s = \frac{R_s^{-\alpha}}{\sum_j R_j^{-\alpha}}$$

Minimal or link-selection model
$\Pi \sim k \rightarrow \gamma = 3$



Duplication models

$$P(k) \sim k^{-\left(1+\frac{1}{\alpha}\right)}$$

Next class ☺!

Let a million variants bloom!

Models with Pref. Attachment

Barabási-Albert Model
$\Pi \sim k \rightarrow \gamma=3$

(…)
(…)
(…)
(…)
(…)

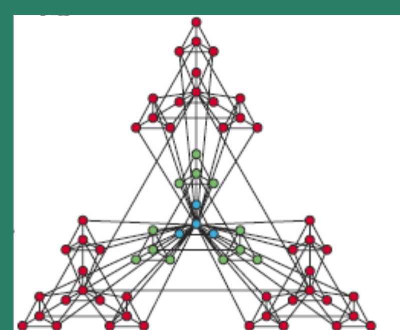
Models without Pref. Attachment

Ranking models

$$\Pi_s = \frac{R_s^{-\alpha}}{\sum_j R_j^{-\alpha}}$$

Minimal or link-selection model
$\Pi \sim k \rightarrow \gamma=3$

Hierarchical nets



Duplication models

$$P(k) \sim k^{-\left(1+\frac{1}{\alpha}\right)}$$