EQUATIONS OF TROPICAL VARIETIES

JEFFREY GIANSIRACUSA[1] AND NOAH GIANSIRACUSA[2]

ABSTRACT. We introduce a scheme-theoretic enrichment of the principal objects of tropical geometry. Using a category of semiring schemes, we construct tropical hypersurfaces as schemes over idempotent semirings such as $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ by writing them as solution sets to explicit systems of tropical equations that are uniquely determined by tropical linear algebra. We then define a tropicalization functor that sends closed subschemes of a toric variety over a ring R with non-archimedean valuation to closed subschemes of the corresponding tropical toric variety. Upon passing to the set of $\mathbb{T}$-points this reduces to Kajiwara-Payne's extended tropicalization, and in the case of a projective hypersurface we show that the scheme structure determines the multiplicities attached to the top-dimensional cells. By varying the valuation, these tropicalizations form algebraic families of $\mathbb{T}$-schemes parameterized by the analytification of $\operatorname{Spec} R$. For projective subschemes, the Hilbert polynomial is preserved by tropicalization, regardless of the valuation. We conclude with some examples and a discussion of tropical bases in the scheme-theoretic setting.

Dedicated to Max and Add(ie)

1. INTRODUCTION

Tropical geometry is a recent tool in algebraic geometry that transforms certain questions into combinatorial problems by replacing a variety with a polyhedral object called a tropical variety. It has had striking applications to a range of subjects, such as enumerative geometry [Mik05, FM10, GM08, AB13], classical geometry [CDPR12, Bak08], intersection theory [Kat09, GM12, OP13], moduli spaces and compactifications [Tev07, HKT09, ACP12, RSS13], mirror symmetry [Gro10, GPS10, Gro11], abelian varieties [Gub07, CV10], representation theory [FZ02, GL12], algebraic statistics and mathematical biology [PS04, Man11] (and many more papers by many more authors). Since its inception, it has been tempting to look for algebraic foundations of tropical geometry, e.g., to view tropical varieties as varieties in a more literal sense and to understand tropicalization as a degeneration taking place in one common algebro-geometric world. However, tropical geometry is based on the idempotent semiring $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, which is an object outside the traditional scope of algebraic geometry.

Motivated by the desire to do algebraic geometry over the *field with one element*, $\mathbb{F}_1$, various authors have constructed extensions of Grothendieck's scheme theory to accommodate geometric objects whose functions form algebraic objects outside the category of rings, such as semirings and monoids—the context of $\mathbb{F}_1$-geometry. The three theories developed in [Dur07, TV09, Lor12] essentially coincide over semirings, where the resulting schemes can be described in familiar terms either as spaces equipped with a sheaf of semirings, or as functors of points extended from rings to the larger category of semirings. While these theories provide distinct categories of $\mathbb{F}_1$-schemes, (split) toric varieties with torus-equivariant morphisms—and a somewhat larger class of naive $\mathbb{F}_1$-schemes that we shall consider—embed as a full subcategory of each, and there are base-change functors from (each version of) $\mathbb{F}_1$-schemes to schemes over any ring or semiring. The above-cited authors have each speculated that the category of schemes over $\mathbb{T}$ might have applications to tropical geometry. However, tropicalization,

[1]DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA SA2 8PP, UK
[2]DEPARTMENT OF MATHEMATICS, UC BERKELEY, 943 EVANS HALL, BERKELEY, CA 94720-3840, USA
E-mail addresses: j.h.giansiracusa@swansea.ac.uk, noahgian@math.berkeley.edu.
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as it currently exists in the literature, produces tropical varieties as sets rather than as solutions to systems of tropical equations, and the set of geometric points of a scheme is very far from determining the scheme, so the challenge is to lift tropicalization to schemes in an appropriate way.

In traditional tropical geometry (e.g., [MS]) one considers subvarieties of a torus defined over a non-archimedean valued field k, usually algebraically closed and complete with respect to the valuation. Tropicalization sends a subvariety Z of the torus $(k^\times)^n$ to a polyhedral subset of the tropical torus $(\mathbb{T}^\times)^n = \mathbb{R}^n$, the Euclidean closure of the image of coordinate-wise valuation. Kajiwara and Payne extended tropicalization to subvarieties of a toric variety, using the stratification by torus orbits [Kaj08, Pay09]. A fan determines a toric scheme X over $\mathbb{F}_1$ and base-change to k yields a familiar toric variety X_k, while base-change to $\mathbb{T}$ yields a tropical toric scheme $X_\mathbb{T}$. The $\mathbb{T}$-points of $X_\mathbb{T}$ form a convex polyhedron, the partial compactification of $N_\mathbb{R}$ dual to the fan, and Kajiwara-Payne tropicalization sends subvarieties of X_k to subsets of $X_\mathbb{T}(\mathbb{T})$.

Theorem A. *Let R be a ring equipped with a non-archimedean valuation (see Definition 2.5.1) $\nu : R \to S$, where S is an idempotent semiring (such as $\mathbb{T}$), and let X be a toric scheme over $\mathbb{F}_1$. There is a tropicalization functor*

$$\mathcal{T}rop^\nu_X\colon \{\textit{closed subschemes of } X_R\} \to \{\textit{closed subschemes of } X_S\}.$$

This is functorial in X with respect to torus-equivariant morphisms, and when $S = \mathbb{T}$ the composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, -)$ *yields the set-theoretic functor of Kajiwara-Payne.*

In the case of projective space, $X = \mathbb{P}^n$, if $Z \subset X_R$ is irreducible of dimension d then the set-theoretic tropicalization admits the structure of a polyhedral complex of pure dimension d and there are integer multiplicities associated to the facets such that the well-known balancing condition is satisfied (see, e.g., [DFS07, §2]). We show (in Corollary 7.2.2) that when Z is a hypersurface, the scheme $\mathcal{T}rop^\nu_X(Z)$ determines the multiplicities, and we expect this to be true for Z of arbitrary codimension.

Theorem B. *Let $\nu : k \to S$ be a valued field with S a totally ordered idempotent semifield. Given a closed subscheme $Z \subset \mathbb{P}^n_k$, the tropicalization $\mathcal{T}rop^\nu_{\mathbb{P}^n}(Z) \subset \mathbb{P}^n_S$ has a well-defined Hilbert polynomial and it coincides with that of Z.*

This suggests that the process of sending a variety to its tropicalization behaves like a flat degeneration.

We briefly explain the idea behind the construction of this scheme-theoretic tropicalization. Due to the nature of $(\max, +)$-algebra, the graph of a tropical polynomial f is piecewise linear; the regions of linearity are where a single monomial in f strictly dominates and the "bend locus," where the function is nonlinear, is the set of points where the maximum is attained by at least two monomials simultaneously. The bend locus (often called a tropical hypersurface or locus of tropical vanishing) is the tropical analogue of the zero locus of a polynomial over a ring. We enrich the bend locus of f with a scheme structure by realizing it as the solution set to a natural system of tropical algebraic equations: the *bend relations of f* (§5.1). These equations are given by equating f with each polynomial obtained from f by deleting a single monomial. By the fundamental theorem of tropical geometry [MS, Theorem 3.2.4] (Kapranov's Theorem in the case of a hypersurface), set-theoretic tropicalization can be reformulated by intersecting the bend loci of the coefficient-wise valuations of all polynomials in the ideal defining an affine variety. Our tropicalization is defined by replacing this set-theoretic intersection with the scheme-theoretic intersection of bend loci. This yields a solution to the implicitization problem for the coordinate-wise valuation map. For a homogeneous ideal defining a projective subscheme, these bend relations are compatible with the grading and essentially reduce tropicalization to the framework of tropical linear algebra, from which the Hilbert polynomial result follows.

Toric varieties are a natural class of varieties where there is a well-behaved class of monomials in each coordinate patch and this allows for a global extension of these affine constructions. We use the language of schemes over $\mathbb{F}_1$ as a convenient way to keep track of monomials and to provide a slight generalization of the ambient toric varieties in which tropicalization takes place.

One can ask how the tropicalization of $Z \subset X_R$ depends on the valuation $\nu : R \to \mathbb{T}$. Set-theoretically, the tropicalizations form a family over the Berkovich analytification of $\operatorname{Spec} R$, and we interpret this as an algebraic family.

Theorem C. *Let R be a ring, X a toric scheme over $\mathbb{F}_1$, and $Z \subset X_R$ a closed subscheme.*

(1) *The moduli space $\mathcal{V}al(R)$ of valuations on R is represented in affine idempotent semiring schemes, and there is a universal valuation $\nu_{\text{univ}} : R \to \Gamma(\mathcal{V}al(R), \mathcal{O}_{\mathcal{V}al(R)})$ through which all others factor uniquely. In particular, $\mathcal{V}al(R)(\mathbb{T}) = (\operatorname{Spec} R)^{an}$ as a set.*
(2) *The fiber of the algebraic family $\mathcal{T}rop_X^{\nu_{\text{univ}}}(Z) \to \mathcal{V}al(R)$ over each $\mathbb{T}$-point $\nu : R \to \mathbb{T}$ is the tropicalization $\mathcal{T}rop_X^{\nu}(Z) \subset X_{\mathbb{T}}$. If $X = \mathbb{P}^n$ and R is a field then the Hilbert polynomials of the fibres exist and are all equal.*

1.1. **Organization of the paper.** We begin in §2 by recalling some standard material on monoids and semirings and then giving our slightly generalized definition of valuation. In §3 we discuss the construction of $\mathbb{F}_1$-schemes and semiring schemes, and in §4 we review some constructions in toric schemes within this setting. The core of the paper is §5, where we define bend loci as schemes, and §6, where we use this to define and study scheme-theoretic tropicalization. In §7 we study the tropical Hilbert function and the multiplicities on the facets of a tropical hypersurface and in §8 we investigate tropical bases.

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2. ALGEBRAIC PRELIMINARIES: MONOIDS, SEMIRINGS, AND VALUATIONS

Throughout this paper all monoids, semirings, and rings will be assumed commutative and unital.

2.1. **Monoids and $\mathbb{F}_1$ algebra.** In this paper we shall work with a naive version of algebra over the so-called "field with one element", $\mathbb{F}_1$, which is entirely described in terms of monoids. More sophisticated notions of $\mathbb{F}_1$ algebra exist, such as Durov's commutative algebraic monads [Dur07], but the naive version recalled here is the one that appears most appropriate for tropical geometry and it provides a convenient language for working with monoids and (semi)rings in parallel. This naive $\mathbb{F}_1$ theory (or a slight variation on it) and its algebraic geometry have been studied by many authors, including [CC10, Dei08, TV09, FW].

Rather than defining an object $\mathbb{F}_1$, one starts by defining the category of modules, $\mathbb{F}_1$-Mod, to be the category of pointed sets. The basepoint of an $\mathbb{F}_1$-module M is denoted 0_M and is called the zero element of M. This category has a closed symmetric monoidal tensor product given by the smash product of pointed sets (take the cartesian product and then collapse the subset $M \times \{0_N\} \cup \{0_M\} \times N$ to the basepoint). The two-point set $\{0,1\}$ is a unit for this tensor product.

An $\mathbb{F}_1$-*algebra* is an $\mathbb{F}_1$-module A equipped with a commutative and unital product map $A \otimes A \to A$ (i.e., it is a commutative monoid in $\mathbb{F}_1$-Mod). Concretely, an $\mathbb{F}_1$-algebra is a commutative and unital monoid with a (necessarily unique) element 0_A such that $0_A \cdot x = 0_A$ for all x; thus $\mathbb{F}_1$-algebras, as defined here, are sometimes called *monoids-with-zero*. The two-point set $\{0,1\}$ admits a multiplication making it an $\mathbb{F}_1$-algebra, and it is clearly an initial object, so we can denote it by $\mathbb{F}_1$ and speak of $\mathbb{F}_1$-algebras without ambiguity.

[1]MacPherson has been developing related ideas in his thesis and has independently discovered the equations for scheme-theoretic tropicalization that we propose here.

Example 2.1.1. The $\mathbb{F}_1$ polynomial algebra $\mathbb{F}_1[x_1,\ldots,x_n]$ is the free abelian monoid-with-zero on n generators. The Laurent polynomial algebra $\mathbb{F}_1[x_1^{\pm},\ldots,x_n^{\pm}]$ is the free abelian group on n generators, $\mathbb{Z}^n$, together with a disjoint basepoint.

An $\mathbb{F}_1$-algebra A is *integral* if the natural map from $A \smallsetminus \{0_A\}$ to its group completion is injective. An *A-module* M is an $\mathbb{F}_1$-module equipped with an associative and unital action of A given by a map $A\otimes M \to M$. Concretely, this is a pointed set with an action of the monoid A such that 0_A sends everything to 0_M. An *A-algebra* is an $\mathbb{F}_1$-algebra morphism $A \to B$

2.2. **Semirings.** Commutative monoids admit a tensor product $\otimes$ generalizing that of abelian groups. A *semiring* is a monoid in the monoidal category of commutative monoids—that is, an object satisfying all the axioms of a ring except for the existence of additive inverses. For a semiring S, an *S-module* is a commutative monoid M equipped with an associative action $S\otimes M \to M$. An *S-algebra* is a morphism of semirings $S \to T$. Polynomial algebras $S[x_1,\ldots,x_n]$, and Laurent polynomial algebras, are defined as they are for rings. The category of semirings has an initial object, $\mathbb{N}$, so the category of semirings is equivalent to the category of $\mathbb{N}$-algebras. A semiring is a *semifield* if every nonzero element admits a multiplicative inverse.

A semiring S is *idempotent* if $a+a=a$ for all $a\in S$. In this case (and more generally, for an idempotent commutative monoid) there is a canonical partial order defined by

$$a \le b \text{ if } a+b=b.$$

The least upper bound of any finite set $\{a_i\}$ of elements exists and is given by the sum $\sum a_i$. If the partial order is actually a total order then $\sum a_i$ is equal to the maximum of the a_i.

From the perspective of tropical geometry, the central example of an idempotent semiring is the semifield of *tropical numbers*, $\mathbb{T}$. As a set,

$$\mathbb{T} := \mathbb{R}\cup\{-\infty\}.$$

The addition operation is defined by the maximum: $a+b=\max\{a,b\}$ if both a and b are finite. Multiplication $a\cdot b$ in $\mathbb{T}$ is defined as the usual addition of real numbers $a+b$ if both are finite. The additive and multiplicative units are $0_{\mathbb{T}} = -\infty$ and $1_{\mathbb{T}} = 0$, respectively, and this defines the extension of addition and multiplication to $-\infty$.

This is a special case of a general construction: given a commutative monoid $(\Gamma,+)$ equipped with a translation-invariant total order, the set $\Gamma\cup\{-\infty\}$ equipped with the operations $(\max,+)$ forms an idempotent semiring, and if Γ is a group then this yields a semifield. The tropical numbers $\mathbb{T}$ are the result when Γ is $(\mathbb{R},+)$ with its canonical total order. Another interesting example of an idempotent semifield comes from $\mathbb{R}^n$ equipped with the lexicographic total order.

Remark 2.2.1. Idempotent totally ordered semifields appear to play much of the role in idempotent algebra and geometry of fields in classical algebra and geometry.

The *boolean semiring* is the subsemiring

$$\mathbb{B} := \{-\infty, 0\} \subset \mathbb{T}.$$

The boolean semiring is initial in the category of idempotent semrings and every $\mathbb{B}$-algebra is idempotent, so $\mathbb{B}$-algebras are the same as idempotent semirings.

2.3. **Scalar extension and restriction.** Given a (semi)ring S, there is an adjoint pair of functors

$$\mathbb{F}_1\text{-Mod} \leftrightarrows S\text{-Mod};$$

the right adjoint sends an S-module to its underlying set with the additive unit as the basepoint, and the left adjoint, denoted $-\otimes S$, sends a pointed set M to the free S-module generated by the non-basepoint elements of M. If M is an $\mathbb{F}_1$-algebra then $M\otimes S$ has an induced S-algebra structure. Note that $-\otimes S$ sends polynomial algebras over $\mathbb{F}_1$ to polynomial algebras over S.

In this paper, S-modules equipped with an $\mathbb{F}_1$-descent datum (i.e., modules of the form $M \otimes S$ for M a specified $\mathbb{F}_1$-module) play a particularly important role. For $f \in M \otimes S$, the *support* of f, denoted $\mathrm{supp}(f)$, is the subset of M corresponding to the terms appearing in f.

Given a semiring homomorphism $\varphi : S \to T$ one obtains an adjoint pair

$$S\text{-Mod} \leftrightarrows T\text{-Mod}$$

in the standard way. As usual, the left adjoint is denoted $- \otimes_S T$, and it sends S-algebras to T-algebras and coincides with the pushout of S-algebras along φ.

2.4. **Ideals, congruences and quotients.** Let A be either an $\mathbb{F}_1$-algebra or a semiring. We can regard A as an A-module and define an *ideal* in A to be a submodule of A. When A is a ring this agrees with the usual definition of an ideal.

Quotients of semirings generally cannot be described by ideals, since a quotient might identify elements f and g without the existence of an element $f - g$ to identify with zero. The same issue arises when constructing quotients of modules over semirings. For this reason, one must instead work with congruences. The omitted proofs in this section are all standard and/or elementary.

Definition 2.4.1. Let S and M be a semiring and S-module respectively. A *semiring congruence* on S is an equivalence relation $J \subset S \times S$ that is a sub-semiring, and a *module congruence* on M is an S-submodule $J \subset M \times M$ that is an equivalence relation. If the type is clear from context, we refer to such an equivalence relation simply as a *congruence*.

Proposition 2.4.2. *Let J be an equivalence relation on a semiring S (or module M over a semiring). The semiring (or module) structure descends to the set of equivalence classes S/J (M/J) if and only if J is a semiring (or module) congruence.*

Definition 2.4.3. Given a morphism of semirings $\varphi : S \to R$, we define the *kernel* congruence

$$\ker \varphi := S \times_R S = \{(f, g) \in S \times S \mid \varphi(f) = \varphi(g)\}.$$

Using congruences in place of ideals, the usual isomorphism theorems extend to semirings:

Proposition 2.4.4.
(1) *Let $\varphi : S \to R$ be a homomorphism of semirings. The image is a semiring, the kernel is a congruence, and $S/\ker \varphi \cong \mathrm{im}\, \varphi$.*
(2) *Let R be a semiring, $S \subset R$ a sub-semiring, I a congruence on R, and let $S + I$ denote the I-saturation of S (the union of all I-equivalence classes that contain an element of S). Then $S + I$ is a sub-semiring of R, I restricts to a congruence I' on $S + I$ and a congruence I'' on S, and there is an isomorphism $(S + I)/I' \cong S/I''$*
(3) *For $J \subset I$ congruences on S, we have a congruence I/J on S/J with $(S/J)/(I/J) \cong S/I$. This yields a bijection between congruences on S/J and congruences on S containing J.*

Since the intersection of congruences is a congruence, for a collection $\{f_\alpha, g_\alpha \in S\}_{\alpha \in A}$ there is a unique smallest (or finest) congruence identifying f_α with g_α for each α; this is the congruence generated by pairs (f_α, g_α). In the case of a semiring congruence, we denote this by $\langle f_\alpha \sim g_\alpha \rangle_{\alpha \in A}$. More generally, for any subset $J \subset S \times S$, we denote by $\langle J \rangle$ the semiring congruence it generates. If $\varphi : S \to R$ is a semiring (or module) homomorphism and J is a congruence on S, then $\varphi(J)$ need not be a congruence on R because transitivity and reflexivity can fail; we denote by $\varphi_* J$ the congruence generated by $\varphi(J)$.

Lemma 2.4.5. *The semiring congruence $\langle f_\alpha \sim g_\alpha \rangle_{\alpha \in A}$ consists of the transitive closure of the sub-semiring of $S \times S$ generated by the elements (f_α, g_α), (g_α, f_α), and the diagonal $S \subset S \times S$. The analogous statement for module congruences also holds.*

Proof. The sub-semiring generated clearly gives a binary relation that is symmetric and reflexive, so it suffices to check that if $R \subset S \times S$ is a sub-semiring, then the transitive closure R' is also a sub-semiring. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$ be sequences of elements in S such that each consecutive pair (x_i, x_{i+1}) and

(y_i, y_{i+1}) is in R. Thus (x_1, x_n) and (y_1, y_k) are in R', and we must show that the product and sum of these are each in R'. We may assume $k \leq n$, and by padding with $n-k$ copies of y_k, we can assume that $k = n$. By adding or multiplying the two sequences term by term we obtain the result. $\square$

Proposition 2.4.6. *There is a bijection between ideals in a semiring S and congruences generated by relations of the form $f \sim 0_S$. The bijection is given by $I \mapsto \langle f \sim 0_S \rangle_{f \in I}$.*

2.5. **Valuations.** The term "non-archimedean valuation" on a ring R usually[2] means a homomorphism of multiplicative monoids $\nu : R \to \mathbb{T}$ satisfying $\nu(0_R) = -\infty$ and the subadditivity condition $\nu(a+b) \leq \nu(a) + \nu(b)$ for all $a, b \in R$. The subadditivity condition appears semi-algebraic but, as observed in [Man11], it can be reformulated as an algebraic condition:

$$\nu(a+b) + \nu(a) + \nu(b) = \nu(a) + \nu(b).$$

We use this observation in §6.5 when constructing the moduli space of valuations on a ring.

It is useful—for example, when studying families of tropical varieties—to allow a more general codomain, so throughout this paper the term "valuation" shall refer to the following generalization. Note that, when passing from $\mathbb{T}$ to an arbitrary idempotent semiring, the total order is replaced by a partial order (cf., §2.2).

Definition 2.5.1. A *valuation* on a ring R is an idempotent semiring S (called the *semiring of values*), and a map $\nu : R \to S$ satisfying

(1) (unit) $\nu(0_R) = 0_S$ and $\nu(\pm 1_R) = 1_S$,
(2) (multiplicativity) $\nu(ab) = \nu(a)\nu(b)$,
(3) (subadditivity) $\nu(a+b) + \nu(a) + \nu(b) = \nu(a) + \nu(b)$.

A valuation ν is said to be *non-degenerate* if $\nu(a) = 0_S$ implies $a = 0_R$.

For $S = \mathbb{T}$ this coincides with the usual notion of a non-archimedean valuation described above. Note that any valuation on a field is automatically non-degenerate. When S is $\mathbb{R}^n \cup \{-\infty\}$ with the lexicographic order then the resulting higher rank valuations and their associated tropical geometry have been studied in [Ban11], and considering these higher rank valuations leads to Huber's "adic spaces" approach to non-archimedean analytic geometry [Hub96].

Remark 2.5.2. When S is totally ordered then the condition $\nu(-1) = 1_S$ holds automatically; however, for more general semirings of values this condition must be imposed separately for the important Lemma 2.5.3 below to hold.

Lemma 2.5.3. *Let $\nu : R \to S$ be a valuation and $a, b \in R$.*

(1) *If $\nu(a) < \nu(b)$ then $\nu(a+b) = \nu(a) + \nu(b)$.*
(2) $\nu(a+b) + \nu(a) = \nu(a+b) + \nu(a) + \nu(b)$.
(3) *If the partial order on S is a total order then the image of ν is a subsemiring of S, and $R \twoheadrightarrow \operatorname{im} \nu$ is a valuation.*

Proof. The first two statements are easy applications of subadditivity. The third statement follows immediately from the first. $\square$

A *valued ring* is a triple $(R, S, \nu : R \to S)$ where R is a ring and ν is a valuation. Valued rings form a category in which a morphism $\varphi : (R, S, \nu) \to (R', S', \nu')$ consists of a ring homomorphism $\varphi_1 : R \to R'$ and a semiring homomorphism $\varphi_2 : S \to S'$ such that $\nu' \circ \varphi_1 = \varphi_2 \circ \nu$. Note that the composition of a valuation $\nu : R \to S$ with a semiring homomorphism $S \to S'$ is again a valuation.

[2] Many authors use the opposite sign convention, and some would call this a "semi-valuation" unless the non-degeneracy condition $\nu^{-1}(-\infty) = 0$ holds.

As an illustration of the utility of considering the general class of valuations defined above, we show that, for a fixed ring R, there exists a *universal valuation* $v^R_{univ} : R \to S^R_{univ}$ on R from which any other valuation can be obtained by composition with a unique semiring homomorphism. This will be used to show that, as one varies the valuation on R, the set of all tropicalizations of a fixed subscheme form an algebraic family over $\operatorname{Spec} S^R_{univ}$ (Theorem C part (1)). Consider the polynomial $\mathbb{B}$-algebra $\mathbb{B}[x_a \,|\, a \in R]$ with one generator x_a for each element $a \in R$. The universal semiring of values S^R_{univ} is the quotient of $\mathbb{B}[x_a \,|\, a \in R]$ by the congruence generated by the relations

(1) $x_0 \sim 0_S$ and $x_1 \sim x_{-1} \sim 1_S$,
(2) $x_a x_b \sim x_{ab}$ for any $a, b \in R$,
(3) $x_{a+b} + x_a + x_b \sim x_a + x_b$ for any $a, b \in R$.

The universal valuation v^R_{univ} sends a to x_a.

Proposition 2.5.4. *Given a valuation $\nu : R \to T$, there exists a unique homomorphism $\phi : S^R_{univ} \to T$ such that $\phi \circ v^R_{univ} = \nu$. Hence valuations with semiring of values T are in bijection with homomorphisms $S^R_{univ} \to T$*

Proof. The homomorphism ϕ is defined by sending each generator x_a to $\nu(a)$. Since the relations in S^R_{univ} correspond exactly to the relations satisfied by a valuation, ϕ is well-defined. Uniqueness is immediate. □

3. $\mathbb{F}_1$-SCHEMES AND SEMIRING SCHEMES

3.1. **Construction of $\mathbb{F}_1$-schemes and semiring schemes.** The papers [TV09], [Lor12], and [Dur07] each construct categories of schemes over semirings and some notion of $\mathbb{F}_1$. For the purposes of the present paper we do not require the full generality of their constructions, so we present below a streamlined construction that follows the classical construction of schemes and yields a category that admits a full embedding into each of their categories.

Remark 3.1.1. Over a semiring, the category of schemes described here is equivalent to that of Toën-Vaquié, and it is a full subcategory of both Lorscheid's blue schemes and Durov's generalized schemes. See [LPL11] for a comparison of these three threories over each of their notions of $\mathbb{F}_1$.

The construction of schemes modelled on $\mathbb{F}_1$-algebras or semirings proceeds exactly as in the classical setting of rings. Let A be a Q-algebra, where Q is either a semiring or an $\mathbb{F}_1$-algebra. A proper ideal in A is *prime* if its complement is closed under multiplication. Given a prime ideal $\mathfrak{p} \subset A$, one can form the localization $A_{\mathfrak{p}}$ via equivalence classes of fractions in the usual way. As a space, the prime spectrum $|\operatorname{Spec} A|$ is the set of prime ideals in A equipped with the Zariski topology in which the open sets are the collections of primes not containing a given ideal (a basis is given by sets of the form $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ for $f \in A$). Any A-module (or algebra) M determines a sheaf $\widetilde{M}$ of Q-modules (or algebras) that sends a principal open set $D(f)$ to the localization $M_f = A_f \otimes M$ in which f is inverted. In particular, A itself gives a sheaf of Q-algebras, and this is the structure sheaf $\mathscr{O}_A$.

An *affine scheme* (over Q) is a pair $(X, \mathscr{O})$ consisting of a topological space X and a sheaf of Q-algebras that is isomorphic to a pair of the form $(|\operatorname{Spec} A|, \mathscr{O}_A)$. A general Q-scheme is a pair that is locally affine. A morphism of schemes is a morphism of pairs that is given in suitable affine patches by a homomorphism of Q-algebras. As explained in [Dur07, 6.5.2], for rings this coincides with the usual construction in terms of locally ringed spaces. The category of affine Q-schemes is equivalent to the opposite of the category of Q-algebras.

Proposition 3.1.2. *Given a Q-algebra A, the category of A-schemes is canonically equivalent to the category of Q-schemes over* $\operatorname{Spec} A$.

An $\mathbb{F}_1$-scheme is *integral* if it admits a cover by affine charts of the form $\operatorname{Spec} M$ with M an integral monoid-with-zero.

Proposition 3.1.3. *An $\mathbb{F}_1$-scheme X is integral if and only if for any open affine* $\operatorname{Spec} M \subset X$ *the monoid-with-zero M is integral.*

Proof. Since an affine scheme has a basis for its topology given by localizations, and any localization of an integral monoid is integral, a standard argument (as in the proof of [Har77, II.3.2]) reduces to proving the following: if $\operatorname{Spec} M = \cup \operatorname{Spec} M_{f_\alpha}$, where each localization M_{f_α} is integral, then M itself is integral. But this is easy: we have $\varnothing = \cap D(f_\alpha)$ so the f_α generate M as an ideal, and for monoids the union of ideals is again an ideal, so $M = \cup (f_\alpha)$ and hence $1 \in (f_\alpha) = M f_\alpha$ for some α. Thus f_α is a unit and $M = M_{f_\alpha}$ is indeed integral. □

3.2. **Base change functors.** The scalar extension and restriction functors of §2.3 admit globalizations that we briefly describe here.

Using the fact that $\mathbb{F}_1$-Mod and S-Mod (for S a semiring) are cocomplete, all fiber products exist in the categories of $\mathbb{F}_1$-schemes and S-schemes and they are constructed in the usual way. In particular, if T is an S-algebra and X is an S-scheme, then $X_T := \operatorname{Spec} T \times_{\operatorname{Spec} S} X$ exists and by Proposition 3.1.2 it can be regarded as a T-scheme. Thus $\operatorname{Spec} T \times_{\operatorname{Spec} S} -$ defines a base change functor from S-schemes to T-schemes, and this is the right adjoint of the forgetful functor (defined using Proposition 3.1.2) that regards a T-scheme as an S-scheme.

For R a ring or semiring, the scalar extension functor $- \otimes R$ clearly sends localizations of $\mathbb{F}_1$-algebras to localizations of R-algebras, so it globalizes to give a base change functor from $\mathbb{F}_1$-schemes to R-schemes. Given an $\mathbb{F}_1$-scheme X, we write X_R for the base change of X to R-schemes. This base change functor is right adjoint to the forgetful functor from R-schemes to $\mathbb{F}_1$-schemes that globalizes the corresponding forgetful functor from R-Mod to $\mathbb{F}_1$-Mod. Given an $\mathbb{F}_1$-scheme X, by a slight abuse of notation, we will write $X(R)$ for the set of R-points of X_R.

3.3. **Closed subschemes.** At a formal level, the classical theory of schemes and the extended theory of semiring schemes are nearly identical when considering open subschemes and gluing. However, novel features appear when considering closed subschemes; this is essentially because the bijection between ideals and congruences breaks down when passing from rings to semirings.

Quasi-coherent sheaves on an S-scheme X are defined exactly as in the classical setting. A *congruence sheaf* $\mathscr{J}$ is a subsheaf of $\mathscr{O}_X \times \mathscr{O}_X$ such that $\mathscr{J}(U)$ is a congruence on $\mathscr{O}(U)$ for each open $U \subset X$. A congruence sheaf is *quasi-coherent* if it is quasi-coherent when regarded as a sub-$\mathscr{O}_X$-module of $\mathscr{O}_X \times \mathscr{O}_X$. We follow Durov's viewpoint on closed subschemes: a *closed immersion* is an affine morphism $\Phi : Y \to X$ such that the sheaf morphism $\Phi^\sharp : \mathscr{O}_X \to \Phi_* \mathscr{O}_Y$ is surjective. In this situation, $\ker \Phi^\sharp$ is a quasi-coherent congruence sheaf.

Curiously, morphisms that are scheme-theoretic closed immersions defined in this way are often not closed embeddings at the level of topological spaces. For instance, a point $\Phi : \operatorname{Spec} \mathbb{T} \to \mathbb{A}^n_{\mathbb{T}}$ corresponding to a $\mathbb{T}$-algebra morphism $\varphi : \mathbb{T}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{T}$ sending each x_i to some finite value $\varphi(x_i) \in \mathbb{R}$ is a closed immersion, but the image of this map is not Zariski closed—in fact, it is a dense point! Indeed, $\varphi^{-1}(-\infty) = \{-\infty\}$, which is contained in all primes, so every point of $|\operatorname{Spec} \mathbb{T}[x_1, \ldots, x_n]|$ is in the closure of the image of the point $|\operatorname{Spec} \mathbb{T}|$.

Remark 3.3.1. One can view the prime spectrum and its Zariski topology as a technical scaffolding whose purpose is to define the functor of points, which is then regarded as the fundamental geometric object as in [TV09]. For instance, as we see in the following example, the $\mathbb{T}$-points of a tropical variety more closely reflect familiar geometry than its prime spectrum.

3.4. **Example: the affine tropical line.** The set of $\mathbb{T}$-points of the affine line $\mathbb{A}^1_{\mathbb{T}} = \operatorname{Spec}\mathbb{T}[x]$ is clearly $\mathbb{T}$ itself, but the ideal-theoretic kernel corresponding to each is trivial except for the point $x \mapsto -\infty$ for which it is maximal. On the other hand, one can of course distinguish all these points using the congruence-theoretic kernel, by the First Isomorphism Theorem.

The semiring $\mathbb{T}[x]$ has a rather intricate structure; however, it admits a quotient with the same set of $\mathbb{T}$-points that behaves more like univariate polynomials over an algebraically closed field:

$$\overline{\mathbb{T}[x]} := \mathbb{T}[x]/\sim, \text{ where } f \sim g \text{ if } f(t) = g(t) \text{ for all } t \in \mathbb{T}.$$

Polynomials in this quotient split uniquely into linear factors. More specifically, if

$$b_t := 0 + t^{-1}x \in \overline{\mathbb{T}[x]} \text{ for } t \in \mathbb{T}^\times = \mathbb{R} \text{ and } b_{-\infty} := x \in \overline{\mathbb{T}[x]},$$

then any element of $\overline{\mathbb{T}[x]}$ can be written uniquely as $c\prod b_{t_i}^{d_i}$ for $c, t_i \in \mathbb{T}$. Nonetheless, the prime spectrum of $\overline{\mathbb{T}[x]}$ is larger than one might guess based on analogy with the case of algebraically closed fields. For any subset $K \subset \mathbb{T}$ we define the ideal $I_K := (\{b_t \mid t \in K\}) \subset \overline{\mathbb{T}[x]}$.

Proposition 3.4.1. *If $K \subset \mathbb{T}$ is an interval (not necessarily closed or open) then $I_K \setminus \{-\infty\}$ is the set of functions that have a bend in K. As a set, $|\operatorname{Spec}\overline{\mathbb{T}[x]}| = \{I_K \mid K \subset \mathbb{T}$ is an interval $\}$. The finitely generated primes correspond to closed intervals and the principal primes to points of $\mathbb{T}$.*

Proof. If $f \in \overline{\mathbb{T}[x]}$ has a bend at $t \in K \subset \mathbb{T}$ then $f \in I_{\{t\}} \subset I_K$. Conversely, if $f \in I_K$ then $f = \sum_{i=1}^n g_i b_{t_i}$ for some $t_i \in K$ and $g_i \in \overline{\mathbb{T}[x]}$. Each summand $g_i b_{t_i}$ has a bend at t_i, and the tropical sum of a function with a bend at t_i and a function with a bend at t_j must have a bend in the closed interval $[t_i, t_j]$. Thus when K is convex (i.e., an interval) we indeed have that the non-constant functions of I_K are precisely the functions with a bend in K.

From this it follows that if K is an interval then I_K is prime: if $f, g \in \overline{\mathbb{T}[x]} \setminus I_K$ then neither f nor g has a bend in K so the same is true of fg, hence $fg \in \overline{\mathbb{T}[x]} \setminus I_K$. Conversely, if $\mathfrak{p} \subset \overline{\mathbb{T}[x]}$ is prime then by the factorization property of $\overline{\mathbb{T}[x]}$, any element of $\mathfrak{p}$ must be divisible by b_t for some $t \in \mathbb{T}$. The identity

$$t_1 r^{-1} b_{t_1} + b_{t_2} = b_r \text{ for any } r \in [t_1, t_2] \subset \mathbb{T}$$

then shows that $\mathfrak{p} = I_K$ where K is the convex hull of all such t. The statement about finitely generated primes and principal primes immediately follows. □

4. Toric varieties, integral $\mathbb{F}_1$-schemes, and their tropical models

4.1. **Toric schemes over $\mathbb{F}_1$ and $\mathbb{T}$.** Let $N \cong \mathbb{Z}^n$ be a lattice with dual lattice M. The datum of a rational polyhedral fan Δ in $N_{\mathbb{R}}$ determines an $\mathbb{F}_1$-scheme as in the usual construction of toric varieties. For each cone $\sigma \in \Delta$, there is a corresponding monoid $M_\sigma = M \cap \sigma^\vee$. If $\tau \subset \sigma$ is a face then M_τ is a localization of M_σ. Hence adjoining zeros to these monoids and taking Spec results in a collection of affine $\mathbb{F}_1$-schemes that glue together according to the incidence relations of the fan Δ to give an $\mathbb{F}_1$-scheme X^Δ. Base change to a ring R yields the usual toric variety over R associated with the fan Δ. The full subcategory of $\mathbb{F}_1$-schemes spanned by the objects of the form X^Δ is equivalent to the category of toric varieties and torus-equivariant morphisms.

Kajiwara [Kaj08] and Payne [Pay09] have each studied toric varieties over $\mathbb{T}$. The $\mathbb{T}$-points of the open torus stratum are canonically identified with the points of $N_{\mathbb{R}}$, and $X^\Delta(\mathbb{T})$ is then the polyhedral partial compactification of $N_{\mathbb{R}}$ dual to the fan Δ, with a codimension i stratum at infinity for each i-dimensional cone. For example, $\mathbb{P}^n(\mathbb{T})$ is an n-simplex.

Remark 4.1.1. Given a toric variety X_k, where k is a valued field, some authors refer to the corresponding tropical scheme $X_{\mathbb{T}}$ as the tropicalization of X_k.

Observe that the toric $\mathbb{F}_1$-schemes X^Δ described above are integral. However, the class of integral $\mathbb{F}_1$-schemes is larger; it allows objects that are non-normal and/or not of finite type. In the scheme-theoretic tropical geometry that we develop in this paper, the class of ambient spaces in which tropicalization makes sense can naturally be enlarged from toric varieties to integral $\mathbb{F}_1$-schemes.

4.2. **Cox's quotient construction.** It is straightforward to see that Cox's construction of (split) toric varieties as quotients of affine space descends to $\mathbb{F}_1$. Let $X = X^\Delta$ be as above and suppose the rays $\Delta(1)$ span $N_\mathbb{R}$, i.e., X has no torus factors. We define the *Cox algebra* as the free $\mathbb{F}_1$-algebra on the set of rays: $\mathrm{Cox}(X) := \mathbb{F}_1[x_\rho \mid \rho \in \Delta(1)]$.

For any field k the toric variety X_k is split and the divisor class group is independent of the field k, so we can formally define $\mathrm{Cl}(X) := \mathbb{Z}^{\Delta(1)}/M$, where

$$M \hookrightarrow \mathbb{Z}^{\Delta(1)},\ m \mapsto (m \cdot u_\rho)_{\rho\in\Delta(1)},$$

and u_ρ denotes the first lattice point on the ray $\rho \in N_\mathbb{R}$.

The Cox algebra has a grading by the divisor class group, via the composition

$$\mathrm{Cox}(X) \setminus \{0\} \cong \mathbb{N}^{\Delta(1)} \hookrightarrow \mathbb{Z}^{\Delta(1)} \twoheadrightarrow \mathrm{Cl}(X),$$

and the graded pieces are the eigenspaces for the action of the dual group

$$G := \mathrm{Hom}(\mathrm{Cl}(X),\mathbb{Z}) \subset \mathrm{Hom}(\mathbb{Z}^{\Delta(1)},\mathbb{Z})$$

on $\mathrm{Spec}\,\mathrm{Cox}(X) \cong \mathbb{A}^{\Delta(1)}_{\mathbb{F}_1}$.

Each ray $\rho \in \Delta(1)$ determines a coherent sheaf on X, the global sections of which are naturally isomorphic to the $\mathbb{F}_1$-module of homogeneous elements in $\mathrm{Cox}(X)$ of degree $[\rho]$. If X is complete then each graded piece is finite and the sections of this $\mathbb{F}_1$-sheaf are naturally the lattice-points in a polytope.

The irrelevant ideal $B \subset \mathrm{Cox}(X)$ is generated by the elements $x_\sigma := \prod_{\rho\notin\sigma(1)} x_\rho$ for all cones $\sigma \in \Delta$. This determines closed and open subschemes, respectively,

$$V(B) = \mathrm{Spec}\,(\mathrm{Cox}(X)/\langle x_\sigma \sim 0\rangle_{\sigma\in\Delta}) \text{ and } U := \mathbb{A}^{\Delta(1)}_{\mathbb{F}_1} \setminus V(B).$$

Indeed, as noted in §3.3 it is not generally true that the complement of a closed immersion is Zariski-open, but for congruences induced by ideals (cf. Proposition 2.4.6) this is the case, so there is an induced $\mathbb{F}_1$-scheme structure on the complement of $V(B)$.

Proposition 4.2.1. *With notation as above, X is the categorical quotient U/G in $\mathbb{F}_1$-schemes.*

Proof. This is an immediate translation of [Cox95, Theorem 2.1] and its proof to the setting of monoids. We cover U by $\mathbb{F}_1$-open affine G-invariant charts $U_\sigma := \mathrm{Spec}\,\mathrm{Cox}(X)[x_\sigma^{-1}]$ and observe that Cox's argument carries over to show that

$$\mathrm{Cox}(X)[x_\sigma^{-1}]^G = \mathrm{Cox}[x_\sigma^{-1}]_0 \cong \sigma^\vee \cap M.$$

This clearly implies that for this chart we have the categorical quotient

$$U_\sigma/G = \mathrm{Spec}\,\sigma^\vee \cap M,$$

and following Cox's argument again we see that the way these affine quotients glue together to yield the categorical quotient U/G is identical to the way the affine charts corresponding to the cones in the fan Δ glue together to produce the toric variety X. □

5. BEND LOCI AS SCHEMES

In this section we define *bend locus* schemes; a bend locus is the tropical analogue of the zero locus of a regular function (or more generally, a section of a line bundle). These tropical bend loci are one of the key elements of the framework developed in this paper, and they are the basic building blocks of scheme-theoretic tropicalization.

Recall that over a ring R, a polynomial $f \in R[x_1, \ldots, x_n]$ determines a zero locus in $\mathbb{A}^n_R$ as the set of points where f vanishes, but it has the additional structure of a scheme over R given by $\operatorname{Spec} R[x_1, \ldots, x_n]/(f)$. There are various heuristic arguments (e.g., [RGST05, §3], [Mik06, §3.1]) that the correct analogue of zero locus in the tropical setting is the locus of points where the piecewise linear graph of a tropical polynomial is nonlinear—i.e., the locus where the graph "bends". The relevant question in this setting is then, how to endow this set with a semiring scheme structure, and to do so in a way that generalizes from affine space to a larger class of $\mathbb{F}_1$ schemes and allows for coefficients in an arbitrary idempotent semiring rather than just $\mathbb{T}$.

Endowing the set-theoretic bend locus with the structure of a closed subscheme means realizing it as the set of solutions to a system of polynomial equations over $\mathbb{T}$—more precisely, we must construct a congruence on the coordinate algebra of the ambient affine scheme (and a quasi-coherent congruence sheaf in the non-affine case) such that the $\mathbb{T}$-points of the quotient form the set-theoretic bend locus. To this end, given an idempotent semiring S, an $\mathbb{F}_1$-algebra M (which is the set of monomials) and $f \in M \otimes S$, we construct a congruence $\langle \mathcal{B}(f) \rangle$ which defines the bend locus $\mathcal{B}end(f)$ of f as a closed subscheme of $\operatorname{Spec} M \otimes S$. The generators of this congruence are called the *bend relations* of f. When the ambient space is a torus and $S = \mathbb{T}$, the $\mathbb{T}$-points of this scheme constitute the set-theoretic bend locus of f.

While the $\mathbb{T}$-points alone are not enough to uniquely determine the scheme structure on $\mathcal{B}end(f)$, the particular scheme structure we propose here appears quite natural and allows for a robust theory of scheme-theoretic tropicalization to be developed. Moreover, it contains strictly more information than the set-theoretic bend locus, including multiplicities (§7.2), and it often determines f up to a scalar (see Lemma 5.1.4 below), which the $\mathbb{T}$-points do not in general.

Remark 5.0.2. A word of caution: the set-theoretic bend locus of a tropical polynomial is often called a "tropical hypersurface," and the set-theoretic tropicalization of a hypersurface is an example of one. However, when enriched with scheme structure, the tropicalization of a hypersurface is usually cut out by more relations than just the bend relations of a single tropical polynomial. We shall define tropicalization, in §6, by taking the bend relations of the coefficient-wise valuations of all elements in an ideal. See §8.1 for further discussion.

5.1. **The bend relations and affine bend loci.** Let S be an idempotent semiring and M an $\mathbb{F}_1$-module (or algebra). Given $f \in M \otimes S$ and $j \in \operatorname{supp}(f)$, we write $f_{\hat{j}}$ for the result of deleting the j term from f.

Definition 5.1.1. The *bend relations* of $f \in M \otimes S$ are the relations $\{f \sim f_{\hat{j}}\}_{j \in \operatorname{supp}(f)}$. We write $\mathcal{B}(f)$ for the module congruence on $M \otimes S$ generated by the bend relations. When M is an $\mathbb{F}_1$-algebra this generates a semiring congruence $\langle \mathcal{B}(f) \rangle$ and we define the *affine bend locus* of f, denoted $\mathcal{B}end(f)$, to be the closed subscheme of $\operatorname{Spec} M \otimes S$ defined by $M \otimes S \twoheadrightarrow M \otimes S / \langle \mathcal{B}(f) \rangle$.

Example 5.1.2. If $f = a_1x_1 + a_2x_2 + a_3x_3 \in S[x_1, x_2, x_3]$ then the bend relations of f are

$$a_1x_1 + a_2x_2 + a_3x_3 \sim a_2x_2 + a_3x_3 \sim a_1x_1 + a_3x_3 \sim a_1x_1 + a_2x_2.$$

Note that if λ is a unit in S then $\mathcal{B}(\lambda f) = \mathcal{B}(f)$, and if M is an $\mathbb{F}_1$-algebra and $u \in M^\times$ then $\langle \mathcal{B}(uf) \rangle = \langle \mathcal{B}(f) \rangle$.

Proposition 5.1.3. *Let M be an $\mathbb{F}_1$-algebra and $f \in M \otimes S$, with S a totally ordered idempotent semiring.*

(1) *The S-points of $\mathcal{B}end(f)$ are the points $p \in (\operatorname{Spec} M \otimes S)(S)$ where either the maximum of the terms of $f(p)$ is attained at least twice or $f(p) = 0_S$.*

(2) *If $S = \mathbb{T}$ and $M = \mathbb{F}_1[x_1^{\pm}, \dots, x_n^{\pm}]$ is the coordinate algebra of an $\mathbb{F}_1$-torus, then the $\mathbb{T}$-points of* $\operatorname{Spec} M \otimes \mathbb{T}$ *are* $(\mathbb{T}^{\times})^n = \mathbb{R}^n$ *and the $\mathbb{T}$-points of $\mathcal{B}end(f)$ are the points at which the function $X(\mathbb{T}) \to \mathbb{T}$ induced by f is nonlinear.*

Proof. A semiring homomorphism $p : M \otimes S \to S$ factors through the quotient by $\langle \mathcal{B}(f) \rangle$ if and only if $f(p) = f_{\widehat{i}}(p)$ for each i. This happens if and only if no single term in $f(p)$ is strictly larger than all others, when $|\mathrm{supp}(f)| \geq 2$, or $f(p) = 0_S$ when f consists of a single monomial.

We now prove the second statement. A homomorphism $p : \mathbb{T}[x_1^{\pm}, \dots, x_n^{\pm}] \to \mathbb{T}$ is determined by the n-tuple of tropical numbers $p(x_1), \dots, p(x_n) \in \mathbb{T}^{\times} = \mathbb{R}$, so we identify p with a point in $\mathbb{R}^n$. This Euclidean space is divided into convex polyhedral chambers as follows. For each term of f there is a (possibly empty) chamber consisting of all p for which that term dominates, the interior consisting of points where this term strictly dominates. Since f is the tropical sum (Euclidean maximum) of its terms, the chamber interiors are where the graph of f is linear and the walls are where the maximum is attained at least twice and hence the graph is nonlinear. □

In general one cannot recover a tropical polynomial from its set-theoretic bend locus (consider, e.g., $x^2 + ax + 0 \in \mathbb{T}[x]$ as $a \in \mathbb{T}$ varies). In the case of homogeneous polynomials this is manifest as the statement that the tropicalization of the Hilbert scheme of projective hypersurfaces is not a parameter space for set-theoretically tropicalized hypersurfaces (see [AN13, §6.1]). The following result says in particular that when enriched with its scheme structure, one can indeed recover, up to a scalar, a homogeneous tropical polynomial from its bend locus.

Lemma 5.1.4. *Suppose S is a semifield and $f \in M \otimes S$.*

(1) *The congruence $\mathcal{B}(f)$ determines f uniquely up to a scalar.*
(2) *If M is an $\mathbb{F}_1$-algebra that admits a grading by an abelian group such that $M_0 = 0_M$ and f is homogeneous, then $\langle \mathcal{B}(f) \rangle$ determines f up to a scalar.*

Remark 5.1.5. The hypotheses for (2) are satisfied by the Cox algebra of a toric scheme X over $\mathbb{F}_1$ whose base change to a ring is proper. We show below in §5.4 that a homogeneous polynomial in $\mathrm{Cox}(X_{\mathbb{T}})$ defines a closed subscheme of $X_{\mathbb{T}}$, generalizing the case of a homogeneous polynomial (in the usual sense) defining a tropical hypersurface in projective space.

Proof. For (1), write $f = \sum_{i=1}^{n} a_i m_i$ with $a_i \in S, m_i \in M$. If $n = 1$ then the result is obvious, otherwise consider the elements ϕ of the dual module $\mathrm{Hom}(M \otimes S, S)$ of the form $m_i \mapsto 0_S$ for all i except two indices, say j_1 and j_2. Such a homomorphism descends to the quotient by $\mathcal{B}(f)$ if and only if $a_{j_1}\phi(m_{j_1}) = a_{j_2}\phi(m_{j_2})$. In this way we recover the ratio of each pair of coefficients a_{j_1}, a_{j_2}, and hence the vector of all coefficients $(a_1, \dots, a_n)$ up to a scalar. Item (2) follows from (1) since the hypotheses guarantee that $\langle \mathcal{B}(f) \rangle_{\deg(f)} = \mathcal{B}(f)$, where the latter is viewed as a congruence on the module $M_{\deg(f)} \otimes S$. □

The following result expresses the functoriality of bend loci and is used throughout the sequel.

Lemma 5.1.6. *If $\varphi : M \otimes S \to N \otimes S$ is induced by an $\mathbb{F}_1$-morphism (i.e., map of pointed sets) $M \to N$ and $f \in M \otimes S$, then*

$$\varphi_* \mathcal{B}(f) \subset \mathcal{B}(\varphi(f)) \text{ and } \varphi_* \langle \mathcal{B}(f) \rangle \subset \langle \mathcal{B}(\varphi(f)) \rangle$$

with equality when φ is injective.

Proof. Since $\varphi_* \mathcal{B}(f)$ is generated by the image of the generators of $\mathcal{B}(f)$, it suffices to show that any relation of the form $\varphi(f) \sim \varphi(f_{\widehat{i}})$ is implied by a relation of the form $\varphi(f) \sim \varphi(f)_{\widehat{j}}$. Let g_0 be the term of f whose support is i and let $g_1, \cdots, g_n$ be the terms of f whose supports are identified with i by φ. The relation $\varphi(f)_{\widehat{\varphi(i)}} \sim \varphi(f)$ implies

$$\begin{aligned} \varphi(f_{\widehat{i}}) = \varphi(f)_{\widehat{\varphi(i)}} + \varphi(g_1 + \cdots + g_n) &\sim \varphi(f) + \varphi(g_1 + \cdots + g_n) \\ &= \varphi(f + g_1 + \cdots + g_n) = \varphi(f), \end{aligned}$$

where the last equality follows from the idempotency of addition in S. When φ is injective it is clear that $\varphi(f_{\widehat{i}}) = \varphi(f)_{\widehat{\varphi(i)}}$. The semiring statement follows from the module statement. □

5.2. **Unicity of the bend relations.** Here we show that the bend relations of a tropical linear form f are uniquely determined as the dual module of the tropical hyperplane defined by f. In this section we let S be a totally ordered idempotent semifield and M a finitely generated $\mathbb{F}_1$-module (i.e., a finite pointed set). Then $M \otimes S$ is a free S-module of finite rank and it is canonically isomorphic to its dual $(M \otimes S)^\vee$ and hence also its double dual. We can thus think of $M \otimes S$ as the space of linear functions on its dual. A quotient of $M \otimes S$ dualizes to a submodule; however, a general submodule $W \subset M \otimes S$ only dualizes to a quotient if every linear map $W \to S$ extends to a linear map $M \otimes S \to S$.

Tropical hyperplanes were defined in [SS04] as the set-theoretic tropicalization of classical hyperplanes. The following definition is from [Fre13, Chapter 4].

Definition 5.2.1. Given $f \in M \otimes S$, let $L_f \subset (M \otimes S)^\vee$ denote the set of points x where either $f(x) = 0_S$ or the maximum of the terms of $f(x)$ is attained at least twice (this set is sometimes called the "tropical vanishing locus" of f). A *tropical hyperplane* in $(M \otimes S)^\vee$ is a subset of the form L_f for some $f \in M \otimes S$.

Note that, by Proposition 5.1.3, for $f \in M \otimes S$ we have $(M \otimes S/\mathcal{B}(f))^\vee = L_f$.

Theorem 5.2.2. *The canonical map from $M \otimes S/\mathcal{B}(f)$ to its double dual $L_f^\vee$ is an isomorphism.*

Proof. Tropical hyperplanes are finitely generated as S-modules [Fre13, p. 69], so by [WJK13, Theorem 3.4], every linear map $L_f \to S$ extends to a linear map $(M \otimes S)^\vee \to S$ and hence $L_f^\vee = (M \otimes S/\mathcal{B}(f))^{\vee\vee}$ is a quotient of $(M \otimes S)^{\vee\vee}$. It follows that the bottom arrow of the following diagram is surjective:

$$\begin{tikzcd} M \otimes S \arrow[r, "\cong"] \arrow[d, two heads] & (M \otimes S)^{\vee\vee} \arrow[d, two heads] \\ M \otimes S/\mathcal{B}(f) \arrow[r] & M \otimes S/\mathcal{B}(f)^{\vee\vee}. \end{tikzcd}$$

To show injectivity, we will show that if $g, g' \in M \otimes S$ are equal at each point $p \in L_f$, then they are equal in the quotient by $\mathcal{B}(f)$. It is immediate that g and g' coincide outside $\mathrm{supp}(f)$, so we must show that they agree modulo $\mathcal{B}(f)$ over $\mathrm{supp}(f)$. If f is a monomial then this is trivial, so assume $|\mathrm{supp}(f)| \geq 2$.

For any $a \in M$, let $\chi_a \in (M \otimes S)^\vee$ denote the map sending a to 1_S and all other basis elements to 0_S. For any pair of distinct elements $a, b \in \mathrm{supp}(f)$, consider the element $p_{ab} \in (M \otimes S)^\vee$ given by the formula

$$p_{ab} = \left(\frac{1_S}{\chi_a(f)}\right)\chi_a + \left(\frac{1_S}{\chi_b(f)}\right)\chi_b.$$

Idempotency of addition implies that p_{ab} factors through the quotient by $\mathcal{B}(f)$, i.e., $p_{ab} \in L_f$. Write $g_a := \chi_a(g)/\chi_a(f)$ and likewise for g', and let m and m' denote the minimum of the g_a and g'_a, respectively. By hypothesis, $p_{ab}(g) = p_{ab}(g')$ for all $a, b \in \mathrm{supp}(f)$, which yields the set of equations

$$(\mathscr{R}_{ab}) : g_a + g_b = g'_a + g'_b. \tag{5.2.1}$$

Modulo the congruence $\mathcal{B}(f)$, we may assume the minima m and m' are each attained at least twice; for, if m is attained only once by some g_a and g_b is the minimum of the remaining non-minimal terms, then

$$\begin{aligned} g &= g + g_b f_{\widehat{a}} \\ &\sim g + g_b f \end{aligned}$$

and in the final expression the minimum is equal to g_b and is attained at least twice, so we replace g with this and likewise for g'. Now, for a and b such that $g_a = g_b = m$, the equation $(\mathscr{R}_{ab})$ implies that $m \geq m'$, and choosing a and b such that $g'_a = g'_b = m'$ we likewise see that $m \leq m'$. Hence $m = m'$.

Now let $a_1,\dots,a_n$ be the elements of $\mathrm{supp}(f)$ ordered so that $g_{a_1}=g_{a_2}\leq\cdots\leq g_{a_n}$. Since $m=m'$, the equation $(\mathscr{R}_{a_1a_2})$ implies that $g_{a_1}=g_{a_2}=g'_{a_1}=g'_{a_2}$. For any $k>2$, g_{a_k} and g'_{a_k} are both greater than or equal to m, and so the equation $(\mathscr{R}_{a_1a_k})$ implies that $g_{a_k}=g'_{a_k}$. Thus we have shown that g and g' are equal in $M\otimes S/\mathcal{B}(f)$. □

5.3. **Global bend loci.** Let X be an integral $\mathbb{F}_1$-scheme, $\mathscr{L}$ a line bundle on X (i.e., a locally free sheaf of rank one), and f a global section of the line bundle $\mathscr{L}\otimes S$ on X_S. We now show how to associate a global bend locus $\mathcal{B}end(f)$ to this section by patching together the affine bend loci defined in §5.1.

Let $\{U_\alpha\}$ be a covering of X by affine opens, with $\{U'_\alpha\}$ the induced cover of X_S, and choose local trivializations $\psi_\alpha:\mathscr{L}|_{U_\alpha}\cong\mathscr{O}_X|_{U_\alpha}$ (which induce local trivializations after base change to S that we denote by the same symbol). For each α, the congruence $\langle\mathcal{B}(\psi_\alpha(f))\rangle$ on $\mathscr{O}_{X_S}(U'_\alpha)$ defines a quasi-coherent congruence sheaf on U'_α.

Proposition 5.3.1. *The congruence sheaves $\langle\mathcal{B}(\psi_\alpha(f))\rangle$ glue together to define a quasi-coherent congruence sheaf $\langle\mathcal{B}(f)\rangle$ on X_S that is independent of the covering and local trivializations, and hence a well-defined closed subscheme $\mathcal{B}end(f)\subset X_S$.*

Proof. On the intersection $U'_\alpha\cap U'_\beta$ the trivializations ψ_α and ψ_β differ by a unit u in the monoid $\mathscr{O}_X(U_\alpha\cap U_\beta)$ (i.e., an invertible monomial). Hence $\langle\mathcal{B}(\psi_\alpha(f))\rangle$ and $\langle\mathcal{B}(\psi_\beta(f))\rangle$ are equal when restricted to the intersection. By Lemma 5.1.6 and Proposition 3.1.3, formation of the bend relations commutes with restriction to $\mathbb{F}_1$-open subschemes, since X is integral, so the resulting congruence sheaf is invariant under refinement of the original cover. □

Proposition 5.3.2. *Let $\varphi:X\to Y$ be a morphism of integral $\mathbb{F}_1$-schemes, $\mathscr{L}$ a line bundle on Y, and $f\in\Gamma(Y_S,\mathscr{L}\otimes S)$. Then $\varphi\otimes S:X_S\to Y_S$ maps $\mathcal{B}end(\varphi^*f)$ into $\mathcal{B}end(f)$.*

Proof. It suffices to check on affine patches, where the result follows from Lemma 5.1.6. □

5.4. **Tropical Proj and Cox.** If M is an $\mathbb{N}$-graded monoid-with-zero then $M\otimes S$ is an $\mathbb{N}$-graded S-algebra and we can form the scheme $\mathrm{Proj}\,M\otimes S$ in the usual way. For $f\in M\otimes S$ homogeneous of degree d, the congruence $\langle\mathcal{B}(f)\rangle$ is homogeneous in that $M\otimes S/\langle\mathcal{B}(f)\rangle$ inherits the grading. The bend locus of $f\in\Gamma(\mathrm{Proj}\,M\otimes S,\mathscr{O}(d))$ is the image of the morphism

$$\mathrm{Proj}\,M\otimes S/\langle\mathcal{B}(f)\rangle\to\mathrm{Proj}\,M\otimes S;$$

this is a special case of Proposition 5.4.1 below.

More generally, let $X=X^\Delta$ be a toric variety over S without torus factors and consider its $\mathrm{Cl}(X)$-graded algebra $\mathrm{Cox}(X)=S[x_\rho\mid\rho\in\Delta(1)]$. As mentioned in §4.2, global sections of line bundles correspond to homogeneous polynomials in the Cox algebra. Recall that $X=U/G$, where U is the complement of the vanishing of the irrelevant ideal and $G=\mathrm{Hom}(\mathrm{Cl}(X),\mathbb{Z})$.

Proposition 5.4.1. *If $f\in\mathrm{Cox}(X)$ is homogeneous with degree given by the class of a line bundle, then the bend locus determined by the corresponding global section is the categorical quotient*

$$(\mathcal{B}end(f)\cap U)/G.$$

Proof. For each cone $\sigma\in\Delta$, let $x_{\hat\sigma}:=\prod_{\rho\notin\sigma(1)}x_\rho$. The restriction of $\mathcal{B}end(f)$ to the affine open $\mathrm{Spec}\,\mathrm{Cox}(X)[x_{\hat\sigma}^{-1}]\subset U$ is defined by $\iota_*\langle\mathcal{B}(f)\rangle$, where $\iota:\mathrm{Cox}(X)\to\mathrm{Cox}(X)[x_{\hat\sigma}^{-1}]$ is the localization map. The subalgebra of G-invariants on this chart is the degree zero piece

$$(\mathrm{Cox}(X)[x_{\hat\sigma}^{-1}]/\iota_*\langle\mathcal{B}(f)\rangle)_0=\mathrm{Cox}(X)[x_{\hat\sigma}^{-1}]_0/\iota_*\langle\mathcal{B}(f)\rangle_0,$$

so this defines the restriction of $(\mathcal{B}end(f)\cap U)/G\subset U/G$ to the affine open

$$X_\sigma:=\mathrm{Spec}\,(\mathrm{Cox}(X)[x_{\hat\sigma}^{-1}]_0)\subset X=U/G.$$

On the other hand, a trivialization on X_σ of a line bundle $\mathscr{L}$ on X for which f is a section corresponds to a choice of unit $g \in \mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]$ with $\deg(g) = \deg(f)$. Then the bend locus of $f \in \Gamma(X, \mathscr{L})$ is defined on this affine patch by $\langle \mathcal{B}(\frac{f}{g})\rangle = \iota_* \langle \mathcal{B}(f)\rangle_0$, exactly as above. □

6. Scheme-theoretic tropicalization

Given an integral $\mathbb{F}_1$-scheme X and a valued ring $\nu : R \to S$, we define a tropicalization map $\mathcal{T}rop_X^\nu$ (or simply $\mathcal{T}rop$ if these parameters are clear from the context) from the poset of closed subschemes of X_R to the poset of closed subschemes of X_S. It sends Z to the scheme-theoretic intersection of the bend loci of the coefficient-wise valuations of all functions in the defining ideal. This is functorial in X, compatible with the Cox construction, and when $S = \mathbb{T}$ the composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, -)$ recovers the extended tropicalization functor of Kajiwara-Payne. Moreover, these tropicalizations form an algebraic family as the valuation varies.

6.1. **Construction of the tropicalization functor.** We first construct $\mathcal{T}rop_X^\nu$ in the case when $X = \mathrm{Spec}\,M$ is an integral affine $\mathbb{F}_1$-scheme and then glue these together to define it in general. To construct the affine tropicalization functor, we first define a more general linear tropicalization functor that we think of as tropicalizing linear subspaces; the affine tropicalization is then obtained by applying linear tropicalization to an ideal.

A valuation $\nu : R \to S$ induces a set map $M \otimes R \to M \otimes S$, also denoted ν, given by coefficient-wise valuation. Note that this is *not* a semiring homomorphism.

Definition 6.1.1 (Linear tropicalization). Given an $\mathbb{F}_1$-module M and an R-submodule $N \subset M \otimes R$, let $\mathcal{T}rop^\nu(N)$ denote the module congruence on $M \otimes S$ generated by $\mathcal{B}(\nu(f))$ for all $f \in N$.

Remark 6.1.2. One should think of this $\mathcal{T}rop$ as sending the R-submodule

$$(M \otimes R/N)^\vee \subset (M \otimes R)^\vee$$

to the S-submodule

$$(M \otimes S/\mathcal{T}rop(N))^\vee \subset (M \otimes S)^\vee,$$

although if R is not a field then there might not be a well-defined induced map $\mathscr{T}$ from the set of submodules of $(M \otimes R)^\vee$ to the set of submodules of $(M \otimes S)^\vee$ — for example, there might be two submodules having the same dual but distinct tropicalizations. However, when $R = k$ is a field then the map $\mathscr{T}$ *is* well-defined; in this case, if $|M \setminus 0_M| = n$ and $S = \mathbb{T}$, we will see that $\mathscr{T}$ sends linear subspaces of k^n to tropical linear subspaces (in the sense of [SS04]) of $\mathbb{T}^n$ and coincides with the classical definition of tropicalization of linear spaces.

Lemma 6.1.3. *If M is an integral $\mathbb{F}_1$-algebra and $I \subset M \otimes R$ is an ideal then the module congruence $\mathcal{T}rop^\nu(I)$ is in fact a semiring congruence.*

Proof. By Lemma 2.4.5, it suffices to show that $\mathcal{T}rop(I)$ is closed under multiplication by generating relations. Since M is an integral $\mathbb{F}_1$-algebra, multiplying any generating relation by a monomial yields another generating relation, so the S-module congruence $\mathcal{T}rop(I)$ is actually an $M \otimes S$-module congruence. Suppose $g \sim h$ is an arbitrary relation in $\mathcal{T}rop(I)$ and $f \sim f_{\hat{i}}$ is a generating relation. Then by the observation that $\mathcal{T}rop(I)$ is an $M \otimes S$-submodule, the two relations

$$gf \sim hf \text{ and } hf \sim hf_{\hat{i}}$$

are both in $\mathcal{T}rop(I)$, and hence, by transitivity, the product relation $gf \sim hf_{\hat{i}}$ is as well. □

In light of Lemma 6.1.3 above, the following definition makes sense.

Definition 6.1.4 (Affine tropicalization). If M is an $\mathbb{F}_1$-algebra and $Z \subset \mathrm{Spec}\,M \otimes R$ is the closed subscheme corresponding to an ideal I, then we define $\mathcal{T}rop^\nu(Z) \subset \mathrm{Spec}\,M \otimes S$ to be closed subscheme determined by the semiring congruence $\mathcal{T}rop^\nu(I)$.

Proposition 6.1.5. $\mathcal{T}rop^{\nu}(Z) = \bigcap_{f\in I} \mathcal{B}end(\nu(f))$.

Proof. By Lemma 6.1.3, both sides correspond to the semiring congruence $\langle \mathcal{B}(\nu(f))\rangle_{f\in I}$. □

Now let X be an arbitrary integral $\mathbb{F}_1$-scheme and $Z \subset X_R$ a closed subscheme. Choose a covering $\{U_\alpha\}$ of X by affine $\mathbb{F}_1$-schemes, and let $\{U_{\alpha,R}\}$ and $\{U_{\alpha,S}\}$ be the induced coverings of X_R and X_S.

Lemma 6.1.6. *There is an equality $\mathcal{T}rop(Z\cap U_{\alpha,R})\cap U_{\beta,S} = U_{\alpha,S}\cap \mathcal{T}rop(Z\cap U_{\beta,R})$ of closed subschemes of the open subscheme $U_{\alpha,S}\cap U_{\beta,S} \subset X_S$. Thus, as one varies α, the subschemes $\mathcal{T}rop(Z\cap U_{\alpha,R}) \subset U_{\alpha,S}$ glue together to determine a closed subscheme*

$$\mathcal{T}rop_X(Z) \subset X_S,$$

which is independent of the choice of cover.

Proof. This follows immediately from Lemma 5.1.6, since integrality implies all localization maps are injective. □

6.2. Basic properties of $\mathcal{T}rop^{\nu}_X$.

Proposition 6.2.1. *For X an integral $\mathbb{F}_1$-scheme, $\mathcal{T}rop_X(X_R) = X_S$.*

Proof. When I is the zero ideal, $\mathcal{T}rop(I)$ is clearly the trivial congruence. □

Thus, one can view the tropical model X_S of X as a canonical tropicalization of X_R.

Lemma 6.2.2. *Let $W \subset X$ be a locally closed integral subscheme of an integral $\mathbb{F}_1$-scheme X such that W is locally defined by equations of the form $x \sim 0$. Then $\mathcal{T}rop_X(Z)\cap W_S = \mathcal{T}rop_W(Z\cap W_R)$. In particular, $\mathcal{T}rop_X(W_R) = W_S$.*

Proof. It suffices to show this in the affine case. By Lemma 5.1.6, tropicalization commutes with restriction to an open subscheme defined over $\mathbb{F}_1$, so we are reduced to the case when W is a closed subscheme, and then the result follows from Lemma 8.1.4 below. The equality $\mathcal{T}rop_X(W_R) = W_S$ then follows from Proposition 6.2.1 □

For X an $\mathbb{F}_1$-scheme and R a (semi)ring, a morphism $\operatorname{Spec} R \to X_R$ is given locally by a multiplicative map from a monoid to R. Thus, a valuation $\nu : R \to S$ determines a map $\widetilde{\nu} : X(R) \to X(S)$. In particular, if $X = \mathbb{A}^n_{\mathbb{F}_1}$ then $\widetilde{\nu} : R^n \to S^n$ is coordinate-wise valuation.

Proposition 6.2.3. *The tropicalization of a point is the image of the point under $\widetilde{\nu}$; more precisely, if $Z \subset X_R$ is the closed subscheme corresponding to a point $p \in X(R)$, then $\mathcal{T}rop^{\nu}_X(Z)$ is the closed subscheme corresponding to the point $\widetilde{\nu}(p) \in X(S)$.*

Proof. Locally, $X = \operatorname{Spec} M$; let $\{x_i\}_{i\in A}$ be a set of generators for the monoid M. The point p is determined by the collection $\{p(x_i)\}_{i\in A}$ of elements of R, and Z is defined by the ideal $I := (x_i - p(x_i))_{i\in A}$. On the other hand, $\widetilde{\nu}(p)$ is determined by the elements $\widetilde{\nu}(p)(x_i) = \nu(p(x_i))$ of S, and since $\mathcal{B}(\nu(x_i - p(x_i)))$ is generated by the relation $x_i \sim \nu(p(x_i))$, it follows from Proposition 8.1.3 that $\mathcal{T}rop(I)$ is generated by these relations for all $i \in A$ and hence defines the subscheme $\widetilde{\nu}(p)$. □

6.3. **Relation to the Kajiwara-Payne extended tropicalization functor.** We now show that the above scheme-theoretic tropicalization recovers the Kajiwara-Payne extended tropicalization functor [Kaj08, Pay09] upon composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\operatorname{Spec}\mathbb{T},-)$.

Let X be a toric variety over $\mathbb{F}_1$, and let k be an algebraically closed field equipped with a non-trivial valuation $\nu : k \to \mathbb{T}$. The Kajiwara-Payne extended tropicalization is a map of posets

$$\underline{\mathcal{T}rop}_X : \{\text{subvarieties of } X_k\} \to \{\text{subsets of } X(\mathbb{T})\}$$

sending $Z \subset X_k$ to the Euclidean closure of the image of $Z(k)$ under the map $\widetilde{\nu} : X(k) \to X(\mathbb{T})$.

Theorem 6.3.1. *The set of* $\mathbb{T}$*-points of* $\mathcal{T}rop_X(Z)$ *coincides with* $\underline{\mathcal{T}rop}_X(Z)$ *as a subset of* $X(\mathbb{T})$.

Proof. By [Pay09, Prop. 3.4], the set-theoretic tropicalization can be computed stratum by stratum. I.e., if W is a torus orbit in X then $\underline{\mathcal{T}rop}_X(Z) \cap W(\mathbb{T}) = \underline{\mathcal{T}rop}_W(Z \cap W_k)$. By the Fundamental Theorem of tropical geometry [MS, Theorem 3.2.4] (a.k.a Kapranov's Theorem in the case of a hypersurface), $\underline{\mathcal{T}rop}_W(Z \cap W_k)$ is the subset of points in $W(\mathbb{T}) \cong \mathbb{R}^n$ where the graph of each nonzero function in the ideal defining $Z \cap W_k$ is nonlinear. By Proposition 5.1.3 and Lemma 6.2.2, this is equal to the set of $\mathbb{T}$-points of $\mathcal{T}rop_X(Z) \cap W_{\mathbb{T}}$. $\square$

6.4. **Functoriality.** We now examine the functoriality properties of the scheme-theoretic tropicalization map

$$\mathcal{T}rop_X^\nu : \{\text{closed subschemes of } X_R\} \to \{\text{closed subschemes of } X_S\}.$$

We show that it is functorial in X in the sense below, and under certain additional hypotheses it is functorial in the valuation ν.

For a (semi)ring R, let $\mathcal{P}(R)$ denote the category of pairs

$$(X \text{ an integral } \mathbb{F}_1\text{-scheme}, Z \subset X_R \text{ a closed subscheme}),$$

where a morphism $(X,Z) \to (X',Z')$ is a morphism $\Phi : X \to X'$ such that $\Phi(Z) \subset Z'$.

Proposition 6.4.1. *The tropicalization maps* $\{\mathcal{T}rop_X^\nu\}$ *determine a functor* $\mathcal{T}rop^\nu : \mathcal{P}(R) \to \mathcal{P}(S)$ *sending* (X,Z) *to* $(X, \mathcal{T}rop_X^\nu(Z))$.

Proof. Given an arrow $(X,Z) \to (X',Z')$ in $\mathcal{P}(R)$, we must show that $\Phi(\mathcal{T}rop_X(Z)) \subset \mathcal{T}rop_{X'}(Z')$. It suffices to show this in the affine case: $X = \operatorname{Spec} M$, $X' = \operatorname{Spec} M'$, the map Φ is given by a monoid homomorphism $\varphi : M' \to M$, and Z and Z' are given by ideals $I \subset M \otimes R$ and $I' \subset M' \otimes R$ with $\varphi(I') \subset I$. The claim is now that $\varphi_* \mathcal{T}rop(I') \subset \mathcal{T}rop(I)$, and for this it suffices to show that $\varphi_* \mathcal{B}(\nu(f)) \subset \mathcal{B}(\nu(\varphi(f)))$ for any $f \in M' \otimes R$. In fact, we will show that each generating relation

$$\varphi(\nu(f)) \sim \varphi(\nu(f)_{\widehat{i}}) \tag{6.4.1}$$

of $\varphi_* \mathcal{B}(\nu(f))$ is implied by the corresponding relation

$$\nu(\varphi(f)) \sim \nu(\varphi(f))_{\widehat{\varphi(i)}} \tag{6.4.2}$$

in $\mathcal{B}(\nu(\varphi(f)))$ by adding the RHS of (6.4.1) to both sides. We show this by comparing coefficients term-by-term. For $\ell \in \operatorname{supp}(f)$, let $a_\ell \in R$ denote the coefficient of ℓ. For each $m \in \operatorname{supp}(\varphi(f))$ with $m \neq \varphi(i)$, the coefficients of m on boths sides of (6.4.1) are equal to

$$\sum_{\ell \in \varphi^{-1}(m)} \nu(a_\ell). \tag{6.4.3}$$

The coefficients of m on either side in (6.4.2) are both equal to

$$\nu\left(\sum_{\ell \in \varphi^{-1}(m)} a_\ell\right). \tag{6.4.4}$$

By the subadditivity property of the valuation, adding (6.4.4) to (6.4.3) yields (6.4.3).

We now examine the coefficients of $\varphi(i)$ in (6.4.1) and (6.4.2); they are, respectively,

(6.4.5)
$$\underset{\textit{(LHS)}}{\sum_{\ell\in\varphi^{-1}(\varphi(i))} \nu(a_\ell)}, \qquad \underset{\textit{(RHS)}}{\sum_{\ell\in\varphi^{-1}(\varphi(i))\smallsetminus\{i\}} \nu(a_\ell)}$$

and

(6.4.6)
$$\underset{\textit{(LHS)}}{\nu\left(\sum_{\ell\in\varphi^{-1}(\varphi(i))} a_\ell\right)} \qquad \underset{\textit{(RHS)}}{0_S}.$$

By Lemma 2.5.3 part (2), adding the RHS of (6.4.5) to both sides of (6.4.6) yields (6.4.5). $\square$

We now turn to the dependence on ν.

Proposition 6.4.2. *Let $\nu : R \to S$ be a valuation and $\varphi : S \to T$ a map of semirings. Then*
$$\mathcal{T}rop_X^{\varphi\circ\nu}(Z) = \mathcal{T}rop_X^{\nu}(Z) \times_{\operatorname{Spec} S} \operatorname{Spec} T$$
as subschemes of X_T.

Proof. It suffices to prove this in the case X is affine, so assume $X = \operatorname{Spec} A$ for some $\mathbb{F}_1$-algebra A, and let $I \subset A \otimes R$ be the ideal defining $Z \subset X$.

Given a module congruence J on $M \otimes S$, the canonical isomorphism $(M \otimes S) \otimes_S T \cong M \otimes T$ descends to an isomorphism
$$\big(M \otimes S/J\big) \otimes_S T \cong M \otimes T/\varphi_* J.$$
The claim follows from this by taking $M = A$, $J = \mathcal{T}rop^{\nu}(I)$ and observing that $\varphi_*\mathcal{T}rop^{\nu}(I) = \mathcal{T}rop^{\varphi\circ\nu}(I)$. $\square$

6.5. **Moduli of valuations and families of tropicalizations.** Let $\mathcal{V}al(R) := \operatorname{Spec} S^R_{univ}$ be the affine $\mathbb{B}$-scheme corresponding to the semiring of values associated with the universal valuation on R defined in §2.5. By Proposition 2.5.4, $\mathcal{V}al(R)$ represents the functor on affine $\mathbb{B}$-schemes,
$$\operatorname{Spec} S \mapsto \{\text{valuations } R \to S\}.$$
Thus $\mathcal{V}al(R)$ is the moduli scheme of valuations on R. This is a refinement of the observation of Manon [Man11] that the set of all valuations with semiring of values $\mathbb{T}$ forms a fan. In particular, the $\mathbb{T}$-points of $\mathcal{V}al(R)$ are the usual non-archimedean valuations on R—i.e., $\mathcal{V}al(R)(\mathbb{T})$ is equal to the underlying set of the Berkovich analytification $(\operatorname{Spec} R)^{an}$.

As a special case of Proposition 6.4.2 we have the following (Theorem C part (1) from the introduction).

Theorem 6.5.1. *Given an integral $\mathbb{F}_1$-scheme X, a ring R, and a subscheme $Z \subset X_R$, the tropicalization of Z with respect to the universal valuation, $\mathcal{T}rop_X^{\nu^R_{univ}}(Z)$, forms an algebraic family of $\mathbb{B}$-schemes over $\mathcal{V}al(R)$ such that the fibre over each valuation ν is $\mathcal{T}rop_X^{\nu}(Z)$.*

6.6. **Compatibility with Cox's quotient construction.** Let $X = X^\Delta$ be a toric scheme over $\mathbb{F}_1$ and recall (§4.2) that $X = U/G$, where $U \subset \mathbb{A}^{\Delta(1)}$ is the complement of the vanishing of the irrelevant ideal and $G = \operatorname{Hom}(\operatorname{Cl}(X), \mathbb{Z})$. A homogeneous ideal $I \subset \operatorname{Cox}(X_R) = R[x_\rho \mid \rho \in \Delta(1)]$ determines a closed subscheme $Z \subset X_R$, and if Δ is simplicial then every closed subscheme arises in this way [Cox95, Theorem 3.7]. The scheme Z is the categorical quotient of the G-invariant locally closed subscheme $\widetilde{Z} \cap U_R \subset \mathbb{A}_R^{\Delta(1)}$, where $\widetilde{Z} := V(I)$. In other words, we have
$$Z = (\widetilde{Z} \cap U_R)/G \subset U_R/G = X_R.$$

Theorem 6.6.1. *Tropicalization commutes with the Cox quotient:*

$$\mathcal{T}rop_X(Z) = (\mathcal{T}rop_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \cap U_S)/G \subset U_S/G = X_S.$$

Proof. In the notation of §5.4, we can cover X by open affines X_σ for $\sigma \in \Delta$. The subscheme $Z \subset X_R$ is defined in each such chart as $Z_\sigma := \operatorname{Spec}(\operatorname{Cox}(X_R)[x_{\widehat{\sigma}}^{-1}]_0/I'_0)$, where I' denotes the image of I in this localization and I'_0 its degree zero part. The tropicalization $\mathcal{T}rop_X(Z)$ is then obtained by gluing the affine tropicalizations

$$\mathcal{T}rop_{X_\sigma}(Z_\sigma) = \operatorname{Spec}\operatorname{Cox}(X_S)[x_{\widehat{\sigma}}^{-1}]_0/\mathcal{T}rop(I'_0).$$

Since the valuation preserves degree and taking quotients commutes with taking degree zero part, this is the spectrum of $(\operatorname{Cox}(X_S)[x_{\widehat{\sigma}}^{-1}]/\mathcal{T}rop(I'))_0$. As in §4.2, taking degree zero here coincides with taking the subalgebra of G-invariants, and by Lemma 5.1.6 tropicalization commutes with $\mathbb{F}_1$-localization, so this is the categorical quotient of $\mathcal{T}rop_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \setminus V(x_{\widehat{\sigma}})$. In the usual way, these categorical quotients patch together to yield the categorical quotient of $\mathcal{T}rop_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \cap U_S$. □

7. Numerical invariants

Here we show that there is a natural way to define Hilbert polynomials for the class of tropical projective subschemes that arise as tropicalizations, and that tropicalization preserves the Hilbert polynomial. We also show that for a projective hypersurface, the multiplicities (sometimes called weights) decorating the facets of its tropicalization, which are frequently used in tropical intersection theory, are encoded in the tropical scheme structure.

7.1. **The Hilbert polynomial.** First recall the classical setup. Let k be a field, $A := \mathbb{F}_1[x_0, \dots, x_n]$ the homogeneous coordinate algebra of $\mathbb{P}^n_{\mathbb{F}_1}$, and $Z \subset \mathbb{P}^n_k$ a subscheme defined by a homogeneous ideal $I \subset A \otimes k$. The Hilbert function of $A \otimes k/I$ is usually defined to be the map $d \mapsto \dim_k(A \otimes k/I)_d$; however, one could equally well replace $\dim_k(A \otimes k/I)_d$ with $\dim_k(A \otimes k/I)_d^\vee$, an observation that will be relevant when we consider semifields. All homogeneous ideals defining Z have the same saturation, so the corresponding Hilbert functions coincide for $d \gg 0$ and this determines the Hilbert polynomial of $Z \subset \mathbb{P}^n_k$.

To define a tropical Hilbert function for a homogeneous congruence J on $A \otimes S$ one first needs an appropriate definition of the dimension of an S-module for S an idempotent semiring. We assume here S is a totally ordered semifield. The following definition is from [MZ08], in the case $S = \mathbb{T}$.

Definition 7.1.1. Let S be a totally ordered semifield and L an S-module.

(1) A collection $v_1, \dots, v_k \in L$ is *linearly dependent* if any linear combination of the v_i can be written as a linear combination of a proper subset of the v_i; otherwise it is *linearly independent*.
(2) The *dimension* of L, denoted $\dim_S L$, is the largest number d such that there exists a set of d linearly independent elements in L.

Lemma 7.1.2. *Let $\varphi : S \to T$ be a homomorphism of totally ordered idempotent semifields. If L is an S-submodule of a finitely generated free module then* $\dim_S L = \dim_T L \otimes_S T$.

Proof. It suffices to prove the result when $T = \mathbb{B}$ and φ is the unique homomorphism to $\mathbb{B}$, defined by sending all nonzero elements to $1_{\mathbb{B}}$. Moreover, since L is a submodule of a finitely generated free module, it suffices to show that a set $v_1, \dots, v_d \in S^n$ is linearly independent (in the sense of the above definition) if and only if the set $\varphi(v_1), \dots, \varphi(v_d)$ is linearly independent. Clearly if the v_i are S-linearly dependent then their images under φ are $\mathbb{B}$-linearly dependent. Conversely, suppose $\varphi(v_1), \dots, \varphi(v_d)$ are $\mathbb{B}$-linearly dependent, so that (without loss of generality)

$$\sum_{i=1}^{d} \varphi(v_i) = \sum_{i=1}^{d-1} \varphi(v_i).$$

This condition says that, for each k, if the k^{th} component of v_i vanishes for $i < d$ then it does so for v_d as well. Since S is a totally ordered semifield, given any notrivial elements $a, b \in S$, there exists $c \in S$ such that $ca \geq b$. Hence for each $i < d$, we may choose an $a_i \in S$ large enough so that each component of $a_i v_i$ is greater than or equal to the corresponding component of each v_d. By construction we then have

$$\sum_{i=1}^{d} a_i v_i = \sum_{i=1}^{d-1} a_i v_i,$$

which shows that the v_i are S-linearly dependent. □

Lemma 7.1.3. *If $L \subset S^n$ is a tropical linear space of rank r (in the sense of* [Fre13] *or* [SS04, Spe08]*) then* $\dim_S L = r$.

Proof. Let $\psi : S \to \mathbb{T}$ be any homomorphism (for example, one can take the unique homomorphism to $\mathbb{B}$ followed by the unique homomorphism $\mathbb{B} \hookrightarrow \mathbb{T}$). The base change $L \otimes_S \mathbb{T}$ is a tropical linear space of rank d in $\mathbb{T}^n$ (this can easily be seen in terms of the corresponding valuated matroids). By Lemma 7.1.2, $\dim_S L = \dim_{\mathbb{T}} L \otimes_S \mathbb{T}$ and by [MZ08, Proposition 2.5], $\dim_{\mathbb{T}} L \otimes_S \mathbb{T}$ is equal to the maximum of the local topological dimensions of the polyhedral set underlying $L \otimes_S \mathbb{T}$. The statement now follows from the fact that a tropical linear space in $\mathbb{T}^n$ is a polyhedral complex of pure dimension equal to its rank. □

Definition 7.1.4. Given a homogenous congruence J on $A \otimes S = S[x_0, \ldots, x_n]$, the *tropical Hilbert function* of J is the map $d \mapsto \dim_S (A \otimes S/J)_d^{\vee}$.

Two homogeneous congruences (cf. §5.4) define the same projective subscheme if and only if they coincide in all sufficiently large degrees; it follows that the Hilbert function of any tropical projective subscheme is well-defined for sufficiently large values of d. Since modules over a semiring do not form an abelian category, it does not appear automatic that the Hilbert function of an arbitrary tropical projective subscheme is eventually polynomial, but remarkably, this is the case for schemes in the image of the tropicalization functor.

Theorem 7.1.5. *Let $\nu : k \to S$ be a valued field. If $I \subset A \otimes k$ is a homogenous ideal then the Hilbert function of I coincides with the tropical Hilbert function of $\mathcal{T}rop(I)$. Consequently, for any subscheme $Z \subset \mathbb{P}^n_k$, the tropical subscheme $\mathcal{T}rop^{\nu}_{\mathbb{P}^n}(Z) \subset \mathbb{P}^n_S$ has a well-defined Hilbert polynomial and it coincides with that of Z.*

Proof. Linear tropicalization commutes with restriction to the degree d graded piece, so

$$(A \otimes S/\mathcal{T}rop(I))_d = A_d \otimes S/\mathcal{T}rop(I_d).$$

By Propositions 5.1.3 and 6.1.5, the dual, $(A_d \otimes S/\mathcal{T}rop(I_d))^{\vee}$ is the tropical linear space in $(A_d \otimes S)^{\vee}$ that is the tropicalization of the linear subspace $(A_d \otimes k/I_d)^{\vee} \subset (A_d \otimes k)^{\vee}$. Since the tropicalization of a subspace of dimension r is a rank r tropical linear space, the statement that tropicalization preserves the Hilbert function now follows from Lemma 7.1.3. The statement about the Hilbert polynomials then follows since, by Theorem 6.6.1, $\mathcal{T}rop^{\nu}_{\mathbb{P}^n}(Z)$ is defined by the homogeneous congruence $\mathcal{T}rop(I)$.

□

Recall that classically a family of projective subschemes is flat if and only if the Hilbert polynomials of the fibres are all equal. The above result therefore suggests that if one views tropicalization as some kind of degeneration of complex structures on a variety, then the numerical behavior is that of a flat degeneration. Moreover, this next result (Theorem C part (2)) shows that the family of all tropicalizations of a projective subscheme Z has the numerical behaviour of a flat family.

Corollary 7.1.6. *For S a totally ordered idempotent semifield, the Hilbert polynomial of the fibre of the family $\mathcal{T}rop^{\nu^k_{univ}}(Z) \to \mathcal{V}al(k)$ over any S-point is equal to the Hilbert polynomial of Z.*

Proof. This follows directly from Theorems 6.5.1 and 7.1.5 since the Hilbert polynomials of the fibres are all equal to the Hilbert polynomial of Z. □

7.1.1. *Example: points on the line.* Assume $k = \overline{k}$ is algebraically closed and recall that the Hilbert scheme of m points on the projective line is $\mathbb{P}^m_k = \mathbb{P}\Gamma(\mathbb{P}^1_k, \mathcal{O}(m))$. A k-point of this scheme is a degree m binary form and the associated hypersurface in $\mathbb{P}^1_k$ is a zero-dimensional length m scheme. Although it is not clear how much of the theory of Hilbert schemes carries over to the tropical setting, one can at least consider the set $\mathcal{H}$ of subschemes of $\mathbb{P}^1_{\mathbb{T}}$ with constant Hilbert polynomial m. Theorem 7.1.5 implies that tropicalization yields a map $\varphi : \mathbb{P}^m_k(k) \to \mathcal{H}$. However, φ is not surjective, since there are many tropical schemes that lie outside the image of the tropicalization functor. For instance, consider the family

$$Z := \operatorname{Proj} \mathbb{T}[x,y,t]/\langle x^2 \sim x^2 + txy, y^2 \sim -\infty\rangle \subset \mathbb{P}^1_{\mathbb{T}} \times \mathbb{A}^1_{\mathbb{T}} \to \mathbb{A}^1_{\mathbb{T}},$$

where t is the parameter on $\mathbb{A}^1_{\mathbb{T}}$.

Proposition 7.1.7. *The fiber Z_t over $t \in \mathbb{T}$ has Hilbert polynomial* 2*, yet it is in the image of φ (i.e., the tropicalization of a length 2 subscheme of $\mathbb{P}^1_k$) if and only if $t = -\infty$.*

Proof. Since $Z_{-\infty} = \operatorname{Proj} \mathbb{T}[x,y]/\langle y^2 \sim -\infty\rangle$, this is the tropicalization of a double point at the origin of $\mathbb{P}^1_k$. On the other hand, it follows from the proof of Theorem 7.1.5 that if Z_t were a tropicalization then the dual $\mathbb{T}$-module of each graded piece of the homogeneous coordinate semiring would be a tropical linear subspace, and we claim this is not the case for any $t \in \mathbb{R}$. The degree 2 piece is a quotient of the rank 3 free $\mathbb{T}$-module $\mathbb{T}x^2 \oplus \mathbb{T}xy \oplus \mathbb{T}y^2$. If we set $X := x^2, Y := y^2, Z := xy$, then the dual of this quotient is a $\mathbb{T}^\times$-invariant subset of $\mathbb{A}^3_{\mathbb{T}}$ which on the affine chart $\{Z = 0\} = \mathbb{A}^2_{\mathbb{T}}$ is defined by the equations $X = X + t$ and $Y = -\infty$. For any $t \neq -\infty$ this is a line segment and hence not a tropical linear space. That the Hilbert function stabilizes at 2 for any $t \in \mathbb{T}$ is obvious. □

7.2. **Recovering the multiplicities and defining polynomial of a tropical hypersurface.**

Proposition 7.2.1. *For any valued ring $\nu : R \to S$ such that S is a semifield, and any projective hypersurface $Z = V(f) \subset \mathbb{P}^n_R$, the tropical scheme $\mathcal{T}rop(Z) \subset \mathbb{P}^n_S$ determines the defining homogeneous polynomial $\nu(f) \in (A \otimes S)_d$ uniquely up to scalar.*

Proof. Since $\mathcal{T}rop(Z) = \operatorname{Proj} A \otimes S/\langle \mathcal{B}(\nu(g))\rangle_{g \in (f)}$, and this homogeneous congruence in degree d coincides with the congruence $\mathcal{B}(\nu(f))$, the result follows from Lemma 5.1.4. □

Corollary 7.2.2. *For an algebraically closed valued field $\nu : k \to \mathbb{T}$, and an irreducible projective hypersurface $Z \subset \mathbb{P}^n_k$, the scheme $\mathcal{T}rop(Z) \subset \mathbb{P}^n_{\mathbb{T}}$ determines the multiplicities on the facets of its $\mathbb{T}$-points.*

Proof. This follows immediately from Proposition 7.2.1, since the multiplicities for a tropical hypersurface are lattice lengths in the Newton polytope of f [DFS07, §2]. □

Remark 7.2.3. It would be interesting to see whether the multiplicities are determined by the tropical scheme structure for arbitrary codimension irreducible subvarieties.

8. Hypersurfaces and tropical bases

We have seen that associated to a tropical polynomial is a bend locus scheme whose $\mathbb{T}$-points are what have traditionally been referred to as a tropical hypersurface. On the other hand, given a polynomial with coefficients in a valued ring there is a classical hypersurface which can be tropicalized to produce another geometric object that could justifiably be referred to as a tropical hypersurface. In this section we explain how these two notions are only compatible in special situations. The discrepancy can be understood in terms of Theorem 7.1.5: the tropicalization of a projective hypersurface must have enough relations in its homogeneous coordinate algebra to yield the Hilbert polynomial of a codimension one subscheme, but the bend relations of a single tropical polynomial do not typically suffice for this numerical constraint.

This discussion leads naturally to the notion of a scheme-theoretic tropical basis, a term we introduce as a replacement for the usual set-theoretic notion considered in the tropical literature.

8.1. **Bend loci versus tropical hypersurfaces.** To illustrate concretely that the tropicalization of a hypersurface $V(f)$ may need more relations than those provided by the bend relations of the coefficient-wise valuation $\nu(f)$, consider the following example.

Example 8.1.1. Let $f = x^2 + xy + y^2 \in k[x,y]$, where $\nu : k \to \mathbb{T}$ is a valued field. One can see as follows that the tropicalization of the principal ideal generated by f, denoted $\mathcal{T}rop(f)$, is a congruence that is strictly larger than the semiring congruence generated by $\mathcal{B}(\nu(f))$. This latter congruence is generated by the degree 2 relations $x^2 + y^2 \sim x^2 + xy \sim xy + y^2$. The degree 3 part of $\langle \mathcal{B}(\nu(f)) \rangle$ is generated (as a module congruence) by the relations $\mathcal{B}(x^3 + x^2y + xy^2)$ and $\mathcal{B}(x^2y + xy^2 + y^3)$. Since arithmetic in $\mathbb{T}[x,y]$ can only enlarge the support of a polynomial, we see that any nontrivial degree 3 relation in $\langle \mathcal{B}(\nu(f)) \rangle$ involves only polynomials with at least 2 terms. However, $(x-y)f = x^3 - y^3$, and this gives the degree 3 monomial relation $x^3 \sim y^3$ in $\mathcal{T}rop(f)$.

The behaviour above, where $\mathcal{B}(\nu(f))$ does not generate all the relations in the tropicalization of the principal ideal generated by f, appears to be generic. Suppose now that $f = x^2 + xy + ty^2$ for some $t \neq 0, 1$. The degree 3 part of $\langle \mathcal{B}(\nu(f)) \rangle$ is generated as a module congruence by the bend relations of $\nu(xf) = x^3 + x^2y + \nu(t)xy^2$ and $\nu(yf) = x^2y + xy^2 + \nu(t)y^3$. However, in $\mathcal{T}rop(f)$ one also has the bend relations of $\nu((x-ty)f) = x^3 + \nu(1-t)x^2y + \nu(t)^2y^3$; among these is the relation

$$x^3 + \nu(t)^2y^3 \sim x^3 + \nu(1-t)x^2y$$

which cannot be obtained from $\mathcal{B}(\nu(xf))$ and $\mathcal{B}(\nu(yf))$. In fact, one can check that these relations now generate all relations in the degree 3 part of $\mathcal{T}rop(f)$.

In general, when passing from $\langle \mathcal{B}(\nu(f)) \rangle$ to $\mathcal{T}rop(f)$, the additional relations appearing in $\mathcal{T}rop(f)$ are not uniquely determined by the single tropical polynomial $\nu(f)$, so the tropicalization of a hypersurface is not uniquely determined by the bend locus of the valuation of a defining polynomial. The following is a simple example illustrating this: two polynomials with the same valuation but whose associated hypersurfaces have unequal tropicalizations.

Example 8.1.2. Let $k = \mathbb{C}$ with the trivial valuation $\nu : k \to \mathbb{B}$, and consider the polynomials in $\mathbb{C}[x,y]$,

$$f = a_1x^2 + a_2xy + a_3y^2, \text{ and } g = x^2 + xy + y^2,$$

where the coefficients in f do not satisfy the quadratic relation $a_2^2 = a_1a_3$. Clearly $\nu(f) = \nu(g)$, and as seen in Example 8.1.1, $\mathcal{T}rop(g)$ contains the relation $x^3 \sim y^3$. However, for any nonzero linear form $h = b_1x + b_2y \in \mathbb{C}[x,y]$ the polynomial fh has at least three terms, so $\mathcal{T}rop(f)$ cannot contain the relation $x^3 \sim y^3$.

There are, however, certain nice situations where the tropicalization of an ideal is equal to the intersection of the bend loci of a set of generators of the ideal.

Proposition 8.1.3. *Let M be a torsion-free integral monoid-with-zero, and suppose S is totally ordered. If $f = ax + by$ is a binomial ($a, b \in R$, and $x, y \in M$) then $\mathcal{T}rop(f) = \langle \mathcal{B}(\nu(f)) \rangle$.*

Proof. We must show that $\langle \mathcal{B}(\nu(f)) \rangle$ implies $\langle \mathcal{B}(\nu(fg)) \rangle$ for any $g \in M \otimes R$. Since f is a binomial, $\langle \mathcal{B}(\nu(f)) \rangle$ is generated by the single relation $\nu(a)x \sim \nu(b)y$.

We define a binary relation $\to$ on $\mathrm{supp}(g)$ as follows. $z_1 \to z_2$ if $z_1x = z_2y$. This generates an equivalence relation; let $\{C_i\}$ be the set of equivalence classes. Note that $C_ix \cup C_iy$ is necessarily disjoint from $C_jx \cup C_jy$ if $i \neq j$. Hence we can, without loss of generality, assume that $\mathrm{supp}(g)$ consists of just a single equivalence class C. If C consists of a single element then the claim holds trivially, so we assume that C consists of at least 2 elements.

Since M is integral and torsion free, C must consist of a sequence of elements $z_1, \ldots, z_n$ such that $z_ix = z_{i+1}y$ (having a loop would imply that xy^{-1} is a torsion element in the group completion of M, and the integral condition implies that if $x \to y$ and $x \to y'$ then $y = y'$).

Let c_i be the coefficient of z_i in g. We then have

$$\nu(fg) = \nu(ac_n)z_nx + \nu(ac_{n-1} + bc_n)z_{n-1}x + \cdots + \nu(ac_1 + bc_2)z_1x + \nu(bc_1)z_1y.$$

We first show that the relation $\nu(a)x \sim \nu(b)y$ allows the first term, $\nu(ac_n)z_nx$, to be absorbed into one of the terms to its right. First,

$$\nu(ac_n)z_nx \sim \nu(bc_n)z_ny = \nu(bc_n)z_{n-1}x.$$

Either $\nu(ac_{n-1} + bc_n) = \nu(ac_{n-1}) + \nu(bc_n)$, in which case we are done, or $\nu(ac_{n-1}) = \nu(bc_n)$, in which case $\nu(bc_n)z_{n-1}x = \nu(ac_{n-1})z_{n-1}x \sim \nu(bc_{n-1})z_{n-2}x$. We continue in this fashion until the term absorbs or we reach the end of the chain, at which point it will be absorbed into the final term $\nu(bc_1)z_1y$. Working from right to left instead, the final term can be absorbed into the terms to its left by the same argument.

Finally, given a middle term, $\nu(ac_{i-1} + bc_i)z_{i-1}x$, we have that $\nu(ac_{i-1})z_{i-1}x$ and $\nu(bc_i)z_{i-1}x$ are both larger, and so the above argument in reverse allows us to replace the term $\nu(ac_{i-1} + bc_i)z_{i-1}x$ with $\nu(ac_{i-1})z_{i-1}x + \nu(bc_i)z_{i-1}x$. Then the above argument in the forward direction allows these two terms to be absorbed into the terms to the right and left respectively. □

Lemma 8.1.4. *Suppose I is an ideal generated by elements $f_1, \ldots, f_n$. If $\mathcal{Trop}(I) = \langle \mathcal{B}(f_i)\rangle_{i=1\ldots n}$, and J is the ideal generated by I and a monomial f_0, then $\mathcal{Trop}(J) = \langle \mathcal{B}(f_i)\rangle_{i=0\ldots n}$.*

Proof. We will show that the generating relations of $\mathcal{Trop}(J)$ are all contained in the sub-congruence $\langle \mathcal{B}(\nu(f_i))\rangle_{i=0\ldots n}$. Let $g = \sum_{i=0}^n h_i f_i \in J$, with $h_i \in M \otimes R$. Since $\langle \mathcal{B}(f_0)\rangle = \langle f_0 \sim 0_S\rangle$, for any $F \in M \otimes S$, the congruence $\langle \mathcal{B}(f_0)\rangle$ contains the relation $F \cdot f_0 \sim 0_S$. This means that if $F_1, F_2 \in M \otimes S$ are identical outside of $f_0 \cdot M$, then the relation $F_1 \sim F_2$ is contained in $\langle \mathcal{B}(f_0)\rangle$.

The tropical polynomials $F := \nu(g)$ and $F' := \nu(\sum_{i=1}^n h_i f_i)$ dffer only outside of $f_0 \cdot M$, as do $F_{\hat{j}}$ and $F'_{\hat{j}}$. In $\langle \mathcal{B}(\nu(f_i))\rangle_{i=0\ldots n}$ we thus have the relations

$$\begin{aligned} F &\sim F' && \text{from } \mathcal{B}(f_0) \\ &\sim F'_{\hat{j}} && \text{from } \mathcal{Trop}(I) = \langle \mathcal{B}(\nu(f_i))\rangle_{f=1\ldots n} \\ &\sim F_{\hat{j}} && \text{from } \mathcal{B}(f_0), \end{aligned}$$

This completes the proof. □

If f_0 is instead a binomial then the analogue of the above lemma can fail.

Example 8.1.5. Consider $f_1 = x - y$ and $f_0 = x + y$ in $R[x,y]$ and the trivial valuation $\nu : R \to \mathbb{B}$. By Proposition 8.1.3, $\mathcal{Trop}(f_1) = \langle \mathcal{B}(f_1)\rangle$. However, $\langle \mathcal{B}(\nu(f_0)), \mathcal{B}(\nu(f_1))\rangle = \langle x \sim y\rangle$ is not the tropicalization of the ideal (f_0, f_1), since the latter contains the bend relation of $\nu(f_0 + f_1) = x$, namely $\langle x \sim -\infty\rangle$, which is not implied by the former.

8.2. **Tropical bases.** It is well-known that the set-theoretic tropical variety associated to an ideal is not necessarily equal to the set-theoretic intersection of the tropical hypersurfaces associated with generators of this ideal. A set of generators for which this holds is called a *tropical basis* in [MS], where this notion is studied and related to Gröbner theory. We use the term *set-theoretic tropical basis* for this concept to distinguish it from the following notion of tropical basis that arises when considering scheme-theoretic tropicalization.

Definition 8.2.1. Let $\nu : R \to S$ be a valued ring, X an integral $\mathbb{F}_1$-scheme, and $Z \subset X_R$ a closed subscheme. A *scheme-theoretic tropical basis* for Z is a set $\beta = \{Y_1, Y_2, \ldots\}$ of hypersurfaces in X_R containing Z such that the following scheme-theoretic intersections hold:

$$Z = \bigcap_i Y_i \quad \text{and} \quad \mathcal{Trop}(Z) = \bigcap_i \mathcal{Trop}(Y_i).$$

In the affine case, say $X = \operatorname{Spec} M$ and $Z = \operatorname{Spec} M \otimes R/I$, a scheme-theoretic tropical basis is a generating set $\{f_1, f_2, \ldots\}$ for the ideal I such that the corresponding congruences $\mathcal{T}rop(f_i)$, obtained by tropicalizing the principal ideals (f_i), generate the congruence $\mathcal{T}rop(I)$. Note that this is generally a weaker requirement than the requirement that the bend relations of the f_i generate $\mathcal{T}rop(I)$. For instance, for a principal ideal $I = (f)$ it is automatic that $\{f\}$ is a tropical basis, whereas it is not always the case, as discussed above, that $\mathcal{T}rop(I) = \langle \mathcal{B}(f) \rangle$.

Not surprisingly, being a scheme-theoretic tropical basis is a stronger requirement than being a set-theoretic tropical basis.

Example 8.2.2. Let $R = k[x,y,z]$ with the trivial valuation. As discussed in [MS, Example 3.2.2], the elements $x+y+z$ and $x+2y$ do not form a tropical basis for the ideal I they generate, since $y-z \in I$ tropically yields the relation $y \sim z$ which is not contained in $\langle \mathcal{B}(x+y+z), \mathcal{B}(x+y) \rangle$. This can be rectified by adding the element $y-z$, and indeed these three polynomials form a set-theoretic tropical basis for I. However, if we instead add the element $(y-z)^2 \in I$ then the corresponding congruence has the same $\mathbb{T}$-points, so this is still a set-theoretic tropical basis, but it is no longer a scheme-theoretic tropical basis since the relation $y \sim z$ is still missing.

Remark 8.2.3. It is known that subvarieties of affine space defined over an algebraically closed field with non-trivial valuation admit *finite* set-theoretic tropical bases (see [SS04, Corollary 2.3] and [MS, Corollary 3.2.3]). It would be interesting to see if this also holds scheme-theoretically.

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